INVERSE LIMIT SEQUENCES WITH COVERING MAPS

BY

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1. **Introduction.** The purpose of this paper is two-fold. We define a class of spaces called *solenoidal* spaces, which generalize the solenoids of van Dantzig [13], and study their structure (§§4, 5). Then we use part of the structure developed to prove a theorem on the homogeneity of certain solenoidal spaces (§6). A solenoidal space is the limit of an inverse limit sequence of "nice" spaces (the precise definition is below) where the bonding maps are regular covering maps (2). We will see that these spaces still have many of the properties of the classical solenoids.

A space X is called *homogeneous* if for each pair x, y of points of X there is a homeomorphism of (X, x) onto (X, y). M. K. Fort, Jr. [8] asked the general question, "When is an inverse limit space homogeneous?" The pseudo-arc, which is known to be homogeneous (see Bing [1]), can be described roughly as an inverse limit of arcs where the bonding maps become sufficiently "crooked." Perhaps one can obtain some result on homogeneity in which "crookedness" of the bonding maps is one of the assumptions. (See Brown [4] for a precise definition of *ϵ-crooked map*.) Probably one should restrict himself to the case where the factor spaces are 1-dimensional, for the following reason. Brown [4] has shown that an inverse limit of locally connected continua with sufficiently crooked bonding maps is hereditarily indecomposable; and Bing [2] has shown that if X is an n-dimensional, hereditarily indecomposable continuum and $n > 1$, then X is not homogeneous. The word "probably" was used two sentences ago because of the fact that dimension may be lowered by taking an inverse limit of continua with "onto" bonding maps (although it can never be raised).

The theorem in §6 goes in the opposite direction from the preceding suggestion, by assuming that the bonding maps are "smooth." The assumption of local smoothness in the sense of differentiability will of course do no good. A kind of global smoothness is needed; this is why covering maps are appropriate. Case [5] has taken an inverse limit of universal curves where the bonding maps are regular covering maps to get a new example of a 1-dimensional homogeneous continuum containing arcs. Two of our theorems almost generalize two of his, but, as they stand, do not imply his.

For notation and terminology on inverse limit sequences refer to [7]. We

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Received by the editors July 12, 1963.

(1) This research was partially supported by the Air Force under SAR G AF AFOSR 62-20.

(2) In [14] van Heemert dealt with inverse limits of manifolds where the bonding maps are covering maps.

consider only the case where the directed set of indices is the positive integers. If (X, f) is an inverse limit sequence, the maps f_m^n: $X_n \to X_m$ $(m \leqq n)$ are called *bonding maps*. If $x \in X_\infty$, then x_n will often be used to denote the nth coordinate $f_n(x)$ of x. The symbol ❙ will be used to indicate the ends of proofs.

2. **Motivation: The P-adic solenoid.** If $P = (p_1, p_2, \cdots)$ is a sequence of prime numbers (1 not being included as a prime), the P-adic solenoid Σ_P is defined as the limit of the inverse limit sequence (X, f), where for each $n, X_n = \{z: |z| = 1\}$ (unit circle in the complex plane), and where each bonding map f_n^{n+1}: $X_{n+1} \to X_n$ is given by $f_n^{n+1}(z) = z^{p^n}$. Call prime sequences P and Q *equivalent* (written $P \sim Q$) if a finite number of terms can be deleted from each sequence so that every prime number occurs the same number of times in the deleted sequences. Bing [3] remarked that if $P \sim Q$ then Σ_P is homeomorphic to Σ_Q and suggested, "Perhaps the converse of this is true." One can see that $P \sim Q$ if and only if Σ_P is homeomorphic to Σ_Q (written $\Sigma_P \equiv \Sigma_Q$) as follows(3): From the continuity theorem for Čech cohomology [7, p. 261] one sees that $H^1(\Sigma_P)$ is isomorphic to the group F_P of P-adic rationals (all rationals of the form $k/(p_1 p_2 \cdots p_n)$ where k is an integer and n is a positive integer). Also it can be seen that Σ_P, as a topological group, is topologically isomorphic to the character group of F_P (written $\Sigma_P \approx_{\text{top}} \hat{F}_P$). By number-theoretic considerations one can see that F_P is isomorphic to F_Q (written $F_P \approx F_Q$) if and only if $P \sim Q$. Thus

$$\Sigma_P \equiv \Sigma_Q \Longrightarrow H^1(\Sigma_P) \approx H^1(\Sigma_Q) \Longrightarrow F_P \approx F_Q \Longrightarrow P \sim Q.$$

Conversely,

$$P \sim Q \Longrightarrow F_P \approx F_Q \Longrightarrow \hat{F}_P \approx_{\text{top}} \hat{F}_Q \Longrightarrow \Sigma_P \approx_{\text{top}} \Sigma_Q.$$

3. **Solenoidal spaces.** For basic notions of covering space theory refer to [9] or [11].

Definition 3.1. A *solenoidal sequence* is an inverse limit sequence (X, f) such that (1) each space X_n is *nice* in the sense that it is connected, locally pathwise connected, and semi-locally simply connected, and (2) each bonding map f_1^n: $X_n \to X_1$ is a regular covering map. The limit X_∞ will be called a *solenoidal* space.

Remark 3.2. The spaces X_n are assumed to be *nice* in order to guarantee the constructions of covering space theory. In particular, they could be polyhedra.

Remark 3.3. Condition (2) implies that each bonding map f_m^n: $X_n \to X_m$ $(m \leqq n)$ is a regular covering map.

