Inverse Optimal Adaptive Control for Attitude Tracking of Spacecraft

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***Abstract*—The attitude tracking control problem of a rigid spacecraft with external disturbances and an uncertain inertia matrix is addressed using the adaptive control method. The adaptive control laws proposed in this paper are optimal with respect to a family of cost functionals. This is achieved by the inverse optimality approach, without solving the associated Hamilton–Jacobi–Isaacs partial differential (HJIPD) equation directly. The design of the optimal adaptive controllers is separated into two stages by means of integrator backstepping, and a control Lyapunov argument is constructed to show that the inverse optimal adaptive controllers achieve H_∞ disturbance attenuation with respect to external disturbances and global asymptotic convergence of tracking errors to zero for disturbances with bounded energy. The convergence of adaptive parameters is also analyzed in terms of invariant manifold. Numerical simulations illustrate the performance of the proposed control algorithms.**

***Index Terms*—Adaptive control, attitude tracking control, disturbance attenuation, integrator backstepping, inverse optimal control, nonlinear system.**

I. INTRODUCTION

ATTITUDE control systems are required to provide the present generation of spacecraft with attitude maneuver, tracking and pointing capabilities. The equations that govern attitude maneuvers and attitude tracking are nonlinear and coupled, thus, the attitude control system must consider these nonlinear dynamics. Various nonlinear control algorithms, such as nonlinear feedback control [1], [2], variable structure control [3], [4], sliding control [5] and optimal control [6], etc., have been proposed for solving the attitude tracking control problem for spacecraft with known parameters. However, in a practical situation, the mass properties of the spacecraft may be uncertain or may change due to onboard payload motion, rotation of solar arrays or fuel consumption. Therefore, the nonlinear attitude control system should be able to adapt to uncertainties in the mass properties and have robust capability to attenuate external disturbances.

Adaptive control method [7] is a natural choice to deal with uncertain parameters and has been applied to the attitude tracking control problem of spacecraft. In [8], an adaptive tracking law was developed; however, it is not globally valid because of a singularity of the attitude representation using Rodriguez parameters. In [9], a passivity-based adaptive control scheme was designed to achieve attitude tracking with a global convergence. Using a singularity-free representation of spacecraft attitude and based on control Lyapunov functions, the authors of [10] and [11] developed adaptive feedback control laws for a zero-disturbance spacecraft to achieve asymptotic attitude tracking with a global convergence of the tracking errors to zero. In [12] and [13], an integrated power and attitude control system was studied using flywheels and control moment gyroscopes, respectively, and adaptive tracking controllers were designed for the power/attitude tracking problem. The degree of optimality of these adaptive controllers were not stated explicitly. Also, the disturbance attenuation problem was not involved in designing these adaptive attitude controllers.

The optimal control of nonlinear systems without disturbances boils down to the solvability of a Hamilton–Jacobi–Bellman (HJB) equation. By solving the HJB equation directly, an optimal controller [6] was designed for a spacecraft to track a constant attitude trajectory. Due to its inherent robustness with respect to external disturbances and uncertainties, nonlinear H_∞ optimal control [14] is a potential approach for solving the optimal attitude tracking control problem with external disturbances. However, the practical applications of H_∞ optimal control remain open due to the difficulty in solving the associated Hamilton–Jacobi–Isaacs partial differential (HJIPD) equation. Various techniques have been proposed to study particular H_∞ suboptimal control problems. These techniques were based on solving the associated HJIPD inequality by algebraic and geometric tools [15], [16], power series [17], and other numerical methods [18], [19].

An alternative approach to the design of robust optimal feedback controllers is the so-called *inverse optimal control approach* [20], [21], which circumvents the task of solving the HJIPD equation and results in a feedback controller that is optimal with respect to a set of meaningful cost functionals. The application of this approach to the attitude control problem was first presented by Bharadwaj *et al.* [22] and Krstić [23], who designed an inverse optimal feedback controller for the attitude regulation problem of a rigid spacecraft without external disturbances and uncertainties in the inertia matrix. Krstić [23] also used Rodriguez parameters to represent the spacecraft attitude, which is only a regional solution because the attitude representation using Rodriguez parameters has a singularity.

In this paper, the attitude of spacecraft is represented by the unit quaternion, which is singularity-free. The adaptive control method and the inverse optimal control approach are combined

Manuscript received February 8, 2005; revised June 10, 2005. Recommended by Associate Editor J. Huang. The work presented in this paper was supported by the NTU AcRF under Project RG 9/00. Part of this paper was submitted to the 12th Mediterranean Conference on Control and Automation (MED'04).

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Digital Object Identifier 10.1109/TAC.2005.858694

to account for the uncertainty in the inertia matrix of the spacecraft, the disturbance attenuation, and the optimality of the attitude controllers, for the attitude tracking control problem. The method of integrator backstepping [20] is used to construct a control Lyapunov function and stabilizing control laws. The main contributions of this paper relative to other works are as follows.

1) By means of an adaptive control Lyapunov function, the nonadaptive inverse optimal control approach in [21] is extended to uncertain nonlinear systems with exogenous disturbances. An inverse optimal adaptive control algorithm is presented and then applied to the attitude tracking control problem. The attitude tracking controllers derived in this paper are global.
2) For the zero-disturbance case, the proposed inverse optimal adaptive controller achieves asymptotic attitude tracking with a global convergence of tracking errors to zero for all initial conditions. In comparison with the work of [8]–[13], the adaptive control law in this paper is inverse optimal with respect to a meaningful cost functional involving tracking errors and control efforts.
3) When external disturbances are considered, an adaptive attitude tracking controller is designed that is inverse optimal and achieves H_∞ disturbance attenuation without solving the associated HJIPD equation directly. The closed-loop attitude system under the inverse optimal adaptive tracking controller is input-to-state stable, therefore bounded (and persistent) external disturbances are allowed in the attitude control system and will lead to bounded tracking errors. In comparison with the H_∞ suboptimal controllers in [15], [16], [24], [25] that were designed for the attitude stabilizing problem and required the $\mathcal{L}_2$-gain γ to be larger than certain values, the inverse optimal adaptive controller presented in this paper allows the disturbance attenuation level γ of the closed-loop system to be chosen sufficiently small so as to achieve any level of $\mathcal{L}_2$ disturbance attenuation at the cost of a larger control effort.

The remaining of the paper is organized as follows. In Section II, important results on the inverse optimal adaptive control problem are presented. In Section III, the attitude tracking control problem of a rigid spacecraft is formulated using the unit quaternion to represent the attitude orientation. In Section IV, we present our main results on designing inverse optimal adaptive control laws to solve the attitude tracking control problem. Numerical simulations are shown in Section V to demonstrate the performance of the adaptive feedback control algorithms. Finally, conclusions follow in Section VI.

II. Inverse Optimal Adaptive Control

In this section, the inverse optimal adaptive control problem is formulated and some important results on optimal adaptive controller design are presented. First, the following notations are introduced. For a vector $x \in \mathcal{R}^n$, let $\|x\| = \sqrt{x^T x}$ denote the Euclidean norm of x and let $\|x\|_Q^2$ represent the quadratic form $x^T Q x$ for a positive definite symmetric matrix $Q \in \mathcal{R}^{n\times n}$. For a matrix $A \in \mathcal{R}^{m\times n}$, we use the standard notation $\|A\| = \sqrt{\lambda_{\max}(A^T A)}$ to denote the induced 2-norm of A, where $\lambda_{\max}(A^T A)$ denotes the maximal eigenvalue of $A^T A$. Let $L_f V$ denote the Lie derivative of the Lyapunov function $V(x)$ with respect to $f(x)$, that is, $L_f V = (\partial V(x)/\partial x) f(x)$. A continuous function $\rho : \mathcal{R}_+ \to \mathcal{R}_+$ is said to belong to class $\mathcal{K}$ if it is positive definite, strictly increasing and $\rho(0) = 0$. It is of class $\mathcal{K}_\infty$ if $\rho \in \mathcal{K}$ and $\rho(r) \to \infty$ as $r \to \infty$. A function $\phi : \mathcal{R}^+ \times \mathcal{R}^+ \to \mathcal{R}^+$ is of class $\mathcal{KL}$ if, for each fixed $t \geq 0$, $\phi(\cdot, t)$ is of class $\mathcal{K}$ and, for each fixed $s \geq 0$, $\lim_{t\to\infty} \phi(s,t) = 0$. For a positive integer p, the set $\mathcal{L}_p[0,\infty)$ is a linear space consisting of square integrable $\mathcal{R}^m$-valued functions, i.e., $v \in \mathcal{L}_p[0,\infty)$ implies that $[\int_0^\infty \|v(t)\|^p dt]^{1/p}$ is finite. The commonly used cases are $p = 1, 2$.

Consider the nonlinear uncertain system

$$\dot{x} = f(x) + F(x)\theta + g(x)u \tag{1}$$

where $x \in \mathcal{R}^n$, $u \in \mathcal{R}^m$, the mappings $f(x)$, $F(x)$ and $g(x)$ are smooth, $\theta \in \mathcal{R}^p$ is a constant unknown parameter vector. Let $\hat{\theta}$ denote an estimate of θ with the estimation error $\tilde{\theta} = \theta - \hat{\theta}$, and $\|\tilde{\theta}\|_{\Gamma^{-1}}^2 = \tilde{\theta}^T \Gamma^{-1} \tilde{\theta}$ for a positive definite symmetric matrix $\Gamma \in \mathcal{R}^{p\times p}$.

Definition 1: The *adaptive control problem* for (1) is solvable if there exist a function $\alpha(x,\hat{\theta})$ smooth on $(\mathcal{R}^n \setminus \{0\}) \times \mathcal{R}^p$ with $\alpha(0,\hat{\theta}) \equiv 0$, a smooth function $\tau(x,\hat{\theta})$ and a positive definite symmetric matrix $\Gamma \in \mathcal{R}^{p\times p}$ such that the dynamic feedback controller

$$u = \alpha(x,\hat{\theta}) \tag{2a}$$
$$\dot{\hat{\theta}} = \Gamma \tau(x,\hat{\theta}) \tag{2b}$$

guarantees that the solution $(x(t), \hat{\theta}(t))$ is globally bounded, and $x(t) \to 0$ as $t \to \infty$, for all $\theta \in \mathcal{R}^p$.

Definition 2: [26] A smooth function $V(x,\theta) : \mathcal{R}^n \times \mathcal{R}^p \to \mathcal{R}_+$, positive definite and radially unbounded in x for each θ, is called an *adaptive control Lyapunov function* (**aclf**) for (1) if there exist a positive–definite symmetric matrix $\Gamma \in \mathcal{R}^{p\times p}$, a continuous function $W(x,\theta)$ positive definite in x for each $\theta \in \mathcal{R}^p$ and a control $u = \alpha(x,\theta)$ smooth on $(\mathcal{R}^n \setminus \{0\}) \times \mathcal{R}^p$ with $\alpha(0,\theta) \equiv 0$ such that $V(x,\theta)$ satisfies

$$\frac{\partial V}{\partial x}\left[f + F\left(\theta + \Gamma\left(\frac{\partial V}{\partial \theta}\right)^T\right) + gu\right] \leq -W(x,\theta) \tag{3}$$

for the auxiliary system

$$\dot{x} = f(x) + F(x)\left(\theta + \Gamma\left(\frac{\partial V}{\partial \theta}\right)^T\right) + g(x)u. \tag{4}$$

The approach adopted in Definition 2 to stabilize (1) is to first replace the problem of adaptive stabilization of (1) by a problem of nonadaptive stabilization of an auxiliary system (4), and then design an adaptive controller by applying the results got from the auxiliary system and the concept of "certainty equivalence" [7]. This approach allows us to study adaptive stabilization in the framework of control Lyapunov functions.

If $V(x,\theta)$ is an aclf and $\alpha(x,\theta)$ is a stabilizing control law of the auxiliary system (4), we can construct a new Lyapunov function candidate $V_2(x,\hat{\theta}) = V(x,\hat{\theta}) + (1/2)\tilde{\theta}^T \Gamma^{-1} \tilde{\theta}$ and choose the tuning function $\tau(x, \hat{\theta})$ of (2b) as $\tau(x, \hat{\theta}) = (L_F V)^T$. The control law $\alpha(x, \theta)$ stabilizes the auxiliary system (4) but may not stabilize the original system (1). However, its certainty equivalence form $\alpha(x, \hat{\theta})$ is an adaptive stabilizing control law for the original uncertain system (1). To see this, replacing θ by the estimate $\hat{\theta}$ obtained from the parameter update law (2b) with the tuning function $\tau(x, \hat{\theta}) = (L_F V)^T$ and applying the inequality (3), we have

$$\begin{aligned} \dot{V}_2(x, \hat{\theta}) &= \frac{\partial V}{\partial x}\left[f(x) + F(x)\hat{\theta} + g(x)\alpha(x, \hat{\theta})\right] + \frac{\partial V}{\partial \hat{\theta}}\Gamma\tau(x, \hat{\theta}) \\ &\le -W(x, \hat{\theta}). \end{aligned} \tag{5}$$

Then the "certainty equivalence" controller $\alpha(x, \hat{\theta})$ prevents $\tau(x, \hat{\theta})$ from destroying the nonpositivity of the Lyapunov derivative $\dot{V}_2(x, \hat{\theta})$. Based on the inequality (5), one can easily construct a continuous weighting function $l(x, \hat{\theta})$ positive definite in x for each $\hat{\theta} \in \mathcal{R}^p$, as expressed in Theorems 1 and 3, for the following inverse optimal adaptive control problem.

