

## INVERSE FACTORIAL-SERIES SOLUTIONS OF DIFFERENCE EQUATIONS

A. B. OLDE DAALHUIS

*School of Mathematics, King's Buildings, University of Edinburgh,  
Edinburgh EH9 3JZ, UK (adri@maths.ed.ac.uk)*

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*Abstract* We obtain inverse factorial-series solutions of second-order linear difference equations with a singularity of rank one at infinity. It is shown that the Borel plane of these series is relatively simple, and that in certain cases the asymptotic expansions incorporate simple resurgence properties. Two examples are included. The second example is the large  $a$  asymptotics of the hypergeometric function  ${}_2F_1(a, b; c; x)$ .

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### 1. Introduction

In this paper we study the resurgence of inverse factorial-series solutions of the second-order linear difference equation

$$w(z+2) + f(z)w(z+1) + g(z)w(z) = 0, \quad (1.1)$$

where  $f(z)/z$  and  $g(z)/z^2$  are analytic for  $|z| \geq \mathcal{A} > 2$ . The special case where  $f(z)$  is a linear function of  $z$  and where  $g(z)$  is a quadratic function of  $z$  is studied in [9]. This paper can be seen as a generalization of that paper.

The usual large  $z$  asymptotic expansions of this difference equation are of the form\*

$$\rho^z \sum_{s=0}^{\infty} \frac{c_s}{z^{s+\mu}} \quad (1.2)$$

(see, for example, [5]). In [5] it is shown that the Borel transform of the formal series (1.2) has infinitely many singularities in the complex Borel plane, and these give rise to infinitely many exponentially small terms in the complete asymptotic expansion that has

\* The solutions of (1.1) grow like factorials. Asymptotic expansion (1.2) is the expansion for the function  $y(z) = w(z)/\Gamma(z)$ .

(1.2) at its highest level. For an example of infinitely many exponentially small terms in a complete asymptotic expansion see [2] and [11], where the exponentially improved asymptotic expansion of the gamma function is discussed.

The original reason that we study inverse factorial-series solutions of the form

$$w_j(z) \sim \rho_j^z \sum_{s=0}^{\infty} a_{s,j} \Gamma(z - \alpha_j - s), \quad j = 1, 2, \quad (1.3)$$

is that these expansions show up in the asymptotic expansions of late coefficients in asymptotic expansions for differential equations and integrals. In [7] and [8] it is shown that inverse factorial series are the natural asymptotic basis in these problems, and in the second example we show that they are the natural basis for asymptotics of hypergeometric functions with large parameters, that is, the coefficients in the expansions are very simple. Note that an expansion of the form (1.2) can always be converted in an expansion of the form (1.3), and vice versa. The (normal) factorial-series solutions of (1.1) are of the form (2.1) and are discussed in [3] and [4]. These factorial series converge (slowly) in half-planes.

We will study the Borel transform of (1.3) and show that it has only three singularities in the complex Borel plane, located at  $1/\rho_1$ ,  $1/\rho_2$  and the origin. The singularities at  $1/\rho_j$  are simple in that they correspond to the two formal series solutions of the form (1.3) of (1.1). The singularity at the origin is more complicated.

In the case in which  $1/\rho_1$  is closer to  $1/\rho_2$  than it is to the origin, we give an asymptotic expansion for the coefficients  $a_{s,1}$ , as  $s \rightarrow \infty$ . The coefficients in this expansion are  $a_{n,2}$ . This phenomenon is called resurgence. In this case the divergent inverse factorial series (1.3), with  $j = 1$ , can also be optimally truncated and re-expanded in a new series. The coefficients in the re-expansion are again  $a_{n,2}$ . This exponentially improved asymptotic expansion determines the solution uniquely. The case in which  $1/\rho_1$  is closer to the origin than it is to  $1/\rho_2$  is more complicated and will be discussed in a future paper.

The structure of this paper is as follows. In §2 we state the main results, which are

- (i) formulae to compute all the coefficients on the right-hand side of (1.3);
- (ii) Theorem 2.1, which states that solutions of the form (1.3) exist;
- (iii) the large  $s$  asymptotics of the coefficients  $a_{s,j}$  in Theorem 2.2; and
- (iv) Theorem 2.3, which contains the exponentially improved version of expansion (1.3).

The Borel transform is introduced in §3, where we prove that the complex Borel plane contains only three singularities. Two of these singularities are relatively simple, that is, we are able to obtain local expansions near these singularities. For the Borel transforms that are defined near one of these ‘simple’ singularities, we determine the local behaviour near the other ‘simple’ singularity, and the growth at infinity.

The proofs of Theorems 2.1–2.3 are based on the results of §3 and are given in §4. Some remarks on the excluded cases are given in §5, and since the proofs of all of the

results of this paper are very technical, we also include §6, which contains two examples. The second example is the large  $a$  asymptotics of the Gauss hypergeometric function  ${}_2F_1(a, b; c; x)$ . In this case the inverse factorial series are convergent in certain  $x$ -regions. However, the sum of the convergent series is in general not equal to the special solutions of the difference equation that are defined via the Borel transform, that is, the complete asymptotic expansion of the sum of the convergent inverse factorial series contains more terms than just the convergent series itself!

**2. The main results**

Recall that we assume that  $f(z)/z$  and  $g(z)/z^2$  are analytic for  $|z| \geq \mathcal{A} > 2$ . The factorial-series expansions for these functions are

$$f(z) = f_0z + f_1 + \frac{f_2}{z+1} + \frac{f_3}{(z+1)(z+2)} + \frac{f_4}{(z+1)(z+2)(z+3)} + \dots, \tag{2.1 a}$$

$$g(z) = g_0(z-1)z + g_1z + g_2 + \frac{g_3}{z+1} + \frac{g_4}{(z+1)(z+2)} + \frac{g_5}{(z+1)(z+2)(z+3)} + \dots. \tag{2.1 b}$$

These series converge absolutely for  $\text{Re } z > \mathcal{A}$ . The integral representations for the coefficients are

$$f_k = \frac{1}{2\pi i} \oint_{C(\mathcal{A})} \frac{\Gamma(z+k-1)}{\Gamma(z+1)} f(z) dz, \quad g_k = \frac{1}{2\pi i} \oint_{C(\mathcal{A})} \frac{\Gamma(z+k-2)}{\Gamma(z+1)} g(z) dz, \tag{2.2}$$

$k = 0, 1, 2, \dots$ , where the contour of integration is the circle  $|z| = \mathcal{A}$ . We will need the estimates

$$|f_k| \leq \frac{\Gamma(\mathcal{A} + k - 1)}{\Gamma(\mathcal{A})} M(f, \mathcal{A}), \quad |g_k| \leq \frac{\Gamma(\mathcal{A} + k - 2)}{\Gamma(\mathcal{A})} M(g, \mathcal{A}), \tag{2.3}$$

where  $M(f, \mathcal{A}) = \max\{|f(z)| \mid |z| = \mathcal{A}\}$ . Note that the condition  $\mathcal{A} > 2$  guarantees that (2.3) holds for  $k = 0, 1$ .

Difference equation (1.1) has a formal solution of the form (1.3), where  $\rho$  is a solution of

$$\rho^2 + f_0\rho + g_0 = 0 \tag{2.4}$$

and

$$\alpha = 1 - \frac{f_1\rho + g_1}{f_0\rho + 2g_0}. \tag{2.5}$$

We call these solutions  $\rho_1$  and  $\rho_2$ , and we assume that  $g_0 \neq 0$  and  $\rho_1 \neq \rho_2$ . Let  $\alpha_1, \alpha_2$  correspond, respectively, to  $\rho_1, \rho_2$ . Then by using (2.4) we obtain

$$\alpha_1 = 1 + \frac{f_1\rho_1 + g_1}{\rho_1(\rho_1 - \rho_2)}, \quad \alpha_2 = 1 + \frac{f_1\rho_2 + g_1}{\rho_2(\rho_2 - \rho_1)}. \tag{2.6}$$

In the proofs of Theorems 2.2 and 2.3 we will assume that for  $j = 1, 2$ ,  $\text{Re } \alpha_j > 1$  and  $\alpha_j$  is a non-integer. If this is not the case, then we can replace the independent variable

$z$  by  $z - q$ , where we choose  $q$  such that for  $j = 1, 2$ ,  $\operatorname{Re}(\alpha_j + q) > 1$  and  $\alpha_j + q$  is a non-integer.

The coefficients are given by the recurrence relations

$$\begin{aligned} \rho_1(\rho_1 - \rho_2)(s + 1)a_{s+1,1} &= (g_0(\alpha_1 + s - 1)(\alpha_1 + s) + g_1(\alpha_1 + s) + g_2)a_{s,1} \\ &+ \sum_{q=0}^s \rho_1 f_{q+2} \sum_{k=0}^{s-q} (-)^k a_{s-k-q,1} \binom{k+q}{k} \frac{\Gamma(s + \alpha_1 + 1)}{\Gamma(s - k + \alpha_1 + 1)} \\ &+ \sum_{q=0}^{s-1} g_{q+3} \sum_{k=0}^{s-1-q} (-)^k a_{s-1-k-q,1} \binom{k+q}{k} \frac{\Gamma(s + \alpha_1 + 1)}{\Gamma(s - k + \alpha_1 + 1)} \end{aligned} \quad (2.7 a)$$

and

$$\begin{aligned} \rho_2(\rho_2 - \rho_1)(s + 1)a_{s+1,2} &= (g_0(\alpha_2 + s - 1)(\alpha_2 + s) + g_1(\alpha_2 + s) + g_2)a_{s,2} \\ &+ \sum_{q=0}^s \rho_2 f_{q+2} \sum_{k=0}^{s-q} (-)^k a_{s-k-q,2} \binom{k+q}{k} \frac{\Gamma(s + \alpha_2 + 1)}{\Gamma(s - k + \alpha_2 + 1)} \\ &+ \sum_{q=0}^{s-1} g_{q+3} \sum_{k=0}^{s-1-q} (-)^k a_{s-1-k-q,2} \binom{k+q}{k} \frac{\Gamma(s + \alpha_2 + 1)}{\Gamma(s - k + \alpha_2 + 1)}. \end{aligned} \quad (2.7 b)$$

Usually, the recurrence relations for the coefficients in (inverse) factorial series are very complicated. A consequence of taking factorial-series expansions (2.1) for the functions  $f(z)$  and  $g(z)$  is that here we have a relatively simple recurrence relation.

**Theorem 2.1.** *The difference equation (1.1) has solutions  $w_1(z)$  and  $w_2(z)$  with the properties*

$$w_1(z) \sim \rho_1^z \sum_{s=0}^{\infty} a_{s,1} \Gamma(z - \alpha_1 - s), \quad (2.8 a)$$

$$w_2(z) \sim \rho_2^z \sum_{s=0}^{\infty} a_{s,2} \Gamma(z - \alpha_2 - s), \quad (2.8 b)$$

as  $z \rightarrow +\infty$ , provided that  $\rho_1 \neq \rho_2$ .

**Theorem 2.2.** *As  $s \rightarrow \infty$ ,*

$$a_{s,1} \sim K_1 \left( \frac{\rho_2}{\rho_1 - \rho_2} \right)^{s+\alpha_1} \sum_{j=0}^{\infty} a_{j,2} \Gamma(s - j + \alpha_1 - \alpha_2) \left( \frac{\rho_1 - \rho_2}{\rho_1} \right)^{j+\alpha_2} \quad (2.9)$$

provided that

$$\left| \frac{1}{\rho_1} - \frac{1}{\rho_2} \right| < \left| \frac{1}{\rho_1} \right|, \quad (2.10)$$

and

$$a_{s,2} \sim K_2 \left( \frac{\rho_1}{\rho_2 - \rho_1} \right)^{s+\alpha_2} \sum_{j=0}^{\infty} a_{j,1} \Gamma(s - j + \alpha_2 - \alpha_1) \left( \frac{\rho_2 - \rho_1}{\rho_2} \right)^{j+\alpha_1} \tag{2.11}$$

provided that

$$\left| \frac{1}{\rho_2} - \frac{1}{\rho_1} \right| < \left| \frac{1}{\rho_2} \right|. \tag{2.12}$$

The  $K_j, j = 1, 2$ , are constants.

