Invasion Percolation and Eden Growth: Geometry and Universality

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The mapping of optimal paths in the strong disorder limit to the strands of invasion percolation clusters is shown to lead to a new universal property of these clusters. We suggest that the corresponding strands arising in the annealed Eden growth process are in the same universality class as directed polymers in weak quenched disorder with an upper critical dimension ≤ 6. [S0031-9007(96)00188-3]

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The dynamics and the resulting geometries of nonequilibrium growth phenomena have been a subject of much recent study. It is now recognized that the simplest nonlinear continuum growth equation that captures the physics of ballistic deposition, flux lines in superconductors, as well as Eden growth is that due to Kardar, Parisi, and Zhang (KPZ) [1,2]. The KPZ equation in $1 + 1$ dimensions has an exact solution. The exponent describing the power-law dependence of the saturation width of the rough interface on the lateral system size $\alpha_{\mathrm{KPZ}} = \frac{1}{2}$, the exponent characterizing the temporal growth of the roughness $\beta_{\mathrm{KPZ}} = \frac{1}{3}$, and the dynamical exponent of the saturation time as a function of the lateral size $z_{\mathrm{KPZ}} = \alpha/\beta = 3/2$ [2]. Strikingly, the KPZ equation with *annealed* noise (uncorrelated in space and time and random) can be mapped to the problem of a directed polymer (DP) in a random medium [3] or equivalently, in two dimensions ($D = 2$), the pinned domain wall problem in random exchange ferromagnets [4]. Both these situations correspond to *quenched* randomness arising from a fixed (in time) disordered environment. Yet the DP has a self-affine geometry with the exponent characterizing the end-to-end displacement α_{DP} equal to $1/z_{\mathrm{KPZ}}$ [2,4]. The upper critical dimension of the KPZ equation is believed [5] to be 5 (i.e., $4 + 1$).

The geometry of an undirected polymer (path) at zero temperature in a strongly disordered medium has been recently considered [6] in $D = 2$. As before, the bonds of the medium are quenched random variables. All possible configurations of the polymer are considered that start and end at given sites (the length of the polymer is not fixed)—the optimal one is that with the lowest cost. The cost of a particular configuration is assumed simply to be equal to the largest bond within it. In the case of a tie, the next largest bond is used as a tiebreaker and so on. Strikingly, a new universality class was found in this case: The polymer is no longer self-affine but is self-similar with a fractal dimension, $D_f \approx 1.2$ in 2D [6].

We address several issues in this Letter. What are the geometries of the optimal polymer in a strongly disordered medium in higher dimensions? What is the upper critical dimensionality for this problem? Is the geometry of the polymer in a strongly disordered medium universal? Our study is carried out in the context of a growing invasion percolation cluster. In this procedure, the bonds of the lattice are assigned strengths in a quenched random manner, and a cluster grows by invading the weakest interfacial bond. We will show that both bond and site variants of percolation lead to the same universality class. We then go on to study an analogous model with *annealed* instead of *quenched* disorder. In this case all interfacial bonds have an equal probability of being invaded. We present arguments and numerical evidence that even though the randomness is annealed, the effects of quenched disorder are *self-generated* within the model leading to geometries that are self-affine and characterized by the roughness exponent α_{DP}. Thus, within the same process, the interface of the Eden cluster is characterized by a dynamical exponent z_{KPZ}, whereas the static wandering exponent of the strands of the cluster is given by $1/z_{\mathrm{KPZ}}$ (a strand is defined as the unique path that excludes dead ends from an arbitrary site to a central seed site). Our results have a wide range of applicability—the strong disorder limit is relevant up to a correlation length in a variety of situations [7] including transport in amorphous semiconductors at low temperatures, electrical conduction and fluid flow in porous rocks, and the magnetic properties of doped semiconductors. Further, there are novel forms of percolation that are equivalent to the problem of the optimal polymer in a strongly disordered environment [6]. Our prediction of the self-generated quenched randomness ought to be observable in Eden growth and other random invasion processes.

We begin with an alternative way to view the geometry of the polymer in a strongly disordered environment. We

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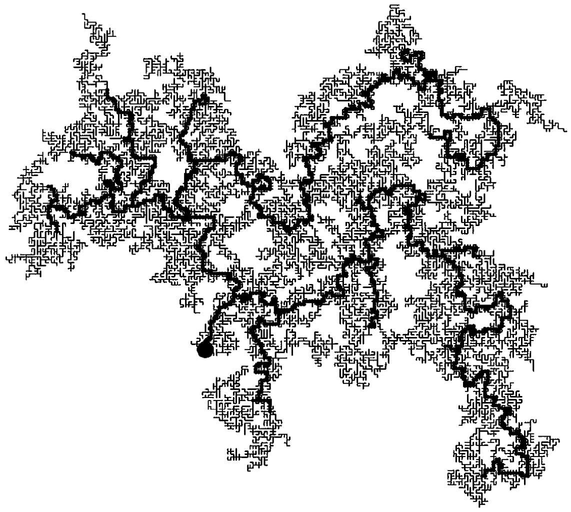

FIG. 1. A cluster generated in a 2D invasion percolation process on a square lattice with a central seed site, indicated by a larger circle. Here, the growth has been stopped when the maximal horizontal distance reached is equal to 128 lattice constants. The bonds picked during the selection of growth sites are indicated. They form a loopless network of strands. The circles indicate sites with a significant overlap of the individual strands.