Definition 3: The *inverse optimal adaptive control problem* for the system (1) is solvable if there exist a positive constant β, a smooth nonnegative function $E(x) \ge 0$, a positive–definite symmetric matrix $R(x, \hat{\theta})$, a real-valued function $l(x, \hat{\theta})$ positive definite in x for each $\hat{\theta}$, and a dynamic feedback law (2) that solves the adaptive control problem and also minimizes the cost functional

$$J_a = \lim_{T\to\infty}\left\{\beta\left\|\tilde{\theta}(T)\right\|^2_{\Gamma^{-1}} + E\left(x(T)\right) + \int_0^T \left[l(x, \hat{\theta}) + u^T R(x, \hat{\theta})u\right] dt\right\} \tag{6}$$

for each $\theta \in \mathcal{R}^p$.

Definition 3 is a bit different from [26, Def. 5.12] in that a smooth nonnegative function $E(x(T))$ that penalizes the terminal state $x(T)$ is introduced in the cost functional (6) to avoid imposing an assumption that $x(T) \to 0$ as $T \to \infty$. In the next two theorems we design inverse optimal adaptive controllers for the uncertain nonlinear system (1) in the sense of Definition 3.

Theorem 1: Suppose there exist an **aclf** $V(x, \theta)$ for (1), a positive definite symmetric matrix $\Gamma \in \mathcal{R}^{p\times p}$, a positive definite symmetric matrix $R(x, \theta) \in \mathcal{R}^{m\times m}$ and a feedback control law

$$u = \tilde{\alpha}(x, \theta) = -R(x, \theta)^{-1}\left(\frac{\partial V}{\partial x}g\right)^T$$

that stabilizes the auxiliary system (4). Then, the dynamic feedback control law

$$u = \tilde{\alpha}^*(x, \hat{\theta}) = \beta\tilde{\alpha}(x, \hat{\theta}), \qquad \beta \ge 2$$

together with the parameter update law

$$\dot{\hat{\theta}} = \Gamma\tau(x, \hat{\theta}) = \Gamma\left(\frac{\partial V}{\partial x}F\right)^T$$

minimizes the cost functional J_a in (6) with $E(x) \equiv 0$, where

$$\begin{aligned} l(x, \hat{\theta}) = -2\beta&\left[\frac{\partial V}{\partial x}f + \frac{\partial V}{\partial x}F\left(\hat{\theta} + \Gamma\left(\frac{\partial V}{\partial \hat{\theta}}\right)^T\right) + \frac{\partial V}{\partial x}g\tilde{\alpha}\right] \\ &+ \beta(\beta - 2)\frac{\partial V}{\partial x}gR^{-1}\left(\frac{\partial V}{\partial x}g\right)^T. \end{aligned}$$

Proof: It is a straightforward extension of [26, Th. 5.13] to the multiple-input–multiple-output (MIMO) case with some necessary modifications. Therefore, the proof is omitted. ■

Theorem 2: Suppose the nonlinear system (1) is globally adaptively stabilizable with an **aclf** $V(x, \theta)$, a smooth control law $\alpha(x, \theta)$ and a smooth tuning function $\tau(x, \theta)$, and (3) is satisfied with $W(x, \theta) = x^T\Omega(x, \theta)x$, where $\Omega(x, \theta) \in \mathcal{R}^{n\times n}$ is positive definite and symmetric for all x and θ. Assume that $f(x)$, $F(x)$ and $F_1(x, \xi)$ are smooth and vanish at $x = 0$. Then, the inverse optimal adaptive control problem with $E(x, \xi) \equiv 0$ for the augmented system

$$\dot{x} = f(x) + F(x)\theta + g(x)\xi \tag{7a}$$
$$\dot{\xi} = u + F_1(x, \xi)\theta \tag{7b}$$

is also solvable with a smooth dynamic feedback control law.

Proof: It is a straightforward extension of [26, Lemma 5.20] and [7, Lemma 4.7, Cor. 4.9] to the MIMO case with some necessary modifications. The proof is omitted here. ■

Remark 1: It should be noted that the augmented system in [26, Lemma 5.20] is augmented by an integrator $\dot{\xi} = u$, which is different from (7) and can be considered as a special case of the system (7). A nonlinear adaptive controller is designed for the augmented system (7) in [7, Lemma 4.7 and Corollary 4.9] using the nonlinearity cancellation technique, which is in general not guaranteed to be inverse optimal. Theorem 2 establishes the inverse optimality for the augmented system (7). □

In Theorems 1 and 2, we have addressed the inverse optimal adaptive control problem for the zero-disturbance nonlinear system (1). We then proceed to consider the inverse optimal adaptive control problem for uncertain systems with disturbances. The next theorem establishes an inverse optimal adaptive feedback controller for such systems.

Theorem 3: Consider the nonlinear system with disturbances

$$\dot{x} = f(x) + F(x)\theta + g_1(x)d + g_2(x)u \tag{8}$$

and the auxiliary system

$$\begin{aligned} \dot{x} = f(x) + F(x)&\left[\theta + \Gamma\left(\frac{\partial V}{\partial \theta}\right)^T\right] \\ &+ g_1(x)\ell_\rho\left(2\left\|L_{g_1}V\right\|\right)\frac{(L_{g_1}V)^T}{\|L_{g_1}V\|^2} + g_2(x)u \end{aligned} \tag{9}$$

where $V(x, \theta)$ is a Lyapunov function candidate; ρ is a class $\mathcal{K}_\infty$ function whose derivative ρ' is also a class $\mathcal{K}_\infty$ function; ℓ_ρ denotes the transform $\ell_\rho(r) = \int_0^r (\rho')^{-1}(s)ds$ where $(\rho')^{-1}(r)$ stands for the inverse function of $d\rho(r)/dr$. Suppose that there exists a real matrix $R_2(x,\theta) = R_2^T(x,\theta) > 0$ such that the control law

$$u = \alpha(x,\theta) = -R_2(x,\theta)^{-1}(L_{g_2}V)^T \tag{10}$$

asymptotically stabilizes (9) with respect to $V(x,\theta)$. Then, the dynamic feedback control

$$u = \alpha^*(x,\hat{\theta}) = -\beta R_2(x,\hat{\theta})^{-1}(L_{g_2}V)^T \tag{11a}$$

$$\dot{\hat{\theta}} = \Gamma\tau(x,\hat{\theta}) = \Gamma(L_F V)^T \tag{11b}$$

with $\beta \geq 2$ solves the inverse optimal adaptive control problem for the nonlinear system (8) by minimizing the cost functional

$$J_a = \sup_{d\in\mathcal{D}}\left\{\lim_{T\to\infty}\left[\beta\left\|\tilde{\theta}(T)\right\|^2_{\Gamma^{-1}} + 2\beta V\left(x(T),\hat{\theta}(T)\right) + \int_0^T \left(l(x,\hat{\theta}) + u^T R_2(x,\hat{\theta})u - \beta\lambda\rho\left(\frac{\|d\|}{\lambda}\right)\right)dt\right]\right\} \tag{12}$$

for any $\lambda \in (0,2]$, where

$$l(x,\hat{\theta}) = -2\beta L_f V - 2\beta L_F V\left[\hat{\theta} + \Gamma\left(\frac{\partial V}{\partial\hat{\theta}}\right)^T\right] - \beta\lambda\ell_\rho(2\|L_{g_1}V\|) + \beta^2 L_{g_2}VR_2^{-1}(L_{g_2}V)^T \tag{13}$$

and $\mathcal{D}$ is the set of locally bounded functions of x.

Remark 2: If the parameter θ is known, the control problem is reduced to a nonadaptive inverse optimal problem with no Γ, which was considered in [21, Th. 3.1]. Theorem 3 is an important extension of [21, Th. 3.1], as the adaptive control problem of uncertain parameters is also considered to form an inverse optimal adaptive control problem. The proof here is based on that of [21, Th. 3.1], with certain significant modifications for the adaptive case.

Proof: Since the control law $u = \alpha(x,\theta)$ in (10) stabilizes the auxiliary system (9), it follows from Definition 2 that there exists a continuous function $W(x,\theta)$ positive definite in x for each $\theta \in \mathcal{R}^p$ such that

$$L_f V + L_F V\left[\theta + \Gamma\left(\frac{\partial V}{\partial\theta}\right)^T\right] + \ell_\rho(2\|L_{g_1}V\|) - L_{g_2}VR_2^{-1}(L_{g_2}V)^T \leq -W(x,\theta) \leq 0$$

which brings

$$l(x,\hat{\theta}) \geq W(x,\hat{\theta}) + \beta(2-\lambda)\ell_\rho(2\|L_{g_1}V\|) + \beta(\beta-2)L_{g_2}VR_2^{-1}(L_{g_2}V)^T.$$

Since $\beta \geq 2$, $R_2(x,\hat{\theta}) > 0$ and $W(x,\hat{\theta})$ is positive definite, $l(x,\hat{\theta})$ is also positive definite in x for each $\hat{\theta} \in \mathcal{R}^p$. Therefore, the cost functional J_a in (12) is a meaningful cost functional, which puts penalties on the state $x(t)$, the control input $u(t)$ and the disturbance $d(t)$.

Substituting $l(x,\hat{\theta})$ in (13) into J_a in (12) and applying the dynamic feedback control law (11), along the trajectories of (8) we get

$$\begin{aligned}
J_a = \sup_{d\in\mathcal{D}}\Bigg\{\lim_{T\to\infty}\Bigg[&\beta\left\|\tilde{\theta}(T)\right\|^2_{\Gamma^{-1}} + 2\beta V\left(x(T),\hat{\theta}(T)\right) \\
&- 2\beta\int_0^T (L_f V + L_F V\theta + L_{g_1}Vd + L_{g_2}Vu)\,dt \\
&+ \int_0^T \Bigg(2\beta L_F V\tilde{\theta} - 2\beta L_F V\Gamma\left(\frac{\partial V}{\partial\hat{\theta}}\right)^T \\
&+ \beta^2 L_{g_2}VR_2^{-1}(L_{g_2}V)^T + 2\beta L_{g_2}Vu + u^T R_2 u\Bigg)dt \\
&+ \beta\int_0^T \left(2L_{g_1}Vd - \lambda\ell_\rho(2\|L_{g_1}V\|) - \lambda\rho\left(\frac{\|d\|}{\lambda}\right)\right)dt\Bigg]\Bigg\} \\
= \sup_{d\in\mathcal{D}}\Bigg\{\lim_{T\to\infty}\Bigg[&\beta\left\|\tilde{\theta}(T)\right\|^2_{\Gamma^{-1}} + 2\beta V\left(x(T),\hat{\theta}(T)\right) \\
&- 2\beta\int_0^T \left(\frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial\hat{\theta}}\dot{\hat{\theta}} - L_F V\tilde{\theta}\right)dt \\
&+ \int_0^T (u-\alpha^*)^T R_2(u-\alpha^*)dt \\
&+ \beta\int_0^T \left(2L_{g_1}Vd - \lambda\ell_\rho(2\|L_{g_1}V\|) - \lambda\rho\left(\frac{\|d\|}{\lambda}\right)\right)dt\Bigg]\Bigg\} \\
= 2\beta V&\left(x(0),\hat{\theta}(0)\right) + \int_0^\infty (u-\alpha^*)^T R_2(u-\alpha^*)dt \\
&+ \beta\left\|\tilde{\theta}(0)\right\|^2_{\Gamma^{-1}} + \beta\sup_{d\in\mathcal{D}}\Bigg\{\int_0^\infty \Bigg(2L_{g_1}Vd + \lambda\rho\left(\frac{\|d^*\|}{\lambda}\right) \\
&- \lambda\rho'\left(\frac{\|d^*\|}{\lambda}\right)\frac{\|d^*\|}{\lambda} - \lambda\rho\left(\frac{\|d\|}{\lambda}\right)\Bigg)dt\Bigg\}
\end{aligned}$$

where $d^* = \lambda(\rho')^{-1}(2\|L_{g_1}V\|)((L_{g_1}V)^T/\|L_{g_1}V\|)$ is the "worst-case" disturbance and we have made use of the property $\ell_\rho(r) = r(\rho')^{-1}(r) - \rho((\rho')^{-1}(r))$. (See [21, Lemma A1].)

It was shown in the proof of [21, Th. 3.1] that

$$\sup_{d\in\mathcal{D}}\left\{\int_0^\infty \left[2L_{g_1}Vd + \lambda\rho\left(\frac{\|d^*\|}{\lambda}\right) - \lambda\rho'\left(\frac{\|d^*\|}{\lambda}\right)\frac{\|d^*\|}{\lambda} - \lambda\rho\left(\frac{\|d\|}{\lambda}\right)\right]dt\right\} \leq 0$$

and the equal sign "=" is satisfied if and only if $d = d^*$. Hence, the minimum of the cost functional J_a in (12) is reached with $u = \alpha^*$, and the dynamic feedback control law $u = \alpha^*(x,\hat{\theta})$ in (11) with the tuning function $\tau(x,\hat{\theta}) = (L_F V)^T$ minimizes the cost functional (12). ∎

The following nonlinear H_∞ problem, which is similar to [21, Def. 5.1] but extended to adaptive control, is a special case of the inverse optimal control problem in Theorem 3 if we choose $\rho(r) = (\lambda/\beta)\gamma^2 r^2$.

Definition 4: The H_∞ *inverse optimal adaptive control problem* for the system (8) is solvable if there exist a positive constant β, a smooth nonnegative function $E(x) \geq 0$, a positive–definite symmetric matrix $R(x, \hat{\theta})$, a real-valued function $l(x, \hat{\theta})$ positive definite in x for each $\hat{\theta}$, and a dynamic feedback law (2) that solves the adaptive control problem and also minimizes the cost functional

$$J_a(u) = \sup_{d\in\mathcal{D}} \left\{ \lim_{T\to\infty} \left[\beta \left\| \tilde{\theta}(T) \right\|^2_{\Gamma^{-1}} + E(x(T)) + \int_0^T \left(l(x,\hat{\theta}) + u^T R(x,\hat{\theta})u - \gamma^2 \|d\|^2 \right) dt \right] \right\} \quad (14)$$

for each $\theta \in \mathcal{R}^p$, where $\mathcal{D}$ is the set of locally bounded functions of x.