Remark 3.4. If each X_n is a continuum then X_∞ is a continuum.

(3) Essentially the same result was stated by van Dantzig.

EXAMPLE 1. For each n, let X_n be the r-dimensional torus $T^r = S^1 \times \cdots \times S^1$ (r times). Take each f_n^{n+1}: $X_{n+1} \to X_n$ to be of the form $f_n^{n+1}(z_1, \cdots, z_r) = (z_1^{p_1}, \cdots, z_r^{p_r})$, where the p_i's are positive integers.

EXAMPLE 2. For each n, X_n is obtained as follows. Take two disjoint copies of S^1 and identify them at the 2^{n-1} points $\exp(2\pi ik/2^{n-1})$, $k = 0, \cdots, 2^{n-1} - 1$. ($X_1$ is a figure eight.) Then the map s: $S^1 \to S^1$ given by $s(z) = z^2$ induces a covering map f_n^{n+1}: $X_{n+1} \to X_n$. The regularity of f_1^n follows from the fact that its covering transformations act transitively on the fibers.

EXAMPLE 3. If Y_1 is any nice space, F_1 is the fundamental group $\pi(Y_1, b_1)$, and $(F_2, F_3, \cdots)$ is a decreasing sequence of normal subgroups of F_1, then we can construct a solenoidal sequence (Y, g) with base points b_n in Y_n such that

$$(g_1^n)_*(\pi(Y_n, b_n)) = F_n.$$

EXAMPLE 4. As a special case of the method of Example 3, we obtain a solenoidal sequence of closed 3-manifolds as follows. Let (X, f) be as in Example 2. Let Y_1 be the connected sum of the 3-manifold $S^1 \times S^2$ with itself. Then $F_1 = \pi(Y_1, b_1)$ is a free group on two generators, hence isomorphic to $\pi(X_1, 1)$. The decreasing sequence $((f_1^n)_* \pi(X_n, 1))$, $n = 2, \cdots$, of normal subgroups of $\pi(X_1, 1)$ then defines a decreasing sequence $(F_2, F_3, \cdots)$ of normal subgroups of F_1. Hence we may obtain (Y, g). One might say that the solenoidal sequence (X, f) of 1-polyhedra serves as a *model* for constructing the solenoidal sequence (Y, g) of 3-manifolds.

4. **A lemma on covering transformations.** The result of this section will be used twice in § 5. Suppose we are given a commutative diagram

$$\begin{array}{ccc} (X_3, b_3) & & \\ \downarrow f_1^3 & \searrow f_2^3 & \\ (X_1, b_1) & \xleftarrow{f_1^2} & (X_2, b_2), \end{array} \tag{4.1}$$

where the three maps are regular covering maps and the base points b_k are fixed throughout the discussion. Let F_1 be the fundamental group $\pi(X_1, b_1)$ and for $k = 2, 3$ let $F_k = (f_1^k)_*(\pi(X_k, b_k))$. Thus $F_1 \supset F_2 \supset F_3$ and F_2, F_3 are normal in F_1. Let G_k be the covering transformation group of f_1^k. There is a canonical isomorphism ϕ_k of $Q_k = F_1/F_k$ onto G_k. Since $F_3 \subset F_2$ there is a natural homomorphism ν: $Q_3 \to Q_2$ (given by $\nu(aF_3) = aF_2$). Now define μ: $G_3 \to G_2$ by commutativity in

$$\begin{array}{ccc} G_2 & \xleftarrow{\mu} & G_3 \\ \uparrow \phi_2 & & \uparrow \phi_3 \\ Q_2 & \xleftarrow{\nu} & Q_3 \end{array} \tag{4.2}$$

LEMMA 4.1. (a) *If* $g_3 \in G_3$ *and* $g_2 = \mu(g_3)$ *then the diagram*

$$(4.3)\qquad \begin{array}{ccc} X_2 & \xleftarrow{f_2^3} & X_3 \\ \downarrow{\scriptstyle g_2} & & \downarrow{\scriptstyle g_3} \\ X_2 & \xleftarrow{f_2^3} & X_3 \end{array}$$

is commutative. (b) *If* $g_2 \in G_2$, $g_3 \in G_3$, *and the diagram* (4.3) *is commutative at some point* $x_3 \in X_3$, *then* $g_2 = \mu(g_3)$.

Proof. (a) Let $h_3 = [\alpha_1]F_3$ be the element of Q_3 such that $\phi_3(h_3) = g_3$ (where α_1 is a loop on b_1), and let $h_2 = \nu(h_3) = [\alpha_1]F_2$. By commutativity of (4.2) we have $\phi_2(h_2) = g_2$. Now lift α_1 by f_1^2 to the path α_2 starting at b_2. Then from the definition of ϕ_2, g_2 is the unique element of G_2 such that $g_2(b_2) = \alpha_2(1)$. Lift α_2 by f_2^3 to the path α_3 starting at b_3. But $f_1^3\alpha_3 = f_1^2 f_2^3 \alpha_3 = f_1^2\alpha_2 = \alpha_1$, so that $g_3 = \phi_3(h_3)$ satisfies $g_3(b_3) = \alpha_3(1)$. We conclude that $f_2^3 g_3(b_3) = f_2^3\alpha_3(1) = \alpha_2(1) = g_2(b_2)$.