**Theorem 2.3.** *Provided that (2.10) holds, the difference equation (1.1) has a unique solution  $w_1(z)$ , determined by*

$$\begin{aligned} w_1(z) \sim & \rho_1^z \sum_{s=0}^{N_1-1} a_{s,1} \Gamma(z - \alpha_1 - s) + K_1 \rho_1^z \Gamma(z - \alpha_1 - N_1 + 1) \left( \frac{\rho_2}{\rho_1 - \rho_2} \right)^{N_1+\alpha_1} \\ & \times \sum_{j=0}^{\infty} \frac{a_{j,2} \Gamma(N_1 - j + \alpha_1 - \alpha_2)}{z - \alpha_2 - j} \left( \frac{\rho_1 - \rho_2}{\rho_1} \right)^{j+\alpha_2} \\ & \times {}_2F_1 \left( 1, N_1 - j + \alpha_1 - \alpha_2; \frac{\rho_1}{\rho_1 - \rho_2} \right), \end{aligned} \tag{2.13}$$

as  $z \rightarrow \infty$ . The optimum number of terms in the first sum of (2.13) is

$$N_1 = \left( 1 + \left| \frac{\rho_2}{\rho_1 - \rho_2} \right| \right)^{-1} |z|. \tag{2.14}$$

Provided that (2.12) holds, the difference equation (1.1) has a unique solution  $w_2(z)$ , determined by

$$\begin{aligned} w_2(z) \sim & \rho_2^z \sum_{s=0}^{N_2-1} a_{s,2} \Gamma(z - \alpha_2 - s) + K_2 \rho_2^z \Gamma(z - \alpha_2 - N_2 + 1) \left( \frac{\rho_1}{\rho_2 - \rho_1} \right)^{N_2+\alpha_2} \\ & \times \sum_{j=0}^{\infty} \frac{a_{j,1} \Gamma(N_2 - j + \alpha_2 - \alpha_1)}{z - \alpha_1 - j} \left( \frac{\rho_2 - \rho_1}{\rho_2} \right)^{j+\alpha_1} \\ & \times {}_2F_1 \left( 1, N_2 - j + \alpha_2 - \alpha_1; \frac{\rho_2}{\rho_2 - \rho_1} \right), \end{aligned} \tag{2.15}$$

as  $z \rightarrow \infty$ . The optimum number of terms in the first sum of (2.15) is

$$N_2 = \left( 1 + \left| \frac{\rho_1}{\rho_2 - \rho_1} \right| \right)^{-1} |z|. \tag{2.16}$$

The constants  $K_j, j = 1, 2$ , are the same as in Theorem 2.2.

In this paper we shall make repeated use of the integral representation for the beta integral:

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^{\infty} \frac{t^{x-1}}{(1+t)^{x+y}} dt, \quad \text{Re } x > 0, \text{ Re } y > 0. \tag{2.17}$$

### 3. The Borel transform

In this section we introduce the Borel transform. Locally it can be defined via dividing each term of the divergent asymptotic series by a suitable factorial (see (3.5)). For a more global representation we use integrals. In the case of differential equations, one usually uses a Laplace transform to define the Borel transform (see, for example, [7]). For difference equations it is more natural to use a Mellin transform.

We write the solutions of (1.1) as a Mellin transform

$$w_j(z) = -\frac{z!}{2\pi i} \int_{\infty}^{(+1/\rho_j)} Y_j(t) t^{-z-1} dt, \quad (3.1)$$

where the contour of integration starts at  $\infty$ , encircles the point  $t = 1/\rho_j$  once in the positive sense, and returns to its starting point. The difference equation for  $w_j(z)$  translates to the differential–integral equation

$$\begin{aligned} (1 - \rho_1 t)(1 - \rho_2 t) Y_j''(t) + (g_1 t + f_1) Y_j'(t) + \left(g_2 + \frac{f_2}{t}\right) Y_j(t) \\ + \sum_{s=0}^{\infty} \frac{t^{-s-2}}{s!} (g_{s+3} t + f_{s+3}) \int_{1/\rho_j}^t (t - \tau)^s Y_j(\tau) d\tau = 0. \end{aligned} \quad (3.2)$$

Estimates (2.3) show us that the sum in (3.2) converges in the half-plane

$$P_j = \left\{ t \mid \left| 1 - \frac{1}{t\rho_j} \right| < 1 \right\}. \quad (3.3)$$

In this half-plane we can interchange the integration and summation and use integral representations (2.2). We obtain

$$\begin{aligned} (1 - \rho_1 t)(1 - \rho_2 t) Y_j''(t) + (g_1 t + f_1) Y_j'(t) + \left(g_2 + \frac{f_2}{t}\right) Y_j(t) \\ + \frac{1}{2\pi i} \int_{1/\rho_j}^t Y_j(\tau) \oint_{C(\mathcal{A})} \frac{t^z}{\tau^{z+1}} \left(g(z) + \frac{z+1}{\tau} f(z)\right) dz d\tau = 0. \end{aligned} \quad (3.4)$$

This differential–integral equation can be used in the entire complex  $t$ -plane.

It is easy to check that

$$Y_j(t) = \sum_{s=0}^{\infty} a_{s,j} \Gamma(-\alpha_j - s) (1 - \rho_j t)^{s+\alpha_j} \quad (3.5)$$

is a formal solution for these differential–integral equations.

Let  $\tilde{P}_1$  be the half-plane  $P_1$ , with, in the case that  $1/\rho_2 \in P_1$ , a branch cut from  $1/\rho_2$  to  $\infty \exp(i \operatorname{ph}(1/\rho_2 - 1/\rho_1))$ . The definition for  $\tilde{P}_2$  is similar.

**Theorem 3.1.** *Differential–integral equation (3.2) has a unique solution  $Y_j(t)$  in  $\tilde{P}_j$  such that  $Y_j(t) \sim a_{0,j} \Gamma(-\alpha_j) (1 - \rho_j t)^{\alpha_j}$  as  $t \rightarrow 1/\rho_j$ . For this solution (3.5) is a convergent series expansion that converges in a neighbourhood of  $t = 1/\rho_j$ , and there exists a constant  $M_j$  such that  $Y_j(t) = \mathcal{O}(t^{M_j})$  as  $|t| \rightarrow \infty$  in  $\tilde{P}_j$ .*

**Proof.** Let  $Y_1(t) = (1 - \rho_1 t)^{\alpha_1} (1 + h(1 - \rho_1 t))$  and  $\beta = 1 - (\rho_1/\rho_2)$ . Then

$$\begin{aligned} g_0 t(t - \beta)h''(t) + g_0(\alpha_1 + 1)(t - \beta)h'(t) \\ = -\psi_0(t)(1 + h(t)) - \psi_1 t h'(t) \\ + \sum_{s=0}^{\infty} \frac{g_{s+3} + (\rho_1 f_{s+3}/(1-t))}{(1-t)^{s+1} s!} \int_0^t (x-t)^s \left(\frac{x}{t}\right)^{\alpha_1} (1+h(x)) dx, \end{aligned} \tag{3.6}$$

where

$$\psi_0(t) = \alpha_1(\alpha_1 - 1)g_0 + \alpha_1 g_1 + g_2 + \frac{\rho_1 f_2}{1-t} \quad \text{and} \quad \psi_1 = (\alpha_1 - 1)g_0 + g_1.$$

We get

$$\begin{aligned} h(t) = \int_0^t \frac{K(t, \tau)}{\tau - \beta} \left[ -\psi_0(\tau)(1 + h(\tau)) - \psi_1 \tau h'(\tau) \right. \\ \left. + \sum_{s=0}^{\infty} \frac{g_{s+3} + (\rho_1 f_{s+3}/(1-\tau))}{(1-\tau)^{s+1} s!} \int_0^\tau (x-\tau)^s \left(\frac{x}{\tau}\right)^{\alpha_1} (1+h(x)) dx \right] d\tau, \end{aligned} \tag{3.7}$$

where

$$K(t, \tau) = \frac{1 - (\tau/t)^{\alpha_1}}{g_0 \alpha_1}.$$

The  $\tau$ -integration in (3.7) will be along straight lines. Since  $\text{Re } \alpha_1 > 0$  we have

$$|K(t, \tau)| \leq \frac{2}{|g_0 \alpha_1|} \quad \text{and} \quad \left| \frac{\partial}{\partial t} K(t, \tau) \right| \leq \frac{1}{|t g_0|}. \tag{3.8}$$

Note that

$$\left| \sum_{s=0}^{\infty} \frac{g_{s+3} + (\rho_1 f_{s+3}/(1-\tau))}{(1-\tau)^{s+1} s!} \int_0^\tau (x-\tau)^s \left(\frac{x}{\tau}\right)^{\alpha_1} dx \right| \leq \psi_2(\tau), \tag{3.9}$$

where

$$\psi_2(\tau) = \sum_{s=0}^{\infty} \frac{\Gamma(\text{Re } \alpha_1 + 1)}{\Gamma(\text{Re } \alpha_1 + s + 2)} \left( |g_{s+3}| + \frac{|\rho_1 f_{s+3}|}{|1-\tau|} \right) \left( \frac{|\tau|}{|1-\tau|} \right)^{s+1}. \tag{3.10}$$

The final sum converges for  $\text{Re } \tau < \frac{1}{2}$ .

Let

$$\Psi_0(t) = \int_0^t \frac{|\psi_0(\tau)| + \psi_2(\tau)}{|1-\tau|} |d\tau|, \quad \Psi_1(t) = \int_0^t \frac{|\psi_1|}{|1-\tau|} |d\tau| \quad \text{and} \quad \mathcal{K}(t) = \sup_{\tau \in (0,t)} \left| \frac{1-\tau}{\tau-\beta} \right|. \tag{3.11}$$

We take  $h_0(t) = 0$  and, for  $p = 0, 1, 2, \dots$ ,

$$\begin{aligned} h_{p+1}(t) = \int_0^t \frac{K(t, \tau)}{\tau - \beta} \left[ -\psi_0(\tau)(1 + h_p(\tau)) - \psi_1 \tau h'_p(\tau) \right. \\ \left. + \sum_{s=0}^{\infty} \frac{g_{s+3} + (\rho_1 f_{s+3}/(1-\tau))}{(1-\tau)^{s+1} s!} \int_0^\tau (x-\tau)^s \left(\frac{x}{\tau}\right)^{\alpha_1} (1+h_p(x)) dx \right] d\tau. \end{aligned} \tag{3.12}$$

The reader can check that for  $p = 0, 1, 2, \dots$

$$\left. \begin{aligned} |h_{p+1}(t) - h_p(t)| &\leq \frac{2\mathcal{K}(t)}{|g_0\alpha_1|} \Psi_0(t) \frac{1}{p!} \left( \frac{2\mathcal{K}(t)}{|g_0\alpha_1|} \Psi_0(t) + \frac{\mathcal{K}(t)}{|g_0|} \Psi_1(t) \right)^p, \\ |t| |h'_{p+1}(t) - h'_p(t)| &\leq \frac{\mathcal{K}(t)}{|g_0|} \Psi_0(t) \frac{1}{p!} \left( \frac{2\mathcal{K}(t)}{|g_0\alpha_1|} \Psi_0(t) + \frac{\mathcal{K}(t)}{|g_0|} \Psi_1(t) \right)^p. \end{aligned} \right\} \quad (3.13)$$

Now let

$$h(t) = \sum_{p=0}^{\infty} (h_{p+1}(t) - h_p(t)). \quad (3.14)$$

Then

$$|h(t)| \leq \frac{2\mathcal{K}(t)}{|g_0\alpha_1|} \Psi_0(t) \exp \left( \frac{2\mathcal{K}(t)}{|g_0\alpha_1|} \Psi_0(t) + \frac{\mathcal{K}(t)}{|g_0|} \Psi_1(t) \right). \quad (3.15)$$

Let  $P(t)$  be the half-plane  $\operatorname{Re} t < \frac{1}{2}$ , with, in the case where  $\operatorname{Re} \beta < \frac{1}{2}$ , a branch cut from  $\beta$  to  $\infty \exp(i \operatorname{ph} \beta)$ . The sum in (3.14) converges uniformly in any compact set in  $P(t)$ . Hence,  $h(t)$  is a solution of (3.6) that is analytic in  $P(t)$ . From (3.10) and (2.3) it follows that in the case  $\operatorname{Re} \alpha_1 > \mathcal{A} + 1$  and  $t$  restricted to the half-plane  $\operatorname{Re} t \leq \frac{1}{2}$ , function  $\psi_2(t)$  is bounded. Since  $|\psi_0(t)|$  is also bounded in this half-plane, it follows from the definition of  $\Psi_j(t)$  that  $\Psi_j(t) = \mathcal{O}(\ln |t|)$  as  $|t| \rightarrow \infty$  in  $P(t)$ . Thus there exists a constant  $M$  such that

$$h(t) = \mathcal{O}(t^M), \quad \text{as } |t| \rightarrow \infty \text{ in } P(t). \quad (3.16)$$

To obtain this result we needed the assumption  $\operatorname{Re} \alpha_1 > \mathcal{A} + 1$ . If this is not the case, then we take an integer  $m$  such that  $\operatorname{Re} \alpha_1 + m > \mathcal{A} + 1$ . Let

$$v(z) = \frac{w(z)}{z(z-1)\cdots(z-m+1)}.$$

This new function is a solution of the difference equation

$$v(z+2) + \tilde{f}(z)v(z+1) + \tilde{g}(z)v(z) = 0,$$

where

$$\tilde{f}(z) = \frac{z-m+2}{z+2} f(z) \quad \text{and} \quad \tilde{g}(z) = \frac{(z-m+2)(z-m+1)}{(z+2)(z+1)} g(z).$$

Hence,  $\tilde{f}(z)/z$  and  $\tilde{g}(z)/z^2$  are analytic for  $|z| \geq \mathcal{A} > 2$ .