Remark 3: If we let $\lambda = 2$, $\beta = 2$ and $\rho(r) = \gamma^2 r^2$, the Lyapunov function $V(x, \theta)$ in Theorem 3 solves the following HJI equation:

$$\frac{\partial V}{\partial x}\left[f + F\theta + F\Gamma \left(\frac{\partial V}{\partial \theta}\right)^T \right] + \frac{\partial V}{\partial x}\left[-\frac{1}{\gamma^2} g_1 g_1^T + g_2 R_2^{-1} g_2^T \right] \left(\frac{\partial V}{\partial x}\right)^T + \frac{1}{4} l(x,\theta) = 0.$$

Replacing θ by the estimate $\hat{\theta}$ and applying the parameter update law $\dot{\hat{\theta}} = \Gamma[L_F V(x, \hat{\theta})]^T$, we see that the "certainty equivalence" controller $u = -2R_2(x, \hat{\theta})^{-1}[L_{g_2} V(x, \hat{\theta})]^T$ achieves γ-level of H_∞ disturbance attenuation [14], [15] given by

$$\int_0^T \left[l(x,\hat{\theta}) + u^T R(x,\hat{\theta})u \right] dt \leq \gamma^2 \int_0^T \|d\|^2 dt + 4V\left(x(0), \hat{\theta}(0)\right) + 2\left\| \theta - \hat{\theta}(0) \right\|^2_{\Gamma^{-1}}$$

for all $T \geq 0$ and for each $\theta \in \mathcal{R}^p$. Precisely, since $l(x, \hat{\theta})$ is positive definite in x for each $\hat{\theta}$, we may define an output function $y = h(x, \hat{\theta})$ such that $h^T h = l(x, \hat{\theta})$. Hence, the closed-loop system has an $\mathcal{L}_2$-gain γ from the disturbance d to the block vector $\binom{h}{R^{1/2}u}$. However, it should be noted that the above was derived with a fixed γ. In other words, h and R vary with γ in general. Therefore, a different γ corresponds to a different H_∞ problem and a smaller γ does not imply a better disturbance attenuation. Fortunately, for our attitude control problem in Section IV, we are able to prove a bound of the $\mathcal{L}_2$-gain from d to x that is indeed in the order of γ, which can then be made arbitrarily small at the cost of a larger u. See Remark 9 for details. □

III. Attitude Tracking Control Problem

The spacecraft is modeled as a rigid body with actuators that provide torques about three mutually perpendicular axes that define a body-fixed frame $\mathcal{B}_s$. The attitude kinematics and dynamics of a rigid spacecraft can be modeled as (see [27, Ch. 4])

$$\dot{q}_v = \frac{1}{2}\left(q_4 I_3 + q_v^\times\right)\Omega \quad (15a)$$

$$\dot{q}_4 = -\frac{1}{2} q_v^T \Omega \quad (15b)$$

$$J\dot{\Omega} = -\Omega^\times J\Omega + u + d \quad (15c)$$

where $(q_v, q_4) \in \mathcal{R}^3 \times \mathcal{R}$ denotes the unit quaternion representing the attitude orientation of the spacecraft in the body frame $\mathcal{B}_s$ with respect to an inertial frame $\mathcal{I}$ and satisfies the constraint $q_v^T q_v + q_4^2 = 1$; $\Omega \in \mathcal{R}^3$ is the angular velocity of the spacecraft with respect to the inertial frame $\mathcal{I}$ and expressed in the frame $\mathcal{B}_s$:

$$J = \begin{bmatrix} J_{11} & J_{12} & J_{13} \\ J_{12} & J_{22} & J_{23} \\ J_{13} & J_{23} & J_{33} \end{bmatrix} \quad (16)$$

is the constant, positive–definite inertia matrix of the spacecraft and expressed in $\mathcal{B}_s$; $u \in \mathcal{R}^3$ and $d \in \mathcal{R}^3$ denote the control torques and the external disturbances respectively; I_3 is the 3 × 3 identity matrix; the operator $a^\times$ denotes a skew-symmetric matrix acting on the vector $a = [a_1,\ a_2,\ a_3]^T$ and has the form

$$a^\times = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$$

which satisfies the following important properties:

$$a^\times b = -b^\times a, \quad a^\times a = 0, \quad a^T b^\times a = 0$$
$$a^\times b^\times = ba^T - a^T b I_3, \quad (a^\times b)^\times = ba^T - ab^T. \quad (17)$$

In the case of tracking a desired attitude motion, the attitude tracking problem is formulated similarly as in the related work [2], [3], [11]. The target attitude of the spacecraft in the body-fixed frame $\mathcal{B}_c$ with respect to the frame $\mathcal{I}$ is described by the unit quaternion $(q_{cv}, q_{c4}) \in \mathcal{R}^3 \times \mathcal{R}$ that satisfies $q_{cv}^T q_{cv} + q_{c4}^2 = 1$. Let $\nu \in \mathcal{R}^3$ be the desired angular velocity of $\mathcal{B}_c$ with respect to $\mathcal{I}$ and be expressed in the frame $\mathcal{B}_c$. The following assumptions are made about $\nu(t)$ and $\dot{\nu}(t)$.

Assumption 1: The desired angular velocity $\nu(t)$ and its derivative $\dot{\nu}(t)$ are bounded for all $t \geq 0$, i.e., there exist some finite constants $c_1 > 0$ and $c_2 > 0$ such that $\|\nu(t)\| \leq c_1$ and $\|\dot{\nu}(t)\| \leq c_2$ for all $t \geq 0$.

Let $(\epsilon, \eta) \in \mathcal{R}^3 \times \mathcal{R}$ be the unit quaternion representing the orientation error of $\mathcal{B}_s$ relative to $\mathcal{B}_c$. The error quaternion (ϵ, η) satisfies the constraint $\epsilon^T \epsilon + \eta^2 = 1$ and is related to (q_v, q_4) and (q_{cv}, q_{c4}) by quaternion multiplication [27, App. A]. The corresponding direction cosine matrix $C = C(\epsilon, \eta) \in SO(3)$ relating $\mathcal{B}_s$ to $\mathcal{B}_c$ is given by

$$C = (\eta^2 - \epsilon^T \epsilon) I_3 + 2\epsilon\epsilon^T - 2\eta\epsilon^\times \quad (18)$$

where $SO(3)$ is the Lie group of orthogonal matrices with determinant 1. It follows from [27, Ch. 4] that $C^T C = 1$, $\|C\| = 1$, $\det(C) = 1$ and $\dot{C} = -\omega^\times C$. Note that both (ϵ, η) and

$(-\epsilon, -\eta)$ stand for exactly the same physical attitude orientation, resulting in the same $C(\epsilon, \eta) \in SO(3)$. The angular velocity error $\omega \in \mathcal{R}^3$ of the frame $\mathcal{B}_s$ with respect to $\mathcal{B}_c$ is then represented by $\omega = \Omega - C\nu$.

Definition 5: Under Assumption 1, the *attitude tracking control problem* is to find a continuous dynamic feedback control $u = \alpha(\epsilon, \eta, \omega, \nu, \dot{\nu})$ such that $C(\epsilon, \eta) \to I_3$ and $\omega \to 0$ as $t \to \infty$.

From (18) and the constraint $\epsilon^T\epsilon + \eta^2 = 1$, it follows that $C(\epsilon, \eta) \to I_3$ if and only if $\epsilon \to 0$. Therefore, the attitude tracking problem is solved if and only if $\epsilon \to 0$ and $\omega \to 0$ as $t \to \infty$. The attitude tracking control problem is thus transformed into the problem of stabilizing the error system ϵ and ω, and the equations that govern their motion are given by [2], [11]

$$\dot{\epsilon} = \frac{1}{2}[\eta I_3 + \epsilon^{\times}]\omega \tag{19a}$$

$$\dot{\eta} = -\frac{1}{2}\epsilon^T\omega \tag{19b}$$

$$J\dot{\omega} = -(\omega + C\nu)^{\times}J(\omega + C\nu) + J[\omega^{\times}C\nu - C\dot{\nu}] + u + d. \tag{19c}$$

IV. Inverse Optimal Adaptive Attitude Tracking

In this section, we present adaptive feedback control laws to solve the inverse optimal adaptive control problem for the attitude tracking of spacecraft. The inverse optimality approach used herein requires the knowledge of a control Lyapunov function and a feedback control law of a particular form. We construct both of them via the method of *integrator backstepping* [7], [20].

Observe that the error system in (19) is a nonlinear cascade interconnection, that is, the kinematics subsystem (19a) and (19b) is stabilized only indirectly through the angular velocity vector ω. Stabilizing control laws for cascade systems can be efficiently designed using the method of *integrator backstepping*. By this method, ω in (19a) and (19b) is considered as a *virtual control* input and a control law ω_d is designed to stabilize the kinematics subsystem. Subsequently, the actual control u is designed to stabilize the dynamics subsystem (19c) without destabilizing the kinematics subsystem (19a) and (19b).

Step 1) *Control of the kinematics subsystem*: Consider ω in the kinematics subsystem (19a) and (19b) as a virtual control input and design the control law

$$\omega_d = -K\epsilon \tag{20}$$

where $K = K^T \in \mathcal{R}^{3\times 3}$ is positive definite. On the convergence of ϵ and η, we have the following lemma.

Lemma 1: With the control law (20), the vector ϵ in the kinematics subsystem converges to zero asymptotically for all initial conditions $\epsilon(0)$, and $\eta \to 1$ as $t \to \infty$ whenever the initial condition $\eta(0) \neq -1$.

Proof: We first proceed to show that $\eta \to 1$ as $t \to \infty$ whenever $\eta(0) \neq -1$. Let k_1 and k_2 be the minimum and maximum eigenvalues of K, i.e., $k_1 = \lambda_{\min}(K)$ and $k_2 = \lambda_{\max}(K)$. Applying the virtual control law (20) to (19b) and using the condition $\epsilon^T\epsilon + \eta^2 = 1$, we have

$$\frac{1}{2}k_1(1-\eta^2) \le \dot{\eta} = \frac{1}{2}\epsilon^T K\epsilon \le \frac{1}{2}k_2(1-\eta^2).$$

It follows from the comparison principle [28, Lemma 2.5, p. 85] that $\eta(t)$ satisfies the inequalities

$$1 - \frac{2\left[1-\eta(0)\right]e^{-k_1 t}}{1+\eta(0)+\left[1-\eta(0)\right]e^{-k_1 t}} \le \eta(t) \le 1 - \frac{2\left[1-\eta(0)\right]e^{-k_2 t}}{1+\eta(0)+\left[1-\eta(0)\right]e^{-k_2 t}}$$

for all $t \ge 0$. (The first one is obtained from $\dot{\tilde{\eta}} \le -(1/2)k_1(1-\tilde{\eta}^2)$ by letting $\tilde{\eta} = -\eta$.) Hence, $\eta(t) \equiv -1$ for all $t \ge 0$ if $\eta(0) = -1$. Otherwise, $\eta(t) > -1$ for all $t \ge 0$ and $\eta(t) > 0$ for all $t \ge \max\{0, (1/k_1)\ln((1-\eta(0))/(1+\eta(0)))\}$ with $\lim_{t\to\infty}\eta(t) = 1$. In particular, when $k_1 = k_2$ and $\eta(0) \neq -1$, $\eta(t)$ is strictly increasing for all $t \ge 0$, i.e., $\eta(t_1) < \eta(t_2)$ for all $0 \le t_1 < t_2$.

Applying the fact that $\eta(t) \to 1$ whenever $\eta(0) \neq -1$, we have that $\epsilon(t) \to 0$ as $t \to \infty$. Furthermore, we can show the global asymptotic stability of $(\epsilon, \eta) = (0, 1)$ under the control law (20) by selecting the following Lyapunov function for the kinematics subsystem:

$$V_a = c\epsilon^T\epsilon + c(1-\eta)^2 \tag{21}$$

where the constant $c > 0$. The derivative of V_a is given by

$$\dot{V}_a = c\epsilon^T\omega_d = -c\epsilon^T K\epsilon \le -ck_1\|\epsilon\|^2 \le 0. \tag{22}$$

Hence, the global asymptotic stability of $(\epsilon, \eta) = (0, 1)$ follows for all initial conditions $(\epsilon(0), \eta(0))$ except $(\epsilon(0), \eta(0)) = (0, -1)$ ∎

Since both $(\epsilon, \eta) = (0, +1)$ and $(\epsilon, \eta) = (0, -1)$ represent exactly the same physical attitude orientation, we can practically conclude that the kinematics subsystem of attitude motion under the control law (20) is globally asymptotically stable.

