Inverse and Implicit Functions in Domain Theory

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Abstract

We construct a domain-theoretic calculus for Lipschitz and differentiable functions, which includes addition, subtraction and composition. We then develop a domain-theoretic version of the inverse function theorem for a Lipschitz function, in which the inverse function is obtained as a fixed point of a Scott continuous functional and is approximated by step functions. In the case of a C^1 function, the inverse and its derivative are obtained as the least fixed point of a single Scott continuous functional on the domain of differentiable functions and are approximated by two sequences of step functions, which are effectively computed from two increasing sequences of step functions respectively converging to the original function and its derivative. In this case, we also effectively obtain an increasing sequence of polynomial step functions whose lower and upper bounds converge in the C^1 norm to the inverse function. A similar result holds for implicit functions, which combined with the domain-theoretic model for computational geometry, provides a robust technique for construction of curves and surfaces.

1. Introduction

The earliest systematic method of computing increasingly better lower and upper bounds to π is credited to Archimedes, who obtained his approximation by circumscribing and inscribing the circle with regular polygons having 96 sides. In the 15th century, Jamshid Kashani, the Iranian mathematician, further developed this recursive method by using regular polygons with 805306368 sides, obtaining 16 decimal digits of π with a correct estimation of the error [1]. In today's mathematical language, we say that the two sequences of refining n-sided regular polygons circumscribing and inscribing the unit circle for $n = 3, 4, 5, \ldots$, as in Figure 1, converge in the C^1 norm respectively from outside and inside to the circle, a 1-dimensional closed manifold given by the implicit curve $x^2 + y^2 - 1 = 0$. This well-known and ancient technique describes the essence of our work in a simple and accessible way. The present paper aims to develop a systematic and recursive method of approximating an implicitly given surface with two sequences of piecewise linear or piecewise polynomial surfaces, which converge locally in the C^1 norm from inside and outside to the implicit surface. A main application of this method is in geometric modelling and computer aided design, where curves and surfaces are usually defined implicitly [2]. Currently, there are no robust methods to approximate an implicit surface and the most reliable technique provided by interval analysis [12] is only able to approximate the implicit surface without approximating its derivative. The paper thus presents a framework for a robust CAD system, where implicitly given surfaces can be effectively obtained up to the C^1 precision required by the user. In more mathematical language, we will use domain theory to provide a recursion-theoretic account of the inverse function and the implicit function theorems, which are the main fundamental tools in multi-variable differential calculus and the theory of manifolds. In [7], a domain-theoretic framework for differential calculus of one variable was developed which in particular provides an effectively given domain for Lipschitz or differentiable functions. Later on, domain-theoretic techniques for solving initial value problems were obtained in [5, 9], which enable us to approximate the unique solution of an initial value problem given by a Lipschitz vector field up to the precision required by the user. In [8], the domain-theoretic model was extended to multi-variable differential calculus, resulting in the no-

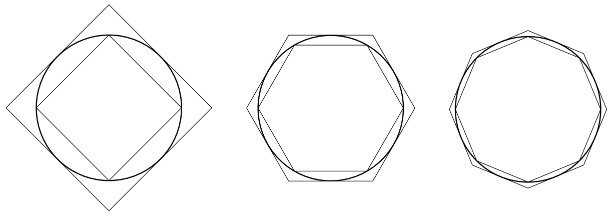

Figure 1. Approximations of the unit circle

tion of a domain-theoretic derivative, which for Lipschitz functions gives the smallest hyper-rectangle containing the Clark gradient [3], as well as a domain for multi-variable differentiable functions. In this paper, we will illustrate the first main application of the domain of multi-variable differentiable functions, by using it to obtain the inverse and implicit functions of classical analysis as the fixed point of a functional that we construct on the domain of Lipschitz and differentiable functions. We first obtain, as a fixed point, the inverse of a function which differs from the identity map by a contraction. Then, based on the domain-theoretic derivative, we introduce the notion of mean differential and use it to solve the problem of inverse function for Lipschitz in a more general setting. The domain-theoretic inverse and implicit function theorems for C^1 functions have the following distinguished features compared to other approaches:

(i) The inverse and implicit functions are obtained as fixed points of Scott continuous functionals.

(ii) From two increasing sequences of linear step functions converging to the function and its derivative, we can effectively obtain two increasing sequences of linear and polynomial step functions converging to the inverse of the function and its derivative. Similarly for the implicit function.

(iii) From two increasing sequences of linear step functions converging to the function and its derivative, we can effectively obtain an increasing sequences of polynomial step functions converging in the C^1 norm to the inverse of the function. Similarly, for the implicit function.

1.1 Related work

We have already mentioned the work on interval analysis regarding implicit surfaces; it gives approximations to the surface by voxel sets but there is no approximation of the derivative of the surface [12]. We here state the classical theorem on inverse functions for Lipschitz maps of Euclidean spaces [11, p. 108], which has an elementary but non-constructive proof, in the version of [10, Theorem 3.1].

For $x \in \mathbb{R}^n$ and any norm $\|\cdot\|$ on $\mathbb{R}^n$, we write $B_r = \{y \in \mathbb{R}^n \mid \|y\| < r\}$ for the open ball around the origin with radius r and denote the closed ball by $\overline{B}_r = \{y \in \mathbb{R}^n \mid \|y\| \leq r\}$.

Theorem 1.1 Inverse Function Theorem for Lipschitz maps *[10] Let $\overline{B}_r$ be a closed ball containing the origin in $\mathbb{R}^n$ and let $f : \overline{B}_r \to \mathbb{R}^n$ with $f(0) = 0$, so that for some invertible linear map $L : \mathbb{R}^n \to \mathbb{R}^n$ and some $\rho < 1$*

$$\|L^{-1}f(x_2) - L^{-1}f(x_1) - (x_2 - x_1)\| \leq \rho\|x_2 - x_1\|$$

holds for all $x_1, x_2 \in \overline{B}_r$. Then, for all $x_1, x_2 \in \overline{B}_r$

$$\frac{1-\rho}{\|L^{-1}\|}\|x_2 - x_1\| \leq \|f(x_2) - f(x_1)\|$$

$$\leq \|L\|(1+\rho)\|x_2 - x_1\|,$$

i.e. f is injective on $\overline{B}_r$. The restriction $f|_{B_r} : B_r \to \mathbb{R}^n$ of f to the open ball B_r is an open map so that $V := f[B_r]$ is open. Thus $f|_{B_r} : B_r \to \mathbb{R}^n$ is Lipschitz with Lipschitz inverse and V contains the open ball B_s where $s = \frac{\rho(1-\rho)}{\|L^{-1}\|}$.

Note that the theorem requires the existence of a linear map satisfying the above inequalities. We will actually see in the domain-theoretic Inverse Function Theorem 5.2 that, using the mean differential of the domain-theoretic derivative, we are able to compute a linear map L with the above property. A more general, but still non-constructive result, for the existence of an inverse of a Lipschitz function follows by using the Clarke gradient [3, p. 253], which is non-elementary. In Bishop's framework of constructive analysis, a constructive proof of the existence of the inverse function (or the implicit function) for a C^1 function is obtained by approximation but no approximations to the derivative of the inverse in provided [4]. In none of these approaches, the inverse or the implicit function is obtained as a fixed point of a functional.

2. Preliminaries

We briefly recall the essential notions of the domain-theoretic framework for multi-variable calculus from [8]. We write $\mathbf{I}\mathbb{R}$ for the interval domain $\{[\underline{a}, \overline{a}] \mid \underline{a} \leq \overline{a}, \underline{a}, \overline{a} \in \mathbb{R}\} \cup \{\mathbb{R}\}$, ordered by reverse inclusion; we write $\perp = \mathbb{R}$ for the least element of $\mathbf{I}\mathbb{R}$. The n-fold product of $\mathbf{I}\mathbb{R}$ with itself is denoted by $\mathbf{I}\mathbb{R}^n$, and we write $\mathbf{I}\mathbb{R}^n_s$ for the n-fold smash product of $\mathbf{I}\mathbb{R}$. The same convention applies to $n \times k$ interval matrices. For $A \in \mathbf{I}\mathbb{R}^n$, we have the sub-domain $\mathbf{I}A = \{a \in \mathbf{I}\mathbb{R}^n \mid A \sqsubseteq a\}$ with inherited ordering. We denote by $(A \to \mathbf{I}B)$ (resp. $(A \to \mathbf{I}\mathbb{R}^n_s)$) the set of Scott continuous functions of type $A \to \mathbf{I}B$ (resp. $A \to \mathbf{I}\mathbb{R}^n_s$), where $A, B \in \mathbf{I}\mathbb{R}^n$. For brevity, we put $D^0(A) = (A \to \mathbf{I}\mathbb{R})$ and $D^0 = D^0([0,1]^n)$.

