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INVERSE H-SEMIGROUPS AND t-SEMISIMPLE INVERSE H-SEMIGROUPS(1)

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Abstract. An H-semigroup is a semigroup such that both its right and left congruences are two-sided. A semigroup is t-semisimple provided the intersection of all its maximal modular congruences is the identity relation. We prove that a semigroup is an inverse H-semigroup if and only if it is a semilattice of disjoint Hamiltonian groups. Using the set E of idempotents of S as the semilattice, we show that an inverse H-semigroup S is t-semisimple if and only if for each pair of groups G_e, G_f in the semilattice, with $f \geqq e$ in E, the homomorphism $\varphi_{f,e}$ on G_f into G_e, defined by $a\varphi_{f,e} = ae$, is a monomorphism; and for each e in E, for each $a \neq e$ in G_e, there exists a subsemigroup T_p of S such that $a \notin T_p$ and, for each f in E, $T_p \cap G_f = H_f$, where $H_f = G_f$ or H_f is a maximal subgroup of prime index p in G_f.

Introduction. In this paper we adopt the definition of a Hamiltonian semigroup presented by R. H. Oehmke [6]. Let σ be an equivalence relation on a semigroup S. If a is equivalent to b we shall write $a \, \sigma \, b$. The σ-class containing a will be denoted by σ_a. An equivalence relation σ on a semigroup S is a right (left) congruence provided $a, b, c \in S$ and $a \, \sigma \, b$ imply $(ac) \, \sigma \, (bc)$ $((ca) \, \sigma \, (cb))$. If an equivalence relation is both a right and a left congruence, we shall call it a two-sided congruence or, more briefly, a congruence. We use the natural partial ordering on relations and say that $\sigma \leqq \rho$ if and only if $a, b \in S$ and $a \, \sigma \, b$ imply $a \, \rho \, b$. Clearly, the identity relation ι and the universal relation ν are congruences and $\iota \leqq \sigma \leqq \nu$ for each congruence σ on S. A congruence $\sigma \neq \nu$ is called maximal if for each congruence σ' on S such that $\sigma \leqq \sigma' \leqq \nu$, either $\sigma = \sigma'$ or $\sigma' = \nu$. An H-semigroup S is defined to be a semigroup such that every right congruence and every left congruence is a two-sided congruence on S. Since a subgroup of a group is normal if and only if its corresponding right (left) congruence is two-sided, then the class of H-semigroups contains the Hamiltonian groups in addition to the commutative semigroups, where we include all commutative groups in the set of all Hamiltonian groups. An inverse H-semigroup is a semigroup that is an inverse semigroup as well as an H-semigroup.

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Using the above definitions we prove in §2 that a semigroup is an inverse H-semigroup if and only if it is a semilattice of disjoint Hamiltonian groups.

We define τ to be the intersection of all the maximal modular congruences on a semigroup S, where a congruence σ is called modular if there is an element e of S such that $(ea)\,\sigma\, a$ and $(ae)\,\sigma\, a$ for all a in S. The element e is called an identity for σ. We refer to τ as the t-radical of S. S is said to be t-semisimple if $\tau=\iota$ [7]. In §3, we give necessary and sufficient conditions for an inverse H-semigroup S to be t-semisimple. This result has several nontrivial corollaries.

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1. **Preliminary definitions and results.** An element b of a semigroup S is an inverse of an element a of S provided $aba=a$ and $bab=b$. S is an inverse semigroup provided every element of S has a unique inverse. The inverse of an element a of an inverse semigroup S will be denoted by a^{-1} so that $aa^{-1}a=a$ and $a^{-1}aa^{-1}=a^{-1}$. The preceding definitions are taken from [1].

We also make use of the following results which have been proved in [1, pp. 23–30]. Let S be an inverse semigroup. The set E of idempotents of S is a semilattice, i.e., a commutative idempotent semigroup with the induced ordering $e\leqq f$ if and only if $ef=e$. If $a, b \in S$ then $(a^{-1})^{-1}=a$ and $(ab)^{-1}=b^{-1}a^{-1}$. Every principal right ideal and every principal left ideal of S has a unique idempotent generator. The idempotent $e=aa^{-1}$ $(f=a^{-1}a)$ is the unique idempotent generator of aS (Sa).

A left (right) zero of a semigroup S is an element a of S such that $as=a$ $(sa=a)$, for each $s \in S$ [1]. Let c be a left (right) zero of an inverse semigroup S. Then for each $s \in S$, $csc=c$ implies $scs=sc$ $(scs=cs)$ is an inverse of c. But c has a unique inverse, namely c, so that, for each $s \in S$, $sc=c$ $(cs=c)$. Hence c is a right (left) zero of S and S has at most one (left, right) zero.