Step 2) *Control of the full rigid-body models*: We consider that the inertia matrix $J \in \mathcal{R}^{3\times 3}$ is constant, but is unknown or poorly known. In this case, we can replace it by an estimate and update the estimate by an adaptive scheme. To isolate the uncertain parameter, a linear operator $L : \mathcal{R}^3 \to \mathcal{R}^{3\times 6}$ acting on the vector $a = [a_1,\ a_2,\ a_3]^T$ is defined by

$$L(a) = \begin{bmatrix} a_1 & 0 & 0 & 0 & a_3 & a_2 \\ 0 & a_2 & 0 & a_3 & 0 & a_1 \\ 0 & 0 & a_3 & a_2 & a_1 & 0 \end{bmatrix} \tag{23}$$

and the parameter vector $\theta \in \mathcal{R}^6$ is defined by

$$\theta = [J_{11} \quad J_{22} \quad J_{33} \quad J_{23} \quad J_{13} \quad J_{12}]^T \tag{24}$$

then it follows that

$$Ja = L(a)\theta. \tag{25}$$

Let $\hat{\theta}$ denote the parameter estimate of θ and $\tilde{\theta}$ be the estimation error defined by $\tilde{\theta} = \theta - \hat{\theta}$. We also make the following notations:

$$z = \omega - \omega_d = \omega + K\epsilon \tag{26}$$
$$u_c = H(\nu, \dot{\nu})\hat{\theta} \tag{27}$$
$$u_e = u + u_c = u + H(\nu, \dot{\nu})\hat{\theta}. \tag{28}$$

Then, it follows that

$$J\dot{z} = [F(\epsilon, \eta, \omega, \nu, \dot{\nu}) + G(\epsilon, \eta, \omega)]\,\theta + H(\nu, \dot{\nu})\tilde{\theta} + u_e + d \tag{29}$$

where $H(\nu, \dot{\nu}) \in \mathcal{R}^{3\times 6}$, $F(\epsilon, \eta, \omega, \nu, \dot{\nu}) \in \mathcal{R}^{3\times 6}$ and $G(\epsilon, \eta, \omega) \in \mathcal{R}^{3\times 6}$ are given by

$$H = -\nu^{\times}L(\nu) - L(\dot{\nu}) \tag{30a}$$
$$\begin{aligned} F =\,& L(\omega^{\times}C\nu) - (\omega + C\nu)^{\times}L(\omega) - (\omega + C_1\nu)^{\times}L(C\nu) \\ & - \nu^{\times}L(C_1\nu) - L(C_1\dot{\nu}) \end{aligned} \tag{30b}$$
$$G = \frac{1}{2}L\left(K(\eta I_3 + \epsilon^{\times})\omega\right). \tag{30c}$$

$C(\epsilon, \eta) \in \mathcal{R}^{3\times 3}$ is defined by (18), and $C_1(\epsilon, \eta) \in \mathcal{R}^{3\times 3}$ is given by

$$C_1(\epsilon, \eta) = C(\epsilon, \eta) - I_3 = -2\epsilon^T\epsilon I_3 + 2\epsilon\epsilon^T - 2\eta\epsilon^{\times}. \tag{31}$$

Hence, the stabilizing control problem of ω in (19c) with the control input u is transformed into the stabilizing control problem of z in (29) with an auxiliary control input u_e. When $z \to 0$, we have that $\omega \to \omega_d$ and then the kinematics subsystem (19a) and (19b) is asymptotically stable as analyzed in Lemma 1, that is, $\epsilon \to 0$ and subsequently $\omega \to 0$ as $t \to \infty$ according to (26).

Once u_e is designed, we can also obtain the actual control input $u = u_e - u_c$ by (27) and (28). u_c is independent of the tracking errors ϵ and ω.

In summary, we need to design a dynamic feedback control law

$$u_e = \alpha(\epsilon, \eta, \omega, \hat{\theta}, \nu, \dot{\nu})$$

and an adaptive parameter update law

$$\dot{\hat{\theta}} = \Gamma\tau(\epsilon, \eta, \omega, \hat{\theta}, \nu, \dot{\nu})$$

to stabilize the full-model system (19a), (19b), and (29) with an uncertain parameter θ.

A. Zero-Disturbance Case

First, we consider the zero-disturbance case, $d = 0$, which is a special case, but has some interesting properties such as optimality, asymptotic property and global convergence. Next theorem presents an adaptive feedback controller that achieves the global asymptotic attitude tracking in the sense of Definition 5.

Theorem 4: Suppose that Assumption 1 is satisfied and the external disturbance $d(t)$ in (29) is $d = 0$. Let $K \in \mathcal{R}^{3\times 3}$, $K_1 \in \mathcal{R}^{3\times 3}$ and $\Gamma \in \mathcal{R}^{6\times 6}$ be constant, positive definite and symmetric and let $c > 0$. Then, the dynamic feedback control law

$$\begin{aligned} u_e =\,& \alpha(\epsilon, \eta, \omega, \hat{\theta}, \nu, \dot{\nu}) \\ =\,& -R(\epsilon, \eta, \omega, \hat{\theta}, \nu, \dot{\nu})^{-1}(\omega + K\epsilon) \end{aligned} \tag{32a}$$
$$\begin{aligned} \dot{\hat{\theta}} =\,& \Gamma\tau(\epsilon, \eta, \omega, \nu, \dot{\nu}) \\ =\,& \Gamma\left[F(\epsilon, \eta, \omega, \nu, \dot{\nu}) + G(\epsilon, \eta, \omega) + H(\nu, \dot{\nu})\right]^T \\ & \times(\omega + K\epsilon) \end{aligned} \tag{32b}$$

with $R(\epsilon, \eta, \omega, \hat{\theta}, \nu, \dot{\nu})$ satisfying

$$R^{-1} = K_1 + \frac{\Psi_1^T\Psi_1}{2} + \frac{\Psi_2 K_1^{-1}\Psi_2}{2} \tag{33}$$

solves the adaptive attitude tracking control problem asymptotically, that is, $\epsilon \to 0$ and $\omega \to 0$ as $t \to \infty$. Furthermore, $\hat{\theta}$ is bounded for all $t \geq 0$ and $\dot{\hat{\theta}} \to 0$ as $t \to \infty$.

Here, the smooth matrix functions $\Psi_1(\epsilon, \eta, \omega, \hat{\theta}, \nu, \dot{\nu}) \in \mathcal{R}^{3\times 3}$, $\Psi_2(\epsilon, \eta, \omega, \hat{\theta}) \in \mathcal{R}^{3\times 3}$, $C_2(\epsilon, \eta, \nu) \in \mathcal{R}^{3\times 3}$ and $C_3(\epsilon, \eta, \dot{\nu}) \in \mathcal{R}^{3\times 3}$ are defined by

$$\begin{aligned} \Psi_1 = c^{-\frac{1}{2}}K^{\frac{1}{2}}\Big[&(\omega + C\nu)^{\times}\hat{J} + \hat{J}(C\nu)^{\times} - (\hat{J}C\nu)^{\times} \\ & - \frac{1}{2}\hat{J}K(\eta I_3 + \epsilon^{\times}) + cK^{-1} + (\hat{J}C\nu)^{\times}C_2K^{-1} \\ & - \nu^{\times}\hat{J}C_2K^{-1} - \hat{J}C_3K^{-1}\Big]^T \end{aligned} \tag{34}$$
$$\begin{aligned} \Psi_2 =\,& \frac{1}{4}\hat{J}K(\eta I_3 + \epsilon^{\times}) - \frac{1}{2}\omega^{\times}\hat{J} \\ & + \left[\frac{1}{4}\hat{J}K(\eta I_3 + \epsilon^{\times}) - \frac{1}{2}\omega^{\times}\hat{J}\right]^T \end{aligned} \tag{35}$$
$$C_2 = 2\epsilon^T\nu I_3 - 2\nu\epsilon^T + 2\eta\nu^{\times} \tag{36}$$
$$C_3 = 2\epsilon^T\dot{\nu} I_3 - 2\dot{\nu}\epsilon^T + 2\eta\dot{\nu}^{\times} \tag{37}$$

the matrix $\hat{J}$ is of the form (16), obtained from the estimate $\hat{\theta}$; the matrices $H(\nu, \dot{\nu})$, $F(\epsilon, \eta, \omega, \nu, \dot{\nu})$ and $G(\epsilon, \eta, \omega)$ are given as in (30).

Proof: We define an adaptive control Lyapunov function $V_2(\epsilon, \eta, \omega, \hat{\theta})$ for the nonlinear system (19a), (19b) and (29) with an unknown parameter θ as follows:

$$V_2 = c\epsilon^T\epsilon + c(1 - \eta)^2 + \frac{1}{2}z^TJz + \frac{1}{2}\tilde{\theta}^T\Gamma^{-1}\tilde{\theta}. \tag{38}$$

Along the solutions of (19a), (19b), (29), and (32b) we have

$$\begin{aligned} \dot{V}_2 =\,& 2c\epsilon^T\dot{\epsilon} - 2c(1 - \eta)\dot{\eta} + z^TJ\dot{z} - \tilde{\theta}^T\Gamma^{-1}\dot{\hat{\theta}} \\ =\,& c\epsilon^T(z - K\epsilon) + z^T\left[u_e + (F + G)\hat{\theta}\right] \\ & + \tilde{\theta}^T\left[(F + G + H)^Tz - \tau\right] \\ =\,& -c\epsilon^TK\epsilon + z^T\left[c\epsilon + u_e + (F + G)\hat{\theta}\right]. \end{aligned} \tag{39}$$

To render $\dot{V}_2$ negative, one natural choice like the adaptive feedback control laws in [10], [11] is

$$u_e = -K_2z - (F + G)\hat{\theta} - c\epsilon$$

which cancels all the nonlinear terms in (39), where $K_2 \in \mathcal{R}^{3\times 3}$ is a positive–definite symmetric matrix. However, this feedback control law based on nonlinearity cancellation is not guaranteed to be inverse optimal in general. To design an inverse optimal adaptive controller to solve the attitude tracking problem, we employ the "nonlinear damping" technique as follows.

From (24) and (25), it follows that $L(a)\hat{\theta} = \hat{J}a$ for all $a \in \mathcal{R}^3$. Suppose that the matrices $C_2(\epsilon, \eta, \nu)$ and $C_3(\epsilon, \eta, \dot{\nu})$ are defined as in (36) and (37), respectively, such that

$$C_1\nu = C_2\epsilon \quad C_1\dot{\nu} = C_3\epsilon.$$

Applying the tuning function $\tau(\epsilon, \eta, \omega, \nu, \dot{\nu})$ in (32b), the matrix $F(\epsilon, \eta, \omega, \nu, \dot{\nu})$ in (30b), the matrix $G(\epsilon, \eta, \omega)$ in (30c) and some properties of $a^\times$ in (17), we can rewrite (39) as

$$\begin{aligned}\dot{V}_2 =& -c\epsilon^T K\epsilon \\ &+ z^T\Big[c\epsilon + u_e - (\omega + C\nu)^\times \hat{J}(z - K\epsilon) \\ &\quad + (\hat{J}C\nu)^\times(z - K\epsilon) - \hat{J}C_1\dot{\nu} - \hat{J}(C\nu)^\times(z - K\epsilon) \\ &\quad + \frac{1}{2}\hat{J}K(\eta I_3 + \epsilon^\times)(z - K\epsilon) \\ &\quad - \nu^\times \hat{J}C_1\nu + (\hat{J}C\nu)^\times C_1\nu\Big] \\ =& -c\epsilon^T K\epsilon + z^T u_e \\ &+ z^T\Big(cK^{-1} + (\omega + C\nu)^\times \hat{J} - (\hat{J}C\nu)^\times \\ &\quad - \frac{1}{2}\hat{J}K(\eta I_3 + \epsilon^\times) - \nu^\times \hat{J}C_2K^{-1} + \hat{J}(C\nu)^\times \\ &\quad + (\hat{J}C\nu)^\times C_2K^{-1} - \hat{J}C_3K^{-1}\Big)K\epsilon \\ &+ z^T\left(\frac{1}{2}\hat{J}K(\eta I_3 + \epsilon^\times) - \omega^\times \hat{J}\right)z + z^T H_1 z\end{aligned}$$

where $H_1 = (\hat{J}\nu)^\times - (C\nu)^\times \hat{J} - \hat{J}(C\nu)^\times$. It can be easily shown that $H_1 = -H_1^T$ due to the fact that $\hat{J}$ is symmetric and $(\hat{J}\nu)^\times$ and $(C\nu)^\times$ are skew-symmetric. Therefore, $z^T H_1 z = 0$ for all $z \in \mathcal{R}^3$. Introducing two smooth matrices $\Psi_1(\epsilon, \eta, \omega, \hat{\theta}, \nu, \dot{\nu}) \in \mathcal{R}^{3\times 3}$ as in (34) and $\Psi_2(\epsilon, \eta, \omega, \hat{\theta}) \in \mathcal{R}^{3\times 3}$ as in (35), we have

$$\dot{V}_2 = -c\epsilon^T K\epsilon + z^T u_e + \sqrt{c}z^T\Psi_1^T K^{\frac{1}{2}}\epsilon + z^T\Psi_2 z. \tag{40}$$

The choice

$$u_e = \alpha(\epsilon, \eta, \omega, \hat{\theta}, \nu, \dot{\nu}) = -R(\epsilon, \eta, \omega, \hat{\theta}, \nu, \dot{\nu})^{-1}z$$

with $R(\epsilon, \eta, \omega, \hat{\theta}, \nu, \dot{\nu})$ satisfying (33) renders

$$\begin{aligned}\dot{V}_2 =& -\frac{c}{2}\epsilon^T K\epsilon - \frac{1}{2}z^T K_1 z - \frac{1}{2}\left\|\sqrt{c}K^{\frac{1}{2}}\epsilon - \Psi_1 z\right\|^2 \\ &- \frac{1}{2}z^T(K_1 - \Psi_2)^T K_1^{-1}(K_1 - \Psi_2)z \\ \leq& -\frac{c}{2}\epsilon^T K\epsilon - \frac{1}{2}z^T K_1 z\end{aligned} \tag{41}$$

which shows that $\dot{V}_2$ is negative semidefinite, where $K_1 \in \mathcal{R}^{3\times 3}$ is positive definite and symmetric such that the smooth matrix $R(\epsilon, \eta, \omega, \hat{\theta}, \nu, \dot{\nu}) \in \mathcal{R}^{3\times 3}$ is positive definite and symmetric. Since $V_2(\epsilon, \eta, \omega, \hat{\theta})$ is nonincreasing and bounded below, i.e.,

$$V_2\left(\epsilon(t), \eta(t), \omega(t), \hat{\theta}(t)\right) \leq V_2\left(\epsilon(0), \eta(0), \omega(0), \hat{\theta}(0)\right)$$

for all $t \geq 0$, and since θ is a constant, it follows that the signals $\epsilon(t)$, $\eta(t)$, $\omega(t)$, $z(t)$, $\tilde{\theta}(t)$ and $\hat{\theta}(t)$ are all bounded for all $t \geq 0$. As $\nu(t)$ and $\dot{\nu}(t)$ are bounded by Assumption 1, it follows that $H(\nu, \dot{\nu})$, $G(\epsilon, \eta, \omega)$ and $F(\epsilon, \eta, \omega, \nu, \dot{\nu})$ are bounded and consequently $\dot{\epsilon}(t)$ and $\dot{z}(t)$ are bounded for all $t \geq 0$, which implies that $\epsilon(t)$ and $z(t)$ are uniformly continuous functions.