We identify a real number $x \in \mathbb{R}$ with the maximal element $\{x\} \in \mathbf{I}\mathbb{R}$ and a classical function $f : A \to \mathbb{R}$ with the function $x \mapsto \{f(x)\} : A \to \mathbf{I}\mathbb{R}$, for $A \in \mathbf{I}\mathbb{R}^n$.

The Scott continuous function $f : [0,1]^n \to \mathbf{I}\mathbb{R}$ has *an interval Lipschitz constant* $b \in (\mathbf{I}\mathbb{R})^{1\times n}_s$ in $a \in (\mathbf{I}[0,1])^n$ if for all $x, y \in a^\circ$ we have: $b(x-y) \sqsubseteq f(x) - f(y)$, where we use the canonical extension of basic arithmetic operations from real numbers to real intervals. The *single-step tie* $\delta(a,b) \subseteq D^0([0,1]^n)$ of a with b is the collection of all functions in $D^0([0,1]^n)$ which have an interval

Lipschitz constant b in a. If $b \neq \perp$, then $\delta(a,b)$ consists only of functions which in the interior a° are classical Lipschitz functions. The Scott continuous *primitive map* $\int : ([0,1]^n \to (\mathbf{I}\mathbb{R})_s^{1\times n}) \to (\mathcal{P}(D^0), \supseteq)$ is given by $\int(\bigsqcup_{i\in I} a_i \searrow b_i) = \bigcap_{i\in I} \delta(a_i, b_i)$, where $\mathcal{P}$ is the power set functor. The *domain-theoretic derivative* of a continuous function $f : [0,1]^n \to \mathbf{I}\mathbb{R}$ is the well-defined and Scott continuous map $\frac{df}{dx} = \bigsqcup_{f\in\delta(a,b)} a \searrow b : [0,1]^n \to (\mathbf{I}\mathbb{R})_s^{1\times n}$. For a classical C^1 function f, we have $\frac{df}{dx} = f'$, where f' is the classical derivative of f, a notation which is used throughout the paper. For a classical Lipschitz map f, the domain-theoretic derivative $\frac{df}{dx}(x_0)$ at x_0 is the smallest hyper-rectangle containing the Clarke gradient [3] at x_0, which is a non-empty compact and convex set. The domain for Lipschitz functions is the continuous Scott subdomain

$$D^1([0,1]^n \to \mathbf{I}\mathbb{R}) \subset ([0,1]^n \to \mathbf{I}\mathbb{R}) \times ([0,1]^n \to \mathbf{I}\mathbb{R}_s^n)$$

of *consistent pairs* defined by $(f,g) \in D^1([0,1]^n \to \mathbf{I}\mathbb{R})$ iff $\uparrow f \cap \int g \neq \emptyset$, which is equivalent to the existence of a classical continuous function h : dom$(g) \to \mathbb{R}$ with $g \sqsubseteq \frac{dh}{dx}$ and $f \sqsubseteq h$. For $f \in D^0$, we have $(f, \frac{df}{dx}) \in D^1([0,1]^n \to \mathbf{I}\mathbb{R})$, where $\frac{df}{dx}$ is, as always in this paper, the domain-theoretic derivative of f. In [8], it is shown that consistency is decidable on the rational step functions of $([0,1]^n \to \mathbf{I}\mathbb{R}) \times ([0,1]^n \to \mathbf{I}\mathbb{R}_s^n)$ and thus $D^1([0,1]^n \to \mathbf{I}\mathbb{R})$ can be given an effective structure.

The same results hold, if we replace $[0,1]^n$ with an arbitrary $A \in \mathbf{I}\mathbb{R}^n$. Given $B \in \mathbf{I}\mathbb{R}^n$ and $C \in \mathbf{I}\mathbb{R}_s^{n\times n}$, we write $D^1(A \to \mathbf{I}B, A \to \mathbf{I}C)$ (resp. $D^1(A \to \mathbf{I}B, A \to \mathbf{I}\mathbb{R}_s^{n\times n})$) for the sub-domain of consistent pairs in $(A \to \mathbf{I}B) \times (A \to \mathbf{I}C)$ (resp. $(A \to \mathbf{I}B) \times (A \to \mathbf{I}\mathbb{R}_s^{n\times n})$. For brevity, we write $D^1(A \to \mathbf{I}B) = D^1(A \to \mathbf{I}B, A \to \mathbf{I}\mathbb{R}_s^{n\times n})$ and $D^1 = D^1([0,1]^n)$. The space of classical C^1 functions of type $A \to B$ is denoted by $C^1(A \to B)$.

The framework can be extended in a straightforward way to functions of type $\mathbf{I}[0,1]^n \to \mathbf{I}\mathbb{R}$ and of type $\mathbf{I}[0,1]^n \to \mathbf{I}\mathbb{R}^m$: for $a \in \mathbf{I}[0,1]^n$ and $b \in \mathbf{I}\mathbb{R}_s^n$, we say that $f : \mathbf{I}[0,1]^n \to \mathbf{I}\mathbb{R}$ has *interval Lipschitz constant* b *in* a, if $b(x-y) \sqsubseteq f(x) - f(y)$ for all $x, y \ll a$. The collection of all $f : \mathbf{I}[0,1]^n \to \mathbf{I}\mathbb{R}$ with interval Lipschitz constant b in a is denoted by $\delta_i(a,b)$.

If X is a set and $f : X \to \mathbf{I}\mathbb{R}^n$, we define the *width* of f as $w(f) := \max_{1\le i\le m} sup_{x\in X} w(f_i(x))$ where $w([c,d]) = d - c$ is the width of an interval.

If we write D_{i}^0 for the space of Scott continuous functions of type $\mathbf{I}[0,1]^n \to \mathbf{I}\mathbb{R}$, we have the Scott continuous map $\mathcal{E} : D^0 \to D_{\mathsf{i}}^0$ with $\mathcal{E}(f)(x) = \bigsqcap\{f(y) \mid y \in x\}$, which sends any map in D^0 to its maximal extension in D_{i}^0. For convenience, we sometimes write $\mathbf{I}f := \mathcal{E}(f)$. For functions $f : A \to \mathbf{I}B$ and $g : B \to \mathbf{I}C$ we write $g \circ f$ for $\mathbf{I}g \circ f$. We also recall the notion of a piecewise polynomial step function [7]. Let $r \in \mathbf{I}[-1,1]^n$ and let $p, q : r \to \mathbb{R}$ be piecewise polynomial functions (i.e. each given by a finite number of polynomials) satisfying $p(x) \le q(x)$ for $x \in r^\circ$. Note that p and q can be discontinuous in r°. The piecewise polynomial single-step functions $r \searrow [p,q] : [0,1]^n \to \mathbf{I}\mathbb{R}$ and $r \searrow_{\mathsf{i}} [p,q] : \mathbf{I}[0,1]^n \to \mathbf{I}\mathbb{R}$ are, respectively, given by

$$(r \searrow [p,q])(x) = \begin{cases} [p,q](x) & \text{if } x \in r^\circ \\ \perp & \text{otherwise.} \end{cases}$$

$$(r \searrow_{\mathsf{i}} [p,q])(x) = \begin{cases} \mathbf{I}[p,q](x) & \text{if } r \ll x \\ \perp & \text{otherwise.} \end{cases}$$

A piecewise polynomial step function is the lub $\bigsqcup_{i\in I} r_i \searrow [p_i, q_i]$ (or $\bigsqcup_{i\in I} r_i \searrow_{\mathsf{i}} [p_i, q_i]$) of a consistent finite set of piecewise polynomial single-step functions. If r_i, p_i and q_i are defined over rationals, then we say that the polynomial step function is rational. From now on we will be dealing with step functions of type $A \to \mathbf{I}B$ or $\mathbf{I}A \to \mathbf{I}B$ where $A, B \in \mathbf{I}\mathbb{R}^n$. A polynomial step function, say, of type $[0,1]^n \to \mathbf{I}\mathbb{R}^n$ will be given by a vector polynomial step function of type:

$$(r \searrow [p,q])(x) = \begin{cases} [p,q](x) & \text{if } x \in r^\circ \\ \perp & \text{otherwise.} \end{cases},$$

where $[p,q]$ is now a vector of n interval-valued functions given by polynomials $p_j \le q_j$ for $j = 1, \cdots, n$. The composition $f \circ g$ of two piecewise linear step functions $f \in \mathbf{I}\mathbb{R}^n \to \mathbf{I}\mathbb{R}^n$ and $g \in [0,1]^n \to \mathbf{I}\mathbb{R}^n$ is a piecewise linear step function of type $[0,1]^n \to \mathbf{I}\mathbb{R}^n$. The collection of rational piecewise linear (or polynomial) step functions gives a basis of $(D^0)_s^n$ and $(D_{\mathsf{i}}^0)_s^n$.