In [6] Oehmke proved the following result which we state as a lemma.

LEMMA 1.1. *If S is an H-semigroup and I is a right (left) ideal of S then, for any b in S, $bI\subseteq I$ $(Ib\subseteq I)$ or $bI=\{c\}$ where c is a left zero ($Ib=\{c\}$ where c is a right zero).*

We use this result to show that every one-sided ideal of an inverse H-semigroup S is two-sided and thus we obtain that S is a semilattice of disjoint groups.

LEMMA 1.2. *A right (left) ideal of an inverse H-semigroup S is two-sided.*

Proof. Let I be a right ideal of S and $b \in S$. By Lemma 1.1, either $bI\subseteq I$ or $bI=\{c\}$ where c is a left zero. If the latter is true, then, since c is also a right zero and I is a right ideal, we have $\{c\}=Ic\subseteq I$ so that $bI=\{c\}\subseteq I$ and I is a left ideal in either case. By a similar proof, any left ideal of S is a right ideal of S.

Let S be an inverse H-semigroup and e an idempotent of S. Since Se is an ideal and $e \in Se$, it follows that $Se=eS$. Then for any $a \in S$ we have $ae=a$ if and only if $ea=a$. But for $a \in S$ there exists a unique element $a^{-1} \in S$ such that aa^{-1} and

$a^{-1}a$ are idempotents. Thus we have $(aa^{-1})a=a$ so that $a(aa^{-1})=a$, and also $a=a(a^{-1}a)$ so that $(a^{-1}a)a=a$. Hence

$$a^{-1}a = a^{-1}(aaa^{-1}) = (a^{-1}aa)a^{-1} = aa^{-1}.$$

It is well known that if every element of an inverse semigroup S commutes with its inverse then S is a union of disjoint groups. Thus we have the following lemma:

LEMMA 1.3. *If S is an inverse H-semigroup then S is a union of disjoint groups.*

If Y is a semilattice such that $S=\bigcup \{S_\alpha : \alpha \in Y\}$ is a decomposition of S such that, for every pair of elements α, β of Y, there is an element γ of Y such that $S_\alpha S_\beta \subseteq S_\gamma$, we say that S is the union of the semilattice Y of semigroups S_α, $\alpha \in Y$. We also abbreviate the expression and say that S is a semilattice of semigroups of type $\mathscr{C}$ to mean that S is the union of the semilattice of semigroups S_α, $\alpha \in Y$, where each S_α is of type $\mathscr{C}$.

Let S be an inverse H-semigroup and let $G_e=\{b \in S : bb^{-1}=e\}$. It readily follows that G_e is a maximal subgroup of S and $S=\bigcup \{G_e : e \in E\}$, where $G_e \cap G_f=\varnothing$ for $e \neq f$. Using Lemma 1.3, we obtain the result that E is contained in the center of S [1, pp. 127–128] so that Theorem 1 follows.

THEOREM 1. *If S is an inverse H-semigroup then S is a semilattice of disjoint groups, and if $f \geqq e$ in E, the mapping $\varphi_{f,e}$, defined by $a\varphi_{f,e}=ae$ where $a \in G_f$, is a homomorphism of G_f into G_e. Also, $\varphi_{f,f}$ is the identity mapping of G_f and if $f \geqq e \geqq g$, then $\varphi_{f,e}\varphi_{e,g}=\varphi_{f,g}$. Moreover, every product in S is known, since for $a \in G_f$ and $b \in G_e$, $ab=(a\varphi_{f,fe})(b\varphi_{e,fe})$.*

2. In this section we shall obtain a characterization of inverse H-semigroups, namely:

THEOREM 2. *A semigroup S is an inverse H-semigroup if and only if S is a semilattice of disjoint Hamiltonian groups.*

Proof. Let δ be a right congruence on an inverse H-semigroup S. Let G_e be a maximal subgroup of S and let δ' be the restriction of δ to G_e. By a straightforward argument, it can be shown that there is a subgroup H_e of G_e such that δ' is the right congruence on G_e induced by H_e. On the other hand, for any e in E, let H_e be any subgroup of G_e. Let σ be the right congruence induced by H_e on G_e. If $f<e$, let $H_f=G_f$. If e and f are not comparable, written $e\ ?\ f$, let $H_f=G_f$. If $f \geqq e$, let $H_f=(H_e)\varphi_{f,e}$ where $\varphi_{f,e}$ is the homomorphism on G_f into G_e. Let $a, b \in S$. Write

$$a\,\sigma'\,b \Leftrightarrow a, b \in G_f \text{ and } ab^{-1} \in H_f \quad \text{for some } f \in E.$$