Now take an arbitrary point x_3 in X_3 and let $x_2 = f_2^3(x_3)$. We can determine $g_2(x_2)$ and $g_3(x_3)$ as follows. Take a path β_3 from b_3 to x_3, and let $\beta_2 = f_2^3\beta_3$ and $\beta_1 = f_1^2\beta_2 = f_1^3\beta_3$. Since β_2 is a path from b_2 to x_2 we see (e.g. from [11, p. 196]) that if we lift β_1 by f_1^2 to the path β_2' starting at $g_2(b_2)$, then $g_2(x_2) = \beta_2'(1)$. Since $f_2^3 g_3(b_3) = g_2(b_2)$ (by the preceding paragraph) we may lift β_2' by f_2^3 to the path β_3' starting at $g_3(b_3)$. Then, since $f_1^3\beta_3' = \beta_1 = f_1^3\beta_3$, we have $g_3(x_3) = \beta_3'(1)$. Thus $f_2^3 g_3(x_3) = f_2^3\beta_3'(1) = \beta_2'(1) = g_2(x_2)$, which shows the commutativity of (4.3).

(b) We are supposing that at some point x_3, $g_2 f_2^3(x_3) = f_2^3 g_3(x_3)$. By part (a), then, $g_2 f_2^3(x_3) = \mu(g_3) f_2^3(x_3)$. But an element in G_2 is determined by its value on a single point, so that $g_2 = \mu(g_3)$.

REMARK 4.2. This lemma shows that the definition of μ is independent of the choice of the base points b_k.

5. **The structure of solenoidal spaces.** We assume in this section that we are given an arbitrary solenoidal sequence (X, f). To avoid triviality we assume that for each n the covering $f_n^{n+1}: X_{n+1} \to X_n$ is k_n-to-1 where k_n is a cardinal greater than 1.

Let us choose once and for all a base point $b = (b_1, b_2, \cdots)$ in X_∞. For each n, let $F_n = (f_1^n)_*(\pi(X_n, b_n))$ so that we have a descending sequence of groups $F_1 \supset F_2 \supset \cdots$, each F_n being normal in $F_1 = \pi(X_1, b_1)$. Let $Q_n = F_1/F_n$ and let ϕ_n be the canonical isomorphism of Q_n onto the covering transformation group G_n of $f_1^n: X_n \to X_1$. Defining homomorphisms $\nu_n^{n+1}: Q_{n+1} \to Q_n$ and $\mu_n^{n+1}: G_{n+1} \to G_n$ according to the prescription of the preceding section, we get inverse limit sequences of groups (Q, ν) and (G, μ) with limit groups Q_∞ and G_∞. Since from the definition of μ_n^{n+1}, $\phi_n \nu_n^{n+1} = \mu_n^{n+1}\phi_{n+1}$, the sequence (ϕ_n) induces an isomorphism $\phi_\infty: Q_\infty \to G_\infty$. Consider each G_n as a discrete topological group, and give G_∞ the inverse limit topology.

LEMMA 5.1. G_∞ *is totally disconnected and perfect. If the coverings* f_n^{n+1} *are* *finite-to-one then G_∞ is homeomorphic to the Cantor set.*

Proof. It is easy to see that an inverse limit of totally disconnected spaces is totally disconnected. From the fact that for each g_n in G_n, $(\mu_n^{n+1})^{-1}(g_n)$ contains $k_n > 1$ elements, one can see that G_∞ is perfect. If each k_n is finite, then each G_n is finite, so that G_∞ is compact metric and is therefore homeomorphic to the Cantor set. ∎

LEMMA 5.2. *G_∞ acts on X_∞ as an effective topological transformation group.*

Proof. Suppose $g = (g_1, g_2, \cdots) \in G_\infty$. By Lemma 4.1 we have for each n the commutativity relation $g_n f_n^{n+1} = f_n^{n+1} g_{n+1}$, so that g induces a homeomorphism of X_∞ onto itself, which we still denote by g, i.e., for

$$x = (x_1, x_2, \cdots) \in X_\infty,$$

$g(x) = (g_1(x_1), g_2(x_2), \cdots)$. Obviously we have (1) $(g \cdot g')(x) = g(g'(x))$ and (2) the identity element of G_∞ is the unique element of G_∞ which acts as the identity transformation on X_∞. Now we want to show that the map $G_\infty \times X_\infty \to X_\infty$ given by $(g, x) \to g(x)$ is continuous. Suppose (g^0, x^0) is given and U is a neighborhood of $g^0(x^0)$. By [7, p. 218] we may assume that $U = f_n^{-1}(U_n)$ where U_n is a neighborhood of $g_n^0(x_n^0)$. Since G_n is discrete, $V = \mu_n^{-1}(g_n^0)$ is a neighborhood of g^0. Since $g_n^0 f_n$ is continuous, there is a neighborhood W of x^0 such that $g_n^0 f_n(W) \subset U_n$. Then if $(g, x) \in V \times W$, $f_n g(x) = g_n f_n(x) = g_n^0 f_n(x) \in U_n$, so that $g(x) \in U$. ∎

LEMMA 5.3. *If x and x' are in X_∞ and $f_1(x) = f_1(x')$, then there is one and only one g in G_∞ such that $g(x) = x'$.*

Proof. Let $x = (x_1, x_2, \cdots)$ and $x' = (x_1', x_2', \cdots)$ where $x_1 = x_1'$. Since $f_1^n(x_n) = f_1^n(x_n')$ and f_1^n is regular, there is a unique $g_n \in G_n$ such that $g_n(x_n) = x_n'$. Since

$$g_n f_n^{n+1}(x_{n+1}) = g_n(x_n) = x_n' = f_n^{n+1}(x_{n+1}') = f_n^{n+1} g_{n+1}(x_{n+1}),$$

Lemma 4.1(b) gives that $g_n = \mu_n^{n+1}(g_{n+1})$. Thus $g = (g_1, g_2, \cdots) \in G_\infty$, and from the definition of the action of G_∞, $g(x) = x'$.