The new difference equation has formal solution

$$\rho_1^z \sum_{k=0}^{\infty} b_{k,1} \Gamma(z - \alpha_1 - m - k),$$

where the coefficients  $b_{k,1}$  are related to  $a_{s,1}$ :

$$a_{s,1} = \sum_{q=0}^m \binom{m}{q} \frac{(\alpha_1 + s)!}{(\alpha_1 + s - q)!} b_{s-q,1}.$$



Let

$$Y_{1,m}(t) = \sum_{k=0}^{\infty} b_{k,1} \Gamma(-\alpha_1 - m - k) (1 - \rho_1 t)^{k + \alpha_1 + m}.$$

It follows from the analysis that gave us (3.16) that there exists a constant  $M_1$  such that

$$Y_{1,m}(t) = \mathcal{O}(t^{M_1}), \quad \text{as } |t| \rightarrow \infty \text{ in } P(t).$$

The reader can check that

$$Y_1(t) = t^m Y_{1,m}^{(m)}(t).$$

Hence, also in the case  $\text{Re } \alpha_1 \leq \mathcal{A} + 1$  we have the estimate

$$Y_1(t) = \mathcal{O}(t^{M_1}), \quad \text{as } |t| \rightarrow \infty \text{ in } P(t).$$

When we compute the ‘Taylor’ series expansion of  $Y_1(t)$  at  $t = 1/\rho_1$  we see that (3.5), with  $j = 1$ , is the unique series expansion.  $\square$

**Theorem 3.2.** *The function  $Y_j(t)$  of Theorem 3.1 is analytic in the entire complex  $t$ -plane, except for the branch points at  $t = 0$ ,  $t = 1/\rho_1$  and  $t = 1/\rho_2$ . The function  $Y_1(t)$  is bounded as  $t$  approaches  $t = 1/\rho_2$  and the function  $Y_2(t)$  is bounded as  $t$  approaches  $t = 1/\rho_1$ .*

**Proof.** We integrate the  $\tau$ -integral in (3.4) by parts, and use the fact that  $Y_1(1/\rho_1) = 0$ . The result can be written as

$$\begin{aligned} & \frac{d}{dt} [(1 - \rho_1 t)^{1-\alpha_1} (1 - \rho_2 t)^{1-\alpha_2} Y_1'(t)] \\ &= \frac{-1}{2\pi i} (1 - \rho_1 t)^{-\alpha_1} (1 - \rho_2 t)^{-\alpha_2} \int_{1/\rho_1}^t Y_1'(x) \oint_{C(\mathcal{A})} \left(\frac{t}{x}\right)^z \left(\frac{g(z)}{z} + \frac{f(z)}{x}\right) dz dx. \end{aligned} \quad (3.17)$$

Write  $Y_1'(t) = (1 - \rho_1 t)^{\alpha_1 - 1} (1 + h(1 - \rho_1 t))$  and again  $\beta = 1 - (\rho_1/\rho_2)$ . The condition  $h(0) = 0$  gives us the integral equation

$$\begin{aligned} h(t) = (\beta - t)^{\alpha_2 - 1} \int_0^t (\beta - \tau)^{-\alpha_2} \left[ 1 - \alpha_2 + \frac{\tau^{-\alpha_1}}{\rho_1 \rho_2 2\pi i} \int_0^\tau (1 + h(x)) x^{\alpha_1 - 1} \right. \\ \left. \times \oint_{C(\mathcal{A})} \left(\frac{1 - \tau}{1 - x}\right)^z \left(\frac{g(z)}{z} + \frac{\rho_1 f(z)}{1 - x}\right) dz dx \right] d\tau. \end{aligned} \quad (3.18)$$

The domain  $\Omega$  is  $\mathbb{C}$  minus the half-lines  $[1, +\infty)$ ,  $[\beta, \infty \exp(i \text{ph } \beta))$ , and let  $D$  be a compact convex domain in  $\Omega$  containing the origin. We also define

$$\begin{aligned} M_D = \sup \left\{ \left| \left(\frac{x}{\tau}\right)^{\alpha_1 - 1} \right| \left| \frac{1}{|\rho_1 \rho_2| 2\pi} \oint_{C(\mathcal{A})} \left(\frac{1 - \tau}{1 - x}\right)^z \right| \right. \\ \left. \times \left( \left| \frac{g(z)}{z} \right| + \left| \frac{\rho_1 f(z)}{1 - x} \right| \right) |dz| \mid t \in D, \tau \in [0, t], x \in [0, \tau] \right\}, \end{aligned} \quad (3.19 a)$$

$$N_D = \sup \left\{ |\beta^{1-\alpha_2}| \int_C |(\beta - t + st)^{\alpha_2 - 2}| |ds| \mid t \in D \right\}, \quad (3.19 b)$$

where  $\mathcal{C}$  is the contour  $\tau(\beta - t)/(\beta - t\tau)$ ,  $\tau \in [0, 1]$ . We use in the following derivation the substitution  $\tau = s\beta/(\beta - t + st)$  and obtain

$$\begin{aligned} |(\beta - t)^{\alpha_2 - 1}| \int_0^t |(\beta - \tau)^{-\alpha_2}| |d\tau| &= |t| |(\beta - t)^{\alpha_2 - 1}| \int_0^1 |(\beta - \tau t)^{-\alpha_2}| d\tau \\ &= \frac{\beta}{|\beta^{\alpha_2}|} |t| \frac{\beta - t}{|\beta - t|} \int_{\mathcal{C}} |(\beta - t + st)^{\alpha_2}| \frac{ds}{(\beta - t + st)^2} \\ &\leq |t| N_D. \end{aligned} \quad (3.20)$$

Again, we take  $h_0(t) = 0$  and, for  $p = 0, 1, 2, \dots$ ,

$$\begin{aligned} h_{p+1}(t) &= (\beta - t)^{\alpha_2 - 1} \int_0^t (\beta - \tau)^{-\alpha_2} \left[ 1 - \alpha_2 + \frac{\tau^{-\alpha_1}}{\rho_1 \rho_2 2\pi i} \int_0^\tau (1 + h_p(x)) x^{\alpha_1 - 1} \right. \\ &\quad \left. \times \oint_{\mathcal{C}(\mathcal{A})} \left( \frac{1 - \tau}{1 - x} \right)^z \left( \frac{g(z)}{z} + \frac{\rho_1 f(z)}{1 - x} \right) dz dx \right] d\tau. \end{aligned} \quad (3.21)$$

We use (3.20) and obtain

$$\begin{aligned} |h_1(t)| &\leq |(\beta - t)^{\alpha_2 - 1}| \int_0^t |(\beta - \tau)^{-\alpha_2}| \left( |1 - \alpha_2| + \frac{M_D}{|\tau|} \int_0^\tau |dx| \right) |d\tau| \\ &\leq (M_D + |1 - \alpha_2|) N_D |t|, \end{aligned} \quad (3.22)$$

and

$$\begin{aligned} |h_2(t) - h_1(t)| &\leq M_D |(\beta - t)^{\alpha_2 - 1}| \int_0^t |(\beta - \tau)^{-\alpha_2}| \frac{1}{|\tau|} \int_0^\tau |h_1(x)| |dx| |d\tau| \\ &\leq (M_D + |1 - \alpha_2|) M_D N_D |(\beta - t)^{\alpha_2 - 1}| \int_0^t |(\beta - \tau)^{-\alpha_2}| \frac{1}{|\tau|} \int_0^{|\tau|} x dx |d\tau| \\ &= (M_D + |1 - \alpha_2|) M_D N_D |(\beta - t)^{\alpha_2 - 1}| \int_0^t |(\beta - \tau)^{-\alpha_2}| \frac{1}{2} |\tau| |d\tau| \\ &\leq \frac{1}{2} |t| (M_D + |1 - \alpha_2|) M_D N_D |(\beta - t)^{\alpha_2 - 1}| \int_0^t |(\beta - \tau)^{-\alpha_2}| |d\tau| \\ &\leq \frac{1}{2} |t|^2 (M_D + |1 - \alpha_2|) M_D N_D^2. \end{aligned} \quad (3.23)$$

The reader can check that for  $p = 0, 1, 2, \dots$

$$|h_{p+1}(t) - h_p(t)| \leq \frac{1}{(p+1)!} |t|^{p+1} (M_D + |1 - \alpha_2|) M_D^p N_D^{p+1}. \quad (3.24)$$

Now let

$$h(t) = \sum_{p=0}^{\infty} (h_{p+1}(t) - h_p(t)). \quad (3.25)$$

Then

$$|h(t)| \leq \frac{M_D + |1 - \alpha_2|}{M_D} (e^{M_D N_D |t|} - 1). \quad (3.26)$$

Hence,  $h(t)$  is analytic in  $\Omega$ .

To show that  $h(t)$  is bounded as  $t \rightarrow \beta$  we take  $D = [0, \beta)$ . Then

$$\begin{aligned} N_D &= \sup_{t \in (0, \beta)} |\beta^{1-\alpha_2}| \int_0^1 |(\beta - t + st)^{\alpha_2-2}| ds \\ &= \sup_{|t| \in (0, |\beta|)} |\beta|^{1-\operatorname{Re} \alpha_2} \int_0^1 (|\beta| - |t| + s|t|)^{\operatorname{Re} \alpha_2-2} ds \\ &= \sup_{|t| \in (0, |\beta|)} \frac{1}{|t|(\operatorname{Re} \alpha_2 - 1)} \left( 1 - \left( \frac{|\beta| - |t|}{|\beta|} \right)^{\operatorname{Re} \alpha_2-1} \right) = \frac{1}{|\beta|}. \end{aligned} \tag{3.27}$$

Thus

$$|h(t)| \leq \frac{M_D + |1 - \alpha_2|}{M_D} (e^{M_D|t|/|\beta|} - 1). \tag{3.28}$$

Hence,  $h(t)$  is bounded as  $t \rightarrow \beta$ , that is,  $Y_1(t)$  is bounded as  $t \rightarrow 1/\rho_2$ . □

**Theorem 3.3.** *Differential–integral equation (3.2) has a solution  $\tilde{Y}_j(t)$  that is analytic for  $t$  in a neighbourhood of  $1/\rho_j$ . This function is uniquely determined when we take  $\tilde{Y}_j(1/\rho_j) = 1$ .*

The proof of this theorem is very similar to the proof of Theorem 3.5. The main difference is that the right-hand side in (3.34) is zero. Thus all the  $c_s$  in the proof of Theorem 3.5 are zero. We omit the details.