It readily follows that σ' is an equivalence relation on S. Assume $a\,\sigma'\,b$ and let $c \in G_k$. If (1) $k<e, f<e$ or (2) $k<e, f\ ?\ e$ then either $fk<e$ or $fk\ ?\ e$ so that $H_{fk}=G_{fk}$. In these cases $H_f=G_f$ and $H_k=G_k$. Hence $H_f \cdot H_k \subseteq H_{fk}$. A similar argument obtains the same result in each of the remaining cases so that $(ac)\,\sigma'\,(bc)$ and σ'

is a (right) congruence on S. Therefore, if $a, b \in G_e$ and $a \, \sigma \, b$, then $ab^{-1} \in H_e$ implies σ is the restriction of σ' to G_e so that σ is a congruence on G_e. Then H_e is a normal subgroup of G_e and G_e is Hamiltonian. Hence, if S is an inverse H-semigroup, then S is a semilattice of disjoint Hamiltonian groups.

Let Y be any semilattice and to each element α of Y assign a group G_α such that G_α and G_β are disjoint if $\alpha \neq \beta$ in Y. To each pair of elements α, β of Y such that $\alpha > \beta$, assign a homomorphism $\varphi_{\alpha,\beta}$ of G_α into G_β such that if $\alpha > \beta > \gamma$, then $\varphi_{\alpha,\beta}\varphi_{\beta,\gamma} = \varphi_{\alpha,\gamma}$. Let $\varphi_{\alpha,\alpha}$ be the identity automorphism of G_α. Let S be the union of all the groups G_α, $\alpha \in Y$, and define the product of any two elements a_α, b_β of S (a_α in G_α, b_β in G_β) by $a_\alpha b_\beta = (a_\alpha \varphi_{\alpha,\alpha\beta})(b_\beta \varphi_{\beta,\alpha\beta})$. Then S is an inverse semigroup which is a union of groups [1, p. 128].

Assume the groups G_α, $\alpha \in Y$, are Hamiltonian. It remains to show that S is an H-semigroup. Let σ be a right congruence on S. For each G_α, σ restricted to G_α induces a right congruence σ_α on G_α. Since G_α is Hamiltonian then σ_α is two-sided so that σ determines a normal subgroup H_α of G_α. Let e_α be the identity of G_α. Recall that E is in the center of S where E is the semilattice of idempotents of S. Then we have

$$\begin{aligned} a_\alpha \, \sigma \, b_\beta &\Rightarrow (a_\alpha e_\alpha e_\beta) \, \sigma \, (b_\beta e_\alpha e_\beta) \Rightarrow (a_\alpha e_\beta) \, \sigma \, (b_\beta e_\alpha) \\ &\Rightarrow (a_\alpha e_\beta) \, \sigma_{\alpha\beta} \, (b_\beta e_\alpha) \Rightarrow a_\alpha e_\beta (b_\beta e_\alpha)^{-1} = a_\alpha b_\beta^{-1} \in H_{\alpha\beta}. \end{aligned}$$

Further,

$$a_\alpha \, \sigma \, b_\beta \Rightarrow (a_\alpha b_\beta^{-1}) \, \sigma \, e_\beta \Rightarrow (a_\alpha b_\beta^{-1}) \, \sigma \, e_{\alpha\beta}$$

so that $e_{\alpha\beta} \, \sigma \, e_\beta$ and, by symmetry, $e_{\alpha\beta} \, \sigma \, e_\alpha$ so that $e_\beta \, \sigma \, e_{\alpha\beta} \, \sigma \, e_\alpha$.

Conversely,

$$\begin{aligned} a_\alpha b_\beta^{-1} \in H_{\alpha\beta} \text{ and } e_\beta \, \sigma \, e_{\alpha\beta} \, \sigma \, e_\alpha &\Rightarrow (a_\alpha b_\beta^{-1}) \, \sigma \, e_{\alpha\beta} \, \sigma \, e_\beta \text{ and } a_\alpha \, \sigma \, (a_\alpha e_\beta) \\ &\Rightarrow (a_\alpha e_\beta) \, \sigma \, b_\beta \text{ and } a_\alpha \, \sigma \, b_\beta. \end{aligned}$$

Let $c_\gamma \in S$ and assume $a_\alpha \, \sigma \, b_\beta$.