The following gives an algorithm to evaluate $\mathcal{E}$ on a piecewise polynomial step function, which extends the algorithm in [7, Section 2]:

$$\mathcal{E}(\bigsqcup_{i\in I} r_i \searrow [p_i, q_i]) =$$
$$\bigsqcup\{(\sqcap_{j\in J} r_j) \searrow_{\mathsf{i}} [p_J, q_J] \mid J \subseteq I, \bigcup_{j\in J} r_j^\circ \text{ connected}\}, \quad (1)$$

where, for each $J \subset I$, the piecewise polynomials $p_J, q_J : (\sqcap_{j\in J} r_j) \to \mathbb{R}$ are defined for $x \in (\sqcap_{j\in J} r_j)$ by:

$$p_J(x) = \max\{p_j(x) \mid j \in J, x \in \text{dom}(p_j)\}$$

$$q_J(x) = \min\{q_j(x) \mid j \in J, x \in \text{dom}(q_j)\}.$$

Lemma 2.1 *If $\bigsqcup_{i\in I} r_i \searrow [p_i, q_i]$ is a piecewise linear step function, then so is $\mathcal{E}(\bigsqcup_{i\in I} r_i \searrow [p_i, q_i])$.*

Lemma 2.2 *If $u \in D^0$ and $u = \bigsqcup_{i\ge 0} s_i$, where s_i are piecewise linear step functions, then $\mathcal{E}(u) = \bigsqcup_{i\ge 0} \mathcal{E}(s_i)$ is the supremum of piecewise linear step functions.*

We will also need to use step functions made up of single-step functions of the form $c \searrow [p,q]$ and $c \searrow_{\mathrm{i}} [p,q]$ where $c \subseteq \mathbb{R}^n$ is a parallelogram. This class of step functions, say, of type $\mathbb{R}^n \to \mathbf{I}\mathbb{R}^n$ is closed under pre-composition and post-composition with invertible linear maps:

$$\begin{aligned}(c \searrow [p,q]) \circ L &= (L^{-1}c) \searrow [p,q],\\ L \circ (c \searrow [p,q]) &= c \searrow [L \circ p, L \circ q],\end{aligned}$$

and similarly:

$$\begin{aligned}(c \searrow_{\mathrm{i}} [p,q]) \circ L &= (L^{-1}c) \searrow_{\mathrm{i}} [p,q]\\ L \circ (c \searrow_{\mathrm{i}} [p,q]) &= c \searrow_{\mathrm{i}} [L \circ p, L \circ q].\end{aligned}$$

We will use the max norm for vectors and matrices in $\mathbb{R}^n$, so in particular classical Lipschitz constant are always meant to be with respect to the max norm. The max norm is extended to interval valued vectors and matrices as follows. For $b \in \mathbf{I}\mathbb{R}^{m\times n}$, we define its maximum norm by $\|b\| = \max_{1\le i\le m} \sum_{j=1}^{n} \max\{|b_{ij}{}^-|, |b_{ij}^+|\}$, where $b_{ij} = [b_{ij}^-, b_{ij}^+]$. We write the maximum norm of a vector $s \in \mathbb{R}^n$ by $\|s\| := \max_{1\le i\le n} |s_i|$.

Definition 2.3 We say $f : \mathbf{I}A \to \mathbf{I}\mathbb{R}^m$ is *interval Lipschitz* in the open set $O \subset A$, if there exists $\ell \ge 0$ such that $w(f(x)) \le \ell w(x)$ for all $x \in \mathbf{I}A$ with $x \subset O$. We say that f is *interval contracting* if it is interval Lipschitz with an interval Lipschitz constant less than one.

Proposition 2.4 *If $A \in \mathbf{I}\mathbb{R}^n$ and $f : A \to \mathbb{R}$ has Lipschitz constant $c \ge 0$, then $\mathbf{I}f$ has interval Lipschitz constant c.*

Proof As in [9, Proposition 6].

Lemma 2.5 *Suppose $A \in \mathbf{I}\mathbb{R}^n$ and $f : A \to \mathbb{R}$ satisfies $c \sqsubseteq \frac{df}{dx}$ on A where $c = (c_1, \ldots, c_n) \in \mathbf{I}\mathbb{R}^n$. Then f has Lipschitz constant $\sum_{i=1}^{n} \|c_i\|$.*

Proposition 2.6 (i) *For $h \in ([0,1]^n \to \mathbf{I}\mathbb{R})$ and $g \in ([0,1]^n \to \mathbf{I}\mathbb{R}^n_s)$ we have $h \in \int g$ iff, for all $x, y \in [0,1]^n$ we have $\mathbf{I}g(x \sqcap y)(x - y) \sqsubseteq h(x) - h(y)$.*

(ii) *For $h \in (\mathbf{I}[0,1]^n \to \mathbf{I}\mathbb{R})$ and $g \in (\mathbf{I}[0,1]^n \to \mathbf{I}\mathbb{R}^n_s)$ we have $h \in \int g$ iff, for all $x, y \in \mathbf{I}[0,1]^n$ we have $g(x \sqcap y)(x - y) \sqsubseteq h(x) - h(y)$.*

Proof Similar to [7, Proposition 7.7].

Corollary 2.7 *Let $h \in ([0,1]^n \to \mathbf{I}\mathbb{R})$ and $g \in ([0,1]^n \to \mathbf{I}\mathbb{R}^n_s)$. Then $h \in \int g$ iff $\mathbf{I}h \in \int \mathbf{I}g$.*

Proof The "only if" part is trivial. For the "if part", let $X, Y \in \mathbf{I}[0,1]^n$, For $x \in X$ and $y \in Y$ we have by assumption:

$$\mathbf{I}g(x \sqcap y)(x - y) \sqsubseteq \mathbf{I}h(x) - \mathbf{I}h(y).$$

By taking glb over $x \in X$, we get:

$$\mathbf{I}g(X \sqcap y)(X - y) \sqsubseteq \mathbf{I}h(X) - \mathbf{I}h(y).$$

The result follows by taking glb over $y \in Y$. □

3. Domain-theoretic calculus of functions

We note that $f \in D^1([0,1]^n \to \mathbf{I}\mathbb{R}^m)$ iff $f_i \in D^1([0,1]^n \to \mathbf{I}\mathbb{R})$ for $i = 1, \cdots, m$, where $f = \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix}$.

For ease of presentation, we take the following notation: A vector $(x_1, \ldots, x_n) \in \mathbb{R}^n$ is denoted by $(x_i)_i$ a short-hand for $(x_i)_{1\le i\le n}$, i.e. a repeated index always runs from 1 to n,where n is the dimension of the Euclidean space $\mathbb{R}^n$, and describes the components of a vector, and similarly for a vector function. Thus, $f \in D^1(\mathbb{R}^n \to \mathbf{I}\mathbb{R})$ is represented as $(f_0, (f_i)_i)$ where f_0 is the function part of f and $(f_i)_i$ its derivative part. This notation is also used for $n \times n$ matrices, i.e. $(a_{ij})_{ij}$ denotes an $n \times n$ matrix with ij entry a_{ij}. Thus, a vector function $f \in D^1([0,1]^n \to \mathbf{I}\mathbb{R}^n)$ is denoted by $f = ((f_{i0})_i, (f_{ij})_{ij})$, where f_{i0} is the ith function part component of the f and $(f_{ij})_j$ for $j = 1, \ldots, n$ denotes the n derivative components of f_{i0}. The following statement is the extension of the classical chain rule to the domain D^1.

Theorem 3.1 *Suppose $(f_0, (f_i)_i) \in D^1(\mathbb{R}^n \to \mathbf{I}\mathbb{R})$ and $(g_{i0}, g_{i1}, \cdots, g_{im}) \in D^1([0,1]^m \to \mathbf{I}\mathbb{R})$ for $i = 1, \cdots n$. If*

$$h_0 = \mathbf{I}f_0((g_{i0})_i) \qquad h_j = (\sum_{k=1}^{n} \mathbf{I}f_k((g_{i0})_i)) \cdot g_{kj}$$

for $1 \le j \le m$, then $(h_0, (h_j)_j) \in D^1([0,1]^m \to \mathbf{I}\mathbb{R})$.