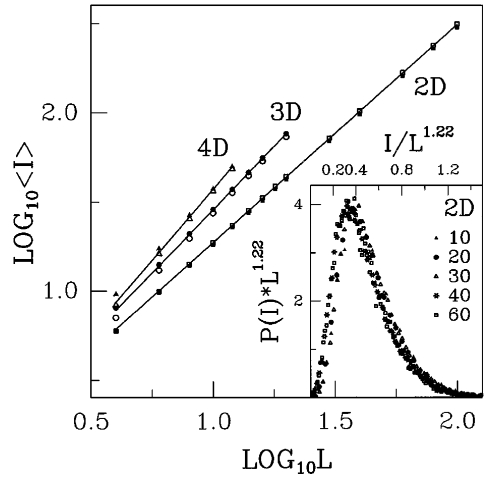

FIG. 2. Average length $\langle I \rangle$ of the optimal polymers (paths) in the strong disorder limit versus the spanning distance L. The dimensionalities of the systems are indicated. The solid and open symbols are for the bond and site disorder problems, respectively. The data points for the site problem have been multiplied by 5/4 so that they almost coincide with the corresponding data points for the bond problem. The slopes indicated correspond to D_f shown in Table I. The number of samples is as follows: (a) square lattice—20 000 for L up to 40 and at least 3000 for larger L's; (b) cubic lattice—20 000 for L up to 10 and at least 5000 for larger L's; (c) 4D hypercubic—between 2000 and 40 000. The slopes shown in Table I are averaged between the site and bond problems. The average transverse displacement scales linearly with L. The inset shows a scaling plot of the distribution of the path lengths, $P(I)$, for the 2D bond problem for the values of L indicated in the inset.

assume that random numbers are assigned in a quenched, random manner to the bonds of a lattice. We now describe an invasion percolation process [8] starting from a central seed site. We consider all possible bonds that the invasion can take place into and pick the bond with, say, the smallest random number assigned to it. The same procedure is used with the enlarged set of interfacial bonds with the invasion proceeding only to previously uninvaded sites. This procedure avoids loops, and the resulting structure is a spanning tree that is the union of all the optimal polymer configurations from the central seed site to each of the other sites on the lattice [9]. Note that when two polymers intersect they overlap the rest of the way to the central site (Fig. 1). Since the upper critical dimension of invasion percolation is 6 [10], this mapping allows us to deduce that the upper critical dimension of our problem is also 6—above $D = 6$, the optimal polymer has a fractal dimension of 2 corresponding to that of an uncorrelated random walk.

We have carried out detailed numerical studies to determine D_f in two, three, and four dimensions on a hypercubic lattice. Figure 2 shows a plot of the mean polymer length versus the distance spanned (L) for both bond and site percolation. The bond and site percolation exponents are consistent with each other in accord with universality, and we find the approximate result $D_f = (D + 4)/5$ for the dimensions studied. The precise values of the numerically determined exponents are shown in Table I. We note that the geometry of the strands of an invasion percolation cluster is a new universal attribute [11] of these clusters and should be experimentally accessible.

We now turn to the annealed version of the invasion percolation problem. We proceed exactly as before except that all interfacial bonds have an equal probability of being invaded again leading to a spanning tree. This procedure is the same as the Eden growth problem that has been well studied in the biological context of the formation of cell colonies such as tissue cultures or bacterial growth. The scaling properties of the interface of the Eden cluster is described by the annealed KPZ equation. The growth process results in each occupied site having a unique path to the seed site with the whole cluster being a union of such paths. A 2D example of the Eden network of paths is shown in Fig. 3.

We have carried out detailed studies of the geometry of the Eden cluster network. We have monitored the length and transverse displacement of the strand that first reaches

TABLE I. Summary of the numerical results obtained in this work on hypercubic lattices.

D	α—Eden strands	D_f—Invasion percolation strands
2	0.66 ± 0.01	1.22 ± 0.01
3	0.62 ± 0.02	1.42 ± 0.02
4	0.59 ± 0.02	1.59 ± 0.02

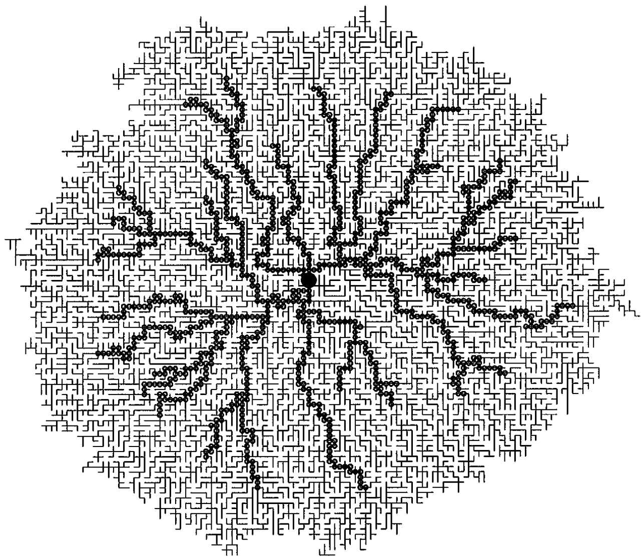

FIG. 3. Same as Fig. 1 but for an Eden growth process stopped when the maximal horizontal distance reached is equal to 64 lattice constants.

