*Geophys. J. R. astr. Soc.* (1980) **62**, 41–48

# Inverse eigenvalue problems for the mantle

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Received 1979 September 19; in original form 1979 April 16

**Summary**. It is shown that if Adams–Williamson's equation is used in the lower mantle and if the mass of the core and the density at the core–mantle boundary are given then the density, the rigidity and the incompressibility are uniquely determined throughout the mantle and in the crust by the velocities of the *P*- and *S*-waves and by one torsional spectrum. The velocity of the *S*-waves in the upper mantle and in the crust can be replaced by an additional torsional spectrum.



## Introduction

The inverse problem for the Earth amounts to determining the density $\rho$, the incompressibility $k$ and the rigidity $\mu$ in the interior of the Earth. We represent the Earth as a sphere with radius $R$ and assume that $\rho$, $k$ and $\mu$ depend only on the distance $r$ to the centre. If the Earth consists of a perfect elastic isotropic material then the velocities $\alpha$ and $\beta$ of the *P*- and the *S*-waves can be expressed in terms of the elastic parameters, specifically $\alpha^2 = (\lambda + 2\mu)/\rho$ and $\beta^2 = \mu/\rho$ where $\lambda = k - (2/3)\mu$. Thus if $\alpha$ and $\beta$ are known then the Lamé parameters $\lambda$ and $\mu$ are determined if the density can be determined independently.

In this paper we will assume that the lower part of the mantle is chemically homogeneous and devoid of phase changes. The density distribution of the lower mantle can therefore be found by the Adams–Williamson's equation provided the mass $m_c$ of the core and the density $\rho_c$ at the core–mantle boundary are given. In Section 1 we will show that the density in the upper part of the mantle and in the crust is uniquely determined by the velocity of the *S*-waves and by one torsional spectrum. The elastic parameters are then completely determined. In Section 2 we will show that the density and the rigidity in the upper part of the mantle can be determined from two torsional spectra. In this case the velocity of the *S*-waves in the upper mantle is not needed. The reader should be warned that the terms upper and lower mantle is used in a very specific way. The terminology is defined in Section 1 and agree approximately with the usual convention.

Since we use Adams–Williamson's equation in the lower mantle, our earth model is closely connected to model A of Bullen. There is one difficulty, namely that the mass of the core and the density at the core–mantle boundary $r = R_c$ are not observable. However, there may be a one to one correspondence between these values and the mass of the Earth and the density at the surface. This can be checked – at least locally – by numerical experiments. The moment of inertia for the Earth cannot be used as a constraint since the moment of inertia for the core is unknown.

It has been pointed out by Anderssen & Chandler (1978) that if $\beta = \pi^{-1} \log (R/R_c) \lim (\omega_l/l) \cdot r$ then the density is not determined even if all torsional modes $\{\omega_l^2\}$ are given, see also Hald (1977). It is therefore the homogeneity assumption in the lower mantle that leads to uniqueness. If this assumption is not used then additional information must be provided to ensure uniqueness. Thus Sabatier (1978) has shown that the density and the rigidity in the mantle are uniquely determined by the eigenvalues and the normalizing constants of two torsional spectra. In his theory $\mu$ may even tend to zero at the core–mantle boundary.

The proofs are based on two ingredients. First we transform the differential equation for the torsional modes to Liouville normal form. To do this we assume that $\rho$, $\mu$ and $\lambda$ are twice differentiable. This assumption is not fulfilled in most earth models. The second ingredient is a slight extension of a theorem by Hochstadt & Lieberman (1978). They have shown that if the potential is known over half the interval and the boundary conditions are given, then the potential is uniquely determined by one spectrum. To determine the potential over half the interval we use the Adams–Williamson's equation. Our extention concerns the boundary conditions and is presented in Section 3.

## 1 One spectrum

In this Section we will show that if Adams–Williamson's equation holds in the lower mantle, then the density $\rho$ and the elastic parameters $\lambda$ and $\mu$ are uniquely determined throughout the mantle and the crust by the velocity of the $P$- and $S$-waves and by one torsional spectrum.