Proof Let $h_i \in \uparrow g_{i0} \cap \int g_{i1}$, $k \in \uparrow f_0 \cap \int (f_1, \cdots, f_n)$ and put $t = k(h_1, \cdots, h_n)$. By Corollary 2.7, we have: $\mathbf{I}k \in \int(\mathbf{I}f_1, \cdots \mathbf{I}f_n)$. We show, by using Proposition 2.6(i) that $t \in \int \sum_{i=1}^{n} f_i(g_{10}, \cdots, g_{n0}) \cdot g_{i1}$. For elements $x, y \in [0,1]^m$ we have:

$$\begin{aligned}
&t(x) - t(y)\\
=&k(h_j(x)_j) - k(h_j(y)_j)\\
\sqsupseteq&\sum_{i=1}^{n} f_i((h_j(x) \sqcap h_j(y))_j) \cdot (h_i(x) - h_i(y))\\
\sqsupseteq&\sum_{i=1}^{n} f_i((h_j(x) \sqcap h_j(y))_j) \cdot g_{i1}(x \sqcap y)(x - y)\\
\sqsupseteq&\sum_{i=1}^{n} f_i((h_j(x \sqcap y))_j)) \cdot g_{i1}(x \sqcap y)(x - y)\\
\sqsupseteq&\sum_{i=1}^{n} f_i(g_{j0}(x \sqcap y)_j) \cdot g_{i1}(x \sqcap y)(x - y),
\end{aligned}$$

which completes the proof.

As the basic arithmetic operations are differentiable, we can use their canonical extensions, together with composition, to obtain versions of the arithmetical operations in D^1.

Composition $(\cdot \circ \cdot)$:

$D^1(\mathbb{R}^n \to \mathbf{I}\mathbb{R}^n) \times D^1([0,1]^n \to \mathbf{I}\mathbb{R}^n) \to D^1([0,1]^n \to \mathbf{I}\mathbb{R}^n)$

where, $f \circ g = ((h_{i0})_i, (h_{ij})_{ij})$ with

$$h_{i0} = \mathbf{I}f_{i0}((g_{m0})_m) \quad h_{ij} = \sum_{k=1}^{n} \mathbf{I}f_{ik}((g_{m0})_m) \cdot g_{kj}$$

for $1 \leq i, j \leq n$.

Addition $(\cdot + \cdot)$:

$D^1(\mathbb{R}^n \to \mathbf{I}\mathbb{R}^m) \times D^1(\mathbb{R}^n \to \mathbf{I}\mathbb{R}^m) \to D^1(\mathbb{R}^n \to \mathbf{I}\mathbb{R}^m)$

where$(f+g)_j = (f_{j0} + g_{j0}, f_{j1} + g_{j1}, \cdots f_{jn} + g_{jn})$.

Negation $(-\cdot)$:

$$D^1(\mathbb{R}^n \to \mathbf{I}\mathbb{R}^n) \to D^1(\mathbb{R}^n \to \mathbf{I}\mathbb{R}^n)$$

where $(-f)_j = (-f_{j0}, -f_{j1}, \ldots, -f_{jn})$.

Multiplication $(\cdot)$:

$$D^1(\mathbb{R}^n \to \mathbf{I}\mathbb{R}) \times D^1(\mathbb{R}^n \to \mathbf{I}\mathbb{R}) \to D^1(\mathbb{R} \to \mathbf{I}\mathbb{R})$$

where $f \cdot g = (f_0 \cdot g_0, (f_i)_i \cdot g_0 + (g_i)_i \cdot f_0)$.

Inversion $(\frac{1}{\cdot})$:

$$D^1(\mathbb{R}^n \to \mathbf{I}\mathbb{R}) \to D^1(\mathbb{R}^n \to \mathbf{I}\mathbb{R})$$

where $\frac{1}{f} = (\frac{1}{f_0}, \frac{-f_1}{f_0^2}, \ldots, \frac{-f_n}{f_0^2})$ and, as usual, for intervals a and b, we put $a/b = \perp$ if $0 \in b$.

Corollary 3.2 *The arithmetic operations are well-defined:*

(i) *If $f \in D^1(\mathbb{R}^n \to \mathbf{I}\mathbb{R}^n)$ and $g \in D^1([0,1]^n \to \mathbf{I}\mathbb{R}^n)$, then $f \circ g \in D^1([0,1]^n \to \mathbf{I}\mathbb{R}^n)$.*

(ii) *If $f \in D^1(\mathbb{R}^n \to \mathbf{I}\mathbb{R}^n)$ and $g \in D^1(\mathbb{R}^n \to \mathbf{I}\mathbb{R}^n)$, then $f + g \in D^1(\mathbb{R}^n \to \mathbf{I}\mathbb{R}^n)$.*

(iii) *If $f \in D^1(\mathbb{R}^n \to \mathbf{I}\mathbb{R}^n)$, then $-f \in D^1(\mathbb{R}^n \to \mathbf{I}\mathbb{R}^n)$.*

(iv) *If $f \in D^1(\mathbb{R}^n \to \mathbf{I}\mathbb{R})$ and $g \in D^1(\mathbb{R}^n \to \mathbf{I}\mathbb{R})$, then $f \cdot g \in D^1(\mathbb{R}^n \to \mathbf{I}\mathbb{R})$.*

(v) *If $f \in D^1(\mathbb{R}^n \to \mathbf{I}\mathbb{R})$, then $\frac{1}{f} \in D^1(\mathbb{R}^n \to \mathbf{I}\mathbb{R})$.*

Moreover these operations are Scott continuous.

Lemma 3.3 Chain Rule. *For any two functions $f, g \in (\mathbb{R}^n \to \mathbf{I}\mathbb{R}^n)$:*

$$\frac{d(f \circ g)}{dx} \sqsupseteq (\frac{df}{dx} \circ g) \cdot \frac{dg}{dx}.$$

Proof Fix $x_0 \in \mathbb{R}^n$. Let $b_1 \ll \frac{dg}{dx}(x_0)$ and $b_2 \ll (\frac{df}{dx}(g(x_0))$. Then for some $a_1, a_2 \in \mathbf{I}\mathbb{R}^n$ with $x_0 \in a_1$ and $g(x_0) \in a_2$ we have: $f \in \delta(a_1, b_1)$ and $g \in \delta(a_2, b_2)$. Since g is Scott continuous, $g^{-1}(\Uparrow a_2)$ is open in $\mathbb{R}^n$. Let $a \in \mathbf{I}\mathbb{R}^n$, with $a \subseteq a_1 \cap g^{-1}(\Uparrow a_2)$ and $g(x_0) \in a^\circ$. Then, for all $x, y \in a^\circ$:

$$f(g(x)) - f(g(y)) \sqsupseteq b_1(g(x) - g(y)) \sqsupseteq b_1 b_2 (x - y).$$

Therefore $f \circ g \in \delta(a, b_1 b_2)$ and thus $\frac{d(f \circ g)}{dx}(x_0) \sqsupseteq b_1 b_2$.

Note that equality may fail in the chain rule. For example let $f : \mathbb{R} \to \mathbb{R}$ be the absolute value function $x \mapsto |x|$ and $g : \mathbb{R} \to \mathbb{R}$ defined by $x \mapsto x$ if $x \leq 0$ and 0 if $x > 0$. Then $\frac{df \circ g}{dx}(0) = [-1, 0]$ whereas $\frac{df}{dx}(g(0)) \cdot \frac{dg}{dx}(0) = \frac{df}{dx}(0) \cdot \frac{dg}{dx}(0) = [-1, 1][0, 1] = [-1, 1]$.

Lemma 3.4 *Let $(g_0, g_1, \cdots, g_n) \in D^1([0,1]^n \to \mathbf{I}\mathbb{R})$ with g_0 real valued. If $g_i(x)$ is a real number for all $i = 1, \cdots, n$ for some $x \in [0,1]^n$, then $g_0'(x)$ exists and $g_0'(x) = (g_1(x), \cdots, g_n(x))$.*

Proof By assumption, there exists $t \in \uparrow g_0 \cap \int (g_1, \cdots, g_n)$, but then $t = g_0$ as g_0 is maximal. Hence $\frac{dg_0}{dx} \sqsupseteq (g_1, \cdots, g_n)$, which implies $\frac{dg_0}{dx}(x) = (g_1(x), \cdots, g_n(x)))$. By [7, Proposition 4.3(ii)], the result follows.