Integrating both sides of (41) with respect to t and applying $V_2(\epsilon, \eta, \omega, \hat{\theta}) \geq 0$, we have

$$\int_0^\infty (c\epsilon^T K\epsilon + z^T K_1 z)dt \leq 2V_2\left(\epsilon(0), \eta(0), \omega(0), \hat{\theta}(0)\right).$$

Using the *Barbalat's Lemma* [28, Lemma 4.2, p. 192], we conclude that $\epsilon \to 0$ and $z \to 0$ as $t \to \infty$, and consequently $\omega \to 0$ as $t \to \infty$. Hence, the dynamic feedback control law (32) stabilizes the attitude error system (19a), (19b), and (29) with an uncertain parameter θ and zero external disturbance, and thus the adaptive attitude tracking control problem is solved and asymptotic tracking is achieved with the tracking errors ϵ, ω converging to zeros. As the matrices $H(\nu, \dot{\nu})$, $G(\epsilon, \eta, \omega)$ and $F(\epsilon, \eta, \omega, \nu, \dot{\nu})$ are bounded and $z \to 0$, it follows that $\dot{\hat{\theta}} \to 0$ as $t \to \infty$. ■

Remark 4: In the absence of external disturbances, $d = 0$, both $(\epsilon, \eta, \omega, \tilde{\theta}) = (0, \pm 1, 0, 0)$ are the equilibrium points of the system (19a), (19b), (29), and (32b) that describes the adaptive attitude tracking control problem. Both of them stand for exactly the same physical attitude orientation. However, it was shown in [10] that the point $(\epsilon, \eta, \omega, \tilde{\theta}) = (0, -1, 0, 0)$ is an unstable equilibrium point. On the other hand, it can be seen from the proof of Theorem 4 that the equilibrium point $(\epsilon, \eta, \omega, \tilde{\theta}) = (0, 1, 0, 0)$ is uniformly stable [1, Th. 4.1] under the dynamic feedback control law (32). □

Remark 5: Under the assumption that $d = 0$ and with the adaptive control law (32), the tracking errors ϵ and ω converge to zeros asymptotically, which ensures that the attitude tracking is achieved with a global convergence for any initial conditions. The parameter update law (32b) represents a scheme for adjusting the adaptive parameter $\hat{\theta}$. Although the derivative value of the adaptive parameter $\dot{\hat{\theta}} \to 0$ as $t \to \infty$, $\tilde{\theta} = \theta - \hat{\theta}$ does not necessarily converge to zero as $t \to \infty$. □

Replacing K by $k_1 I_3$ in (32b), R^{-1} by $k_2 I_3$ and K by $k_3 I_3$ in (32a), where the scalars $k_1, k_2, k_3 > 0$, and omitting some high-order terms in the states ϵ and ω, we obtain a simplified adaptive attitude tracking controller as those in [10] and [11]

$$u_e = -k_2(\omega + k_3\epsilon) \tag{42a}$$

$$\dot{\hat{\theta}} = \Gamma H(\nu, \dot{\nu})^T(\omega + k_1\epsilon). \tag{42b}$$

Using the Lyapunov function

$$V = k_2(k_1 + k_3)\left[(1 - \eta)^2 + \epsilon^T\epsilon\right] + k_1\epsilon^T J\omega + \frac{1}{2}\omega^T J\omega + \frac{1}{2}\tilde{\theta}^T\Gamma^{-1}\tilde{\theta}$$

and applying (17), we have that

$$\begin{aligned}\dot{V} =& k_2(k_1 + k_3)\epsilon^T\omega + (\omega + k_1\epsilon)^T(F\theta + H\tilde{\theta} + u_e) \\ &+ \frac{1}{2}k_1\omega^T J(\eta I_3 + \epsilon^\times)\omega - \tilde{\theta}^T H^T(\omega + k_1\epsilon) \\ \leq& -\left(k_2 - \frac{1}{2}k_1\lambda_j\right)\|\omega\|^2 \\ &- \left[k_1k_2k_3 - k_1\lambda_j(2\mu\lambda_{c2} + \lambda_{c3})\right]\|\epsilon\|^2 \\ &+ \lambda_j(3\mu k_1 + 2\mu\lambda_{c2} + \lambda_{c3})\|\omega\|\|\epsilon\|\end{aligned}$$

where $\lambda_{c2} = \sup\{\|C_2(\epsilon(t), \eta(t), \nu(t))\| : \forall t \geq 0\}$, $\lambda_{c3} = \sup\{\|C_3(\epsilon(t), \eta(t), \dot{\nu}(t))\| : \forall t \geq 0\}$, $\mu = \sup\{\|\nu(t)\| : \forall t \geq 0\}$, λ_j is the largest eigenvalue of the inertia matrix J, and C_2,

C_3 are defined by (36) and (37), respectively. If the controller gains k_1, k_2 and k_3 satisfy the following inequalities:

$$\begin{aligned}&2k_2 - k_1\lambda_j > 0\\&k_2k_3 - \lambda_j(2\mu\lambda_{c2} + \lambda_{c3}) > 0\\&4\left(k_2 - \frac{1}{2}k_1\lambda_j\right)[k_1k_2k_3 - k_1\lambda_j(2\mu\lambda_{c2} + \lambda_{c3})]\\&\quad - \lambda_j^2(3\mu k_1 + 2\mu\lambda_{c2} + \lambda_{c3})^2 > 0\end{aligned}$$

then $\dot{V} < 0$. Thus, asymptotic attitude tracking is achieved for this simplified controller (42). From the foregoing derivations, we see that the adaptive attitude tracking controller (32) relaxes these constraints on the gains of the simplified controller (42) and allows the gains K and K_1 to be other matrices, hence the designer has much more freedom in selecting the controller gains K and K_1. Furthermore, knowledge of the largest eigenvalue λ_j of J is not required in designing the adaptive control law (32).

Based on the state-feedback control law (32a) and the adaptive parameter update law (32b) and applying Theorems 1 and 2, we can easily construct a dynamic feedback control law that solves the inverse optimal adaptive attitude tracking problem with respect to a meaningful cost functional by the following theorem.

Theorem 5: Suppose that the external disturbance $d = 0$ and Assumption 1 is satisfied. Then, the dynamic feedback control law

$$\begin{aligned}u_e &= \alpha^*(\epsilon,\eta,\omega,\hat{\theta},\nu,\dot{\nu}) = \beta\alpha(\epsilon,\eta,\omega,\hat{\theta},\nu,\dot{\nu})\\&= -\beta R^{-1}(\omega + K\epsilon) \qquad (43a)\\\dot{\hat{\theta}} &= \Gamma\tau(\epsilon,\eta,\omega,\nu,\dot{\nu})\\&= \Gamma\left[F(\epsilon,\eta,\omega,\nu,\dot{\nu}) + G(\epsilon,\eta,\omega) + H(\nu,\dot{\nu})\right]^T\\&\quad\times(\omega + K\epsilon) \qquad (43b)\end{aligned}$$

with any $\beta \geq 2$, solves the inverse optimal assignment problem for the attitude tracking control system (19a), (19b), and (29) by minimizing the cost functional

$$J_a = \lim_{T\to\infty}\left[\beta\left\|\tilde{\theta}(T)\right\|^2_{\Gamma^{-1}} + 4\beta c\left[1 - \eta(T)\right] + \int_0^T\left(l(\epsilon,\eta,\omega,\hat{\theta},\nu,\dot{\nu}) + u_e^TRu_e\right)dt\right] \qquad (44)$$

for each $\theta \in R^6$, where

$$l(\epsilon,\eta,\omega,\hat{\theta},\nu,\dot{\nu}) = -2\beta\left[c\epsilon^T\omega + (\omega + K\epsilon)^T(F+G)\hat{\theta}\right] + \beta^2(\omega + K\epsilon)^TR^{-1}(\omega + K\epsilon) \qquad (45)$$

and $R(\epsilon,\eta,\omega,\hat{\theta},\nu,\dot{\nu})$ is defined by (33); $\Gamma \in \mathcal{R}^{6\times6}$ is constant, positive definite, and symmetric; the matrices $H(\nu,\dot{\nu})$, $F(\epsilon,\eta,\omega,\nu,\dot{\nu})$ and $G(\epsilon,\eta,\omega)$ are given as in (30).

Proof: From the proof of Theorem 4, we have that

$$c\epsilon^T\omega + z^T\left[(F+G)\hat{\theta} - R^{-1}z\right] \leq -W(\epsilon,\eta,\omega)$$

with $W(\epsilon,\eta,\omega) = (c/2)\epsilon^TK\epsilon + (1/2)z^TK_1z$ positive definite in ϵ and z, which implies that

$$l(\epsilon,\eta,\omega,\hat{\theta},\nu,\dot{\nu}) \geq 2\beta W(\epsilon,\eta,\omega) + \beta(\beta - 2)z^TR^{-1}z$$

from which we observe that $l(\epsilon,\eta,\omega,\hat{\theta},\nu,\dot{\nu})$ is positive definite in ϵ and ω, i.e., it is positive whenever $(\epsilon,\omega) \neq (0,0)$, for each $\hat{\theta} \in \mathcal{R}^6$. Therefore, the cost functional J_a in (44) is a meaningful cost functional, penalizing both the tracking errors ϵ and ω as well as the control effort u_e.

Substituting $l(\epsilon,\eta,\omega,\hat{\theta},\nu,\dot{\nu})$ in (45), $z = \omega + K\epsilon$ and

$$v = u_e - \alpha^* = u_e - \beta\alpha = u_e + \beta R^{-1}z$$

into the cost functional J_a in (44) and applying the fact that $2(1-\eta) = (1-\eta)^2 + \epsilon^T\epsilon$, we get the following expression of J_a along the solutions of the attitude tracking error system (19a), (19b), (29) and the adaptive parameter update law (43b):

$$\begin{aligned}J_a &= \lim_{T\to\infty}\Bigg\{\beta\left\|\tilde{\theta}(T)\right\|^2_{\Gamma^{-1}} + 4\beta c\left[1-\eta(T)\right]\\&\quad+\int_0^T\Big[\beta^2z^TR^{-1}z - 2\beta z^T(F+G)\hat{\theta} - 2\beta c\epsilon^T\omega\\&\qquad+(v - \beta R^{-1}z)^TR(v - \beta R^{-1}z)\Big]dt\Bigg\}\\&= \lim_{T\to\infty}\Bigg\{\beta\left\|\tilde{\theta}(T)\right\|^2_{\Gamma^{-1}} + 4\beta c\left[1-\eta(T)\right]\\&\quad+2\beta\int_0^T\left[z^T(v - \beta R^{-1}z) + \beta z^TR^{-1}z - v^Tz\right]dt\\&\quad-2\beta\int_0^T\Big[c\epsilon^T\omega + z^T\left[(F+G)\theta + H\tilde{\theta} + u_e\right]\\&\qquad-z^T(F+G+H)\tilde{\theta}\Big]dt + \int_0^T v^TRvdt\Bigg\}\\&= \lim_{T\to\infty}\Bigg\{\beta\left\|\tilde{\theta}(T)\right\|^2_{\Gamma^{-1}} + 4\beta c\left[1-\eta(T)\right]\\&\quad-2\beta\int_0^T\frac{d}{dt}\left(c(1-\eta)^2 + c\epsilon^T\epsilon + \frac{1}{2}z^TJz\right)dt\\&\quad-\beta\int_0^T\frac{d}{dt}[\tilde{\theta}^T\Gamma^{-1}\tilde{\theta}]dt + \int_0^T v^TRvdt\Bigg\}\\&= \beta\left\|\tilde{\theta}(0)\right\|^2_{\Gamma^{-1}} + 4\beta c\left[1-\eta(0)\right] + \beta z(0)^TJz(0)\\&\quad-\beta\lim_{T\to\infty}z(T)^TJz(T) + \int_0^\infty v^TRvdt.\end{aligned}$$

Substituting $u_e = \beta\alpha(\epsilon,\eta,\omega,\hat{\theta},\nu,\dot{\nu})$ into $\dot{V}_2$ in (40), we can see that the dynamic feedback control law (43) also solves the adaptive attitude tracking problem of the system (19a), (19b) and (29), i.e., $\lim_{t\to\infty}\epsilon(t)=0$, $\lim_{t\to\infty}z(t)=0$ and $\lim_{t\to\infty}\omega(t)=0$. It follows that $\lim_{t\to\infty}z(t)^TJz(t)=0$. Hence, the minimum of the cost functional J_a is reached only if $v=0$. In other words, the control law $u_e=\alpha^*=\beta\alpha$ in (43a) is inverse optimal and minimizes the cost functional (44). The value function of the cost functional (44) is given by $J_a^*(\epsilon,\eta,\omega,\hat{\theta})=2\beta V_2(\epsilon,\eta,\omega,\hat{\theta})$. ■

Remark 6: When the external disturbance d is assumed zero, under the adaptive feedback control law (43), the attitude tracking errors ϵ and ω converge to zeros asymptotically for any initial conditions, which ensures that asymptotic attitude tracking is achieved with a global convergence. □

Remark 7: The parameter $\beta\geq 2$ in Theorem 5 represents a degree of freedom for the design. It also follows from the proof of Theorem 5 that for the inverse optimal adaptive control law (43)

$$\int_0^\infty\left[c\epsilon^TK\epsilon+z^TK_1z+\frac{2(\beta-1)}{\beta^2}u_e^TRu_e\right]dt \leq\tilde{\theta}(0)^T\Gamma^{-1}\tilde{\theta}(0)+4c\left[1-\eta(0)\right]+z(0)^TJz(0).$$

Maximizing the left-hand side over β gives an $\mathcal{L}_2$ bound on the attitude tracking errors ϵ,ω and the control efforts u_e

$$\int_0^\infty\left[c\epsilon^TK\epsilon+z^TK_1z+\frac{1}{2}u_e^TRu_e\right]dt \leq\tilde{\theta}(0)^T\Gamma^{-1}\tilde{\theta}(0)+4c\left[1-\eta(0)\right]+z(0)^TJz(0)$$

which implies that $\epsilon,\omega\in\mathcal{L}_2[0,\infty)$. □

B. With External Disturbances

In Theorems 4 and 5, we have presented dynamic feedback control laws that solve the adaptive control problem and the inverse optimal adaptive control problem, respectively, for the attitude tracking problem of the system (19) without external disturbances, $d=0$. When external disturbances $d(t)$ exist, employing the inverse optimal approach [21] and Theorems 1–3, we can present a dynamic feedback control law that solves a robust inverse optimal control problem by the following theorem.