THEOREM 1

Let $R_c$ and $R$ be given and assume that $\rho$, $\mu$ and $\lambda$ are positive in the interval $R_c \leqslant r \leqslant R$. Consider the eigenvalue problem

$$-(r^4 \mu u')' + \frac{(n+2)(n-1)}{r^2} r^4 \mu u = \omega^2 r^4 \rho u \tag{1}$$

$$u'(R_c) = u'(R) = 0.$$

Assume that $\alpha^2 = (\lambda + 2\mu)/\rho$ and $\beta^2 = \mu/\rho$ are given and determine the constant $r_0$ in the interval $R_c < r < R$ such that

$$\int_{R_c}^{r_0} \beta^{-1} \, dr = \int_{r_0}^{R} \beta^{-1} \, dr. \tag{2}$$

Let $\Phi = \alpha^2 - (4/3)\beta^2$ and assume that $\rho$ and $m$ satisfy the differential equations

$$\rho' = -\frac{Gm\rho}{r^2 \Phi}, \quad m' = 4\pi r^2 \rho \tag{3}$$

for $R_c \leqslant r \leqslant r_0$ with initial conditions $\rho = \rho_c$ and $m = m_c$ at $r = R_c$. Then one spectrum $\{\omega_l^2\}$ determine $\rho(r)$, $\mu(r)$ and $\lambda(r)$ uniquely provided the functions are twice differentiable.

REMARK

Equation (1) is obtained by substituting $U = ru$ in the equation for the torsional modes of a spherically symmetric non-rotating earth, see Alterman, Jarosh & Pekeris (1959). The constant $n$ is called the angular order. It is a positive integer and has its origin in the separation of variables in the equation of motions. The value of $r_0$ corresponds to a depth of approximately 1200 km. The part of the mantle which lies below $r_0$ is here called the lower mantle and we denote the remainder as the upper mantle and the crust. Thus the domain in which we assume the validity of the Adams–Williamson's relation, i.e. equation (3), is contained in region D, which conventionally extends from the depth of approximately 1000 km and down to the core–mantle boundary. Finally $G$ is the gravitational constant and $m(r)$ is the mass included in a sphere of radius $r$.

PROOF

By using the Liouville transformation we can transform equation (1) to Liouville normal form. Let $r = R - z$. We introduce the new independent variable

$$x = \psi(z) = \frac{1}{K}\int_0^z \sqrt{\frac{\rho(R-\xi)}{\mu(R-\xi)}}\,d\xi = \frac{1}{K}\int_0^z \beta^{-1}(R-\xi)\,d\xi \tag{4}$$

$$K = \frac{1}{\pi}\int_0^{R-R_c} \sqrt{\frac{\rho(R-\xi)}{\mu(R-\xi)}}\,d\xi = \frac{1}{\pi}\int_{R_c}^{R} \beta^{-1}(r)\,dr. \tag{5}$$

Since $\rho$ and $\mu$ are positive, $\psi$ has an inverse function which we denote by $z = \phi(x)$. Because $\beta$ is given we see that the constant $K$ and the functions $\psi$ and $\phi$ are uniquely determined by the data. Instead of $u$ we introduce the dependent variable $y$ by

$$y(x) = f(x)u(r)$$

$$f(x) = r^2\ \sqrt[4]{\rho(r)\mu(r)}$$

where $r = R - \phi(x)$. Thus the Liouville transformation leads to the eigenvalue problem

$$-y'' + \left[\frac{f''(x)}{f(x)} + K^2\,\frac{(n+2)(n-1)}{r^2}\,\frac{\mu}{\rho}\right]y = \omega^2 K^2 y \tag{6}$$

on the interval $0 \leqslant x \leqslant \pi$ and with boundary conditions

$$y'(0) - hy(0) = y'(\pi) + Hy(\pi) = 0.$$

Here $h = f'(0)/f(0)$ and $H = -f'(\pi)/f(\pi)$. The constant $K$ can also be determined by the asymptotic behaviour of the eigenvalues since $\omega_l^2 K^2 = l^2 + 0(1)$.