$$\begin{aligned} a_\alpha \, \sigma \, b_\beta &\Rightarrow a_\alpha b_\beta^{-1} \in H_{\alpha\beta} \Rightarrow (a_\alpha b_\beta^{-1}) \, \sigma_{\alpha\beta} \, e_{\alpha\beta} \Rightarrow (a_\alpha b_\beta^{-1}) \, \sigma \, e_{\alpha\beta} \\ &\Rightarrow (a_\alpha b_\beta^{-1} e_\gamma) \, \sigma \, (e_{\alpha\beta} e_\gamma) \Rightarrow (a_\alpha b_\beta^{-1} e_\gamma) \, \sigma_{\alpha\beta\gamma} \, e_{\alpha\beta\gamma} \Rightarrow a_\alpha b_\beta^{-1} e_\gamma \in H_{\alpha\beta\gamma}. \end{aligned}$$

Since $H_{\alpha\beta\gamma}$ is a normal subgroup of $G_{\alpha\beta\gamma}$ then

$$(c_\gamma e_\alpha e_\beta) a_\alpha b_\beta^{-1} e_\gamma (c_\gamma e_\alpha e_\beta)^{-1} = c_\gamma a_\alpha b_\beta^{-1} c_\gamma^{-1} = (c_\gamma a_\alpha)(c_\gamma b_\beta)^{-1} \in H_{\alpha\beta\gamma}$$

and

$$(c_\gamma a_\alpha)^{-1}(c_\gamma b_\beta) \in H_{\alpha\beta\gamma}.$$

Further, $a_\alpha \, \sigma \, b_\beta \Rightarrow e_\beta \, \sigma \, e_{\alpha\beta} \, \sigma \, e_\alpha \Rightarrow e_{\gamma\alpha} \, \sigma \, e_{\gamma\alpha\beta} \, \sigma \, e_{\gamma\beta}$. Thus we have $(c_\gamma a_\alpha) \, \sigma \, (c_\gamma b_\beta)$ and σ is also a left congruence. By an analogous proof, if σ is a left congruence on S, then σ is a right congruence on S. Hence S is an inverse H-semigroup.

3. In this section we first identify the maximal modular congruences of an inverse H-semigroup S, and then obtain necessary and sufficient conditions for S to be t-semisimple.

A homomorphic image of an inverse semigroup is an inverse semigroup. Moreover, in any homomorphism, the inverse of an element is mapped onto the inverse of the image of that element [2, p. 57].

As one might also expect, the homomorphic image of an H-semigroup S is an H-semigroup. For if ψ is a homomorphism from S onto S', S' is obviously a semigroup. And if μ' is any right (left) congruence on S', we can define μ on S by

$$a \mu b \Leftrightarrow (a\psi) \mu' (b\psi).$$

Then μ is a congruence on S and from this it follows that μ' is a left (right) congruence on S'.

Let S be an H-semigroup. Let I be an ideal of S and T a subsemigroup of S such that $I \cup T = S$ and $I \cap T = \varnothing$. Write

$$a \rho b \Leftrightarrow a, b \in I \text{ or } a, b \in T.$$

Then ρ is a maximal modular congruence on S, where each element of T is an identity for ρ and ρ is not cancellative.

Let S be an inverse H-semigroup. For $e \in E$, let $T_e = \bigcup \{G_f : e \leqq f\}$. Define the relation $\rho^{(e)}$ on S by

$$a \rho^{(e)} b \Leftrightarrow a, b \in T_e \text{ or } a, b \in T_e'$$

where $T_e' = S - T_e$. If e is not a minimum idempotent in S, we claim that $\rho^{(e)}$ is a maximal modular congruence on S with identity e and $\rho^{(e)}$ is not cancellative. Since $e \in T_e$, $T_e \neq \varnothing$. Let $d, b \in T_e$, say $d \in G_f$, $b \in G_k$. Then $e \leqq f$ and $e \leqq k$ imply $e \leqq fk$ so that $db \in G_{fk} \subseteq T_e$ and T_e is a subsemigroup of S. Assume e is not minimum in E so that $T_e' \neq \varnothing$. Let $d \in T_e'$, $b \in S$, say $d \in G_f$ and $b \in G_k$. Now $d \in T_e'$ implies $f < e$ or $f \mathrel{?} e$. If $f < e$ then $fk < e$ and $db \in G_{fk} \subseteq T_e'$. If $f \mathrel{?} e$ then $fk < e$ or $fk \mathrel{?} e$ and $db \in T_e'$. Thus T_e' is a right ideal of S. By Lemma 1.2, T_e' is an ideal of S. It follows that $\rho^{(e)}$ is a maximal congruence on S with identity e and $\rho^{(e)}$ is not cancellative.