Now we wish to introduce the (common) universal covering space of the spaces $X_1, X_2, \cdots$. This step is essential to carrying out our program. Recall that we have fixed a base point $b = (b_1, b_2, \cdots)$ in X_∞. Let p_1: $(\tilde{X}, \tilde{b}) \to (X_1, b_1)$ be the universal covering space of X_1 (which exists since X_1 is *nice*). Assuming the covering map p_n: $(\tilde{X}, \tilde{b}) \to (X_n, b_n)$ has been defined, let p_{n+1}: $(\tilde{X}, \tilde{b}) \to (X_{n+1}, b_{n+1})$ be the unique covering map which satisfies the relation $f_n^{n+1} p_{n+1} = p_n$. Because of this relation, the sequence $(p_1, p_2, \cdots)$, so defined, induces a map p_∞: $(\tilde{X}, \tilde{b}) = (X_\infty, b)$; explicitly, $p_\infty(\tilde{x}) = (p_1(x), p_2(x), \cdots)$. In the case of the solenoids, $\tilde{X}$ is a line that p_∞ maps continuously and 1-1 onto a path component of X_∞, which is dense in X_∞. (This follows from general results proved below.)

Now let $Y_\infty = G_\infty \times \widetilde{X}$ and define $P_\infty\colon Y_\infty \to X_\infty$ by $P_\infty(g, \tilde{x}) = gp_\infty(\tilde{x})$. Call a subset $\widetilde{V}$ of $\widetilde{X}$ *simple* if p_1 maps $\widetilde{V}$ homeomorphically onto $p_1(\widetilde{V})$. The following lemma is used in the next two theorems.

LEMMA 5.4. *P_∞ is a local homeomorphism of Y_∞ onto X_∞. In fact, if $\widetilde{V}$ is any simple open subset of $\widetilde{X}$, then P_∞ maps $G_\infty \times \widetilde{V}$ homeomorphically onto $f_1^{-1}(p_1(\widetilde{V}))$.*

Proof. To save a duplication, we first prove the following

SUBLEMMA. *For any n let $g_n^0 \in G_n$, let $W = \mu_n^{-1}(g_n^0)$, and let $\widetilde{V}$ be an open subset of $\widetilde{X}$. Then $P_\infty(W \times \widetilde{V}) = f_n^{-1}[g_n^0 p_n(\widetilde{V})]$, which is an open set.*

Proof. The latter set is open since p_n is open. To show the equality suppose that $x \in f_n^{-1}[g_n^0 p_n(\widetilde{V})]$. Then $x_n = g_n^0 p_n(\tilde{v})$ where $\tilde{v} \in \widetilde{V}$. Since then $x_1 = f_1^n g_n^0 p_n(\tilde{v}) = f_1^n p_n(\tilde{v}) = p_1(\tilde{v})$, there is by Lemma 5.3 a unique g in G_∞ such that $x = gp_\infty(\tilde{v}) = P_\infty(g, \tilde{v})$. But then $g_n p_n(\tilde{v}) = x_n = g_n^0 p_n(\tilde{v})$, so that $g_n = g_n^0$, which means that $g \in W$. Therefore $x \in P_\infty(W \times \widetilde{V})$. The reverse inclusion is obvious.

From the Sublemma it follows that P_∞ is an open map, since sets of the form $W \times \widetilde{V}$ form a basis for the open sets of Y_∞.

Now let $\widetilde{V}$ be a simple open subset of $\widetilde{X}$. In the Sublemma take $n = 1$. (G_1 consists of the identity element alone.) Then we have $P_\infty(G_\infty \times \widetilde{V}) = f_1^{-1}[p_1(\widetilde{V})]$. Furthermore, P_∞ is 1-1 on $G_\infty \times \widetilde{V}$. For suppose $P_\infty(g, \tilde{v}) = P_\infty(g', \tilde{v}')$. Then $p_1(\tilde{v}) = p_1(\tilde{v}')$, and since p_1 is 1-1 on $\widetilde{V}$, $\tilde{v} = \tilde{v}'$. From the uniqueness part of Lemma 5.3, $g = g'$. Finally, since P_∞ is open, P_∞ is a homeomorphism on $G_\infty \times \widetilde{V}$. ∎

THEOREM 5.5. *Y_∞ is a generalized covering space of X_∞ with respect to the map P_∞.*

NOTE. We use the definition in Hu [9, p. 104]. Thus we must show that there is an open covering of X_∞ by sets V such that $P_\infty^{-1}(V)$ can be represented as a disjoint union of open subsets of Y_∞, each of which is mapped homeomorphically onto V by P_∞.