Let $q(x)$ be the potential [...] in equation (6). We will show that $q$ is uniquely determined in $[0, \pi]$. We observe first that the function $r^{-2}\mu/\rho = \beta^2/r^2$ is evaluated at $r = R - \phi(x)$ and thus known for $0 \leqslant x \leqslant \pi$. From equation (3) follows that $\rho(r)$ is uniquely determined in the interval $R_c \leqslant r \leqslant r_0$. The point $r_0$ has been chosen such that it corresponds to $x = \pi/2$. Indeed by using equations (2), (4) and (5) we see that

$$(2/\pi)(x - \pi/2) = \int_r^{r_0} \beta^{-1}(r)\,dr \bigg/ \int_{R_c}^{r_0} \beta^{-1}(r)\,dr.$$

This shows that the interval $R_c \leqslant r \leqslant r_0$ is mapped onto the interval $\pi/2 \leqslant x \leqslant \pi$ in a one to one manner. Since $f(x)$ can be written as $r^2\sqrt{\rho(r)\beta(r)}$ with $r = R - \phi(x)$ we conclude that the function $f(x)$ and consequently also the potential $q(x)$ are uniquely determined in the interval $\pi/2 \leqslant x \leqslant \pi$. Finally the constant $H$ in the boundary condition at $x = \pi$ is equal to $f'(\pi)/f(\pi)$ and thus uniquely determined. We can now use Lemma 1, which will be proved in Section 3. It says that the constant $h$ and the potential $q(x)$ are uniquely determined by one spectrum $\{\omega_l^2\}$ corresponding to a fixed value of $n$. We observe now that the function $f(x)$ involves $\rho$, which is unknown, and $\beta$, which is known. To determine $f(x)$ in the interval $0 \leqslant x \leqslant \pi/2$ we solve the differential equation

$$f''(x) = \left[q(x) - K^2 \frac{(n+2)(n-1)}{[R-\phi(x)]^2} \beta^2 [R - \phi(x)]\right] f(x) \tag{7}$$

with $f$ and $f'$ given at $x = \pi/2$. Since $f(x) = r^2\sqrt{\rho\beta}$ we can recover $\rho(r)$ in the interval $r_0 \leqslant r \leqslant R$ by

$$\rho(r) = \frac{f^2[\psi(R-r)]}{r^4\beta(r)}.$$

Finally $\mu(r)$ and $\lambda(r)$ are obtained by $\mu = \beta^2\rho$ and $\lambda = (\alpha^2 - 2\beta^2)\rho$. This completes the proof.

The above proof is valid as long as $\rho'$ and $\mu'$ are piecewise continuously differentiable. Even these weaker assumptions are not fulfilled in many earth models. In practice the velocities $\alpha$ and $\beta$ of the $P$- and $S$-waves are not known. They are derived from the travel-time curves and may not be uniquely determined, even in principle. Our result is based on Adams–Williamson's equation. This is not essential. Any generalization of this equation can be used as long as it determines $\rho$ in terms of $\alpha$ and $\beta$ or some other data. The proof of Theorem 1 indicates a numerical algorithm, but the data must satisfy one constraint. The asymptotic behaviour of the torsional modes must be in agreement with the determination of $\beta$ and the depth to the core–mantle boundary. So far a numerical method for reconstructing a potential, which is known over half the interval, from one spectrum has not been developed.

## 2 Two spectra

In this section we will show that if Adams–Williamson's equation holds in the lower mantle, then the density and the rigidity are uniquely determined in the upper mantle and in the crust by two torsional spectra. The incompressibility $k$ can then be determined by the velocity of the $P$-waves.