Let σ be a maximal modular congruence on an inverse H-semigroup S. For each $e \in E$, let H_e be the subgroup of G_e induced by σ. Then, as in the proof of Theorem 2, we know that for $a, b \in S$, say $a \in G_f$, $b \in G_k$, $a \sigma b \Leftrightarrow ab^{-1} \in H_{fk}$ and $f \sigma (fk) \sigma k$. Let a be an identity for σ, say $a \in G_f$. Then for each $s \in S$, $(as) \sigma s$ implies $(fas) \sigma (fs)$ so that $(as) \sigma (fs) \sigma s$. Thus f is an identity for σ.

It is generally known that σ is cancellative if and only if $E \subseteq \sigma_e$.

Suppose σ is not cancellative and let $e \in E$ be an identity for σ. If $h \in E$ is an identity for σ, then $h \sigma (eh) \sigma e$ and $h \in \sigma_e$. Since σ is not cancellative there exists $f \in E$ such that $f \notin \sigma_e$, so that f is not an identity for σ. Let $I = \{f \in E : f \text{ is not an identity for } \sigma\}$. Then I is an ideal in E. Let $J = \bigcup \{G_f : f \in I\}$. Then J is an ideal of S and J' is a semigroup of S. Oehmke [7] has shown that if σ is a maximal congruence and J any ideal of S, then either J is contained in a σ-class S_0 (which is also an ideal of S) or J contains an element of each σ-class. If $x \in \sigma_e \cap J$, then $x \sigma e$ and $x \in G_f$, for some $f \in I$. But then $x \sigma (ef)$ so that $e \sigma (ef)$ and, since also

$(ef)\,\sigma f$, then $e\,\sigma f$ and $f \in I$, which is a contradiction. Hence $\sigma_e \cap J = \varnothing$ and there exists a σ-class S_0 such that $J \subseteq S_0$. Suppose there is some $b \in S_0$ such that $b \notin J$, that is, $b \in G_h$ where $h\,\sigma\,e$. Let $f \in I$. Then $b\,\sigma f$ implies $bf \in H_{hf}$ and $h\,\sigma\,(hf)\,\sigma f$, so that $f\,\sigma\,e$. Contradiction. Therefore $J = S_0$. Since J is an ideal and J' is a semigroup, we have the maximal modular congruence σ^* defined by $a\,\sigma^*\,b \Leftrightarrow a, b \in J$ or $a, b \in J'$, where each element of J is an identity for σ^*. Clearly $\sigma \leqq \sigma^*$. Hence $\sigma = \sigma^*$ and we have proved the following lemma.

LEMMA 3.1. *If σ is a maximal modular congruence on an inverse H-semigroup S, then σ is cancellative or σ has exactly two congruence classes, namely the semigroup of identities for σ and the ideal of nonidentities for σ.*

Suppose σ is cancellative. Then $E \subseteq \sigma_e$ where $\sigma_e = \bigcup \{H_f : f \in E\}$. Since σ is maximal, then S/σ has no nontrivial congruences. Therefore S/σ is the semigroup $\{0, 1\}$ or S/σ is simple. Since σ is cancellative, it follows that $S/\sigma \neq \{0, 1\}$. Since S is an inverse H-semigroup, then S/σ is an inverse H-semigroup. Therefore, by Lemma 1.2, every one-sided ideal of S/σ is a two-sided ideal. Hence S/σ is both left and right simple, so that S/σ is a Hamiltonian group [1, p. 6]. Since S/σ has no nontrivial congruences, then S/σ has no nontrivial homomorphisms, so that S/σ is a simple group. Since S/σ is Hamiltonian, then S/σ has no nontrivial subgroups. Hence S/σ is a cyclic group of prime order. Thus there exists a prime number p such that, for every $a \notin \sigma_e$, the σ-classes may be written as $\sigma_a, \sigma_{a^2}, \ldots, \sigma_{a^p}$, where $\sigma_{a^p} = \sigma_e$ and a^p is an identity for σ. In fact, for every $c \in \sigma_e$, $c\,\sigma\,e$ implies $(cs)\,\sigma\,(es)\,\sigma\,s$ and $(sc)\,\sigma\,(se)\,\sigma\,s$ so that c is an identity for σ. By a similar argument, if $a \notin \sigma_e$, then a is not an identity for σ. If $G_e = H_e$ then $G_e \subseteq \sigma_e$. If $G_e \neq H_e$ then σ partitions G_e into cosets of H_e. The number of these cosets must be p, for otherwise S/σ would contain a proper subgroup. Hence the cosets of H_e must form a cyclic group of prime order and H_e must be a maximal subgroup of G_e. We state these results in the following lemma:

LEMMA 3.2. *If σ is a maximal modular cancellative congruence on an inverse H-semigroup S, then S/σ is a cyclic group of prime order p such that for each nonidentity element g for σ, the cosets of S/σ are $\sigma_e, \sigma_g, \ldots, \sigma_{g^{p-1}}$. Moreover, if σ' is the restriction of σ to G_e, then for each e in E, $\sigma' = \nu$ or σ' induces a maximal subgroup H_e of G_e where the cosets of H_e form a cyclic group of prime order p.*

LEMMA 3.3. *If T is a proper subsemigroup of S such that for each $e \in E$, $T \cap G_e = H_e$, where $H_e = G_e$ or H_e is a maximal subgroup of index p in G_e, and for each pair of groups G_e, G_f in the semilattice S, where $e \leqq f$, the homomorphism $\varphi_{f,e}$ on G_f into G_e defined by $a\varphi_{f,e} = ae$ is a monomorphism, then T induces a maximal modular cancellative congruence on S.*

Proof. Define σ on S by $a\,\sigma\,b \Leftrightarrow ab^{-1} \in H_{kf}$, where $a \in G_k$, $b \in G_f$. Clearly, σ is reflexive, symmetric and compatible. That σ is transitive follows from the hypothesis that for each e, f in E where $f \geqq e$, the homomorphism $\varphi_{f,e}$ is a monomorphism. Therefore σ is a congruence on S. It follows immediately from the definition of σ that $T=\sigma_e$ and σ is modular. Then $E \subseteq \sigma_e$ and σ is cancellative. If $\sigma < \sigma'$ where, for each $e \in E$, σ' induces the subgroup K_e in G_e, then there exists $a, b \in S$ such that $a \, \sigma' \, b$ and $a \not\sigma \, b$. Say $a \in G_f$, $b \in G_h$. Then $ab^{-1} \in K_{fh}$ and $ab^{-1} \notin H_{fh}$ which implies that $H_{fh} \subset K_{fh}$ so that $K_{fh} = G_{fh}$, since H_{fh} is then maximal in G_{fh}. If $k > fh$, then $G_k = K_k$. For $k < fh$, $H_k \subset (G_f)\varphi_{fh,k} \subseteq K_k \subseteq G_k$. Assume $K_k \subset G_k$. Since H_k has index p in G_k, then K_k has finite index j in G_k and the index m of H_k in K_k is such that $p = mj$ [4, p. 63]. Then either $m=p$ and $j=1$, or $m=1$ and $j=p$. If $m=1$, then $H_k = K_k$, which is a contradiction. If $j=1$, then $G_k = K_k$, contrary to the assumption. Thus, for $k < fh$ we must have $G_k = K_k$. If $k \; ? \; fh$ then $fhk < fh$ and $G_{fhk} = K_{fhk}$. Since $\varphi_{k,fhk}$ is a monomorphism then $G_k = K_k$. Therefore $\sigma' = \nu$ and σ is maximal. This completes the proof.

Define the relation ρ on S as follows:

$$x \, \rho \, y \Leftrightarrow \text{there exists } e \in E \text{ such that } ex = ey.$$

Clearly ρ is a congruence on S. Then for each e, f in E, $e \, \rho \, f$ since $(ef)e = ef = (ef)f$. Thus $E \subseteq \rho_e$. Once again we note that S/ρ is an inverse H-semigroup containing exactly one idempotent so that S/ρ is a Hamiltonian group. Let σ be any maximal modular cancellative congruence on S and let $a, b \in S$ such that $a \, \rho \, b$. Then

$$\begin{aligned} a \, \rho \, b &\Rightarrow \text{there exists } e \text{ in } E \text{ such that } ea = eb \\ &\Rightarrow (ea) \, \sigma \, (eb) \Rightarrow a \, \sigma \, b. \end{aligned}$$