**Theorem 3.4.** *Let  $Y(t)$  be a solution of differential–integral equation (3.4). Then there are constants  $A$  and  $B$  such that  $Y(t) = AY_j(t) + B\tilde{Y}_j(t)$ , where  $Y_j(t)$  and  $\tilde{Y}_j(t)$  are given in Theorems 3.1 and 3.3.*

**Proof.** Let  $Y_1(t)$  and  $\tilde{Y}_1(t)$  be the functions given in Theorems 3.1 and 3.3. We write  $Y(t) = A(t)Y_1(t)$  and substitute this in (3.4) and obtain

$$\begin{aligned} &(1 - \rho_1 t)(1 - \rho_2 t)Y_1(t)A''(t) + (2(1 - \rho_1 t)(1 - \rho_2 t)Y_1'(t) + (g_1 t + f_1)Y_1(t))A'(t) \\ &+ \frac{1}{2\pi i} \int_{1/\rho_1}^t (A(\tau) - A(t))Y_1(\tau) \oint_{C(A)} \frac{t^z}{\tau^{z+1}} \left( g(z) + \frac{z+1}{\tau} f(z) \right) dz d\tau = 0. \end{aligned} \tag{3.29}$$

We integrate the  $\tau$ -integral by parts and obtain

$$\begin{aligned} &(1 - \rho_1 t)(1 - \rho_2 t)Y_1(t)A''(t) + (2(1 - \rho_1 t)(1 - \rho_2 t)Y_1'(t) + (g_1 t + f_1)Y_1(t))A'(t) \\ &- \frac{1}{2\pi i} \int_{1/\rho_1}^t A'(\tau) \int_{1/\rho_1}^\tau Y_1(x) \oint_{C(A)} \frac{t^z}{x^{z+1}} \left( g(z) + \frac{z+1}{x} f(z) \right) dz dx d\tau = 0. \end{aligned} \tag{3.30}$$

Note that  $A'(t) = y(t) := (d/dt)(\tilde{Y}_1(t)/Y_1(t))$  is a solution. We substitute with respect to  $A'(t) = B(t)y(t)$  and obtain

$$(1 - \rho_1 t)(1 - \rho_2 t)Y_1(t)y(t)B'(t) - \frac{1}{2\pi i} \int_{1/\rho_1}^t (B(\tau) - B(t))y(\tau) \int_{1/\rho_1}^\tau Y_1(x) \oint_{C(\mathcal{A})} \frac{t^z}{x^{z+1}} \left( g(z) + \frac{z+1}{x} f(z) \right) dz dx d\tau = 0. \tag{3.31}$$

Again, we integrate by parts and obtain

$$(1 - \rho_1 t)(1 - \rho_2 t)Y_1(t)y(t)B'(t) = -\frac{1}{2\pi i} \int_{1/\rho_1}^t B'(\tau) \int_{1/\rho_1}^\tau y(x) \int_{1/\rho_1}^x Y_1(\tilde{x}) \oint_{C(\mathcal{A})} \frac{t^z}{\tilde{x}^{z+1}} \left( g(z) + \frac{z+1}{\tilde{x}} f(z) \right) dz d\tilde{x} dx d\tau. \tag{3.32}$$

Note that

$$(1 - \rho_1 t)(1 - \rho_2 t)Y_1(t)y(t) = \mathcal{O}(1) \quad \text{as } t \rightarrow \frac{1}{\rho_1},$$

and

$$\int_{1/\rho_1}^\tau y(x) \int_{1/\rho_1}^x Y_1(\tilde{x}) \oint_{C(\mathcal{A})} \frac{t^z}{\tilde{x}^{z+1}} \left( g(z) + \frac{z+1}{\tilde{x}} f(z) \right) dz d\tilde{x} dx = \mathcal{O}(1 - \rho_1 \tau) \quad \text{as } \tau \rightarrow \frac{1}{\rho_1}.$$

Hence, for  $t$  in a small neighbourhood of  $1/\rho_1$  the only solution of (3.32) is  $B'(t) = 0$ . Thus  $B(t) = B$ , a constant, and  $A(t) = A + B\tilde{Y}_1(t)/Y_1(t)$ , where  $A$  is a constant.  $\square$

**Theorem 3.5.** *Let  $Y_j(t)$ ,  $j = 1, 2$ , be the functions given in Theorem 3.1. Then there exist constants  $K_1$  and  $K_2$  such that*

$$Y_1(t) = K_1 \frac{2\pi i}{1 - e^{2\pi i \alpha_1}} Y_2(t) + \text{reg}(t - 1/\rho_2), \tag{3.33 a}$$

$$Y_2(t) = K_2 \frac{2\pi i}{1 - e^{2\pi i \alpha_2}} Y_1(t) + \text{reg}(t - 1/\rho_1), \tag{3.33 b}$$

where  $\text{reg}(t - 1/\rho_j)$  denotes a function that is analytic in a neighbourhood of  $t = 1/\rho_j$ .

**Proof.** Since  $Y_1(\tau)$  is bounded as  $\tau \rightarrow 1/\rho_2$ , the right-hand side of

$$(1 - \rho_1 t)(1 - \rho_2 t)Y''(t) + (g_1 t + f_1)Y'(t) + \left( g_2 + \frac{f_2}{t} \right)Y(t) + \frac{1}{2\pi i} \int_{1/\rho_2}^t Y(\tau) \oint_{C(\mathcal{A})} \frac{t^z}{\tau^{z+1}} \left( g(z) + \frac{z+1}{\tau} f(z) \right) dz d\tau = \frac{1}{2\pi i} \int_{1/\rho_2}^{1/\rho_1} Y_1(\tau) \oint_{C(\mathcal{A})} \frac{t^z}{\tau^{z+1}} \left( g(z) + \frac{z+1}{\tau} f(z) \right) dz d\tau \tag{3.34}$$

is well defined. Note that  $Y(t) = Y_1(t)$  is a solution. The solutions of the homogeneous version of (3.34) are all of the form  $Y(t) = AY_2(t) + B\tilde{Y}_2(t)$ . We will construct a solution of (3.34) that is analytic in a neighbourhood of  $t = 1/\rho_2$ .

First, we note that the right-hand side of (3.34) is analytic in a neighbourhood of  $t = 1/\rho_2$ . The Taylor series expansion

$$\frac{1}{2\pi i} \int_{1/\rho_2}^{1/\rho_1} Y_1(\tau) \oint_{C(\mathcal{A})} \frac{t^z}{\tau^{z+1}} \left( g(z) + \frac{z+1}{\tau} f(z) \right) dz d\tau = \sum_{s=0}^{\infty} c_s (1 - t\rho_2)^s, \tag{3.35}$$

where

$$c_s = \frac{1}{2\pi i} \int_{1/\rho_2}^{1/\rho_1} Y_1(\tau) \oint_{C(\mathcal{A})} \frac{\Gamma(s-z)\rho_2^{-z}}{s!\Gamma(-z)\tau^{z+1}} \left( g(z) + \frac{z+1}{\tau} f(z) \right) dz d\tau \tag{3.36}$$

converges for  $|1 - t\rho_2| < 1$ .

For  $t$  in a neighbourhood of  $1/\rho_2$  we can expand the  $z$ -integral on the left-hand side of (3.34), compare (3.2) and (3.4). We write (3.34) as

$$\begin{aligned} (1 - \rho_1 t)(1 - \rho_2 t)Y''(t) + (g_1 t + f_1)Y'(t) + \left( g_2 + \frac{f_2}{t} \right)Y(t) \\ + \sum_{p=0}^{\infty} \frac{t^{-p-2}}{p!} (g_{p+3}t + f_{p+3}) \int_{1/\rho_2}^t (t - \tau)^p Y(\tau) d\tau = \sum_{s=0}^{\infty} c_s (1 - t\rho_2)^s. \end{aligned} \tag{3.37}$$

We substitute the series expansion

$$Y(t) = \sum_{s=0}^{\infty} b_s (1 - t\rho_2)^s, \tag{3.38}$$

and obtain for the coefficients the recurrence relation

$$\begin{aligned} \rho_2(\rho_1 - \rho_2)(s+1)(s+1 - \alpha_2)b_{s+1} \\ = (\rho_1\rho_2s(s-1) + g_1s + g_2 + \rho_2f_2)b_s \\ + \sum_{m=0}^{s-1} b_m \left( \sum_{p=0}^{s-m-1} \binom{s-m-1}{p} \frac{m!}{(m+1+p)!} (-)^{p+1} g_{p+3} \right. \\ \left. + \sum_{p=0}^{s-m} \binom{s-m}{p} \frac{m!}{(m+p)!} (-)^p \rho_2 f_{p+2} \right) - c_s. \end{aligned} \tag{3.39}$$

Note that from (2.3) we obtain the estimate

$$\sum_{p=0}^{s-m-1} \binom{s-m-1}{p} \frac{m!}{(m+1+p)!} |g_{p+3}| \leq \frac{M(g, \mathcal{A})}{\Gamma(\mathcal{A})} \sum_{p=0}^{s-m-1} \binom{s-m-1}{p} \frac{\Gamma(\mathcal{A} + p + 1)m!}{(m+1+p)!}. \tag{3.40}$$

Since

$$\sum_{p=0}^{s-m-1} \binom{s-m-1}{p} \frac{\Gamma(\mathcal{A} + p + 1)}{(m + 1 + p)!} = \begin{cases} \frac{1}{\Gamma(m + 1 - \mathcal{A})} \int_0^1 t^{\mathcal{A}}(1 - t)^{m-\mathcal{A}}(1 + t)^{s-m-1} dt & \text{if } \mathcal{A} < m + 1, \\ \Gamma(\mathcal{A} - m) \int_0^1 (1 + e^{2\pi i\theta})^{\mathcal{A}} e^{2\pi i\theta(m-\mathcal{A}+1)} (2 + e^{2\pi i\theta})^{s-m-1} d\theta & \text{if } \mathcal{A} > m, \end{cases} \tag{3.41}$$

we obtain the estimate

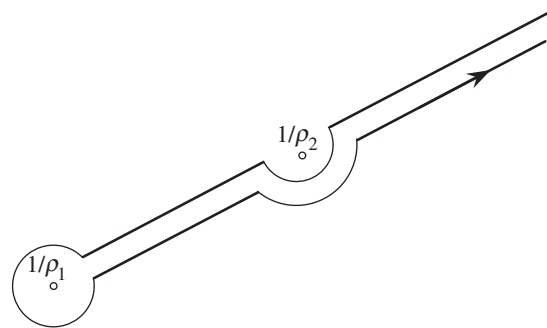
$$\sum_{p=0}^{s-m-1} \binom{s-m-1}{p} \frac{m!}{(m + 1 + p)!} |g_{p+3}| \leq \begin{cases} M(g, \mathcal{A}) \frac{\mathcal{A}}{m + 1} 2^{s-m-1} & \text{if } \mathcal{A} < m + 1, \\ M(g, \mathcal{A}) \frac{m! \Gamma(\mathcal{A} - m)}{\Gamma(\mathcal{A})} 2^{\mathcal{A}} 3^{s-m-1} & \text{if } \mathcal{A} > m. \end{cases} \tag{3.42}$$

Similarly,

$$\sum_{p=0}^{s-m} \binom{s-m}{p} \frac{m!}{(m + p)!} |f_{p+2}| \leq \begin{cases} M(f, \mathcal{A}) \mathcal{A} 2^{s-m} & \text{if } \mathcal{A} < m, \\ M(f, \mathcal{A}) \frac{m! \Gamma(\mathcal{A} - m + 1)}{\Gamma(\mathcal{A})} 2^{\mathcal{A}} 3^{s-m} & \text{if } \mathcal{A} > m - 1. \end{cases} \tag{3.43}$$

Since the Taylor series expansion (3.35) converges for  $|1 - t\rho_2| < 1$ , we can find a constant  $K$  such that  $|c_s| < K3^s$ , for all  $s$ . Now, let  $\beta_s = |b_s|$  for  $s \leq |\alpha_2| - 1$ , and, for  $s > |\alpha_2| - 1$ ,

$$\begin{aligned} & |\rho_2| |\rho_1 - \rho_2| (s + 1) (s + 1 - |\alpha_2|) \beta_{s+1} \\ &= (|\rho_1 \rho_2| s(s - 1) + |g_1| s + |g_2| + |\rho_2 f_2|) \beta_s \\ &+ \sum_{m=0}^{\min(s-1, [\mathcal{A}-1])} \beta_m M(g, \mathcal{A}) \frac{m! \Gamma(\mathcal{A} - m)}{\Gamma(\mathcal{A})} 2^{\mathcal{A}} 3^{s-m-1} \\ &+ \sum_{m=\min(s, [\mathcal{A}])}^{s-1} \beta_m M(g, \mathcal{A}) \frac{\mathcal{A}}{m + 1} 3^{s-m-1} \\ &+ \sum_{m=0}^{\min(s-1, [\mathcal{A}])} \beta_m |\rho_2| M(f, \mathcal{A}) \frac{m! \Gamma(\mathcal{A} - m + 1)}{\Gamma(\mathcal{A})} 2^{\mathcal{A}} 3^{s-m} \\ &+ \sum_{m=\min(s, [\mathcal{A}+1])}^{s-1} \beta_m |\rho_2| M(f, \mathcal{A}) \mathcal{A} 3^{s-m} \\ &+ K3^s, \end{aligned} \tag{3.44}$$