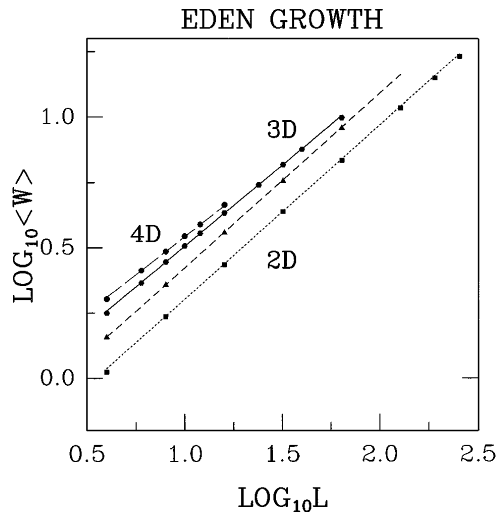

FIG. 4. The transverse end-to-center distance of the most forward paths, generated in the Eden growth in a spherical geometry on L-dimensional lattices, as a function of the longitudinal distance, L. The square, triangle, hexagon, and circular symbols correspond to the square, triangular, cubic, and 4D hypercubic lattices, respectively. The slopes indicated correspond to those shown in Table I. The number of samples is as follows: (a) square lattice—42 000 for L between 4 and 128, 12 000 for $L = 192$ and 256; (b) triangular lattice—27 000; (c) cubic lattice—more than 40 000 for L between 4 and 32, 5000 for $L = 40$, and 3000 for $L = 64$; (d) 4D hypercubic—more than 25 000 for L between 4 and 12, and 11 000 for $L = 16$. The average length of the paths scales linearly with L.

a distance L from the seed along, say, the x direction. The results for the transverse displacement in D-dimensional systems presented in Fig. 4 and summarized in Table I show that the Eden paths are *self-affine.* In particular, in the 2D case, for both the *square* and *triangular* lattices, our results are consistent with $\alpha = \alpha_{\rm DP} = \frac{2}{3}$. On increasing D, α decreases but apparently remains in the DP universality class [2]. This is a quite unexpected and striking result, since the disorder is annealed and not quenched.

In addition to the spherical geometry, we have also studied Eden growth on 1D and 2D substrates with each substrate site acting as a seed. Periodic boundary conditions are imposed in the directions parallel to the substrate. We have determined the transverse displacement, away from a mother seed, for *all* sites in the Eden cluster which are at a vertical distance L from the substrate. The results are then averaged over the sites and growth processes. The resulting geometries are consistent with those arising in spherical growth.

Physically the growing Eden cluster can be thought of as interacting random walks not only growing at the tip, but sprouting out from possibly all of the previously occupied sites. The blocking of the possible new growth sites from the already occupied sites effectively creates a quenched random environment leading to the remarkable coincidence of the geometry with that of the directed polymer in a quenched random environment [12].

Since the invasion percolation case corresponds to quenched disorder, one might expect that the transverse wandering of its strand must be greater than that for the Eden strand of the same total length. (The interaction between the strands is a complicating factor.) Likewise, the transverse wandering of the DP in a quenched random medium ought to be larger than that of a random walk with the same number of steps. These two observations, along with the conjecture [12] that the Eden strand is in the same universality class as a DP in a quenched random environment, lead to $1/D_f \geq \alpha_{\rm DP} \geq \frac{1}{2}$. Since D_f becomes equal to 2 for $D \geq 6$, we deduce that the upper critical dimensionality of the DP problem [5] (and hence the KPZ equation) is ≤ 6 $(= 5 + 1)$.

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[10] See, e.g., D. Stauffer and A. Aharony, *Introduction to Percolation Theory* (Taylor and Francis, London, 1992), 2nd ed.
[11] The geometry of the optimal polymer is distinct from that of the shortest geometrical path on a percolating backbone characterized by a minimum distance $d_{\min}$. [S. Havlin, L. A. N. Amaral, S. V. Buldyrev, S. T. Harrington, and H. E. Stanley, Phys. Rev. Lett. **74**, 4205 (1995)]. When calculating $d_{\min}$, the longer leg of any loop in a percolation backbone is eliminated, whereas in the optimal path either the shorter or the longer leg may carry a smaller overall cost [6]. Indeed, we find numerically that D_f for the optimal path in the strong disorder limit is larger than the exponent characterizing the dependence of $d_{\min}$ on L.
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