Theorem 6: Suppose that Assumption 1 is satisfied. Let the constant matrices $K\in\mathcal{R}^{3\times 3}$, $K_1\in\mathcal{R}^{3\times 3}$ and $\Gamma\in\mathcal{R}^{6\times 6}$ be symmetric and positive definite. Suppose the matrices $H(\nu,\dot{\nu})$, $F(\epsilon,\eta,\omega,\nu,\dot{\nu})$, $G(\epsilon,\eta,\omega)$ and $\Psi_1(\epsilon,\eta,\omega,\hat{\theta},\nu,\dot{\nu})$ are as defined in (30) and (34), respectively. The smooth matrix $\Psi_2(\epsilon,\eta,\omega,\hat{\theta})$ of (35) is redefined as

$$\Psi_2(\epsilon,\eta,\omega,\hat{\theta})=\frac{1}{4}\hat{J}(\eta I_3+\epsilon^\times)-\frac{1}{2}\omega^\times\hat{J} +\left[\frac{1}{4}\hat{J}(\eta I_3+\epsilon^\times)-\frac{1}{2}\omega^\times\hat{J}\right]^T+\frac{1}{\gamma^2}I_3 \quad (46)$$

for some given $\gamma>0$. Then, the dynamic feedback control

$$u_e=\alpha(\epsilon,\eta,\omega,\hat{\theta},\nu,\dot{\nu})=-R(\epsilon,\eta,\omega,\hat{\theta},\nu,\dot{\nu})^{-1}(\omega+K\epsilon) \quad (47)$$

together with the adaptive parameter update law (32b), adaptively stabilizes an auxiliary system that consists of (19a), (19b) and the following equation:

$$J\dot{z}=\left[F(\epsilon,\eta,\omega,\nu,\dot{\nu})+G(\epsilon,\eta,\omega)\right]\theta+H(\nu,\dot{\nu})\tilde{\theta}+\frac{z}{\gamma^2}+u_e \quad (48)$$

with respect to the **aclf** $V(\epsilon,\eta,\omega,\theta)=c\epsilon^T\epsilon+c(1-\eta)^2+(1/2)z^TJz$, that is, $\epsilon\to 0$ and $\omega\to 0$ as $t\to\infty$ for all initial conditions. Furthermore, the dynamic feedback control law

$$\begin{aligned} u_e&=\alpha^*(\epsilon,\eta,\omega,\hat{\theta},\nu,\dot{\nu})=\beta\alpha(\epsilon,\eta,\omega,\hat{\theta},\nu,\dot{\nu})\\ &=-\beta R^{-1}(\omega+K\epsilon) \quad (49a)\\ \dot{\hat{\theta}}&=\Gamma\tau(\epsilon,\eta,\omega,\nu,\dot{\nu})\\ &=\Gamma\left[F(\epsilon,\eta,\omega,\nu,\dot{\nu})+G(\epsilon,\eta,\omega)+H(\nu,\dot{\nu})\right]^T\\ &\quad\times(\omega+K\epsilon) \quad (49b)\end{aligned}$$

with any $\beta\geq 2$ solves an inverse optimal control problem for the adaptive attitude tracking control system (19a), (19b), and (29) by minimizing the cost functional

$$J_a=\sup_{d\in\mathcal{D}}\left\{\lim_{T\to\infty}\left[\beta\left\|\tilde{\theta}(T)\right\|^2_{\Gamma^{-1}}+2\beta V\left(\epsilon(T),\eta(T),\omega(T),\hat{\theta}(T)\right) +\int_0^T\left(l(\epsilon,\eta,\omega,\hat{\theta},\nu,\dot{\nu})+u_e^TRu_e-\frac{\beta\gamma^2}{2}\|d\|^2\right)dt\right]\right\} \quad (50)$$

for each $\theta\in R^6$, where $\mathcal{D}$ is the set of locally bounded functions of z, the weighting matrix $R(\epsilon,\eta,\omega,\hat{\theta},\nu,\dot{\nu})$ is of the same form as (33) with the smooth matrix $\Psi_2(\epsilon,\eta,\omega,\hat{\theta})$ being replaced by (46), and the state weight $l(\epsilon,\eta,\omega,\hat{\theta},\nu,\dot{\nu})$ is given by

$$l(\epsilon,\eta,\omega,\hat{\theta},\nu,\dot{\nu})=-2\beta\left[c\epsilon^T\omega+z^T(F+G)\hat{\theta}\right] +\beta^2z^TR^{-1}z-\frac{2\beta}{\gamma^2}z^Tz. \quad (51)$$

Proof: The first part of the proof is similar to that of Theorem 4 and the second part is analogous to that of Theorem 3. We outline the proof briefly. Considering the adaptive control Lyapunov function $V_2(\epsilon,\eta,\omega,\hat{\theta})$ in (38) and along the solutions of (19a), (19b), (48), and (49b), we have

$$\dot{V}_2=-c\epsilon^TK\epsilon+z^T\left[c\epsilon+u_e+(F+G)\hat{\theta}+\frac{1}{\gamma^2}z\right].$$

Applying the matrices $F(\epsilon,\eta,\omega,\nu,\dot{\nu})$ in (30b), $G(\epsilon,\eta,\omega)$ in (30c), $\Psi_1(\epsilon,\eta,\omega,\hat{\theta},\nu,\dot{\nu})$ in (34) and $\Psi_2(\epsilon,\eta,\omega,\hat{\theta})$ in (46), we can rewrite $\dot{V}_2$ as

$$\dot{V}_2=-c\epsilon^TK\epsilon+z^Tu_e+\sqrt{c}z^T\Psi_1^TK^{\frac{1}{2}}\epsilon+z^T\Psi_2z.$$

Then, the state-feedback control law

$$u_e=\alpha(\epsilon,\eta,\omega,\hat{\theta},\nu,\dot{\nu})=-\left(K_1+\frac{\Psi_1^T\Psi_1}{2}+\frac{\Psi_2K_1^{-1}\Psi_2}{2}\right)z$$

yields

$$\dot{V}_2\leq-\frac{c}{2}\epsilon^TK\epsilon-\frac{1}{2}z^TK_1z=W(\epsilon,z)\leq 0.$$

As analyzed in Theorem 4, we can conclude that, under the adaptive feedback control laws (47) and (32b), the auxiliary system (19a), (19b), and (48) is globally adaptively stable, i.e., $\epsilon \to 0$, $\omega \to 0$ and $z \to 0$ as $t \to \infty$ for any initial condition. Also

$$c\epsilon^T \omega + z^T \left[(F+G)\hat{\theta} + \frac{1}{\gamma^2} z - R^{-1} z \right] \le -W(\epsilon, z)$$

which implies that

$$l(\epsilon, \eta, \omega, \hat{\theta}, \nu, \dot{\nu}) \ge 2\beta W(\epsilon, z) + \beta(\beta - 2) z^T R^{-1} z.$$

Therefore, $l(\epsilon, \eta, \omega, \hat{\theta}, \nu, \dot{\nu})$ is positive definite in ϵ and ω for each $\hat{\theta}$, and the cost functional J_a in (50) is a meaningful cost functional for the attitude tracking control problem, putting penalties on the attitude tracking errors ϵ, ω and the control effort u_e.

Substituting $l(\epsilon, \eta, \omega, \hat{\theta}, \nu, \dot{\nu})$ in (51), $z = \omega + K\epsilon$ and $v = u_e - \alpha^* = u_e + \beta R^{-1} z$ into the cost functional J_a in (50), we obtain the following expression of J_a along the solutions of the attitude control system (19a), (19b), (29), and (49b):

$$\begin{aligned} J_a = \sup_d \Bigg\{ & \lim_{T\to\infty} \Bigg[\beta \left\| \tilde{\theta}(T) \right\|^2_{\Gamma^{-1}} \\ & + 2\beta V \left(\epsilon(T), \eta(T), \omega(T), \hat{\theta}(T) \right) \\ & - \int_0^T \left(\frac{2\beta}{\gamma^2} z^T z - 2\beta z^T d + \frac{\beta\gamma^2}{2} \|d\|^2 \right) dt \\ & - 2\beta \int_0^T \Big(c\epsilon^T \omega + z^T [F\theta + G\theta + H\tilde{\theta} + u_e + d] \\ & - z^T (F+G+H)\tilde{\theta} \Big) dt + \int_0^T v^T R v dt \Bigg] \Bigg\} \\ = \sup_d \Bigg\{ & \lim_{T\to\infty} \Bigg[\beta \left\| \tilde{\theta}(T) \right\|^2_{\Gamma^{-1}} \\ & + 2\beta V \left(\epsilon(T), \eta(T), \omega(T), \hat{\theta}(T) \right) + \int_0^T v^T R v dt \\ & - 2\beta \int_0^T \frac{d}{dt} \left(c(1-\eta)^2 + c\epsilon^T \epsilon + \frac{1}{2} z^T J z + \frac{1}{2} \tilde{\theta}^T J \tilde{\theta} \right) dt \\ & - \frac{\beta\gamma^2}{2} \int_0^T \left\| \frac{2}{\gamma^2} z - d \right\|^2 dt \Bigg] \Bigg\} \\ = & 2\beta V_2 \left(\epsilon(0), \eta(0), \omega(0), \hat{\theta}(0) \right) + \int_0^\infty v^T R v dt \\ & - \frac{\beta\gamma^2}{2} \sup_d \left[\int_0^\infty \left\| \frac{2}{\gamma^2} z - d \right\|^2 dt \right]. \end{aligned}$$

It is clear that

$$\sup_{d \in \mathcal{D}} \left[- \int_0^\infty \left\| \frac{2}{\gamma^2} z - d \right\|^2 dt \right] = 0$$

and the "worst-case" disturbance $d^*(\epsilon, \omega)$ is given by

$$d^*(\epsilon, \omega) = \frac{2}{\gamma^2} z = \frac{2}{\gamma^2} (\omega + K\epsilon). \tag{52}$$

Hence, the minimum of the cost functional J_a is reached only if $v = 0$, i.e., the control law $u_e = \alpha^*(\epsilon, \eta, \omega, \hat{\theta}, \nu, \dot{\nu})$ in (49a) is inverse optimal and minimizes the cost functional (50). The value function of (50) is $J_a^*(\epsilon, \eta, \omega, \hat{\theta}) = 2\beta V_2(\epsilon, \eta, \omega, \hat{\theta})$. ■

Remark 8: The parameter $\beta \ge 2$ in Theorem 6 represents a degree of freedom for the design. Also, applying the inverse optimal adaptive attitude controller (49), we obtain the derivative value $\dot{V}_2$ as

$$\dot{V}_2 \le -\frac{c}{2} \epsilon^T K \epsilon - \frac{1}{2} z^T K_1 z - \frac{1}{\gamma^2} z^T z - (\beta - 1) z^T R^{-1} z + z^T d$$

along the solutions of (19a), (19b) and (29). It follows from the Young's inequality [29] that

$$z^T d \le \frac{\gamma^2}{4} \|d\|^2 + \frac{1}{\gamma^2} z^T z$$

where the "=" sign is satisfied only when $d(t) = d^*(\epsilon, \omega) = (2/\gamma^2) z$. Note that $R^{-1} \ge K_1$. Therefore, we have

$$\dot{V}_2 \le -\frac{c}{2} \epsilon^T K \epsilon - \frac{2\beta - 1}{2} z^T K_1 z + \frac{\gamma^2}{4} \|d\|^2.$$