4. Inverse constructing functional

In many areas of mathematics a construction is first obtained for functions close to the identity map and then extended to more general maps. In this section, we show first that the inverse of a function close to the identity can be obtained as the fixed point of a functional that we introduce here. Later in the paper, we show how this functional can be used to obtain the inverse of a Lipschitz function. For $a > 0$ and $0 < c < 1$, we fix the n-dimensional rectangles

$$A = [-a, a]^n \quad B = [-ca, ca]^n \quad C = [-(1-c)a, (1-c)a]^n$$

and consider the functional

$$T : D^1(A \to \mathbf{I}B) \times D^1(C \to \mathbf{I}B) \to D^1(C \to \mathbf{I}B)$$

defined by

$$T(f, g) = -\mathbf{I}f \circ (I + g)$$

where $I : C \to \mathbf{I}C$ is the identity function $\lambda x.x$. We also put $T_f = T(f, \cdot)$. By Corollary 3.2, T is well-defined. Later, we construct the inverse function of f as the least fixpoint of T_f. The function part of T is given by the functional

$$\begin{array}{llll} R: & (A \to \mathbf{I}B) \times (C \to \mathbf{I}B) & \to & (C \to \mathbf{I}B) \\ & (f, g) & \mapsto & -\mathbf{I}f \circ (I + g) \end{array}$$

Let $R_f = R(f,\cdot)$. For the derivative part, we have the functional S of type

$$(A \to \mathbf{I}\mathbb{R}_s^{n^2}) \times (C \to \mathbf{I}B) \times (C \to \mathbf{I}\mathbb{R}_s^{n^2}) \to (C \to \mathbf{I}\mathbb{R}_s^{n^2})$$

defined by

$$S(h, g_0, g_1) = -\mathbf{I}h \circ (I + g_0) \cdot (\lambda x.\mathrm{Id} + g_1)$$

where $\mathrm{Id} = \frac{dI}{dx}$. Thus,

$$T(f,g) = (R((f_{i0})_i, (g_{i0})_i), S((f_{ij})_{ij}, (g_{i0})_i, (g_{ij})_{ij})$$

if $f = ((f_{i0})_i, (f_{ij})_{ij}) \in D^1(A \to \mathbf{I}B)$ and $g = ((g_{i0})_i, (g_{ij})_{ij}) \in D^1(C \to \mathbf{I}B)$. We also put $S_{(h,g_0)} = S(h, g_0, \cdot)$.

Proposition 4.1 *Suppose $f : A \to B$. Then $R_f : (C \to \mathbf{I}B) \to (C \to \mathbf{I}B)$ is well defined. If moreover f has Lipschitz constant $c < 1$ with respect to the max norm and $f(0) = 0$, we have:*

(i) *The functional R_f has a unique fixed point $(g_{i0})_i$ with $w((g_{i0})_i) = 0$ which is thus a classical C^0 function.*

(ii) *$I + f : [-a,a]^n \to \mathbb{R}^n$ has inverse $I + (g_{i0})_i : Im(I + f) \to [-a,a]^n$.*

(iii) *$(g_{i0})_i(0) = 0$.*

Proof It is easily checked that R_f is well-defined. As f, and hence every f_i, has Lipschitz constant $c < 1$, it follows from Proposition 2.4, that $\mathbf{I}f$, and hence every $\mathbf{I}f_i$, is interval contracting with contractivity factor c.

Let $x \in C$. Then we have

$$\begin{aligned} w(g_{i0}(x)) &= w(\mathbf{I}f_i((x_k + g_{k0}(x)))_k) \\ &\leq c \max_{1\leq i\leq n} w(x_i + g_{i0}(x))) \\ &= c \max_{1\leq i\leq n} w(g_{i0}(x)). \end{aligned}$$

hence $\max_{1\leq i\leq n} w(g_{i0}(x)) = 0$. Thus, the least fixed point g is maximal and hence the unique fixed point.

(ii) We have $(I+f)\circ(I+g) = I + g + f\circ(I+g) = I$ and thus $I+g$ is a right inverse of $I+f$. Since $I+f$ is Lipschitz with $\|f(x) - f(y)\| \leq c\|x-y\|$ it follows by [10, Theorem 3.1] that $I+f$ has an inverse. Thus, $I+g$, being a right inverse, is the inverse of $I+f$.

(iii) We have $(I+f)(0) = 0$ and $(I+f)(g(0)) = (I+f)(0+g(0)) = (I+f)\circ(I+g)(0) = I(0) = 0$, as $I+g$ is the inverse of $I+f$. It follows from injectivity of $I+f$ that $g(0) = 0$.

Now we can obtain a fixed point of $T_{(f,\frac{df}{dx})}$ and examine its properties.

Lemma 4.2 *The partial order $(D^0([0,1] \to \mathbf{I}\mathbb{R}), \sqsupseteq)$ of Scott continuous functions ordered by reverse pointwise ordering is a dcpo with Scott continuous operations for addition $\cdot + \cdot$, multiplication $\cdot \times \cdot$ and negation $-\cdot$.*

Proof Note that any bounded complete dcpo is a dcpo with respect to its opposite order. It is routine to check the Scott continuity of addition and multiplication.

Lemma 4.3 *For $x, y \in \mathbf{I}\mathbb{R}$, we have $w(xy) \leq w(x)\|y\| + \|x\|w(y)$.*

Proposition 4.4 *Suppose $f : A \to B$ satisfies $\|\frac{df}{dx}\| \leq c < 1$ on A and $f(0) = 0$. If $H = [-\frac{c}{1-c}, \frac{c}{1-c}]^{n\times n}$, we have:*

(i) *The functional*

$$T_{(f,\frac{df}{dx})} : D^1(X_0, X_1) \to D^1(X_0, X_1),$$

where $X_0 = (C \to \mathbf{I}B)$ and $X_1 = (C \to \mathbf{I}H)$ is well-defined and has a unique fixed point $g = ((g_{i0})_i, (g_{ij})_{ij})$, such that $(g_{i0})_i$ is the unique fixed point of R_f and $(g_{ij})_{ij}$ is the unique fixed point of $S_{(\frac{df}{dx},(g_{i0})_i)} : X_1 \to X_1$; moreover $(g_{ij})_{ij} = \bigsqcap_{l\geq 0} S^l_{(\frac{df}{dx},(g_{i0})_i)}(\frac{d(g_{i0})_i}{dx}) \sqsubseteq \frac{d(g_{i0})_i}{dx}$.

(ii) *The fixed point g of $T_{(f,\frac{df}{dx})}$ will satisfy $w((g_{ij}(x))_{ij}) \leq \frac{n(1+d)}{1-c} w(\frac{df}{dx}(x_k + g_{k0}(x))_k)$ for any $x \in C$ and thus, $w((g_{ij})_{ij}) \leq \frac{n(1+d)}{1-c} w(\frac{df}{dx})$.*

(iii) *If $w(\frac{df}{dx}((x_k + g_{k0}(x))_k)) = 0$ for some $x \in C$, then $w(g_{ij}(x)) = 0$, for $i,j = 1,\cdots,n$, and $(g_{i1}(x),\cdots,g_{in}(x)) = g'_{i0}(x)$ for $i = 1,\cdots,n$.*

(iv) *If $w(\frac{df}{dx}) = 0$, then $w(g_{i0}) = 0$ for $i = 1,\cdots n$, and $(g_{i1},\cdots,g_{in}) = g'_{i0}$ for $i = 1,\cdots n$.*

(v) *If $\frac{df}{dx}(0) = 0$ then $g_{ij}(0) = 0$ for $i,j = 1,\cdots,n$.*

Proof Suppose $g = ((g_{i0})_i, (g_{ij})_{ij}) \in D^1(X_0, X_1)$. Then by Proposition 4.1 we have $R_f((g_{i0})_i) \in (C \to \mathbf{I}B) = X_0$, hence it suffices to show that $S_{(\frac{df}{dx},(g_{i0})_i)}((g_{ij})_{ij}) \in X_1$. As f satisfies $\|\frac{df}{dx}\| \leq c$ on A, we have, by definition of matrix norm, that the row sum $\sum_{j=1}^n \|(\frac{df}{dx})_{ij}\| \leq c$ for all $i = 1,\ldots,n$. Let $d = \frac{c}{1-c}$. Then, for any $x \in C$, we have $g_{ij}(x) \in [-d,d]$, hence $(g_{ij})_{ij}(x) + \mathrm{Id} \in \mathbf{I}[-d+1, d+1]^{n\times n}$. Letting $(h_{ij})_{ij} = -\mathbf{I}\frac{df}{dx}(I + (g_{i0})_i(x)) \cdot (\mathrm{Id} + (g_{ij})_{ij}(x))$, we obtain for the ij-entry h_{ij} of $(h_{ij})_{ij}$ that $h_{ij} \sqsupseteq \sum_{k=1}^n ((\frac{df}{dx})_{ik} \cdot [-d+1, d+1] \sqsupseteq [-d,d]$, as $\sum_{i=1}^k \|(\frac{df}{dx})_{ik}\| \leq c$. Therefore also the derivative part of $T_{(f,\frac{df}{dx})}$ is well defined. Consistency of function and derivative part follow from Corollary 3.2. By continuity, $T_{(f,\frac{df}{dx})}$ therefore has a least fixpoint $(g_{i0})_i, g_{\min})$.