$0 \circ$

Figure 1. Contour  $\mathcal{L}_1$  with an indentation to the right of  $1/\rho_2$ .

where  $[\mathcal{A}]$  denotes the integer part of  $\mathcal{A}$ . Then  $\beta_s \geq |b_s|$  for all  $s$ . When  $s \geq \max(|\alpha_2|, [\mathcal{A}] + 2)$ , then we re-sum (3.44) and obtain

$$\begin{aligned} &|\rho_2| |\rho_1 - \rho_2| (s + 1)(s + 1 - |\alpha_2|)\beta_{s+1} - 3|\rho_2| |\rho_1 - \rho_2| s(s - |\alpha_2|)\beta_s \\ &= (|\rho_1\rho_2|s(s - 1) + |g_1|s + |g_2| + |\rho_2f_2|)\beta_s \\ &\quad - 3(|\rho_1\rho_2|(s - 1)(s - 2) + |g_1|(s - 1) + |g_2| + |\rho_2f_2|)\beta_{s-1} \\ &\quad + \left(\frac{\mathcal{A}}{s}M(g, \mathcal{A}) + 3\mathcal{A}|\rho_2|M(f, \mathcal{A})\right)\beta_{s-1}. \end{aligned} \tag{3.45}$$

The dominant part of this recurrence relation is

$$|\rho_2| |\rho_1 - \rho_2| \beta_{s+1} - (|\rho_1\rho_2| + 3|\rho_2| |\rho_1 - \rho_2|)\beta_s + 3|\rho_1\rho_2|\beta_{s-1} = 0.$$

The final recurrence relation has solutions  $3^s$  and  $|\rho_1/(\rho_1 - \rho_2)|^s$ . Thus the series  $\sum \beta_s(1 - t\rho_2)^s$  converges on the disc  $|1 - t\rho_2| < \min(\frac{1}{3}, |(\rho_1 - \rho_2)/\rho_1|)$ . Hence, (3.38) converges on the same disc.

We have shown that (3.34) has a solution  $\hat{Y}(t)$  that is analytic in a neighbourhood of  $t = 1/\rho_2$ . Since  $Y(t) = Y_1(t)$  is also a solution, we can find constants  $A$  and  $B$  such that  $Y_1(t) = AY_2(t) + B\tilde{Y}_2(t) + \hat{Y}(t)$ . Hence, (3.33 a) holds.  $\square$

#### 4. The proofs of Theorems 2.1–2.3

In this section we will use the notation  $\infty\beta$  for  $\infty e^{i \text{ph} \beta}$ .

**Proof of Theorem 2.1.** Let  $\mathcal{L}_1$  be the contour that starts at  $\infty/\rho_1$ , encircles the point  $t = 1/\rho_1$  once in the positive sense, and returns to its starting point. In the case that this contour encounters  $1/\rho_2$ , that is  $\rho_1/\rho_2 > 1$ , we have to indent the contour on the left or right of  $1/\rho_2$ . We leave the choice to the reader. See figure 1.

Let

$$w_1(z) = -\frac{z!}{2\pi i} \int_{\mathcal{L}_1} Y_1(t)t^{-z-1} dt. \tag{4.1}$$

First we have to show that  $w_1(z)$  is a solution of (1.1). To ensure that all the sums converge uniformly, we take  $\text{Re } z \geq \mathcal{A} + \delta$ , where  $\delta$  is a positive constant, and obtain

$$\begin{aligned} &w_1(z+2) + f(z)w_1(z+1) + g(z)w_1(z) \\ &= w_1(z+2) + \left(f_0z + f_1 + \frac{f_2}{z+1} + \dots\right)w_1(z+1) + (g_0(z-1)z + g_1z + g_2 + \dots)w_1(z) \\ &= -\frac{z!}{2\pi i} \int_{\mathcal{L}_1} [\text{left-hand side of (3.2)}]t^{-z-1} dt = 0. \end{aligned} \tag{4.2}$$

Second we have to show that (2.8 a) is an asymptotic expansion. We substitute into (4.1) by means of a truncated version of (3.5), with  $j = 1$ , and obtain

$$w_1(z) = \rho_1^z \sum_{s=0}^{N-1} a_{s,1} \Gamma(z - \alpha_1 - s) + R_N(z), \tag{4.3}$$

where

$$R_N(z) = -\frac{z!}{(2\pi i)^2} \int_{\mathcal{L}_1} \oint_{\{t, 1/\rho_1\}} \left(\frac{t - 1/\rho_1}{\tau - 1/\rho_1}\right)^{N+\alpha_1} \frac{t^{-z-1} Y_1(\tau)}{\tau - t} d\tau dt. \tag{4.4}$$

The  $\tau$ -contour of integration is a closed contour that encircles  $t$  and  $1/\rho_1$  once in the positive sense. We collapse  $\mathcal{L}_1$  on to  $[1/\rho_1, \infty/\rho_1)$  and obtain

$$R_N(z) = (1 - e^{2\pi i \alpha_1}) \frac{z!}{(2\pi i)^2} \int_{1/\rho_1}^{\infty/\rho_1} \oint_{\{t, 1/\rho_1\}} \left(\frac{t - 1/\rho_1}{\tau - 1/\rho_1}\right)^{N+\alpha_1} \frac{t^{-z-1} Y_1(\tau)}{\tau - t} d\tau dt. \tag{4.5}$$

Again, in the case  $\rho_1/\rho_2 > 1$ , we have to indent the  $t$ -contour of integration.

Let  $d$  be a fixed positive constant such that  $d < |1 - (\rho_1/\rho_2)|$ . Then

$$\begin{aligned} R_N(z) &= (1 - e^{2\pi i \alpha_1}) \frac{z!}{(2\pi i)^2} \\ &\quad \times \int_{1/\rho_1}^{(1+d)/\rho_1} \oint_{\{t, 1/\rho_1\}} \left(\frac{t - 1/\rho_1}{\tau - 1/\rho_1}\right)^{N+\alpha_1} \frac{t^{-z-1} Y_1(\tau)}{\tau - t} d\tau dt + S_N(z, d), \end{aligned} \tag{4.6}$$

where

$$\begin{aligned} S_N(z, d) &= (1 - e^{2\pi i \alpha_1}) \frac{z!}{(2\pi i)^2} \int_{(1+d)/\rho_1}^{\infty/\rho_1} \oint_{\{t, 1/\rho_1\}} \left(\frac{t - 1/\rho_1}{\tau - 1/\rho_1}\right)^{N+\alpha_1} \frac{t^{-z-1} Y_1(\tau)}{\tau - t} d\tau dt \\ &= \left(\frac{\rho_1}{1+d}\right)^z z! \mathcal{O}(1), \end{aligned} \tag{4.7}$$



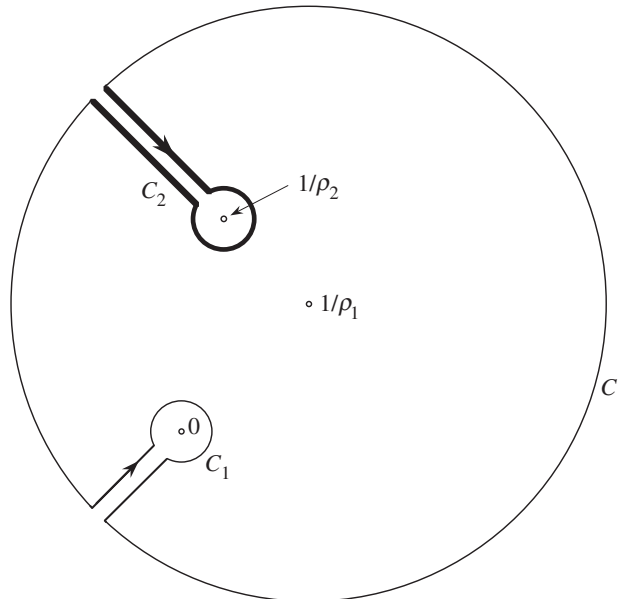


Figure 2. Contours  $C_1$  and  $C_2$  (the ‘bold’ contour).

as  $z \rightarrow +\infty$ . For the first term on the right-hand side of (4.6) we have the estimate

$$\begin{aligned} & \frac{z!}{(2\pi i)^2} \int_{1/\rho_1}^{(1+d)/\rho_1} \oint_{\{t, 1/\rho_1\}} \left( \frac{t - 1/\rho_1}{\tau - 1/\rho_1} \right)^{N+\alpha_1} \frac{t^{-z-1} Y_1(\tau)}{\tau - t} d\tau dt \\ &= \rho_1^z \frac{z!}{(2\pi i)^2} \int_0^d \oint_{[0, d]} \left( \frac{t}{\tau} \right)^{N+\alpha_1} \frac{(1+t)^{-z-1} Y_1((1+\tau)/\rho_1)}{\tau - t} d\tau dt \\ &= \rho_1^z z! \int_0^\infty t^{N+\operatorname{Re} \alpha_1} (1+t)^{-\operatorname{Re} z-1} dt \mathcal{O}(1) \\ &= \rho_1^z \Gamma(\operatorname{Re}(z - \alpha_1) - N) \mathcal{O}(1), \end{aligned} \tag{4.8}$$

as  $z \rightarrow +\infty$ . In the second line of (4.8) the  $\tau$ -contour of integration encircles the interval  $[0, d]$ . Thus for fixed positive integers  $N$  we have

$$R_N(z) = \rho_1^z \Gamma(\operatorname{Re}(z - \alpha_1) - N) \mathcal{O}(1), \quad \text{as } z \rightarrow +\infty. \tag{4.9}$$

□

**Proof of Theorem 2.2.** We use the integral representation

$$\begin{aligned} a_{s,1} &= \frac{-\rho_1}{\Gamma(-\alpha_1 - s)2\pi i} \oint_{\{1/\rho_1\}} \frac{Y_1(t)}{(1 - \rho_1 t)^{s+\alpha_1+1}} dt \\ &= \frac{-\rho_1}{\Gamma(-\alpha_1 - s)2\pi i} \int_{C_2} \frac{Y_1(t)}{(1 - \rho_1 t)^{s+\alpha_1+1}} dt - \frac{\rho_1}{\Gamma(-\alpha_1 - s)2\pi i} \int_{C_1} \frac{Y_1(t)}{(1 - \rho_1 t)^{s+\alpha_1+1}} dt, \end{aligned} \tag{4.10}$$

where  $C_1$  and  $C_2$  are depicted in figure 2. Let  $\varepsilon$  be a small positive constant such that  $\varepsilon < 1 - |\rho_1/\rho_2|$ . For the small loop encircling 0 in figure 2 we take radius  $\varepsilon/|\rho_1|$ . Then  $Y_1(t)(1 - \rho_1 t)^{-\alpha_1 - 1}$  is bounded along  $C_1$  and  $|1 - \rho_1 t|^{-s} \leq (1 - \varepsilon)^{-s}$  for all  $t \in C_1$ . Finally, for the  $C_2$  integral we use (3.33 a) and obtain

$$a_{s,1} = \frac{\rho_1 K_1}{(e^{2\pi i \alpha_1} - 1)\Gamma(-\alpha_1 - s)} \int_{C_2} \frac{Y_2(t)}{(1 - \rho_1 t)^{s+\alpha_1+1}} dt + \frac{(1 - \varepsilon)^{-s}}{\Gamma(-\alpha_1 - s)} \mathcal{O}(1), \quad (4.11)$$

as  $s \rightarrow \infty$ .