Then, there must exist finite constants $c_3 > 0$ and $c_4 > 0$ such that

$$\dot{V}_2 \le -c_3 \|\epsilon\|^2 - c_4 \|\omega\|^2 + \frac{\gamma^2}{4} \|d\|^2 \tag{53}$$

which implies that the closed-loop system under the dynamic feedback control law (49) is d-to-(ϵ, ω)-stable in the sense of input-to-state stability (ISS) [28], [30]. In turn, it follows from the definition of ISS that, if we denote $x = [\epsilon^T, \omega^T]^T$, there exist some continuous functions $\phi(\cdot, \cdot) \in \mathcal{KL}$ and $\chi(\cdot) \in \mathcal{K}_\infty$ such that

$$\|x(t)\| \le \phi(\|x(0)\|, t) + \chi \left(\sup_{0 \le s \le t} \|d(s)\| \right).$$

Therefore, the inverse optimal adaptive control law (49) guarantees the boundedness of the tracking errors ϵ and ω for any bounded (and persistent) external disturbance. We emphasize that the inverse optimal adaptive control (49) is not restricted to disturbances with bounded energy $\int_0^\infty \|d(t)\|^2 dt < \infty$ but any bounded (and persistent) external disturbances are allowed. □

Remark 9: Following the discussion in Remark 3, we can conclude that the inverse optimal adaptive control law (49) shows H_∞ inverse optimality with respect to the external disturbance $d(t)$ and the performance index (50) in the sense of Definition 4. Furthermore, we can obtain a bound of the $\mathcal{L}_2$ attenuation level from $d(t)$ directly to the tracking errors $\binom{\epsilon(t)}{\omega(t)}$ that is in the order of γ. To see this, integrating both sides of (53) with respect to t we can present an $\mathcal{L}_2$ bound on ϵ and ω by the following inequality:

$$\int_0^T \left[4c_3\|\epsilon\|^2 + 4c_4\|\omega\|^2\right] dt \le \gamma^2 \int_0^T \|d\|^2 dt + 4V_2\left(\epsilon(0), \eta(0), \omega(0), \hat{\theta}(0)\right) - 4V_2\left(\epsilon(T), \eta(T), \omega(T), \hat{\theta}(T)\right) \tag{54}$$

for all $T \ge 0$. Hence, the inverse optimal adaptive controller (49) attenuates external disturbances and the $\mathcal{L}_2$-gain from d to $\binom{\epsilon}{\omega}$ is bounded by $\gamma/(2\sqrt{\max\{c_3, c_4\}})$. Moreover, the $\mathcal{L}_2$ disturbance attenuation level can be made arbitrarily small at the cost of a larger u_e. A smaller γ will lead to a larger control u_e because the last term in (46) implies that the value of Ψ_2 is getting larger.

It follows from (54) that if $d \in \mathcal{L}_2[0,\infty)$, the tracking errors $\epsilon, \omega \in \mathcal{L}_2[0,\infty)$ and $V_2(\epsilon(T), \eta(T), \omega(T), \hat{\theta}(T))$ is bounded for all $T \ge 0$, implying that ϵ, η, ω, z, $\tilde{\theta}$ and $\hat{\theta}$ are all bounded signals. As analyzed in the proof of Theorem 4, if ν, $\dot{\nu}$ and d are bounded too, we can conclude that $\dot{\epsilon}$ and $\dot{z}$ are bounded and, hence, ϵ and z are uniformly continuous. Then by (54) and the *Barbalat's lemma* [28, p. 192], $\epsilon \to 0$ and $\omega \to 0$ as $t \to \infty$. In other words, if $d \in \mathcal{L}_2[0,\infty)$ and is bounded, asymptotic attitude tracking is achieved with a global convergence for all initial conditions. Note also that $\dot{\hat{\theta}} \to 0$ consequently. □

C. Convergence of the Adaptive Parameters

As stated in Remark 5, the estimation error $\tilde{\theta} = \theta - \hat{\theta}$ does not necessarily converge to zero as $t \to \infty$. However, $\hat{\theta}$ can converge to its nominal value θ under certain conditions on the references $\nu(t)$ and $\dot{\nu}(t)$.

Proposition 1: Assume that the desired angular velocity $\nu(t)$ is periodic and the external disturbance $d(t)$ is zero. Let

$$\Theta = \{\xi : H(\nu(t), \dot{\nu}(t))\, \xi = 0 \qquad \forall t \ge 0\}. \tag{55}$$

Under the inverse optimal adaptive control law (43), $\theta - \hat{\theta} \to \tilde{\theta}_{ss}$ as $t \to \infty$, where $\tilde{\theta}_{ss}$ is a constant in Θ.

Proof: The proof is similar to that of [11, Th. 2]. Theorem 4 says that $\theta - \hat{\theta}$ converges to a constant. To show that this constant is in Θ, we proceed as follows. With z defined by (26) and $\tilde{\theta} = \theta - \hat{\theta}$, we have the differential equations (19a), (19b), (29), and (43b), where the control input u_e is given by (43a) and the matrices $F(\epsilon, \eta, \omega, \nu, \dot{\nu})$, $G(\epsilon, \eta, \omega)$ and $H(\nu, \dot{\nu})$ are defined by (30). Since $d \equiv 0$ and $\nu(t)$ is periodic, the closed-loop system becomes a periodic system.

Consider the Lyapunov function candidate $V_2(\epsilon, \eta, \omega, \tilde{\theta})$ defined by (38). Applying the matrix R^{-1} defined by (33), along the trajectories of the attitude tracking control system (19a), (19b), (29), and (43b) we obtain the derivative $\dot{V}_2$ as

$$\begin{aligned}\dot{V}_2 = &-\frac{c}{2}\epsilon^T K \epsilon - \frac{1}{2} z^T K_1 z - \frac{1}{2}\left\|\sqrt{c} K^{\frac{1}{2}} \epsilon - \Psi_1 z\right\|^2 \\ &- \frac{1}{2} z^T (K_1 - \Psi_2)^T K_1^{-1} (K_1 - \Psi_2) z \\ &- (\beta - 1) z^T R^{-1} z.\end{aligned}$$

Hence, $\dot{V}_2 = 0$ if and only if $z = \epsilon = 0$. Furthermore, when the latter is true, $\dot{z} = \dot{\epsilon} = 0$ if and only if $H(\nu, \dot{\nu})\tilde{\theta} = 0$ because of (29), (43a) and the fact that $F(\epsilon, \eta, \omega, \nu, \dot{\nu})$, $G(\epsilon, \eta, \omega)$ vanish at $\epsilon = \omega = 0$. Then it follows from LaSalle's result on periodic systems [31, Th. 2.8] that $(z, \epsilon, \eta, \hat{\theta})$ will converge to the set

$$\left\{z = 0, \epsilon = 0, \eta = \{-1, 1\}, \tilde{\theta} \in \Theta\right\}$$

as $t \to \infty$. ■

Proposition 1 states that the adaptive parameter converges to a constant in an invariant manifold Θ under the inverse optimal adaptive control law (43). The following proposition is a straightforward corollary of Proposition 1, stating when the estimate $\hat{\theta}(t)$ can converge to its nominal value θ as $t \to \infty$.

Proposition 2: Assume that the desired angular velocity $\nu(t)$ is periodic and the external disturbance $d(t)$ is zero. Let $0 \le t_1 \le t_2 \le \cdots \le t_n$ and suppose that

$$\text{Rank}\begin{bmatrix} H(\nu(t_1), \dot{\nu}(t_1)) \\ \vdots \\ H(\nu(t_n), \dot{\nu}(t_n)) \end{bmatrix} = 6 \tag{56}$$

where 6 is the dimension of θ. Then, under the adaptive control law (43), $\hat{\theta} \to \theta$ as $t \to \infty$.

Proof: The proof is similar to that of [11, Prop. 2]. Since Proposition 1 implies $\theta - \hat{\theta} \to \Theta$, and (56) implies $\Theta = \{0\}$, we have $\hat{\theta} \to \theta$ as $t \to \infty$. ■

As a result, the inertia matrix J can be completely identified for the zero-disturbance case if the reference signal $\nu(t)$ is periodic and the rank condition (56) is satisfied. We emphasize that when the external disturbance $d(t)$ is persistent and bounded, the adaptive parameter $\hat{\theta}$ might not converge to θ even if the rank condition (56) holds, as shown in the simulations that follow.

V. Simulation Results

An attitude maneuver control problem of a rigid-body microsatellite is simulated to demonstrate the performance of the adaptive feedback attitude tracking controller. The desired attitude motion of the spacecraft is described in the body frame $\mathcal{B}_c$. The spacecraft is assumed to have the inertia matrix of

$$J = \begin{bmatrix} 10 & 1.0 & 0.7 \\ 1.0 & 10 & 0.4 \\ 0.7 & 0.4 & 8 \end{bmatrix} \text{kg}\cdot\text{m}^2$$

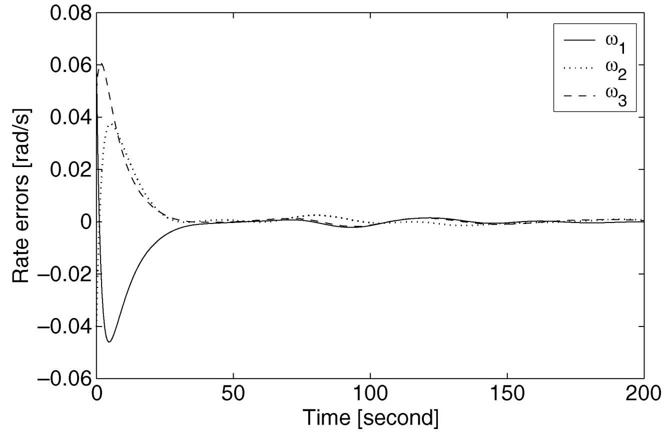

Fig. 1. Relative rate error ω in the zero-disturbance case.

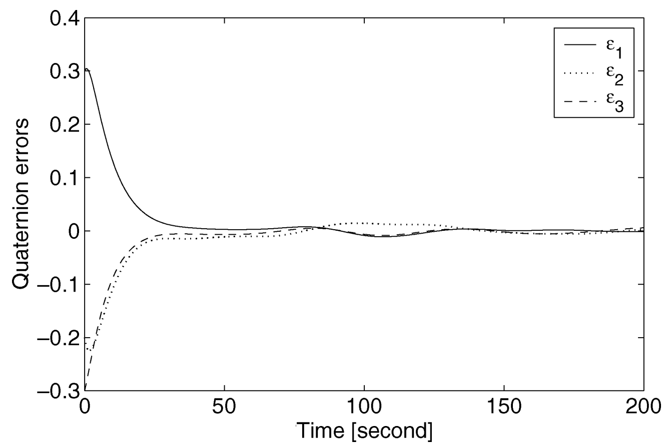

Fig. 2. Orientation error ϵ in the zero-disturbance case.

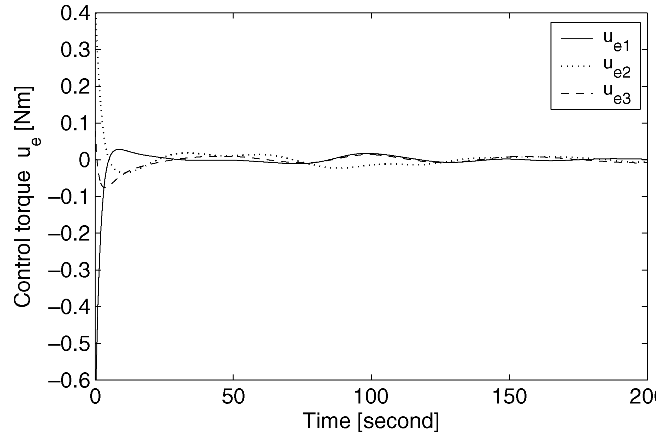

Fig. 3. Control effort u_e given by (43a) with $\beta = 2$.

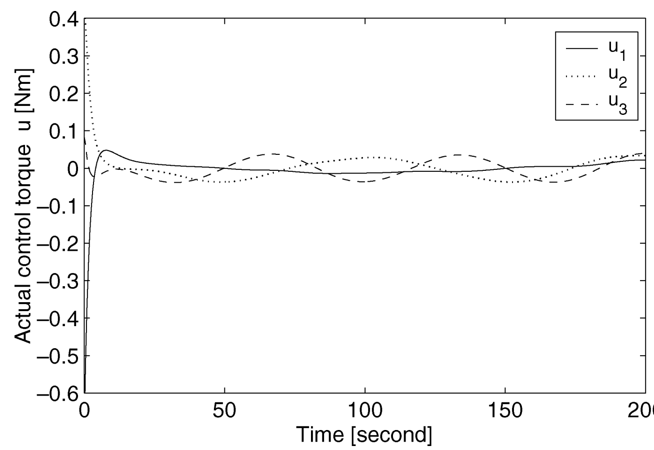

Fig. 4. Actual control effort $u = u_e - u_c$.

which is unknown to the controller in the frame $\mathcal{B}_s$. Arbitrarily, we suppose that the desired angular velocity $\nu(t)$ to be tracked is given in the body frame $\mathcal{B}_c$ by

$$\nu = \begin{bmatrix} 0.05\sin\left(\frac{1\pi t}{100}\right) \\ 0.05\sin\left(\frac{2\pi t}{100}\right) \\ 0.05\sin\left(\frac{3\pi t}{100}\right) \end{bmatrix} \text{ rad/s}.$$

With this choice of the reference signal $\nu(t)$, it is easy to check that (56) is satisfied so that it is possible to verify the convergence of the adaptive parameter $\hat{\theta}$.