Proof. Let V_1 be an open set in X_1 such that $p_1^{-1}(V_1)$ is the disjoint union $\bigcup \widetilde{V}^i$ of open sets in $\widetilde{X}$ such that for each i, p_1 maps $\widetilde{V}^i$ homeomorphically onto V_1. Let $V = f_1^{-1}(V_1)$. Now by Lemma 5.4, P_∞ maps each $G_\infty \times \widetilde{V}^i$ homeomorphically onto V. Furthermore, $\bigcup (G_\infty \times \widetilde{V}^i)$ is a disjoint union of open sets in Y_∞, which we claim is all of $P_\infty^{-1}(V)$. For if $(g, \tilde{x}) \in P_\infty^{-1}(V)$, then $p_1(\tilde{x}) \in V_1$, so that $\tilde{x} \in \widetilde{V}^i$ for some i. It is obvious that the collection of such sets V covers X_∞. ∎

THEOREM 5.6. *$(X_\infty, f_1, X_1, G_\infty)$ is a principal fibre bundle.*

Proof. The three properties (a), (b), (c) established below can be seen to be equivalent to those in Steenrod's [12] definition of a principal fibre bundle; except that, since we have the group G_∞ acting on the left of X_∞, G_∞ will have to act on the right of the fibre (G_∞ itself) in Steenrod's definition. (a) G_∞ acts on X_∞ as a topological transformation group, as has been shown in Lemma 5.2. (b) (X_∞, f_1, X_1) is a locally trivial fibre space with fibre G_∞. Let $\widetilde{V}$ be a simple open subset of $\widetilde{X}$ and let $p_1(\widetilde{V}) = V_1$. Such sets V_1 can be taken as coordinate neighborhoods in X_1. Let q: $V_1 \to \widetilde{V}$ be the inverse of $p_1 | \widetilde{V}$. The map $(g,v) \to (g,q(v))$ is then a homeomorphism of $G_\infty \times V_1$ onto $G_\infty \times \widetilde{V}$. Thus by Lemma 5.4, the map ϕ: $G_\infty \times V_1 \to f_1^{-1}(V_1)$ given by $\phi(g,v) = P_\infty(g,q(v))$ is a homeomorphism. And for every (g,v) in $G_\infty \times V_1$ we have $f_1\phi(g,v) = f_1 g p_\infty q(v) = p_1 q(v) = v$. (c) Finally, the action of G_∞ is compatible with the fibre space structure, for we have for any g and g' in G_∞ and any v in V_1 that $\phi(gg',v) = P_\infty(gg',q(v)) = (gg')p_\infty q(v) = g[g' p_\infty q(v)] = g P_\infty(g',q(v)) = g\phi(g',v)$. ∎

COROLLARY 5.7. *X_∞ is not locally connected at any point.*

Proof. By Theorem 5.6 and Lemma 5.1, X_∞ is locally like a product of a totally disconnected, perfect space with another space. ∎

In order to understand the structure of X_∞ more, we now study its path components. For this the maps p_∞ and P_∞ will be useful.

THEOREM 5.8. *Let K denote the subset $p_\infty(\widetilde{X})$ of X_∞. A subset of X_∞ is a path component of X_∞ if and only if it is a "translate" of K by some element g of G_∞, i.e., it is of the form $g(K)$. Each path component of X_∞ is dense in X_∞.*

Proof. Let us take a translate $g(K)$ and prove it is a path component. First of all, $g(K)$ is pathwise connected since it is a continuous image of the pathwise connected space $\widetilde{X}$. Suppose α is a path from a point x in $g(K)$ to a point y in X_∞. We must show that y also is in $g(K)$. Now $x = gp_\infty(\tilde{x})$ for some $\tilde{x}$ in $\widetilde{X}$; that is, $(g,\tilde{x})$ lies over x with respect to the generalized covering map P_∞ (Theorem 5.5). Thus we may lift α by P_∞ to a path $\tilde{\alpha}$ in Y_∞ starting at $(g,\tilde{x})$. But $\{g\} \times \widetilde{X}$ is a component of Y_∞, and since $\tilde{\alpha}$ begins in this component, it must end in it. Thus $\tilde{\alpha}(1)$ is of the form $(g,\tilde{y})$, and we have $gp_\infty(\tilde{y}) = P_\infty(g,\tilde{y}) = P_\infty\tilde{\alpha}(1) = \alpha(1) = y$, so that $y \in g(K)$.

Thus the sets $g(K)$ are path components. They exhaust all of X_∞, for by Theorem 5.5, P_∞ maps Y_∞ *onto* X_∞.

To show that the sets $g(K)$ are dense in X_∞, it is sufficient to show that K is dense. Open sets of the form $V = f_n^{-1}(V_n)$ where V_n is open in X_n form a basis for the open sets in X_∞. Take any such V_n. Since p_n: $\widetilde{X} \to X_n$ is onto, there is a point $\tilde{v}$ in $\widetilde{X}$ for which $p_n(\tilde{v}) \in V_n$. Then $p_\infty(\tilde{v}) \in K \cap V$. ∎

The path components $g(K)$ are of course disjoint if they are different, but there can be $g \neq g'$ such that $g(K) = g'(K)$. We will determine when this can happen. Since G_∞ is a group, $g(K) = g'(K)$ is equivalent to $g^{-1}g'(K) = K$. So we will determine which g are such that $g(K) = K$. In this direction we let H denote the covering transformation group of p_1: $\widetilde{X} \to X_1$, and we define a certain homomorphism $\sigma\colon H \to G_\infty$. It will turn out that $g(K) = K$ if and only if g is in the image of σ. Also the kernel of σ has significance (Theorems 5.12 and 5.13).