THEOREM 2

Let $R_c$ and $R$ be given and assume that $\rho$, $\mu$ and $\lambda$ are positive and twice differentiable in the interval $R_c \leqslant r \leqslant R$. Consider the eigenvalue problem (1). Let $K = \lim (l/\omega_l)$ where $\{\omega_l^2(n)\}$ is the spectrum for a fixed value of $n$. Assume that $\alpha^2 = (\lambda + 2\mu)/\rho$ is given for $R_c \leqslant r \leqslant R$ and that $\beta^2 = \mu/\rho$ is given for $R_c \leqslant r \leqslant r_0$. Here $r_0$ is determined by

$$\int_{R_c}^{r_0} \beta^{-1}(r)\, dr = (\pi/2)K. \tag{8}$$

If $\rho$ and $m$ satisfy equation (3) for $R_c \leqslant r \leqslant r_0$ then $\rho(r)$, $\mu(r)$ and $\lambda(r)$ are uniquely determined throughout the interval $R_c \leqslant r \leqslant R$ by two spectra $\{\omega_l^2(n_1)\}$ and $\{\omega_l^2(n_2)\}$.

REMARK

By transforming equation (1) to Liouville normal form we see that for each $n, K^2\omega_l^2 = l^2 + 0(1)$. The value of $K$ is therefore independent of the choice of $n$.

PROOF

By using the Liouville transformation (4) and (5) we are lead to equation (6) with $n = n_1$ and $n_2$. Let $q_1(x)$ be the potential corresponding to $n = n_1$ and let $q_2(x)$ be the potential corresponding to $n = n_2$. Since $\beta^2 = \mu/\rho$ is not given in the interval $r_0 < r \leqslant R$ we must determine the functions $\phi$ and $\psi$ indirectly. We will determine $\phi(x)$ as the solution of a differential equation. From equations (4) and (5) follow that $\phi(\pi) = R - R_c$. This is the initial condition. Since $\phi$ is the inverse function of $\psi$, $\psi\,[\phi(x)] = x$. By differentiating both sides of this equation with respect to $x$ and using equation (4) we see that $\phi$ satisfies the differential equation

$$\phi'(x) = K\beta\,[R - \phi(x)]. \tag{9}$$

Since $\beta$ is given for $R_c \leqslant r \leqslant r_0$ we can determine $\phi(x)$ as long as $R - \phi(x) \leqslant r_0$. Assume now that $\phi(x) = R - r_0$. We will show that $x = \pi/2$. From equation (4) follows that

$$x = \frac{1}{K}\int_{r_0}^{R} \beta^{-1}(r)\,dr. \tag{10}$$

By combining equations (5) and (8) we see that even though $\beta$ is not yet determined,

$$\int_{r_0}^{R} \beta^{-1}\,(r)\,dr = \frac{1}{2}\int_{R_c}^{R} \beta^{-1}(r)\,dr.$$

Thus we conclude from equation (10) that $x = \pi/2$. We have therefore determined the function $\phi(x)$ in the interval $\pi/2 \leqslant x \leqslant \pi$.

Let $\rho(r)$ be determined by the differential equation (3). Since $f(x) = r^2\sqrt{\rho(r)\beta(r)}$ where $r = R - \phi(x)$ we see that the potentials $q_1$ and $q_2$ are uniquely determined in the interval $\pi/2 \leqslant x \leqslant \pi$. The constant $H = -f'(\pi)/f(\pi)$ is also uniquely determined. By using Lemma 1 twice we conclude that $q_1(x)$ and $q_2(x)$ are uniquely determined by the spectra $\{\omega_l^2(n_1)\}$ and $\{\omega_l^2(n_2)\}$.