Now we substitute into the integral of (4.11) by means of a truncated version of (3.5), with  $j = 2$ , and obtain

$$\begin{aligned} & \frac{\rho_1 K_1}{(e^{2\pi i \alpha_1} - 1)\Gamma(-\alpha_1 - s)} \int_{C_2} \frac{Y_2(t)}{(1 - \rho_1 t)^{s+\alpha_1+1}} dt \\ &= \sum_{n=0}^{N-1} \frac{\rho_1 K_1 a_{n,2} \Gamma(-\alpha_2 - n)}{(e^{2\pi i \alpha_1} - 1)\Gamma(-\alpha_1 - s)} \int_{C_2} \frac{(1 - \rho_2 t)^{n+\alpha_2}}{(1 - \rho_1 t)^{s+\alpha_1+1}} dt + \mathcal{R}_N(s), \end{aligned} \quad (4.12)$$

where

$$\begin{aligned} \mathcal{R}_N(s) &= \frac{\rho_1 K_1}{2\pi i (e^{2\pi i \alpha_1} - 1)\Gamma(-\alpha_1 - s)} \\ &\quad \times \int_{C_2} \oint_{\{t, 1/\rho_2\}} \frac{1}{(1 - \rho_1 t)^{s+\alpha_1+1}} \left( \frac{1 - \rho_2 t}{1 - \rho_2 \tau} \right)^{N+\alpha_2} \frac{Y_2(\tau)}{\tau - t} d\tau dt. \end{aligned} \quad (4.13)$$

By taking the radius of the outer circle in figure 2 large enough we obtain for the terms in the sum of (4.12)

$$\begin{aligned} & \frac{\rho_1 K_1 a_{n,2} \Gamma(-\alpha_2 - n)}{(e^{2\pi i \alpha_1} - 1)\Gamma(-\alpha_1 - s)} \int_{C_2} \frac{(1 - \rho_2 t)^{n+\alpha_2}}{(1 - \rho_1 t)^{s+\alpha_1+1}} dt \\ &= -\rho_1 K_1 a_{n,2} \frac{\Gamma(-\alpha_2 - n)(e^{2\pi i \alpha_2} - 1)}{\Gamma(-\alpha_1 - s)(e^{2\pi i \alpha_1} - 1)} \int_{1/\rho_2}^{\infty(1/\rho_2 - 1/\rho_1)} \frac{(1 - \rho_2 t)^{n+\alpha_2}}{(1 - \rho_1 t)^{s+\alpha_1+1}} dt \\ &\quad + \frac{(1 - \varepsilon)^{-s}}{\Gamma(-\alpha_1 - s)} \mathcal{O}(1) \\ &= K_1 a_{n,2} \frac{\Gamma(-\alpha_2 - n)(e^{2\pi i \alpha_2} - 1)}{\Gamma(-\alpha_1 - s)(e^{2\pi i \alpha_1} - 1)} \left( \frac{\rho_2 - \rho_1}{\rho_1} \right)^{n+\alpha_2} \left( \frac{\rho_2}{\rho_2 - \rho_1} \right)^{s+\alpha_1} \int_0^\infty \frac{\tau^{n+\alpha_2}}{(1 + \tau)^{s+\alpha_1+1}} d\tau \\ &\quad + \frac{(1 - \varepsilon)^{-s}}{\Gamma(-\alpha_1 - s)} \mathcal{O}(1) \\ &= K_1 \left( \frac{\rho_2}{\rho_1 - \rho_2} \right)^{s+\alpha_1} a_{n,2} \Gamma(s - n + \alpha_1 - \alpha_2) \left( \frac{\rho_1 - \rho_2}{\rho_1} \right)^{n+\alpha_2} + \frac{(1 - \varepsilon)^{-s}}{\Gamma(-\alpha_1 - s)} \mathcal{O}(1), \end{aligned} \quad (4.14)$$

as  $s \rightarrow \infty$ . In this analysis we assumed that  $\text{Re}(s - n + \alpha_1 - \alpha_2) > 0$ .

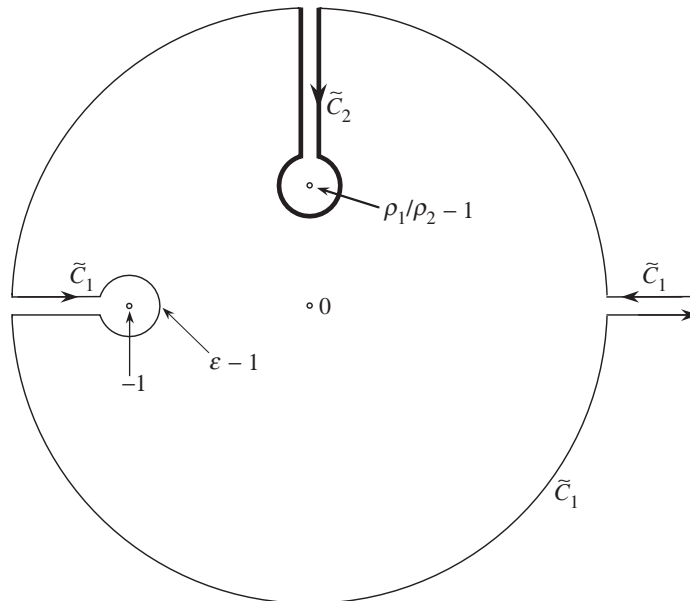


Figure 3. Contours  $\tilde{C}_1$  and  $\tilde{C}_2$  (the ‘bold’ contour).

The proof of

$$\mathcal{R}_N(s) = \left(\frac{\rho_2}{\rho_1 - \rho_2}\right)^{s+\alpha_1} \Gamma(s - N + \text{Re}(\alpha_1 - \alpha_2))\mathcal{O}(1) + \frac{(1 - \varepsilon)^{-s}}{\Gamma(-\alpha_1 - s)}\mathcal{O}(1), \quad (4.15)$$

as  $s \rightarrow \infty$ , is very similar to the proof of (4.9) and we omit the details. Thus we have shown that in the case  $|1 - (\rho_1/\rho_2)| < 1$ , that is (2.10), we have

$$a_{s,1} = K_1 \left(\frac{\rho_2}{\rho_1 - \rho_2}\right)^{s+\alpha_1} \sum_{n=0}^{N-1} a_{n,2} \Gamma(s - n + \alpha_1 - \alpha_2) \left(\frac{\rho_1 - \rho_2}{\rho_1}\right)^{n+\alpha_2} + \left(\frac{\rho_2}{\rho_1 - \rho_2}\right)^{s+\alpha_1} \Gamma(s - N + \text{Re}(\alpha_1 - \alpha_2))\mathcal{O}(1) + \frac{(1 - \varepsilon)^{-s}}{\Gamma(-\alpha_1 - s)}\mathcal{O}(1), \quad (4.16)$$

as  $s \rightarrow \infty$ . Since

$$\left|\frac{\rho_2}{\rho_1 - \rho_2}\right| > \frac{1}{1 - \varepsilon},$$

we can absorb the final terms in (4.16) in the penultimate term in (4.16). Hence, we have shown that (2.9) is an asymptotic expansion.  $\square$

**Proof of Theorem 2.3.** In this proof we assume that (2.10) holds. We also assume that  $0, 1/\rho_1$  and  $1/\rho_2$  are not collinear, that is  $\rho_1/\rho_2 \not\asymp 0$ . If this is not the case, then we have to make indentations in some of the contours of integration, and show that the contributions from the indentations are of the correct size.

In the proof of Theorem 2.1 we obtained integral representation (4.5) for the remainder. We replace the  $\tau$ -contour of integration by the union of  $C_1$  and  $C_2$ , given in figure 2, and then, in the double integral with  $C_1$  as the  $\tau$ -contour, we use the substitutions  $t = (\tilde{t} + 1)/\rho_1$  and  $\tau = (\tilde{\tau} + 1)/\rho_1$ , transforming  $C_1$  into  $\tilde{C}_1$ , given in figure 3. The result is

$$\begin{aligned}
 R_N(z) &= (1 - e^{2\pi i \alpha_1}) \frac{z!}{(2\pi i)^2} \int_{1/\rho_1}^{\infty/\rho_1} \int_{C_2} \left( \frac{t - 1/\rho_1}{\tau - 1/\rho_1} \right)^{N+\alpha_1} \frac{t^{-z-1} Y_1(\tau)}{\tau - t} d\tau dt \\
 &\quad + \rho_1^z (1 - e^{2\pi i \alpha_1}) \frac{z!}{(2\pi i)^2} \int_0^\infty \int_{\tilde{C}_1} \left( \frac{\tilde{t}}{\tilde{\tau}} \right)^{N+\alpha_1} \frac{(\tilde{t} + 1)^{-z-1} Y_1((\tilde{\tau} + 1)/\rho_1)}{\tilde{\tau} - \tilde{t}} d\tilde{\tau} d\tilde{t}.
 \end{aligned}
 \tag{4.17}$$

In this proof we assume that  $N = \lambda z + \mathcal{O}(1)$ , as  $z \rightarrow +\infty$ , where  $\lambda$  is a constant. We estimate the second term on the right-hand side of (4.17) by

$$\begin{aligned}
 &\rho_1^z (1 - e^{2\pi i \alpha_1}) \frac{z!}{(2\pi i)^2} \int_0^\infty \int_{\tilde{C}_1} \left( \frac{\tilde{t}}{\tilde{\tau}} \right)^{N+\alpha_1} \frac{(\tilde{t} + 1)^{-z-1} Y_1((\tilde{\tau} + 1)/\rho_1)}{\tilde{\tau} - \tilde{t}} d\tilde{\tau} d\tilde{t} \\
 &= \rho_1^z (1 - \varepsilon)^{-N} \Gamma(N + \operatorname{Re} \alpha_1 + 1) \Gamma(\operatorname{Re}(z - \alpha_1) - N) \mathcal{O}(1),
 \end{aligned}
 \tag{4.18}$$

as  $z \rightarrow +\infty$ . For the first term of the right-hand side of (4.17) we use (3.33 a) and substitute a truncated version of (3.5), with  $j = 2$ , and obtain

$$\begin{aligned}
 &(1 - e^{2\pi i \alpha_1}) \frac{z!}{(2\pi i)^2} \int_{1/\rho_1}^{\infty/\rho_1} \int_{C_2} \left( \frac{t - 1/\rho_1}{\tau - 1/\rho_1} \right)^{N+\alpha_1} \frac{t^{-z-1} Y_1(\tau)}{\tau - t} d\tau dt \\
 &= \frac{K_1 z!}{2\pi i} \int_{1/\rho_1}^{\infty/\rho_1} \int_{C_2} \left( \frac{t - 1/\rho_1}{\tau - 1/\rho_1} \right)^{N+\alpha_1} \frac{t^{-z-1} Y_2(\tau)}{\tau - t} d\tau dt \\
 &= \frac{K_1 z!}{2\pi i} \sum_{j=0}^{J-1} a_{j,2} \Gamma(-\alpha_2 - j) \int_{1/\rho_1}^{\infty/\rho_1} \int_{C_2} \left( \frac{t - 1/\rho_1}{\tau - 1/\rho_1} \right)^{N+\alpha_1} \frac{t^{-z-1} (1 - \rho_2 \tau)^{j+\alpha_2}}{\tau - t} d\tau dt \\
 &\quad + \frac{K_1 z!}{(2\pi i)^2} \int_{1/\rho_1}^{\infty/\rho_1} \int_{C_2} \oint_{\{\tau, 1/\rho_2\}} \left( \frac{t - 1/\rho_1}{\tau - 1/\rho_1} \right)^{N+\alpha_1} \left( \frac{\tau - 1/\rho_2}{\tilde{\tau} - 1/\rho_2} \right)^{J+\alpha_2} \frac{t^{-z-1} Y_2(\tilde{\tau})}{(\tau - t)(\tilde{\tau} - \tau)} d\tilde{\tau} d\tau dt \\
 &= I_1 - I_2 + I_3,
 \end{aligned}
 \tag{4.19}$$

where

$$\begin{aligned}
 I_1 &= \frac{K_1 z!}{2\pi i} (1 - e^{2\pi i \alpha_2}) \sum_{j=0}^{J-1} a_{j,2} \Gamma(-\alpha_2 - j) \int_{1/\rho_1}^{\infty/\rho_1} \int_{1/\rho_2}^{\infty(1/\rho_2 - 1/\rho_1)} \left( \frac{t - 1/\rho_1}{\tau - 1/\rho_1} \right)^{N+\alpha_1} \\
 &\quad \times \frac{t^{-z-1} (1 - \rho_2 \tau)^{j+\alpha_2}}{\tau - t} d\tau dt,
 \end{aligned}
 \tag{4.20}$$