Let $\lambda\colon F_1 \to H$ be the canonical isomorphism (with respect to the base point $\tilde{b} \in \tilde{X}$). For each n let $\tau_n\colon F_1 \to Q_n = F_1/F_n$ be the natural homomorphism. Since $\nu_n^{n+1}\tau_{n+1} = \tau_n$, the τ_n's induce a homomorphism $\tau\colon F_1 \to Q_\infty$ given by $\tau(a) = (\tau_1(a), \tau_2(a), \cdots)$. Now define σ by commutativity in the diagram

(5.1)
$$\begin{array}{ccc} H & \xrightarrow{\sigma} & G_\infty \\ \uparrow \lambda & & \uparrow \phi_\infty \\ F_1 & \xrightarrow{\tau} & Q_\infty. \end{array}$$

Lemma 5.9. *For each h in H the following diagram is commutative*:

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{h} & \tilde{X} \\ \downarrow p_\infty & & \downarrow p_\infty \\ X_\infty & \xrightarrow{\sigma(h)} & X_\infty \end{array}$$

Proof. Let $g = \sigma(h)$. Proving that $gp_\infty = p_\infty h$ is equivalent to proving that for each n, $g_n p_n = p_n h$ (where $g_n = \mu_n(g)$). To show the latter we use Lemma 4.1. In the preliminary discussion of that section replace diagram (4.1) by the commutative diagram

$$\begin{array}{ccc} (\tilde{X}, \tilde{b}) & & \\ \downarrow p_1 & \searrow p_n & \\ (X_1, b_1) & \xleftarrow{f_1^n} & (X_n, b_n). \end{array}$$

Then diagram (4.2) gets replaced by the diagram

$$\begin{array}{ccc} G_n & \xleftarrow{\sigma_n} & H \\ \uparrow \phi_n & & \uparrow \lambda \\ Q_n & \xleftarrow{\tau_n} & F_1, \end{array}$$

where, according to § 4, σ_n is defined by commutativity in this diagram. Then by Lemma 4.1, $\sigma_n(h)p_n = p_n h$. Thus we need only show that $g_n = \sigma_n(h)$. But $g_n = \mu_n(g) = \mu_n\sigma(h) = \mu_n\phi_\infty\tau\lambda^{-1}(h) = \phi_n\nu_n\tau\lambda^{-1}(h) = \phi_n\tau_n\lambda^{-1}(h) = \sigma_n(h)$. ∎

Theorem 5.10. *For $g \in G_\infty$, $g(K) = K$ if and only if $g = \sigma(h)$ for some $h \in H$.*

Proof. Suppose first $g=\sigma(h)$. Then for each $\tilde{x}$ in $\tilde{X}$, by the preceding lemma, $gp_\infty(\tilde{x})=p_\infty h(\tilde{x})\in K$ so that $g(K)\subset K$, whence $g(K)=K$. Conversely, suppose $g(K)=K$. Then for some $\tilde{x}$ and $\tilde{y}$ in $\tilde{X}$ we have $gp_\infty(\tilde{x}) =p_\infty(\tilde{y})$. Then since $p_1(\tilde{x})=p_1(\tilde{y})$, there is an h in H such that $h(\tilde{x})=\tilde{y}$. But then $gp_\infty(\tilde{x})=p_\infty(\tilde{y})=p_\infty h(\tilde{x})=\sigma(h)p_\infty(\tilde{x})$. Thus g and $\sigma(h)$ are elements of G_∞ which agree on a point, so they are equal. ∎

COROLLARY 5.11. *If the fundamental group F_1 of X_1 is countable (in particular if X_1 is a polyhedron) then X_∞ has uncountably many path components.*

Proof. Let S be the set of path components of X_∞. Define a map w from the space $G_\infty/\sigma(H)$ of left cosets of $\sigma(H)$ to S by $w(g\sigma(H))=g(K)$. By the preceding theorem, w is well defined and 1-1; by Theorem 5.8, w is onto. Now G_∞ is uncountable since each bonding map $\mu_n^{n+1}\colon G_{n+1}\to G_n$ has everywhere nondegenerate point inverses. And since $\sigma(H)=\sigma\lambda(F_1)$ is countable, $G_\infty/\sigma(H)$ is uncountable. ∎

In the cases of solenoids the map p_∞ of $\tilde{X}$ onto K is 1-1. What is the general situation? This question is answered by the following theorem.

THEOREM 5.12. (a) *The map $p_\infty\colon \tilde{X}\to X_\infty$ is* 1-1 *if and only if the homomorphism $\sigma\colon H\to G_\infty$ is* 1-1. (b) *The kernel of $\tau\colon F_1\to Q_\infty$ is $N=\bigcap_{n=1}^{\infty}F_n$.* (c) *Hence p_∞ is* 1-1 *if and only if $N=1$.*

Proof. (a) Suppose p_∞ is 1-1. Then σ must be 1-1, for if $\sigma(h)=1$, we have by Lemma 5.9 that for all $\tilde{x}$ in $\tilde{X}$, $p_\infty h(\tilde{x})=\sigma(h)p_\infty(\tilde{x})=p_\infty(\tilde{x})$. Hence $h(\tilde{x})=\tilde{x}$, and $h=1$. Conversely, suppose σ is 1-1 and suppose $p_\infty(\tilde{x}) =p_\infty(\tilde{y})$. Then since $p_1(\tilde{x})=p_1(\tilde{y})$ there is an h in H such that $h(\tilde{x})=\tilde{y}$; therefore $p_\infty(\tilde{x})=p_\infty(\tilde{y})=p_\infty h(\tilde{x})=\sigma(h)p_\infty(\tilde{x})$. Since $\sigma(h)$ is in G_∞ and has a fixed point, it must be the identity. Thus $h=1$, and $\tilde{x}=h(\tilde{x})=\tilde{y}$.

(b) Suppose $a\in\ker\tau$. This means that $\tau(a)=(aF_1,aF_2,\cdots)=(F_1,F_2,\cdots)$ so that $a\in F_n$ for each n, and $a\in N$. The converse is then obvious.