Thus the intersection α of all the maximal modular cancellative congruences of S is greater than or equal to ρ. Now for any f, e in E, where $f < e$, if the homomorphism $\varphi_{e,f}$ is not a monomorphism then there exist $a \neq b$ in G_e with $fa = fb$ so that $a \, \rho \, b \Rightarrow a \, \alpha \, b$. Suppose that S is t-semisimple. From Lemma 3.1, it is clear that the intersection β of all the maximal modular noncancellative congruences of S separates S into its maximal subgroups. Thus, if $\varphi_{e,f}$ is not a monomorphism then $a \, \alpha \, b$ and $a \, \beta \, b$ imply $a \, \tau \, b$ so that $\tau \neq \iota$, contrary to the supposition. Let $a \neq e$, $a \in G_e$, $e \in E$. Since $a \, \beta \, e$ then there must be a maximal modular cancellative congruence σ on S such that $a \not\sigma \, e$. From Lemma 3.2 it follows that there exists a maximal subgroup H_e of index p in G_e such that $a \notin H_e$. Since we know that, for each $f \in E$, the restriction of σ to G_f induces a subgroup H_f of G_f such that H_f is of index p in G_f or $H_f = G_f$, and since $E \subseteq \sigma_e$, then the union of these subgroups is a proper inverse subsemigroup of S. Let $\mathscr{T}$ be the collection of all inverse subsemigroups T_p of S such that, for each e in E, $T_p \cap G_e = H_e$, where $H_e = G_e$ or H_e is a maximal subgroup of prime index p in G_e. Then we may say that, if S is t-semisimple, then for each e in E, for each $a \neq e$ in G_e, there exists $T_p \in \mathscr{T}$ such that $a \notin T_p$. Conversely, assume that for each f, e in E, where $e < f$, $\varphi_{f,e}$ is a monomorphism; and for each e in E, for each $a \neq e$ in G_e, there exists $T_p \in \mathscr{T}$ such that $a \in T_p$. Suppose $a \, \tau \, b$, where $a \in G_e$, $b \in G_f$. Since β separates S into its maximal subgroups, then $e=f$. If $a\neq b$ then $ab^{-1}\neq e$, and by assumption there exists $T_p\in\mathscr{T}$ such that $ab^{-1}\notin T_p$. By Lemma 3.3, it follows that there exists a maximal modular cancellative congruence σ on S separating a and b. Thus if $a\ \tau\ b$, then $a=b$ and S is t-semisimple. We can now state the main result of this section.

THEOREM 3. *An inverse H-semigroup S is t-semisimple if and only if for each pair of groups G_e, G_f in the semilattice, with $f\geqq e$, the homomorphism $\varphi_{f,e}$ on G_f into G_e, defined by $a\varphi_{f,e}=ae$, is a monomorphism, and for each e in E, for each $a\neq e$ in G_e, there exists a subsemigroup T_p of S such that $a\notin T_p$ and, for each f in E, $T_p\cap G_f=H_f$, where $H_f=G_f$ or H_f is a maximal subgroup of prime index p in G_f.*

COROLLARY 3.1. *S is an inverse H-semigroup all of whose maximal modular congruences are cancellative if and only if S is a Hamiltonian group.*

COROLLARY 3.2. *S is a t-semisimple inverse H-semigroup all of whose nontrivial maximal modular congruences are not cancellative if and only if S is a semilattice.*

COROLLARY 3.3. *If S is a t-semisimple inverse H-semigroup, then S is a semilattice of disjoint t-semisimple Hamiltonian groups.*

Proof. Let $f\in E$. It suffices to show that G_f is t-semisimple. Let a, b be distinct elements of G_f. Since $a\ \beta\ b$ then there must be a maximal modular cancellative congruence σ on S such that $a\not\sigma b$. Let the σ-classes be $\sigma_e, \sigma_g, \ldots, \sigma_{g^{p-1}}$. Let σ' be the restriction of σ to G_f. Then $a\not\sigma' b$. Now $a\ \sigma\ b$ implies either $a\notin\sigma_e$ or $b\notin\sigma_e$. Say $a\notin\sigma_e$. Then the σ-classes may be written as $\sigma_f, \sigma_a, \ldots, \sigma_{a^{p-1}}$ so that the σ'-classes are $\sigma'_f, \sigma'_a, \ldots, \sigma'_{a^{p-1}}$. Further, G_f/σ' is a cyclic group of prime order so that σ' is a maximal congruence on G_f separating a and b. Hence G_f is t-semisimple.

LEMMA 3.4.1. *A t-semisimple Hamiltonian group is commutative.*

Proof. Let G be a t-semisimple Hamiltonian group and assume G is not commutative. Then $G=Q\times A\times B$ where Q is a quaternion group, A a commutative group of exponent two, B a commutative group where each element has odd order. Since Q is a finite p-group then Q is nilpotent, and since A and B are commutative then A and B are nilpotent [3, p. 155, p. 149]. Therefore G is nilpotent [5, p. 212]. Now every maximal subgroup of a nilpotent group is normal, is of prime index, and contains the derived group [3, p. 154]. Hence the intersection Φ of the maximal (normal) subgroups of G contains the derived group G'. But G is t-semisimple so that its Frattini subgroup Φ consists of the identity only. Thus G' contains the identity only and G is commutative. But this contradicts our assumption so that the result follows.

