INVERSE PROBLEM FOR ELECTRICAL NETWORKS VIA TWIST

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ABSTRACT. We construct an electrical-network version of the twist map for the positive Grassmannian, and use it to solve the inverse problem of recovering conductances from the response matrix. Each conductance is expressed as a biratio of Pfaffians as in the inverse map of Kenyon and Wilson; however, our Pfaffians are the more canonical B variables instead of their tripod variables, and are coordinates on the positive orthogonal Grassmannian studied by Henriques and Speyer.

1. INTRODUCTION

Let $G = (V, E)$ be a planar graph embedded in a disk $\mathbb{D}$ with vertices $\{b_1, \ldots, b_n\}$ on the boundary labeled in clockwise cyclic order. A function $c : E(G) \to \mathbb{R}_{>0}$ is called a *conductance*, and a pair (G, c) is called an *electrical network*. In this paper, we focus on *well connected* electrical networks (defined in Section 3.1).

The *Laplacian* on G is the linear operator $\Delta : \mathbb{R}^{V(G)} \to \mathbb{R}^{V(G)}$ defined by

$$(\Delta f)(v) := \sum_{e=uv} c(e)(f(v) - f(u))$$

where the sum is over all edges uv incident to v. A function $f : V(G) \to \mathbb{R}$ is said to be *harmonic* if $(\Delta f)(v) = 0$ for all internal vertices v of G. Given a function $u : \{b_1, \ldots, b_n\} \to \mathbb{R}$ on the boundary vertices, there is a unique extension of u to a harmonic function f, called the *harmonic extension* of u. The linear operator $L : \mathbb{R}^{\{b_1,\ldots,b_n\}} \to \mathbb{R}^{\{b_1,\ldots,b_n\}}$ defined by $L(u) = (-\Delta f)\big|_{\{b_1,\ldots,b_n\}}$ is called the *response matrix*. It is a semidefinite negative symmetric matrix whose rows and columns sum to 0. The space of response matrices was characterized by Colin de Verdière [CdV94] and further studied in [CdVGV96, CMM94, CIM98].

In this paper, we are interested in the *inverse problem* of recovering the conductances from the response matrix. The inverse problem was solved using a recursive procedure by Curtis, Ingerman and Morrow [CIM98] (see also [CM, Joh12, Rus]) and explicit rational formulas were given by Kenyon and Wilson [KW09, KW17]. The inverse problem has also been studied in the cylinder [LP12] and the torus [Geo19].

In the formulas in [KW09, KW17], the conductances are expressed as biratios of certain variables called *tripod variables* which are only defined for special networks called standard networks. On the other hand, there are other more canonical variables associated with the vertices and faces of G called *B variables* such that the conductance of an edge is a biratio of B variables of adjacent vertices and faces. The main goal of this paper is to provide a solution to the inverse problem by computing the B variables from the response matrix.

Our approach is motivated by the construction of the inverse boundary measurement map for the dimer model in $\mathbb{D}$ by Muller and Speyer [MS17] using an automorphism of the positive Grassmannian called the *twist*. Let $\Gamma = (B \sqcup W, E, F)$ be a bipartite graph in $\mathbb{D}$ with vertices

Date: May 18, 2023.

$\{d_1, \ldots, d_n\}$ on the boundary of $\mathbb{D}$ (and with strand permutation $\pi_{k,n}$; see Section 2.2). Associated with Γ is the space $\mathcal{X}_\Gamma$ of edge weights modulo gauge equivalence. Postnikov [Pos06] constructed a parameterization of the totally positive Grassmannian $\mathrm{Gr}_{>0}(k,n)$ using a map $\mathrm{Meas}_\Gamma : \mathcal{X}_\Gamma \to \mathrm{Gr}_{>0}(k,n)$ called *boundary measurement*, where $k := \#W - \#B$. There is another space $\mathcal{A}_\Gamma$ of functions $A : F(\Gamma) \to \mathbb{R}_{>0}$. Scott [Sco06] constructed a function $\Phi_\Gamma : \mathrm{Gr}_{>0}(k,n) \to \mathcal{A}_\Gamma/\mathbb{R}_{>0}$ assigning to each face of Γ a certain Plücker coordinate. The spaces $\mathcal{A}_\Gamma$ and $\mathcal{X}_\Gamma$ are the (positive points of the) $\mathcal{A}$ and $\mathcal{X}$ cluster tori of Fock and Goncharov [FG09], and there is a canonical map $p_\Gamma : \mathcal{A}_\Gamma \to \mathcal{X}_\Gamma$ that assigns to an edge bw incident to faces f, g the weight $\frac{1}{A_f A_g}$ (with some modification for boundary edges). Muller and Speyer, generalizing earlier work of Marsh and Scott [MS16], construct automorphisms $\vec{\tau}$ and $\overleftarrow{\tau}$ of $\mathrm{Gr}_{>0}(k,n)$, called *right* and *left twists*, that sit in the following commutative diagram:

$$\begin{array}{ccc}
\mathcal{A}_\Gamma/\mathbb{R}_{>0} & \xrightarrow[\sim]{p_\Gamma} & \mathcal{X}_\Gamma \\
{\scriptstyle \Phi_\Gamma}\big\uparrow{\scriptstyle \sim} & & {\scriptstyle \sim}\big\downarrow{\scriptstyle \mathrm{Meas}_\Gamma} \\
\mathrm{Gr}_{>0}(k,n) & \underset{\underset{\vec{\tau}}{\longrightarrow}}{\overset{\overset{\overleftarrow{\tau}}{\longleftarrow}}{\sim\,\sim}} & \mathrm{Gr}_{>0}(k,n)
\end{array} .$$

As a consequence, the boundary measurement Meas_Γ has inverse $p_\Gamma \circ \Phi_\Gamma \circ \overleftarrow{\tau}$.