For any fixed point $((g_{i0})_i,(g_{ij})_{ij})$ we have $(g_{ij})_{ij} \sqsubseteq \frac{d(g_{i0})_i}{dx}$ since $((g_{i0})_i,(g_{ij})_{ij})$ is consistent and $(g_{i0})_i$ has zero width. Now for any $h \in X_1$ with $g_{\min} \sqsubseteq h \sqsubseteq \frac{d(g_{i0})_i}{dx}$, we have, by the Chain rule 3.3:

$$g_{\min} \sqsubseteq S_{(\frac{df}{dx},(g_{i0})_i)}(h) \sqsubseteq S_{(\frac{df}{dx},(g_{i0})_i)}(\frac{d(g_{i0})_i}{dx}) \sqsubseteq \frac{d(g_{i0})_i}{dx}.$$

It follows by Lemma 4.2 that, in the lattice of functions in X_1 with $\sqsupseteq$ as the ordering, $g_{\min}$ as the top element and $\frac{d(g_{i0})_i}{dx}$ as the least element, the Scott continuous function $S_{(f,(g_{i0})_i)} : X_1 \to X_1$ has a least fixed point

$$g_{\max} = \sqcap_{l\geq 0} S^l_{(\frac{df}{dx},(g_{i0})_i)}(\frac{d(g_{i0})_i}{dx}),$$

which is the greatest fixed point of $S_{(f,(g_{i0})_i)} : X_1 \to X_1$ and thus induces the greatest fixed point $((g_{i0})_i, g_{\max})$ of $T_{(f,\frac{df}{dx})}$. It remains to show the uniqueness of the fixed point of $S_{(f,(g_{i0})_i)} : X_1 \to X_1$. The fixed point equation

$$g_{ij} = \sum_{m=1}^{n} -\mathbf{I}(\frac{df}{dx})_{im}((\pi_k + g_{k0})_k)\cdot(\lambda x.\delta_{mj} + g_{mj}) \quad (2)$$

evaluated at $x \in C$ yields:

$$g_{ij}(x) = \sum_{m=1}^{n} -\mathbf{I}(\frac{df}{dx})_{im}(x_k + g_{k0}(x))_k \cdot (\delta_{mj} + g_{mj}(x)),$$

or equivalently,

$$[g^-_{ij}(x), g^+_{ij}(x)] = \sum_{m=1}^{n}([c_{im}(x), d_{im}(x)]\cdot [\delta_{mj} + g^-_{mj}(x), \delta_{mj} + g^+_{mj}(x)]), \quad (3)$$

where $g_{kl} = [g^-_{kl}, g^+_{kl}]$ and

$$[c_{im}(x), d_{im}(x)] := -\mathbf{I}(\frac{df}{dx})_{im}(x_k + g_{k0}(x))_k,$$

for $1 \leq i, m \leq n$. For any fixed $x \in C$, consider the system of $2n^2$ equations represented by:

$$[y_{ij}, z_{ij}] = \sum_{m=1}^{n}[c_{im}(x), d_{im}(x)] \cdot [\delta_{mj} + y_{ij}, \delta_{mj} + z_{ij}], \quad (4)$$

for the unknown values y_{ij} and z_{ij} $(i, j = 1, \cdots, n)$. This is a linear system, which can be written in the form:

$$A(x)u = b(x) \quad (5)$$

where $A(x) \in \mathbb{R}^{2n^2\times 2n^2}$ and $u, b(x) \in \mathbb{R}^{2n^2}$, with $u_{0ij} = y_{ij}$ and $u_{1ij} = z_{ij}$, in which the subscripts of u are numbers written in base n, i.e., $kij = kn^2 + in + j$. We already know that this linear system has at least one solution since $S_{(f,(g_{i0})_i)} : X_1 \to X_1$ has at least one fixed point. Suppose, for a contradiction, that the fixed point is not unique. Then $(g_{\min})_{i_0j_0}(x_0) \neq (g_{\max})_{i_0j_0}(x_0)$ for some $x_0 \in C$ and some i_0, j_0 with $1 \leq i_0, j_0 \leq n$. And thus, for $x = x_0$, the linear system (5) will have infinitely many solutions of the form:

$$u_{0ij}(k) = (g_{\min})^-_{ij}(x_0) + \sum_{t=1}^{p} k_t\theta^t_{0ij}$$
$$u_{1ij}(k) = (g_{\min})^+_{ij}(x_0) + \sum_{t=1}^{p} k_t\theta^t_{1ij},$$

where $\theta^t \in \mathbb{R}^{2n^2}$ $(t = 1, \cdots p)$ is a basis for the null-set of $A(x_0)$, $k \in \mathbb{R}^p$ and p is the dimension of the null-set of $A(x_0)$. But $(g_{\min})_{ij}(x_0) \supseteq (g_{\max})_{ij}(x_0)$ and for some $k \in \mathbb{R}^p$, we have:

$$u_{0i_0j_0}(k) = (g_{\max})^-_{i_0j_0}(x_0), \qquad u_{1i_0j_0}(k) = (g_{\max})^+_{i_0j_0}(x_0).$$

Now, using this value of k, we define $h : C \to \mathbf{I}H$ by

$$h_{ij}(x) = \begin{cases} (g_{\min})_{i_0j_0}(x_0) - \sum_{t=1}^{p} k_t[\theta^t_{0i_0j_0}, \theta^t_{1i_0j_0}] \\ \quad \text{if } i = i_0, j = j_0, x = x_0 \\ (g_{\min})_{ij}(x) \quad \text{otherwise.} \end{cases}$$

Then, h is Scott continuous and is a fixed point of $S_{(f,(g_{i0})_i)} : X_1 \to X_1$ with $h \sqsubseteq g_{\min}$ and $h \neq g_{\min}$, which is the required contradiction.

(ii) We let $d = \frac{c}{1-c}$. Using Lemma 4.3, we have

$$\begin{aligned}
&w(g_{ij}(x))\\
=&w(\sum_{m=1}^{n}(-\mathbf{I}(\frac{df}{dx})_{im}(x_k + g_{k0}(x))_k)\cdot(\delta_{mj} + g_{mj}(x)))\\
\leq& \sum_{m=1}^{n} w(\frac{df}{dx})_{im}((x_k + g_{k0}(x))_k)(1+d) + \\
&\quad \|(\frac{df}{dx})_{im}\| \cdot w(g_{mj}(x))\\
\leq& n(1+d)w(\frac{df}{dx})_i((x_k + g_{k0}(x))_k) + \\
&\quad \sum_{m=1}^{n} \|(\frac{df}{dx})_{im}\| \cdot w((g_{mj})_j(x))\\
\leq& n(1+d)w(\frac{df}{dx}((x_k + g_{k0}(x))_k)) + cw((g_{mj})_{mj}(x))
\end{aligned}$$

as $\|\frac{df}{dx}\| \leq c$. Thus, $w((g_{ij})_{ij}(x)) \leq n(1+d)w(\frac{df}{dx}((x_k + g_{k0}(x))_k)) + cw((g_{ij})_{ij}(x))$, which gives: $w((g_{ij})_{ij}(x)) \leq \frac{n(1+d)}{1-c} w(\frac{df}{dx}((x_k + g_{k0}(x))_k))$.

(iii) From (ii), $w(g_{ij}) = 0$ for $i, j = 1, \cdots, n$. Lemma 3.4, gives $(g_{i1}(x), \cdots, g_{in}(x)) = g'_{i0}(x)$ for $i = 1, \cdots, n$.

(iv) Follows from (iii).

(iv) Since $\mathbf{I}\frac{df}{dx}(0) = 0$, from Equation (2) we get: $g_{ij}(0) = 0$ for $i, j = 1, \cdots, n$.

Corollary 4.5 *Suppose $f \in C^1([-a,a]^n \to [-ca,ca]^n)$ with $\|f'(x)\| < 1$ for all $x \in [-a,a]^n$ and assume $f(0) = f'(0) = 0$. Then, there is a unique fixed point $((g_{i0})_i), (g_{ij})_{ij})$ of $T_{(f,f')}$. It will satisfy $w(g_{i0}) = w(g_{ij}) = 0$ and $g_{i0}(0) = g_{ij}(0) = 0$ for $1 \leq i \leq n, 0 \leq j \leq n$, and $(g_{ij})_{ij} = (g_{i0})'_i$.*

In particular, this shows that the least fixpoint g is the classical inverse function.

5. Inverse functions

In this section we prove the following theorem, which gives a constructive version of the classical result on the inverse of Lipschitz functions as in Theorem 1.1.

Definition 5.1 For an interval matrix $b = ([\underline{b}_{ij}, \overline{b}_{ij}])_{ij} \in \mathbf{I}\mathbb{R}^{n\times n}$ the *mid-point matrix* of b is given by $(\frac{\underline{b}_{ij}+\overline{b}_{ij}}{2})_{ij}$. If $f : [-1,1]^n \to \mathbb{R}^n$ is locally Lipschitz, then the *mean differential* of f at $x_0 \in [-1,1]^n$ is the midpoint matrix of $\frac{df}{dx}(x_0)$.