We will now show how the functions $\rho(r)$ and $\mu(r)$ can be reconstructed. It follows from the definitions of $q_1$ and $q_2$ that

$$q_2(x) - q_1(x) = (n_2 - n_1)\,(n_2 + n_1 + 1)\beta^2(r)/r^2 \tag{11}$$

where $r = R - \phi(x)$. At this point there could be several functions $\beta$ and $\phi$ such that equation (11) holds. However, by combining equations (11) and (9) we see that

$$\phi'(x) = K\,[R - \phi(x)]\,\sqrt{\frac{q_2(x) - q_1(x)}{(n_2 - n_1)\,(n_2 + n_1 + 1)}}$$

with $\phi(0) = 0$. Since $q_1$ and $q_2$ are uniquely determined we conclude that $\phi$ is uniquely determined by the data. Thus, by using equation (11) we find that $\beta(r)$ is also uniquely determined. The function $\rho(r)$ can now be reconstructed as in the proof of Theorem 1 by replacing $q$ and $n$ in equation (7) by $q_1$ and $n_1$. This completes the proof.

As mentioned after the proof of Theorem 1, our smoothness assumptions for $\rho$, $\mu$ and $\lambda$ can be weakened considerably. The data used in Theorem 2 to establish the uniqueness is presumably slightly overdetermined. The inverse problem for the cylinder is in some ways similar to the one presented here. However, it can be shown that for the cylinder, the lowest eigenvalue in one of the torsional spectra is not needed to determine the density and the rigidity uniquely, see Hald (1977). The author has been unable to establish a similar result for the problem considered here. It is not known whether the inverse problem for the mantle is well-posed if two torsional spectra are used. The inverse problem for a cylinder is ill-posed if the eigenvalues are slightly perturbed in the least square sense, see Hald (1977). However, it may be well-posed for perturbations which are small in some other norm.

## 3 An inverse Sturm–Liouville problem

In this section we will present and prove the uniqueness result which has been used in the proofs of Theorems 1 and 2.

LEMMA 1

Consider the eigenvalue problems

$$\begin{aligned} &-u'' + q(x)u = \lambda u \\ &hu(0) - u'(0) = 0, \qquad Hu(\pi) + u'(\pi) = 0 \end{aligned} \tag{12}$$

$$\begin{aligned} &-u'' + \tilde{q}(x)u = \tilde{\lambda} u \\ &\tilde{h}u(0) - u'(0) = 0, \qquad \tilde{H}u(\pi) + u'(\pi) = 0 \end{aligned} \tag{13}$$

where $q$ and $\tilde{q}$ are integrable on $[0, \pi]$. Let $\lambda_j$ and $\tilde{\lambda}_j$ be the eigenvalues of equations (12) and (13) and assume that $\lambda_j = \tilde{\lambda}_j$ for all $j$. If $q(x) = \tilde{q}(x)$ for almost all $x$ in the interval $\pi/2 \leqslant x \leqslant \pi$ and if $H = \tilde{H}$, then $q(x) = \tilde{q}(x)$ almost everywhere and $h = \tilde{h}$.

REMARK

This lemma is a strengthened version of a theorem due to Hochstadt & Lieberman (1978). They assume that $h = \tilde{h}$ and $H = \tilde{H}$. The new result is sharp. If $q$ is given and the eigenvalues $\lambda_j$ are slightly perturbed then there exist a potential $\tilde{q}$ and a constant $\tilde{h}$ such that $\lambda_j$ corresponds to the perturbed eigenvalues. The constant $\tilde{h}$ will in general not be equal to $h$. The proof below is based on the theory of translations operators. However, the proof by Hochstadt & Lieberman can also be modified to accommodate the new assumptions.