$$I_2 = \frac{K_1 z!}{2\pi i} (1 - e^{2\pi i \alpha_2}) \sum_{j=0}^{J-1} a_{j,2} \Gamma(-\alpha_2 - j) \int_{1/\rho_1}^{\infty/\rho_1} \int_{1/\rho_2 + P(1/\rho_2 - 1/\rho_1)}^{\infty(1/\rho_2 - 1/\rho_1)} \left(\frac{t - 1/\rho_1}{\tau - 1/\rho_1}\right)^{N+\alpha_1} \times \frac{t^{-z-1}(1 - \rho_2 \tau)^{j+\alpha_2}}{\tau - t} d\tau dt, \tag{4.21}$$

$$I_3 = \frac{K_1 z!}{(2\pi i)^2} (1 - e^{2\pi i \alpha_2}) \int_{1/\rho_1}^{\infty/\rho_1} \int_{1/\rho_2}^{1/\rho_2 + P(1/\rho_2 - 1/\rho_1)} \oint_{\{\tau, 1/\rho_2\}} \left(\frac{t - 1/\rho_1}{\tau - 1/\rho_1}\right)^{N+\alpha_1} \times \left(\frac{\tau - 1/\rho_2}{\tilde{\tau} - 1/\rho_2}\right)^{J+\alpha_2} \frac{t^{-z-1} Y_2(\tilde{\tau})}{(\tau - t)(\tilde{\tau} - \tau)} d\tilde{\tau} d\tau dt, \tag{4.22}$$

where we have collapsed the contour  $C_2$  onto  $[1/\rho_2, 1/\rho_2 + P(1/\rho_2 - 1/\rho_1)]$ , where  $1/\rho_2 + P(1/\rho_2 - 1/\rho_1)$  is the point where  $C_2$  meets  $C_1$ . We can choose  $P$  as large as we want.

For  $I_2$  we give the estimate

$$I_2 = \rho_1^z \left(\frac{\rho_2}{\rho_1 - \rho_2}\right)^{N+\alpha_1} \frac{K_1 z!}{2\pi i} (1 - e^{2\pi i \alpha_2}) \sum_{j=0}^{J-1} a_{j,2} \Gamma(-\alpha_2 - j) \left(\frac{\rho_2 - \rho_1}{\rho_1}\right)^{j+\alpha_2} \times \int_0^\infty \int_P^\infty \left(\frac{t}{\tau + 1}\right)^{N+\alpha_1} \frac{(t + 1)^{-z-1} \tau^{j+\alpha_2}}{\tau + 1 - t\rho_2/(\rho_1 - \rho_2)} d\tau dt = \rho_1^z \left(\frac{\rho_2}{\rho_1 - \rho_2}\right)^N \Gamma(\operatorname{Re}(z - \alpha_1) - N) \Gamma(N + \operatorname{Re} \alpha_1 + 1) (P + 1)^{-N} \mathcal{O}(1), \tag{4.23}$$

as  $z \rightarrow +\infty$ , and for  $I_3$  we give the estimate

$$I_3 = \rho_1^z \left(\frac{\rho_2}{\rho_1 - \rho_2}\right)^{N+\alpha_1} \frac{K_1 z!}{(2\pi i)^2} (1 - e^{2\pi i \alpha_2}) \times \int_0^\infty \int_0^P \oint_{\{\tau, 0\}} \left(\frac{t}{\tau + 1}\right)^{N+\alpha_1} \times \left(\frac{\tau}{\tilde{\tau}}\right)^{J+\alpha_2} \frac{(t + 1)^{-z-1} Y_2(1/\rho_2 + \tilde{\tau}(1/\rho_2 - 1/\rho_1))}{(\tau + 1 - t\rho_2/(\rho_1 - \rho_2))(\tilde{\tau} - \tau)} d\tilde{\tau} d\tau dt = \rho_1^z \left(\frac{\rho_2}{\rho_1 - \rho_2}\right)^N \Gamma(\operatorname{Re}(z - \alpha_1) - N) \Gamma(N - J + \operatorname{Re}(\alpha_1 - \alpha_2)) \mathcal{O}(1), \tag{4.24}$$

as  $z \rightarrow +\infty$ .

Finally, for the terms in the sum in  $I_1$  we use the change of variables

$$t = \frac{1}{\rho_1} (x + 1)(y + 1), \quad \tau = \frac{1}{\rho_1} + \left(\frac{\rho_1 - \rho_2}{\rho_2}\right) \frac{t - 1/\rho_1}{x}, \tag{4.25}$$

and obtain

$$\begin{aligned}
 & \frac{K_1 z!}{2\pi i} (1 - e^{2\pi i \alpha_2}) a_{j,2} \Gamma(-\alpha_2 - j) \int_{1/\rho_1}^{\infty/\rho_1} \int_{1/\rho_2}^{\infty(1/\rho_2 - 1/\rho_1)} \left( \frac{t - 1/\rho_1}{\tau - 1/\rho_1} \right)^{N+\alpha_1} \\
 & \qquad \qquad \qquad \times \frac{t^{-z-1} (1 - \rho_2 \tau)^{j+\alpha_2}}{\tau - t} d\tau dt \\
 & = \rho_1^z \left( \frac{\rho_2}{\rho_1 - \rho_2} \right)^{N+\alpha_1} \left( \frac{\rho_2 - \rho_1}{\rho_1} \right)^{j+\alpha_2} \frac{K_1 z!}{2\pi i} (1 - e^{2\pi i \alpha_2}) a_{j,2} \Gamma(-\alpha_2 - j) \int_0^\infty \frac{y^{j+\alpha_2}}{(y+1)^{z+1}} dy \\
 & \qquad \qquad \qquad \times \int_0^\infty \frac{x^{N-j+\alpha_1-\alpha_2-1} (x+1)^{-z+j+\alpha_2}}{1 + x\rho_2/(\rho_2 - \rho_1)} dx \\
 & = \rho_1^z \left( \frac{\rho_2}{\rho_1 - \rho_2} \right)^{N+\alpha_1} \left( \frac{\rho_2 - \rho_1}{\rho_1} \right)^{j+\alpha_2} \frac{K_1}{2\pi i} (1 - e^{2\pi i \alpha_2}) a_{j,2} \Gamma(-\alpha_2 - j) \Gamma(\alpha_2 + j + 1) \\
 & \qquad \qquad \times \frac{\Gamma(N - j + \alpha_1 - \alpha_2) \Gamma(z - \alpha_1 - N + 1)}{z - \alpha_2 - j} {}_2F_1 \left( 1, N - j + \alpha_1 - \alpha_2; \frac{\rho_1}{\rho_1 - \rho_2} \right) \\
 & = K_1 \rho_1^z \Gamma(z - \alpha_1 - N + 1) \left( \frac{\rho_2}{\rho_1 - \rho_2} \right)^{N+\alpha_1} \frac{a_{j,2} \Gamma(N - j + \alpha_1 - \alpha_2)}{z - \alpha_2 - j} \left( \frac{\rho_1 - \rho_2}{\rho_1} \right)^{j+\alpha_2} \\
 & \qquad \qquad \qquad \times {}_2F_1 \left( 1, N - j + \alpha_1 - \alpha_2; \frac{\rho_1}{\rho_1 - \rho_2} \right), \tag{4.26}
 \end{aligned}$$

where we have obtained the hypergeometric function via the integral representation (3.6.3) in [6]. Since we assume that (2.10) holds, we can combine the estimates (4.18), (4.23) and (4.24) to produce the result

$$\begin{aligned}
 R_N(z) & = K_1 \rho_1^z \Gamma(z - \alpha_1 - N + 1) \left( \frac{\rho_2}{\rho_1 - \rho_2} \right)^{N+\alpha_1} \\
 & \times \sum_{j=0}^{J-1} \frac{a_{j,2} \Gamma(N - j + \alpha_1 - \alpha_2)}{z - \alpha_2 - j} \left( \frac{\rho_1 - \rho_2}{\rho_1} \right)^{j+\alpha_2} \\
 & \times {}_2F_1 \left( 1, N - j + \alpha_1 - \alpha_2; \frac{\rho_1}{\rho_1 - \rho_2} \right) \\
 & + \rho_1^z \left( \frac{\rho_2}{\rho_1 - \rho_2} \right)^N \Gamma(\operatorname{Re}(z - \alpha_1) - N) \Gamma(N - J + \operatorname{Re}(\alpha_1 - \alpha_2)) \mathcal{O}(1), \tag{4.27}
 \end{aligned}$$

as  $z \rightarrow +\infty$ . In this result  $J$  is a fixed positive integer. Recall that we assume that  $N = \lambda z + \mathcal{O}(1)$ . The reader can check that

$$\lambda = \left( 1 + \left| \frac{\rho_2}{\rho_1 - \rho_2} \right| \right)^{-1} \tag{4.28}$$

minimizes the final term in (4.27).

With this choice for  $\lambda$  and  $J$  large enough the final term in (4.27) is  $o(\rho_2^z \Gamma(z - \alpha_2))$ . Hence,  $w_1(z)$  is uniquely determined by (2.13).  $\square$

**5. Some remarks on the excluded cases**

For (2.9) and (2.13) to hold, we need (2.10). The Borel transform  $Y_1(t)$  is via (3.5) defined in a neighbourhood of  $t = 1/\rho_1$ , and it has distant singularities at  $t = 1/\rho_2$  and  $t = 0$ . Condition (2.10) means that  $t = 1/\rho_2$  is the nearest singularity. In the case in which the origin is the nearest singularity we need to know the singular behaviour of  $Y_1(t)$  near  $t = 0$ . This case is much more complicated. The singular behaviour of  $Y_1(t)$  near  $t = 0$  depends on the singularities of  $f(z)$  and  $g(z)$  in the disc  $|z| \leq \mathcal{A}$ .

In the case in which these singularities are only poles, then we can evaluate the  $z$ -integral in (3.4), with  $j = 1$ . In this way we can obtain for  $Y_1(t)$  a higher-order linear ordinary differential equation for which  $t = 0$  is a regular singularity. Hence, the dominant singular behaviour of  $Y_1(t)$  near  $t = 0$  will be of the form

$$Y_1(t) \sim Kt^\beta (\ln t)^M, \quad \text{as } t \rightarrow \infty, \tag{5.1}$$

where  $K, \beta \in \mathbb{C}$  and  $M$  is a non-negative integer. Again, we will be able to obtain an asymptotic expansion for  $a_{s,1}$  as  $s \rightarrow \infty$  and a re-expansion of the form (2.13) for  $w_1(z)$ . The main difference will be that in these expansions the coefficients will not be  $a_{s,2}$ . Hence, it seems that we lose the resurgence property.

In the case in which either  $f(z)$  or  $g(z)$  has an essential singularity in the disc  $|z| \leq \mathcal{A}$  we have no information on the possible singular behaviour of  $Y_1(t)$  near  $t = 0$ .

**6. Two examples**

**Example 6.1.** We take  $\rho_1 = 1$  and  $\rho_2 = \frac{3}{2}$ . Hence,  $f_0 = -\frac{5}{2}$  and  $g_0 = \frac{3}{2}$ . For the other coefficients we take

$$f_k = \frac{-2}{3k}, \quad g_k = \frac{1}{3k}, \quad \text{for } k = 1, \dots, 5, \tag{6.1}$$

and  $f_k = g_k = 0$ , for  $k \geq 6$ . Thus  $\alpha_1 = \frac{5}{3}$  and  $\alpha_2 = \frac{1}{9}$ . To compute the constant  $K_1$  that appears in (2.9) and (2.13) we take  $s = 40$ , compute

$$\left. \begin{aligned} a_{40,1} &= 6.498\,951\,116\,498\,708\,068\,4 \times 10^{67}, \\ \left( \frac{\rho_2}{\rho_1 - \rho_2} \right)^{40+\alpha_1} \sum_{j=0}^{20} a_{j,2} \Gamma(40 - j + \alpha_1 - \alpha_2) \left( \frac{\rho_1 - \rho_2}{\rho_1} \right)^{j+\alpha_2} \\ &= 3.470\,227\,511\,697\,557\,373\,2 \times 10^{68} - 2.911\,866\,625\,166\,670\,549\,6i \times 10^{68}, \end{aligned} \right\} \tag{6.2}$$

via (2.7), and obtain from (2.9)

$$K_1 = 0.109\,898\,877\,075\,266\,717\,90 + 0.092\,216\,107\,220\,653\,536\,44i. \tag{6.3}$$

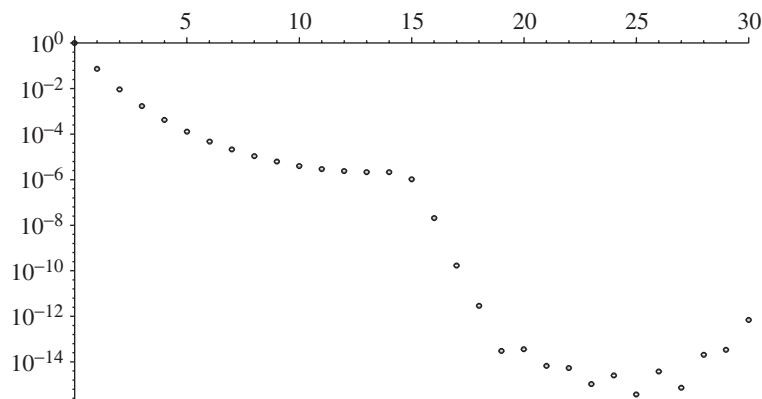


Figure 4. The relative size of the terms in (2.13).