Note that for a locally Lipschitz function, every component of the domain theoretic derivative is $\neq\perp$. With this terminology, we can replace the linear map in the statement of Theorem 1.1 by the mean differential and obtain the following result.

Theorem 5.2 Inverse Function Theorem. *Let $u : [-1,1]^n \to \mathbb{R}^n$ be locally Lipschitz with $u(0) = 0$. Suppose the mean differential at the point 0 (i.e., the linear map represented by the mid-point matrix of $\frac{du}{dx}(0)$), denoted by M, is invertible. Let $\|M^{-1}\frac{du}{dx}(0) - I\| < 1$. Then we have:*

(i) *The map u has a Lipschitz inverse in a neighbourhood of the origin.*

(ii) *If two increasing sequences of linear rational step functions, respectively of type $[-1,1]^n \to \mathbf{I}\mathbb{R}^n$ and $[-1,1]^n \to \mathbf{I}\mathbb{R}_s^{n\times n}$, converging respectively to u and $\frac{du}{dx}$ are given, then we can effectively obtain an increasing sequence of piecewise linear step functions converging to the inverse of u.*

(iii) *If u is C^1 and two increasing sequences of linear rational step functions, respectively of type $[-1,1]^n \to \mathbf{I}\mathbb{R}^n$ and $[-1,1]^n \to \mathbf{I}\mathbb{R}_s^{n\times n}$, converging respectively to u and u' are given, then we can also effectively obtain an increasing sequence of polynomial step functions converging to the derivative of the inverse of u.*

(iv) *With the assumptions in (iii), we can also effectively obtain an increasing sequence of polynomial step functions, whose lower and upper parts are continuous and piecewise polynomial, such that the two sequences of lower and upper parts converge in the C^1 norm, respectively from below and above, to the inverse of u.*

Proof (i) Suppose $N \in \frac{du}{dx}$ is invertible with $\|N^{-1}\frac{du}{dx} - I\| < 1$, e.g. this holds for $N = M$. Let $f = N^{-1}u - I$. Then $\frac{df}{dx} = N^{-1}\frac{du}{dx} - \mathrm{Id}$, where $\mathrm{Id} = \frac{dI}{dx}$. Since $\|N^{-1}\frac{du}{dx}(0) - I\| < 1$, we have: $\|\frac{df}{dx}(0)\| < 1$. It follows that for any c satisfying $\|N^{-1}\frac{du}{dx}(0) - I\| < c < 1$, there exists $a > 0$ such that: $\|\frac{df}{dx}(x_0)\| \leq c$ for all $x_0 \in [-a,a]^n$. It follows from Lemma 2.5 that f has Lipschitz constant $c < 1$. Thus, by Proposition 4.1, if $(g_{i0})_i$ is the unique fixed point of R_f, then $I + (g_{i0})_i : \mathrm{Im}(I+f) \to [-a,a]^n$ is the local inverse of $I + f$. Hence, $(I + (g_{i0})_i) \circ N^{-1} : \mathrm{Im}(u) \to [-a,a]^n$ is the local inverse of u in $[-a,a]^n$. It follows from Proposition 4.4 that the local inverse of u is locally Lipschitz.

(ii) Suppose $u = \bigsqcup_{i\geq 0} s_i$ where $s_i = \bigsqcup_{j\in J_i} r_j \searrow [p_j, q_j]$ is a piecewise linear step function for disjoint finite indexing sets J_i $(i \geq 0)$ and $\frac{du}{dx} = \bigsqcup_{i\geq 0} \beta_i$ for an increasing sequence (β_i) of linear rational step functions. We denote the midpoint matrix of β_i by M_i. As $\|M^{-1}\frac{du}{dx} - I\| < 1$, we can find, by Scott continuity, the smallest $k > 0$ such that M_k is invertible and $\|M_k^{-1}\beta_k(0) - I\| < 1$, from which we obtain $\|M_k^{-1}\frac{du}{dx}(0) - I\| < 1$. We write $N := M_k$ in accordance with item (i) above.

We obtain $N^{-1}u = \bigsqcup_{i\geq 0} N^{-1}s_i$ where $N^{-1}s_i = \bigsqcup_{j\in J_i} r_j \searrow N^{-1}[p_j, q_j]$ and

$$f = N^{-1}u - I = \bigsqcup_{i\geq 0} N^{-1}(s_i - I) = \bigsqcup_{i\geq 0} t_i,$$

where $t_i := N^{-1}s_i - I = \bigsqcup_{j\in J_i} r_j \searrow (N^{-1}[p_j, q_j] - \lambda x.x)$ is a piecewise linear step function for each $i \geq 0$. From $\mathbf{I}f = \bigsqcup_{i\geq 0} \mathcal{E}(t_i)$ we obtain for $\perp_0 = \lambda x.B$, that

$$\begin{aligned} R_f(\perp_0) &= -\mathbf{I}f \circ (I + \perp_0) = -\bigsqcup_{i\geq 0} \mathcal{E}(t_i) \circ (I + \perp_0) \\ &= \bigsqcup_{i\geq 0} R_{t_i}(\perp_0), \end{aligned}$$

and hence,

$$\begin{aligned} (g_{i0})_i &= \bigsqcup_{n\geq 0} R_f^n(\perp_0) = \bigsqcup_{n\geq 0}\bigsqcup_{i\geq 0} R_{t_i}^n(\perp_0) \\ &= \bigsqcup_{n\geq 0} R_{t_n}^n(\perp_0). \end{aligned}$$

For each piecewise linear step functions $t \in A \to \mathbf{I}B$ and $s \in C \to \mathbf{I}B$, the map $R_t(s) = -\mathbf{I}t \circ (I + s)$ is the composition of two piecewise linear step functions and is thus a piecewise linear step function. It follows by a simple induction that for each $n \geq 0$, the map $\alpha_n := R^n_{t_n}(\bot_0)$ is a piecewise linear step function. Finally, we have

$$u^{-1} = (I + g_0) \circ N^{-1} = \bigsqcup_{n\geq 0} (I + \alpha_n) \circ N^{-1},$$

where each $A_l := (I + \alpha_l) \circ N^{-1}$ is a piecewise linear step function, made up of single-step functions with parallelograms as their domains.

(iii) With N as above, by Proposition 4.1, $(g_{i0})_i$ is C^1 and thus from $u^{-1} = (I + (g_{i0})_i) \circ N^{-1}$ we obtain: $(u^{-1})' = (I + (g_{i0})'_i) \circ N^{-1}$, where $(g_{i0})'_i$ is the unique fixed point of $S_{(\frac{df}{dx},(g_{i0})_i)}$. Hence, it is sufficient to show that an increasing sequence of polynomial step functions with lub $(g_{i0})'_i$ can be effectively obtained. Let α_l ($l \geq 0$) be the piecewise linear step function as in (ii) with $(g_{i0})_i = \bigsqcup_{l\geq 0} \alpha_l$. Let β_l ($l \geq 0$) be an increasing sequence of piecewise linear step functions with $u' = \bigsqcup_{l\geq 0} \beta_l$. From $f = I - N^{-1}u$ we obtain: $\frac{df}{dx} = f' = I - N^{-1}u'$. Thus, $f' = \bigsqcup_{l\geq 0}(I - N^{-1}\beta_l)$, where $\gamma_l := I - N^{-1}\beta_l$ is a piecewise linear step function for each $l \geq 0$. It follows that $\mathbf{I}f' = \bigsqcup_{l\geq 0} \mathcal{E}(\gamma_l)$, where $\mathcal{E}(\gamma_l)$ is a piecewise linear step function for each $l \geq 0$. From the Scott continuity of the functional S we get for any $h \in X_1 \to X_1$, where $\mathrm{Id} = \frac{dI}{dx}$:

$$\begin{aligned}
S_{(f',(g_{i0})_i)}(h) &= S(f', (g_{i0})_i, h) \\
&= -\mathbf{I}f' \circ (I + (g_{i0})_i) \cdot (\mathrm{Id} + h) \\
&= \bigsqcup_{l\geq 0} -\mathcal{E}(\gamma_l) \circ (I + \alpha_l) \cdot (\mathrm{Id} + h) \\
&= \bigsqcup_{l\geq 0} S(\gamma_l, \alpha_l, h) \\
&= \bigsqcup_{l\geq 0} S_{(\gamma_l,\alpha_l)}(h).
\end{aligned}$$

Hence, $S^j_{(f',(g_{i0})_i)} = \bigsqcup_{l\geq 0} S^j_{(\gamma_l,\alpha_l)}$ for any $j \geq 0$.