In the numerical simulations of the adaptive attitude tracking controllers, we assume that the initial attitude orientation of the spacecraft in the frame $\mathcal{B}_s$ is given by the unit quaternion $q(0) = [0.3, -0.2, -0.3, 0.8832]^T$, the initial angular velocity of the spacecraft in $\mathcal{B}_s$ is $\Omega(0) = [0.06,\ -0.04,\ 0.05]^T$ rad/s and the initial value of the adaptive parameter $\hat{\theta}$ is given by $\hat{\theta}(0) = [12,\ 12,\ 10,\ 0.8,\ 1,\ 0.5]^T \text{ kg} \cdot \text{m}^2$. The gains of the inverse optimal adaptive control law (43) are chosen to be $K = 0.2I_3$, $K_1 = 2I_3$, $\Gamma = 3000I_6$ and $c = 0.5$. Without loss of inverse optimality, we choose $\beta = 2$.

At first, we consider the zero-disturbance case. Applying the inverse optimal adaptive attitude controller (43), we illustrate the simulation results as Figs. 1–6, from which we conclude that the adaptive attitude tracking is achieved when the inertia matrix J in the body frame $\mathcal{B}_s$ is uncertain. Figs. 1 and 2 depict the time histories of the tracking errors ω and ϵ, which show that the inverse optimal adaptive tracking controller (43) achieves a good performance on the attitude tracking with satisfactory tracking errors ω and ϵ and a rapid convergence. Fig. 3 plots the time history of the control effort u_e given by (43a). The actual control effort $u = u_e - u_c$ that is input to the actual attitude system is shown in Fig. 4, where u_c is given by (27). Figs. 5 and 6 indicate that the estimate of the adaptive parameter $\hat{\theta}$ converges to the nominal value θ, i.e., $\hat{\theta} \to \theta$ in accordance with Proposition 2. It is observed from the numerical simulations that the attitude tracking is achieved rapidly, while the convergence of the adaptive parameter takes a much longer time. The smaller the matrix Γ, the more time it takes for the convergence of the adaptive parameters.

Next, we consider the tracking control problem in the presence of external disturbance $d(t)$. The disturbance model is described by

$$d(t) = 0.01 \times \begin{bmatrix} 2\sin\left(\frac{\pi t}{200}\right) + \sin\left(\frac{2\pi t}{200}\right) - \cos\left(\frac{3\pi t}{200}\right) \\ -2\sin\left(\frac{\pi t}{200}\right) - 2\sin\left(\frac{2\pi t}{200}\right) + \cos\left(\frac{3\pi t}{200}\right) \\ 2\sin\left(\frac{\pi t}{200}\right) + \cos\left(\frac{2\pi t}{200}\right) + \sin\left(\frac{3\pi t}{200}\right) \end{bmatrix} + 0.2 \times \begin{bmatrix} \delta(70,2) \\ \delta(80,2) \\ \delta(90,2) \end{bmatrix} + \begin{bmatrix} 0.005 \\ 0.005 \\ 0.005 \end{bmatrix} \text{ Nm}$$

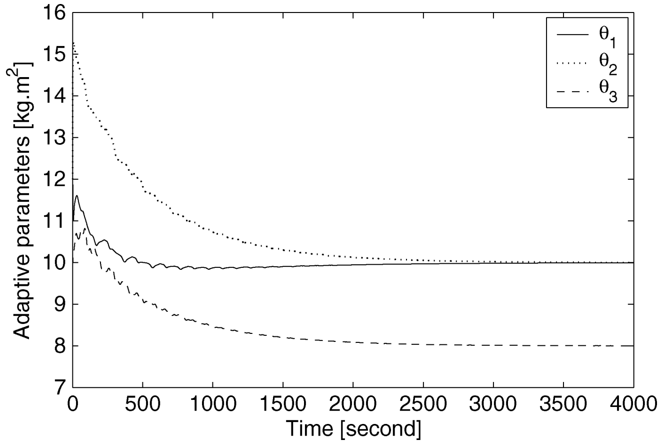

Fig. 5. Adaptive parameters $\hat{\theta}_1$, $\hat{\theta}_2$, and $\hat{\theta}_3$.

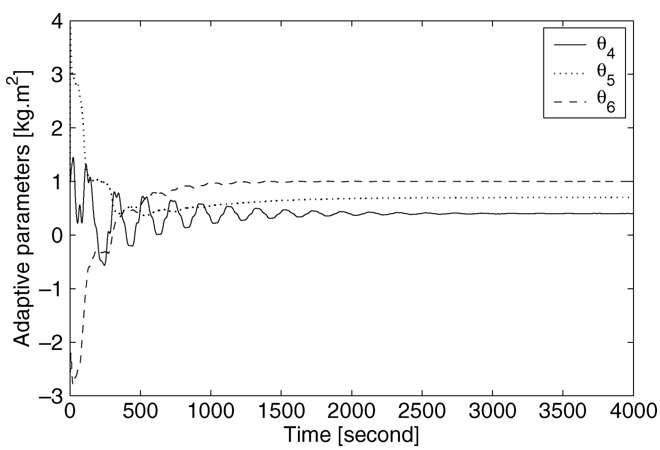

Fig. 6. Adaptive parameters $\hat{\theta}_4$, $\hat{\theta}_5$, and $\hat{\theta}_6$.

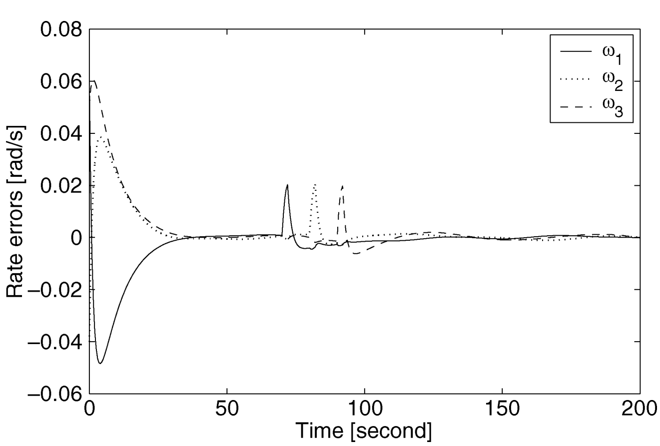

Fig. 7. Relative rate error ω with external disturbances.

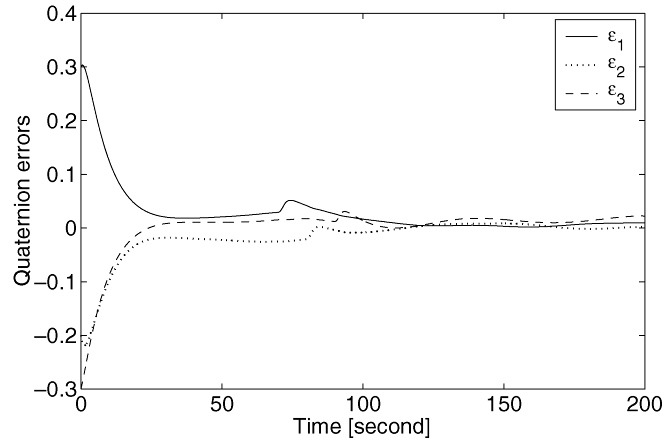

Fig. 8. Orientation error ϵ with external disturbances.

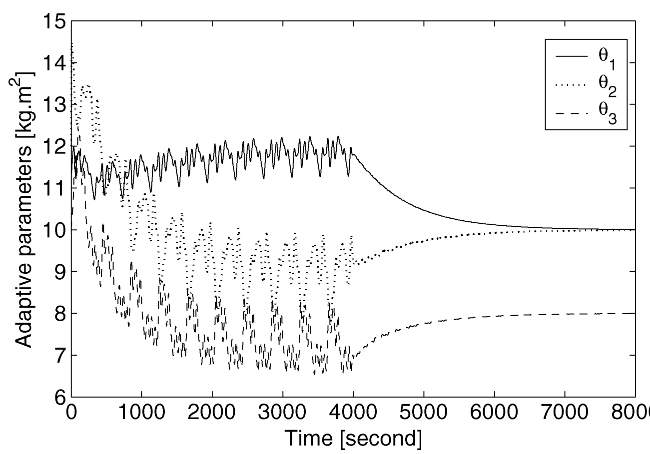

Fig. 9. Adaptive parameters $\hat{\theta}_1$, $\hat{\theta}_2$, and $\hat{\theta}_3$.

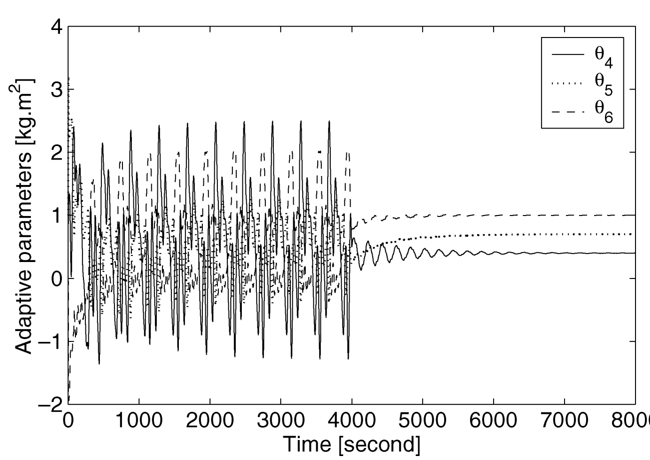

Fig. 10. Adaptive parameters $\hat{\theta}_4$, $\hat{\theta}_5$, and $\hat{\theta}_6$.

where $\delta(t_1, \Delta t_1)$ in the second bracket denotes an impulsive disturbance with magnitude 1 and width Δt_1 seconds, activating at the time point t_1. Letting $\gamma = 1.0$ and applying the robust inverse optimal adaptive control law (49), we present the simulation results as in Figs. 7–10. Figs. 7 and 8 depict the time histories of the rate error ω and the attitude error ϵ, from which we conclude that the adaptive tracking control law (49) can achieve the adaptive attitude tracking with satisfactory tracking errors ω and ϵ and a good convergence even in the presence of external disturbances. Figs. 9 and 10 indicate the estimate of the adaptive parameter $\hat{\theta}$. When $t \leq 4000$ s, the disturbances work persistently and then we can see that $\hat{\theta}$ does not converge to the nominal value θ even using the same adaptive update law. If we get rid of the external disturbances and let $d(t) = 0$ for $t > 4000$ seconds, simulations show that the adaptive parameter estimates will converge back to the nominal value θ.

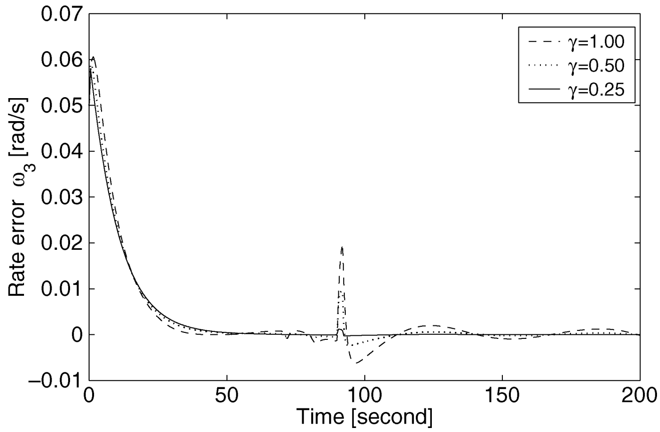

Fig. 11. Relative rate error ω_3 for various attenuation levels.

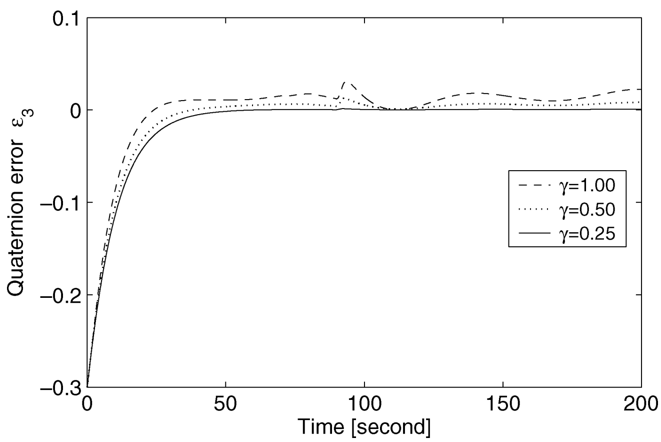

Fig. 12. Tracking error ϵ_3 for various attenuation levels.

Finally, to illustrate the capacity of disturbance attenuation, three different attenuation levels are considered, $\gamma = 1.0$, $\gamma = 0.5$ and $\gamma = 0.25$. The simulation results are shown in Figs. 11 and 12 in terms of ω_3 and ϵ_3, the third components of the tracking errors ω and ϵ. As expected, a smaller γ yields a better attenuation of the external disturbance $d(t)$.

VI. Conclusion

An attitude tracking control system is indeed a nonlinear cascade system. Therefore, stabilizing such a system can be efficiently achieved using the method of backstepping. Employing the adaptive control method and the inverse optimal control approach, this paper has presented inverse optimal adaptive control laws to solve the attitude tracking problem of a rigid spacecraft with an uncertain inertia matrix. In the zero-disturbance case, the inverse optimal adaptive controller proposed in this paper achieves asymptotic attitude tracking of the desired attitude motions with a global convergence for all initial conditions. The control law is inverse optimal with respect to a meaningful cost functional that consists of penalties on both the tracking errors ϵ, ω and the control effort. When external disturbances are considered, we have presented a robust adaptive attitude control law, which is not only inverse optimal with respect to a meaningful cost functional that penalizes the tracking errors and the control effort, but also forms a closed-loop attitude system that has a guaranteed $\mathcal{L}_2$-gain from the external disturbances to the tracking errors. Any given level of $\mathcal{L}_2$ disturbance attenuation can be achieved at the cost of a larger control effort. Such optimal control laws have been obtained without solving the Hamilton-Jacobi-Isaacs equation directly. Numerical simulations have been done to verify the performance of the proposed attitude tracking algorithms.

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