(c) In diagram (5.1) λ and ϕ_∞ are isomorphisms. Hence $\ker\sigma=\lambda(N)$. The statement follows from (a) and (b). ∎

THEOREM 5.13. *The fundamental group of each path component of X_∞ is isomorphic to $N=\bigcap_{n=1}^{\infty}F_n$.*

Proof. By Theorem 5.8 all path components are homeomorphic. We will show that for the map $f_1\colon (X_\infty,b)\to(X_1,b_1)$, $\eta=(f_1)_*$ takes

$$\pi(K,b)=\pi(X_\infty,b)$$

isomorphically onto N. Since $f_1=f_1^nf_n$, the range of η is in N.

To prove that the range of η is all of N, suppose $[\alpha_1]\in N$. Since $[\alpha_1]\in F_n$, the lifting of α_1 by f_1^n to the path α_n starting at b_n is a loop on b_n. And since then $f_n^{n+1}\alpha_{n+1}=\alpha_n$, we may define a loop α on b by $\alpha(t)=(\alpha_1(t),\alpha_2(t),\cdots)$, for which obviously $\eta([\alpha])=[\alpha_1]$. To show that η is 1-1, suppose $\eta([\alpha])=1$.

This means that there is a homotopy h_t^1 of loops on b_1 such that $h_0^1 = \alpha_1$ and $h_1^1 = b_2$. Applying successively the covering homotopy theorem to the maps f_n^{n+1} we may construct a sequence (h_t^n) such that for each n, (1) h_t^n is a homotopy of loops on b_n in X_n, (2) $h_0^n = \alpha_n$, (3) $h_1^n = b_n$, and (4) $f_n^{n+1} h_t^{n+1} = h_t^n$. This sequence then induces a homotopy h_t of loops on b in X_∞ such that $h_0 = \alpha$ and $h_1 = b$. ∎

REMARK 5.14. If (X, f) is an inverse limit sequence where each map f_1^n is a covering map, but is not necessarily regular, the proof above shows that for every x in X_∞, if K_x denotes the path component of x, then $\pi(K_x, x) \approx \bigcap_{n=1}^{\infty} (f_1^n)_* (\pi(X_n, x_n))$.

6. **Homogeneity of certain solenoidal spaces.** Let us call a space X *path-homogeneous* if for every pair of points x, y in X there is a path α from x to y and a homeomorphism h of (X, x) onto (X, y) such that h induces the same isomorphism of $\pi(X, x)$ onto $\pi(X, y)$ as the path α does.

THEOREM 6.1. *If (X, f) is a solenoidal sequence for which X_1 is path-homogeneous, then the solenoidal space X_∞ is homogeneous.*

Proof. Let us use the notation of the preceding section. Suppose x and y are points in X_∞. We wish to show that there is a homeomorphism of (X_∞, x) onto (X_∞, y). Since X_1 is path-homogeneous there is a path α_1 from x_1 to y_1 and a homeomorphism $h_1: (X_1, x_1) \to (X_1, y_1)$ such that $(h_1)_* = (\alpha_1)_*: \pi(X_1, x_1) \to \pi(X_1, y_1)$; that is, $(h_1)_* ([\gamma]) = [h_1 \gamma] = [\alpha_1^{-1} \cdot \gamma \cdot \alpha_1] = (\alpha_1)_* ([\gamma])$ for all $[\gamma] \in \pi(X_1, x_1)$. Now lift α_1 by p_1 to some path $\tilde{\alpha}$ in $\tilde{X}$. Let $u = p_\infty(\tilde{\alpha}(0))$ and $v = p_\infty(\tilde{\alpha}(1))$. Since $u_1 = x_1$ there is, by Lemma 5.3, a (unique) element g of G_∞ such that $g(u) = x$. (In particular g is a homeomorphism of (X_∞, u) onto (X_∞, x).) Similarly there is a g' in G_∞ such that $g'(v) = y$. If we can find a homeomorphism h of (X_∞, u) onto (X_∞, v) we will be through, since then $g' h g^{-1}$ takes x onto y.

We will construct such an h from h_1 by a lifting process which makes use of the path $\tilde{\alpha}$. Let $\alpha_n = p_n \tilde{\alpha}$; this path goes from u_n to v_n. Observe that $f_n^{n+1} \alpha_{n+1} = f_n^{n+1} p_{n+1} \tilde{\alpha} = p_n \tilde{\alpha} = \alpha_n$. Let $\psi_n: \pi(X_n, u_n) \to \pi(X_n, v_n)$ denote the isomorphism induced by α_n. We claim that the following diagram is commutative.

$$
\begin{array}{ccccccc}
\pi(X_1, u_1) & \xleftarrow{(f_1^2)_*} & \pi(X_2, u_2) & \xleftarrow{(f_2^3)_*} & \pi(X_3, u_3) & \xleftarrow{(f_3^4)_*} & \cdots \\
\downarrow \psi_1 & & \downarrow \psi_2 & & \downarrow \psi_3 & & \\
\pi(X_1, v_1) & \xleftarrow{(f_1^2)_*} & \pi(X_2, v_2) & \xleftarrow{(f_2^3)_*} & \pi(X_3, v_3) & \xleftarrow{(f_3^4)_*} & \cdots .
\end{array}
\tag{6.1}
$$