PROOF

Let $u_j$ and $\tilde{u}_j$ be the eigenfunctions corresponding to the eigenvalues $\lambda_j$ and $\tilde{\lambda}_j$ and normalized such that their value at $x = 0$ is 1. If we multiply equation (12) by $\tilde{u}_j$ and equation (13) by $u_j$ and subtract, we obtain after integrating that

$$h - \tilde{h} + \int_0^{\pi} (q - \tilde{q}) u_j \tilde{u}_j \, dx = 0. \tag{14}$$

Here we have used the boundary conditions for $u_j$ and $\tilde{u}_j$. It is well known that the eigenfunctions of a Sturm–Liouville problem are of the form $\cos(jx) + 0(1/j)$, see, e.g. Jörgens

(1964). We replace now $u_j$ and $\tilde{u}_j$ in equation (14) by their asymptotic expansion. Letting $j$ tend to $\infty$ we infer from the Riemann–Lebesque lemma that

$$h - \tilde{h} + \frac{1}{2}\int_0^{\pi} (q - \tilde{q})\, dx = 0.$$

This result can also be derived from the asymptotic expansion of the eigenvalues. Since $q = \tilde{q}$ over half the interval we conclude from equation (14) that

$$\int_0^{\pi/2} (q - \tilde{q})\,(u_j\tilde{u}_j - 1/2)\, dx = 0 \tag{15}$$

for all $j$. We can now show that $q = \tilde{q}$ a.e. It is known that if a function $f$ is orthogonal over the interval $[0, \pi/2]$ to the products $u_j\tilde{u}_j$ for all $j$ then $f(x)$ is zero a.e., see Levitan (1964) Theorem 6.2, p. 78. The remainder of our proof is practically identical to Levitan's proof. It can be shown, by using the theory of translation operators, that if $0 \leqslant x \leqslant \pi/2$ then

$$u_j(x)\tilde{u}_j(x) = \frac{1}{2}\left[1 + u_j(2x) + \int_0^{2x} K(x, t)u_j(t)\, dt\right], \tag{16}$$

see Levitan (1964) p. 75. The kernel $K$ is continuous and can be characterized as the solution of a certain Goursat problem. Let $f = q - \tilde{q}$. By combining equations (15) and (16) we find after interchanging the order of integration that

$$\int_0^{\pi}\left[f\left(\frac{x}{2}\right) + \int_x^{\pi} K\left(\frac{t}{2}, x\right) f\left(\frac{t}{2}\right) dt\right] u_j(x)\, dx = 0$$

for all $j$. Since the eigenfunctions form a complete set we conclude that

$$f\left(\frac{x}{2}\right) + \int_x^{\pi} K\left(\frac{t}{2}, x\right) f\left(\frac{t}{2}\right) dt = 0$$

for almost all $x$ in $[0, \pi]$. But this is a homogeneous Volterra equation and its solution is identically zero. Thus $q = \tilde{q}$ a.e. and from equation (14) follows that $h = \tilde{h}$. This completes the proof.

In the theory of inverse Sturm–Liouville problems a result like Lemma 1 is often accompanied by a theorem in which it is not necessary to use the lowest eigenvalue to infer uniqueness provided the boundary conditions are fixed, see Borg (1946), Hochstadt (1973) and Hald (1978). The author has been unable to establish a similar result in this context. The smoothness of the potentials play a crucial role. If both $q$ and $\tilde{q}$ are continuous then Lemma 1 is still true under the weaker assumption that $\lambda_j = \tilde{\lambda}_j$ for all $j$ except possibly one. The exception need not be the lowest eigenvalue. To prove this result we observe that if $\lambda_j = \tilde{\lambda}_j$ for all $j \neq k$ then

$$f\left(\frac{x}{2}\right) + \int_x^{\pi} K\left(\frac{t}{2}, x\right) f\left(\frac{t}{2}\right) dt = \text{constant} \cdot u_k(x)$$

for all $x$ in $[0, \pi]$. However, because $q$ and $\tilde{q}$ are continuous, $f(\pi/2) = 0$, and since $u_k(\pi)$ is different from zero we see that the constant must be equal to zero. The proof is now completed as before. This indicates that if the eigenvalues corresponding to a smooth potential are slightly perturbed, then the potential corresponding to the perturbed eigenvalues can have a jump discontinuity at $x = \pi/2$. This observation may be of some help in the development of a numerical technique to solve the inverse problems presented in this paper.

## Acknowledgments

The work was carried out while the author was an Alfred P. Sloan Research Fellow.

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