COROLLARY 3.4. *If S is a t-semisimple inverse H-semigroup, then S is commutative.*

COROLLARY 3.5. *If S is an inverse H-semigroup with a minimum idempotent e, then S is t-semisimple if and only if G_e is t-semisimple and, for each group G_f in the semilattice with $f\geqq e$, the homomorphism $\varphi_{f,e}$ on G_f into G_e, defined by $a\varphi_{f,e}$, is a monomorphism.*

Proof. Only the sufficiency requires proof. Let f, $h \in E$ with $f \geqq h$ and assume there exist $a \neq b$ in G_f such that $ah = bh$ in G_h. Then $ae = ahe = bhe = be$ in G_e implies $a = b$, since $\varphi_{f,e}$ is a monomorphism. Contradiction. Hence, for each f, h in E, where $f \geqq h$, $\varphi_{f,h}$ is a monomorphism. Let $f \in E$ and $a \neq f$ in G_f. Assume $a \in \Phi_f$. Since $\varphi_{f,e}$ is injective then $ae \neq e$ in G_e, so that there exists a maximal subgroup H_e in G_e such that $ae \notin H_e$. Thus $a = ae\varphi_{f,e}^{-1} \notin H_e\varphi_{f,e}^{-1} = H_f$ in G_f. But H_f is a maximal subgroup of G_f and $a \notin H_f$ imply $a \notin \Phi_f$. Contradiction. Hence G_f is t-semisimple. It remains to show that for each f in E, for each $a \neq f$ in G_f, there exists a subsemigroup T_p of S such that $a \notin T_p$ and, for each h in E, $T_p \cap G_f = H_h$, where $H_h = G_h$ or H_h is a maximal subgroup of prime index p in G_h. Let $a \neq f$ in G_f. Since G_f is t-semisimple, there exists a maximal subgroup H_f in G_f such that $a \notin H_f$. It follows that G_f/H_f has no nontrivial subgroups and is therefore cyclic of prime order p so that the cosets of H_f may be written as $H_f, H_fa, \ldots, H_fa^{p-1}$. H_fe is a subgroup of G_e which does not contain ea. Let H_e be a subgroup of G_e maximal with respect to not containing ea and such that $H_fe \subseteq H_e$ [8, p. 22]. But then H_e is a maximal subgroup of G_e and G_e/H_e is cyclic of prime order. Since $\varphi_{f,e}$ is a monomorphism from G_f into G_e and H_f is a maximal subgroup of G_f, it follows that $ea^i \notin H_e$, $1 \leqq i \leqq p-1$, and $H_e, H_ea, \ldots, H_ea^{p-1}$ are distinct cosets of H_e. Hence these must be all the cosets of H_e so that H_e is maximal and of index p in G_e. For each h in E, let $H_h = (H_e)\varphi_{h,e}$. It follows that for each H_h either $H_h = G_h$ or H_h is maximal and of index p in G_h. Further, for h, k in E, let $x \in H_h$, $y \in H_k$. Then

$$xe, ye \in H_e \Rightarrow xye \in H_e \Rightarrow xy \in H_{hk}.$$

Thus the union of all H_h, $h \in E$, as defined above, is a subsemigroup T_p with the desired properties and the proof is complete.

COROLLARY 3.6. *If S is a finite inverse H-semigroup, then S is t-semisimple if and only if G_e is t-semisimple, where e is the minimum idempotent of E, and for each subgroup G_f, $f > e$, the homomorphism $\varphi_{f,e}$ from G_f into G_e, defined by $a\varphi_{f,e} = ae$, is a monomorphism.*

COROLLARY 3.7. *If S is a t-semisimple inverse H-semigroup with no nontrivial modular congruences, then S is either a cyclic group of prime order or the unique semilattice of two elements.*

Proof. Since S is t-semisimple, it has maximal modular congruences, that is, ι is a maximal modular congruence. Since there is no nontrivial modular (noncancellative) congruence on S, then S is a group or S is the semilattice of two elements. In the former case, any congruence on S would be modular, so it follows that S has no nontrivial subgroups, hence is cyclic of prime order.

COROLLARY 3.8. *If S is an inverse H-semigroup with zero, then S is t-semisimple if and only if S is a semilattice.*

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