In this example we want to approximate  $w_1(60)$  via (2.8 a) and (2.13). The optimum number of terms on the right-hand side of (2.8 a) is, according to (2.14), 15 terms. We take 15 terms in (2.8 a) and obtain for  $w_1(60)$  the approximation

$$1.465\ 652\ 254\ 306\ 984\ 730\ 9 \times 10^{77}. \tag{6.4}$$

For our second approximation we take in (2.13)  $N_1 = 15$  and 10 terms in the  $j$ -sum and obtain

$$1.465\ 650\ 555\ 270\ 188\ 171\ 3 \times 10^{77}. \tag{6.5}$$

The relative size of the terms in (2.13) is displayed in figure 4. Note that we truncate the original asymptotic expansion at its smallest term.

To compute the ‘exact’ value for  $w_1(60)$  we use 41 terms on the right-hand side of (2.8 a) and obtain

$$\left. \begin{aligned} w_1(161) &= 9.783\ 773\ 690\ 518\ 290\ 058\ 7 \times 10^{280}, \\ w_1(160) &= 6.178\ 267\ 433\ 484\ 865\ 222\ 2 \times 10^{278}, \end{aligned} \right\} \tag{6.6}$$

and use (1.1) in the backwards direction. In this way we obtain for  $w_1(60)$  the approximation

$$1.465\ 650\ 555\ 270\ 195\ 806\ 2 \times 10^{77}. \tag{6.7}$$

**Example 6.2.** As a second example we study the large  $a$  asymptotics of the Gauss hypergeometric function, but first we introduce this function. The Gauss hypergeometric function is defined via the series

$${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; x\right) = \sum_{s=0}^{\infty} \frac{(a)_s (b)_s}{(c)_s s!} x^s, \tag{6.8}$$

where Pochhammer’s symbol  $(a)_s$  is defined by  $(a)_s = \Gamma(a + s)/\Gamma(a)$ . The Gauss series (6.8) converges for all  $|x| < 1$ , and is defined elsewhere by analytic continuation. The



right-hand side of (6.8) can also be seen as an asymptotic expansion of the left-hand side for large  $|c|$ , and in that case the  $x$  domain of validity is much larger. Thus

$${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; x\right) \sim \sum_{s=0}^{n-1} \frac{(a)_s(b)_s}{(c)_s s!} x^s + \mathcal{O}(c^{-n}), \tag{6.9}$$

as  $|c| \rightarrow \infty$ . The precise restrictions on  $a, b$  and  $x$  are discussed in [13].

Let

$$w(a) = \Gamma(a) {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; x\right). \tag{6.10}$$

We will assume that  $x$  is fixed and  $0 < \text{ph } x < 2\pi$ . From the recurrence relation of the hypergeometric function with respect to the parameter  $a$  (see [1]) we obtain

$$w(a+2) + \frac{(2-x)a + 2 - c + (b-1)x}{x-1} w(a+1) + \frac{a(c-a-1)}{x-1} w(a) = 0. \tag{6.11}$$

Thus

$$f_0 = \frac{2-x}{x-1}, \quad f_1 = \frac{2-c+(b-1)x}{x-1}, \quad g_0 = \frac{1}{1-x}, \quad g_1 = \frac{2-c}{1-x}, \tag{6.12}$$

and  $f_n = g_n = 0$  for  $n = 2, 3, \dots$ . The reader can check that

$$\rho_1 = 1, \quad \rho_2 = \frac{1}{1-x}, \quad \alpha_1 = b, \quad \alpha_2 = c-b, \tag{6.13}$$

and that

$$a_{s,1} = \frac{(b)_s(b-c+1)_s}{s!} (-x)^{-s}, \quad a_{s,2} = \frac{(1-b)_s(c-b)_s}{s!} \left(\frac{1-x}{x}\right)^s. \tag{6.14}$$

It follows from (3.2) that since  $f_n = g_n = 0$  for  $n = 2, 3, \dots$  the Borel transforms have no singularities at the origin. Hence, the restrictions (2.10) and (2.12) do not apply in this example and Theorems 2.2 and 2.3 are valid for all non-zero  $\rho_1$  and  $\rho_2$  such that  $\rho_1 \neq \rho_2$ , that is, for all fixed  $x \notin \{0, 1, \infty\}$ .

Theorem 2.1 tells us that there are solutions  $w_1(a)$  and  $w_2(a)$  of (6.11) such that

$$w_1(a) \sim \sum_{s=0}^{\infty} a_{s,1} \Gamma(a-b-s) \sim \Gamma(a-b) \sum_{s=0}^{\infty} \frac{(b)_s(b-c+1)_s}{(b-a+1)_s s!} x^{-s}, \tag{6.15 a}$$

$$\begin{aligned} w_2(a) &\sim (1-x)^{-a} \sum_{s=0}^{\infty} a_{s,2} \Gamma(a+b-c-s) \\ &\sim \frac{\Gamma(a+b-c)}{(1-x)^a} \sum_{s=0}^{\infty} \frac{(1-b)_s(c-b)_s}{(c-a-b+1)_s s!} \left(1 - \frac{1}{x}\right)^s, \end{aligned} \tag{6.15 b}$$

as  $a \rightarrow +\infty$ . Thus

$$\Gamma(a)_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; x\right) \sim C_1(a)\Gamma(a-b)\sum_{s=0}^{\infty} \frac{(b)_s(b-c+1)_s}{(b-a+1)_s s!} x^{-s} + C_2(a)\frac{\Gamma(a+b-c)}{(1-x)^a} \sum_{s=0}^{\infty} \frac{(1-b)_s(c-b)_s}{(c-a-b+1)_s s!} \left(1-\frac{1}{x}\right)^s, \tag{6.16}$$

where  $C_1(a)$  and  $C_2(a)$  are periodic functions in  $a$  with period 1. To find the exact values of these periodic functions we are going to express  $w_1(a)$  and  $w_2(a)$  in terms of hypergeometric functions. The surprising fact is that although the right-hand sides of (6.15) converge in certain  $x$  domains to hypergeometric functions and according to (6.9) these hypergeometric functions have the right-hand sides of (6.15) as their asymptotic expansions, these hypergeometric functions are not equal to  $w_j(a)$ , which are defined via the Borel transform representation (3.1). Thus

$$w_1(a) \neq \Gamma(a-b)_2F_1\left(\begin{matrix} b, b-c+1 \\ b-a+1 \end{matrix}; \frac{1}{x}\right). \tag{6.17}$$

To obtain the correct expressions of  $w_1(a)$  and  $w_2(a)$  in terms of hypergeometric functions we use (6.14) in (3.5) and obtain

$$\left. \begin{aligned} Y_1(t) &= (1-t)^b \Gamma(-b)_2F_1\left(\begin{matrix} b, b-c+1 \\ b+1 \end{matrix}; \frac{1-t}{x}\right), \\ Y_2(t) &= \left(1-\frac{t}{1-x}\right)^{c-b} \Gamma(b-c)_2F_1\left(\begin{matrix} 1-b, c-b \\ c-b+1 \end{matrix}; \frac{t+x-1}{x}\right). \end{aligned} \right\} \tag{6.18}$$

We substitute these results into (3.1), use one integration by parts to simplify the integrand, and obtain

$$\left. \begin{aligned} w_1(a) &= \frac{\Gamma(a)\Gamma(a-c+1)}{\Gamma(a+b-c+1)} {}_2F_1\left(\begin{matrix} b, b-c+1 \\ a+b-c+1 \end{matrix}; 1-\frac{1}{x}\right), \\ w_2(a) &= (1-x)^{-a} \frac{\Gamma(a)\Gamma(a-c+1)}{\Gamma(a-b+1)} {}_2F_1\left(\begin{matrix} 1-b, c-b \\ a-b+1 \end{matrix}; \frac{1}{x}\right). \end{aligned} \right\} \tag{6.19}$$

In this derivation we use contour integral representations of hypergeometric functions (see [12]), but omit the details. The connection relation (3.9.7) in [6] of hypergeometric functions can be rewritten as

$$w_1(a) = \Gamma(a-b)_2F_1\left(\begin{matrix} b, b-c+1 \\ b-a+1 \end{matrix}; \frac{1}{x}\right) + \frac{\pi e^{(c-a-b)\pi i} x^{2b-c} (1-x)^{c-b}}{\Gamma(b)\Gamma(b-c+1) \sin((b-a)\pi)} w_2(a), \tag{6.20}$$

which clearly shows that in general (6.17) is correct. To obtain the period functions  $C_1(a)$  and  $C_2(a)$  in (6.16) we use the connection relation (3.9.13) in [6]

$$w(a) = (-x)^{-b} \frac{\Gamma(c)}{\Gamma(c-b)} w_1(a) + \left(\frac{x}{1-x}\right)^{b-c} \frac{\Gamma(c)}{\Gamma(b)} w_2(a), \tag{6.21}$$

and obtain

$$C_1(a) = (-x)^{-b} \frac{\Gamma(c)}{\Gamma(c-b)}, \quad C_2(a) = \left(\frac{x}{1-x}\right)^{b-c} \frac{\Gamma(c)}{\Gamma(b)}. \tag{6.22}$$

We have shown that for large  $a$  the hypergeometric function has asymptotic expansion (6.16), where  $C_1(a)$  and  $C_2(a)$  are given in (6.22). This result holds for fixed  $b, c$  and  $x$ , where  $0 < \text{ph } x < 2\pi$ . The large  $a$  asymptotics is also discussed in [6]. Equation (7.2.22) in [6] is connection relation (6.21), and the dominant behaviour is determined. However, asymptotic expansion (6.16) seems to be new.

The first part of Theorem 2.2 gives us the expansion

$$\frac{(b)_s(b-c+1)_s}{s!}(-x)^{-s} \sim K_1(-x)^{c-2b-s}(1-x)^{b-c} \sum_{j=0}^{\infty} (-1)^j \frac{(1-b)_j(c-b)_j}{j!} \Gamma(s-j+2b-c), \tag{6.23}$$

as  $s \rightarrow \infty$ . We compare this with the well-known result (see [10, (2.2.42)])

$$\frac{\Gamma(b+s)\Gamma(b-c+1+s)}{s!} \sim \sum_{j=0}^{\infty} (-1)^j \frac{(1-b)_j(c-b)_j}{j!} \Gamma(s-j+2b-c), \tag{6.24}$$

as  $s \rightarrow \infty$ . Hence, we can compute the constant

$$K_1 = \frac{(-x)^{2b-c}(1-x)^{c-b}}{\Gamma(b)\Gamma(b-c+1)} \quad \text{and similarly} \quad K_2 = \frac{x^{c-2b}(1-x)^{b-c}}{\Gamma(1-b)\Gamma(c-b)}. \tag{6.25}$$

We could now use these constants and obtain an exponentially improved version of asymptotic expansion (6.16). The numerical results are similar to those of Example 6.1. The re-expansions are in terms of hypergeometric functions with a large parameter. Hence, the approximants are roughly of the same complexity as the function that we try to approximate. Thus the exponentially improved version of asymptotic expansion (6.16) is not very interesting.

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