For suppose $[\gamma] \in \pi(X_{n+1}, u_{n+1})$. Then

$$\begin{aligned}(f_n^{n+1})_* \psi_{n+1}([\gamma]) &= (f_n^{n+1})_* [\alpha_{n+1}^{-1} \cdot \gamma \cdot \alpha_{n+1}] \\ &= [(f_n^{n+1}\alpha_{n+1})^{-1} \cdot f_n^{n+1}\gamma \cdot f_n^{n+1}\alpha_{n+1}] \\ &= [\alpha_n^{-1} \cdot f_n^{n+1}\gamma \cdot \alpha_n] = \psi_n(f_n^{n+1})_* ([\gamma]).\end{aligned}$$

Looking at the first rectangle in (6.1), using what Hu [9, p. 90] calls the fibre map theorem, and recalling that $(h_1)_* = \psi_1$, we have a unique map

$$h_2: (X_2, u_2) \to (X_2, v_2)$$

which makes the first rectangle in (6.2) below commutative. Furthermore, h_2 is a homeomorphism. For, using ψ_1^{-1} and ψ_2^{-1}, there is a (unique) map h_2': $(X_2, v_2) \to (X_2, u_2)$ such that $f_1^2 h_2' = h_1^{-1} f_1^2$. But then $f_1^2 h_2' h_2 = h_1^{-1} f_1^2 h_2 = h_1^{-1} h_1 f_1^2 = f_1^2$, and from the uniqueness of liftings, $h_2' h_2 =$ identity. Similarly $h_2 h_2' =$ identity. Now since $(f_1^2)_*$ is a monomorphism, $(h_2)_* = \psi_2$. Thus we can use the second rectangle of (6.1) to get a homeomorphism h_3 making the second rectangle of (6.2) commutative. We continue this process, obtaining $(h_1, h_2, h_3, \cdots)$.

$$(6.2)\qquad \begin{array}{ccccccccc} (X_1, u_1) & \xleftarrow{f_1^2} & (X_2, u_2) & \xleftarrow{f_2^3} & (X_3, u_3) & \xleftarrow{f_3^4} & \cdots & \longleftarrow & (X_\infty, u) \\ \downarrow h_1 & & \downarrow h_2 & & \downarrow h_3 & & & & \downarrow h \\ (X_1, v_1) & \xleftarrow{f_1^2} & (X_2, v_2) & \xleftarrow{f_2^3} & (X_3, v_3) & \xleftarrow{f_3^4} & \cdots & \longleftarrow & (X_\infty, v). \end{array}$$

Clearly the induced map h is the desired homeomorphism. ∎

Let us use *n-manifold* in the very general sense of a connected Hausdorff space, each of whose points is contained in an open set homeomorphic to euclidean n-space E^n.

LEMMA 6.2. *Every n-manifold is path-homogeneous.*

COROLLARY 6.3. *If (X, f) is a solenoidal sequence where X_1 is an n-manifold, then X_∞ is homogeneous.*

Proof of Lemma 6.2. By an *isotopic deformation* h_t of a space S let us mean an isotopy h_t: $S \to S$ such that $h_0 =$ identity.

Let M be an n-manifold. First suppose U is an open subset of M for which there is a homeomorphism g: $U \to E^n$, and let $z, w \in U$. Let $\{g(z), g(w)\} \subset \{x \in E^n: \|x\| < r\}$. It is easy to see that there is an isotopic deformation h_t' of E^n such that $h_t'(g(z)) = g(w)$ and for each t, h_t' is the identity on

$$\{x \in E^n: \|x\| \geqq r\}.$$

Define h_t: $M \to M$ by $h_t(m) = g^{-1}h_t'g(m)$ for $m \in U$ and $h_t(m) = m$ for $m \notin U$. Using the fact that M is Hausdorff one can see that h_t is an isotopic deformation of M; and $h_1(z) = w$.

Now let x and y be arbitrary points of M. Since M is connected, there is a chain of open sets $(U_1, U_2, \cdots, U_r)$, each homeomorphic to E^n, such that $x \in U_1$ and $y \in U_r$. For each $i = 2, 3, \cdots, r$ choose $x_i \in U_{i-1} \cap U_i$, and let $x_1 = x$, $x_{r+1} = y$. By the preceding paragraph there is for each $i = 1, 2, \cdots, r$ an isotopic deformation h_t^i of M such that $h_1^i(x_i) = x_{i+1}$. Then the composition $h_t = h_t^r \cdots h_t^2 h_t^1$ is an isotopic deformation of M such that $h_1(x) = y$.

Now it follows from [6, p. 57] that the path α from x to y defined by $\alpha(t) = h_t(x)$ $(0 \leqq t \leqq 1)$ induces the same isomorphism of $\pi(M, x)$ onto $\pi(M, y)$ as h_1 does. ∎

The author suspects that Corollary 6.3 is no longer valid when the bonding maps are nonregular covering maps, even when the X_n's are compact differentiable manifolds. (J. Segal [10] announced in an abstract that if (X, f) is an inverse limit sequence where each X_n is a homogeneous, connected, and locally pathwise connected space and where the bonding maps are covering maps, then X_∞ is homogeneous. However he informed the author of an error in the proof.) The author has shown that such a counterexample can be constructed if a certain group theoretic construction can be made; and, in turn, this can be done provided a certain sequence of graphs and covering maps can be produced. The idea includes using Theorem 5.13 (see Remark 5.14) to show that the continuum has two nonhomeomorphic path components.

Added in proof: Richard M. Schori (Dissertation, State University of Iowa, 1964) has constructed an inverse limit sequence of compact, orientable 2-manifolds such that the bonding maps are (nonregular) covering maps and such that the limit is not homogeneous.

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