If $\bot_1 = \lambda x.H$, then the uniqueness of the fixed point $S_{(f',(g_{i0})_i)}$ gives:

$$\begin{aligned}
(g_{i0})'_i = g_{\min} &= \bigsqcup_{j\geq 0} S^j_{(f',(g_{i0})_i)}(\bot_1) \\
&= \bigsqcup_{j\geq 0}\bigsqcup_{l\geq 0} S^j_{(\gamma_l,\alpha_l)}(\bot_1) \\
&= \bigsqcup_{l\geq 0} S^l_{(\gamma_l,\alpha_l)}(\bot_1).
\end{aligned}$$

Since, for each $l \geq 0$, α_l and γ_l are piecewise linear step functions, it follows from a simple induction that, for each $l, j \geq 0$, $S^j_{(\gamma_l,\alpha_l)}(\bot_1)$ is a polynomial step function. In particular, $B_l := S^l_{(\gamma_l,\alpha_l)}(\bot_1)$ is a polynomial step function.

(iv) Consider the two sequences A_l and B_l converging to u^{-1} and $(u^{-1})'$ respectively as constructed in (ii) and (iii). For each $l \geq 0$, we have $(A_l, B_l) \in D^1(\mathrm{Im}(u) \to [-a,a]^n)$ and the update [7, 8] $\mathrm{Up}(A_l, B_l)$ is a polynomial step function whose lower and upper parts are respectively the least and the greatest functions consistent with both A_l and B_l. It follows that $u^{-1} = \bigsqcup_{i\geq 0} \mathrm{Up}(A_l, B_l)$ and that the lower and upper parts give an increasing and a decreasing sequence of continuous piecewise polynomials converging in the C^1 norm to u^{-1} from below and above respectively.

6. Implicit functions

As in classical theory, the implicit function theorem in the domain-theoretic setting can be deduced from the inverse function theorem.

Theorem 6.1 Implicit Function Theorem.
Let $f : O \to \mathbb{R}^n$, where $O \subseteq \mathbb{R}^{n+k}$, be C^1 with $f(0,0) = 0$ where $(0,0) \in \mathbb{R}^n \times \mathbb{R}^k$. Assume

$$\det(\frac{\partial(f_1,\ldots,f_n)}{\partial(x_1,\ldots,x_n)})(0,0) \neq 0.$$

Then, there exists a k-dimensional open set $W \subseteq \mathbb{R}^k$ with $0 \in W$, and a unique $g : W \to \mathbb{R}^n$ with

(i) $g \in C^1$ on W, (ii) $g(0) = 0$, (iii) $f(g(t),t) = 0$ for $t \in W$.

Moreover, we have:

(iv) *Given two increasing sequences of linear rational step functions, of type $O \to \mathbf{I}\mathbb{R}^n$ and $O \to \mathbf{I}\mathbb{R}^{n\times(n+k)}_s$, converging to f and f' respectively, we can effectively obtain two increasing sequences of, respectively linear and polynomial, step functions converging to g and g' respectively.*

(v) *With the assumption in (iv), we can also effectively obtain an increasing sequence of polynomial step functions, whose lower and upper parts are continuous and piecewise polynomial, such that the two sequences of lower and upper parts converge in the C^1 norm, respectively from below and above, to g.*

Proof Let $u : O \to \mathbb{R}^{n+k}$ with $u_i(x,t) = f_i(x,t)$ for $1 \leq i \leq n$ and $u_i(x,t) = t_i$ for $n+1 \leq i \leq n+k$. Then, $u(0,0) = (0,0)$ and

$$\det u'(0,0) = \det(\frac{\partial(f_1,\ldots,f_n)}{\partial(x_1,\ldots,x_n)})(0,0) \neq 0.$$

By Theorem 5.2, there exists $a > 0$ with $[-a,a]^{n+k} \subseteq O$ such that the restriction $u : [-a,a]^{n+k} \to \mathbb{R}^{n+k}$ has a C^1

inverse $u^{-1} : \mathrm{Im}(u) \to \mathbb{R}^{n+k}$. Put $W = \pi_1(\mathrm{Im}(u))$ where $\pi_1 : \mathbb{R}^{n+k} \to \mathbb{R}^k$ is the projection to $\mathbb{R}^k$. Let $g : W \to \mathbb{R}^n$ with $g = \lambda t.u^{-1}(0,t)$. Then, as in the classical theory, $W \subseteq \mathbb{R}^k$ is open and g is the unique function that satisfies (i), (ii) and (iii). It remains to show (iv) and (v):

(iv) Since $[-a,a]^{n+k} \subseteq O$, we can restrict the domain of the step functions, given in the assumption, to $[-a,a]^{n+k}$. Suppose, therefore, we are given an increasing sequence of linear step functions $\theta_j : [-a,a]^{n+k} \to \mathbf{I}\mathbb{R}^n$, for $j \geq 0$, with $f = \bigsqcup_{j\geq 0} \theta_j$. The functional

$$s : ([-a,a]^{n+k} \to \mathbf{I}\mathbb{R}^n) \to ([-a,a]^{n+k} \to \mathbf{I}\mathbb{R}^{n+k}),$$

with $s(h)_j = h_j$ for $1 \leq j \leq n$ and $s(h)_j = \lambda x.x$ for $n+1 \leq j \leq n+k$ is Scott continuous and preserves linear step functions since the inclusion map $[-a,a]^{n+k} \to \mathbf{I}\mathbb{R}^{n+k}$ is clearly a linear step function. It follows that $u = \bigsqcup_{j\geq 0} s(\theta_j)$ is the lub of an increasing sequence of linear step functions.

Suppose we also have an increasing sequence of linear step functions $\psi_l : [-a,a]^{n+k} \to \mathbf{I}\mathbb{R}_s^{n\times(n+k)}$, for $l \geq 0$, with $f' = \bigsqcup_{l\geq 0} \psi_l$. Note that $u'_i = f'_i$ for $1 \leq i \leq n$ and $(u'_i)_j = \delta_{ij}$ for $n+1 \leq i \leq n+k$. Consider the functional $v : ([-a,a]^{n+k} \to (\mathbf{I}\mathbb{R}^{n\times(n+k)})_s) \to ([-a,a]^{n+k} \to (\mathbf{I}\mathbb{R}^{(n+k)\times(n+k)})_s)$, with $(v(\psi))_i = \psi_i$ for $1 \leq i \leq n$ and $(v(\psi))_{ij} = \delta_{ij}$ for $n+1 \leq i \leq n+k$ and $1 \leq j \leq n+k$. Then, v is Scott continuous and preserves linear step functions. It follows that we can effectively obtain an increasing sequence of piecewise linear step functions $v(\psi_l)$ with $u' = \bigsqcup_{l\geq 0} v(\psi_l)$.

By Theorem 5.2, we can effectively obtain two increasing sequence of linear step functions $(C_j)_{j\geq 0}$ with $u^{-1} = \bigsqcup_{j\geq 0} C_j$ and $(D_j)_{j\geq 0}$ with $(u^{-1})' = \bigsqcup_{l\geq 0} D_l$. The functional

$$t : (\mathrm{Im}(u) \to \mathbf{I}\mathbb{R}^{n+k}) \to (W \to \mathbf{I}\mathbb{R}^n),$$

with $t(h) = \lambda x.h(0,x)$ is Scott continuous and preserves linear step functions, since partial evaluation of a linear step function, on a subset of its arguments, results in a linear step function. Thus, $g = t(u^{-1}) = \bigsqcup_{j\geq 0} t(C_j)$ is obtained effectively as the lub of an increasing sequence of step function.

For the derivative, we obtain that $(g')_i = \lambda x.(u^{-1})'_{n+i}(0,x)$ and thus effectively obtain $(g')_i = \bigsqcup_{l\geq 0} \lambda x.(D_l)_{n+i}(0,x)$, where $\lambda x.(D_l)_{n+i}(0,x)$ is a polynomial step function.

(v) As in the proof of Theorem 5.2(iv).

7 Further work

We will investigate if the domain-theoretic results on the inverse of a Lipschitz function can provide a precise witness for the linear map stipulated in the classical theorem 1.1. The results in the paper is a step toward a theoretical foundation for a robust CAD system. Having obtained domain-theoretic versions of the inverse and implicit function theorems, which in particular provide local C^1 approximations to an implicit surface, the next step would be to be able to patch the local pieces of an implicit surface together to obtain, in particular, a closed connected orientable manifold given by implicit equations such as $f(x,y,z) = 0$ when 0 is a regular value of f. Furthermore, the domain-theoretic framework for geometric modelling developed in [6] combined with the results in this work lead to a domain of orientable closed Lipschitz manifolds. This will synthesize the domain-theoretic framework for geometry and that for differential calculus.

Acknowledgement. We would like to thank Ralph Howard for pointing out the work of F. Nevanlinna and R. Nevanlinna to us.

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