The generalized Temperley's bijection of Kenyon, Propp and Wilson [KPW00] associates to each electrical network (G, c) a weighted bipartite graph $(G_+, [\mathrm{wt}_+])$, giving an embedding $j_G^+ : \mathcal{R}_G \hookrightarrow \mathcal{X}_{G_+}$. Lam [Lam18] studied the composition $\mathrm{Meas}_{G_+} \circ j_G^+ : \mathcal{R}_G \to \mathrm{Gr}_{>0}(n+1, 2n)$ and showed that the image of $\mathcal{R}_G$ is a linear slice of $\mathrm{Gr}_{>0}(n+1,2n)$, which was subsequently identified with a positive Lagrangian Grassmannian $\mathrm{IG}^\Omega_{>0}(n+1,2n)$ of points in $\mathrm{Gr}_{>0}(n+1,2n)$ that are isotropic for a degenerate skew-symmetric bilinear form Ω in [BGKT21, CGS21] (see also [LP15a]). [CGS21, Theorem 1.8] explicitly identifies the space of response matrices with $\mathrm{IG}^\Omega_{>0}(n+1,2n)$.

Therefore, in principle, the inverse problem for electrical networks can be solved using the inverse boundary measurement. However, in practice, the result of inverting the boundary measurement yields a weight on G_+ to which one has to apply a complicated gauge transformation to obtain the conductances.

Like the space $\mathcal{A}_\Gamma$, there is a second space $\mathcal{B}_G$ associated with an electrical network parameterized by the B variables. The space $\mathcal{B}_G$ consists of functions $B : V(G) \sqcup F(G) \to \mathbb{R}_{>0}$, and there is a canonical map $q_G : \mathcal{B}_G \to \mathcal{R}_G$ defined as follows. Let $e = uv$ be an edge of G and let f, g denote the faces of G incident to e. Define $q_G : \mathcal{B}_G \to \mathcal{R}_G$ by $c(e) := \frac{B_u B_v}{B_f B_g}$ (cf. Equation (56) in [GK13, Section 5.3.1]). The space $\mathcal{B}_G$ arises from the study of the cube recurrence, a nonlinear recurrence introduced by Propp [Pro01] whose solutions were characterized combinatorially by Carroll and Speyer [CS04] (see also [FZ02, LP15b]). The cube recurrence was further studied by Henriques and Speyer [HS10], who related it to the orthogonal Grassmannian $\mathrm{OG}(n+1,2n)$ of $(n+1)$-dimensional subspaces that are coisotropic for a certain symmetric bilinear form Q. $\mathrm{OG}(n+1,2n)$ has an embedding in $\mathbb{CP}^{2^{n-1}-1} \times \mathbb{CP}^{2^{n-1}-1}$ giving bihomogeneous coordinates on $\mathrm{OG}(n+1,2n)$ called *Cartan coordinates* (which are given by certain Pfaffians; see Section 3.5). Henriques and Speyer constructed a homeomorphism $\Psi_G : \widetilde{\mathrm{OG}}_{>0}(n+1,2n) \xrightarrow{\sim} \mathcal{B}_G$ assigning to each vertex and face of G a Cartan coordinate, where $\widetilde{\mathrm{OG}}(n+1,2n)$ is the "affine cone" over $\mathrm{OG}(n+1,2n)$ and $\widetilde{\mathrm{OG}}_{>0}(n+1,2n)$ is the subset where all Cartan coordinates are positive. Our first main result is the following.

Theorem 1.1 (cf. Theorem 4.2)**.** *There is a map $\vec{\tau}_{\mathrm{elec}}$, which we call the electrical right twist, such that the following diagram commutes.*

$$\begin{array}{ccc} \mathcal{B}_G & \xrightarrow{\quad q_G \quad} & \mathcal{R}_G \\ {\scriptstyle \Psi_G}\Big\uparrow{\scriptstyle \sim} & & {\scriptstyle \sim}\Big\downarrow{\scriptstyle \mathrm{Meas}_{G_+} \circ j_G^+} \\ \widetilde{\mathrm{OG}}_{>0}(n+1,2n) & \xrightarrow{\vec{\tau}_{\mathrm{elec}}} & \mathrm{IG}^{\Omega}_{>0}(n+1,2n) \end{array}.$$

To solve the inverse problem, we need to invert the electrical right twist. Our second main result is:

Theorem 1.2 (cf. Theorem 5.6)**.** *There are actions of $\mathbb{R}^{n+1}_{>0}$ on $\mathcal{B}_G$ and $\widetilde{\mathrm{OG}}_{>0}(n+1,2n)$ compatible with Ψ_G such that upon taking quotients, q_G and $\vec{\tau}_{\mathrm{elec}}$ are invertible. The inverse $\overleftarrow{\tau}_{\mathrm{elec}}$ is called the electrical right twist, and the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{B}_G/\mathbb{R}^{n+1}_{>0} & \xrightarrow[\sim]{\qquad q_G \qquad} & \mathcal{R}_G \\ {\scriptstyle \Psi_G}\Big\uparrow{\scriptstyle \sim} & & {\scriptstyle \sim}\Big\downarrow{\scriptstyle \mathrm{Meas}_{G_+} \circ j_G^+} \\ \widetilde{\mathrm{OG}}_{>0}(n+1,2n)/\mathbb{R}^{n+1}_{>0} & \begin{array}{c} \xleftarrow[\sim]{\overleftarrow{\tau}_{\mathrm{elec}}} \\ \xrightarrow[\vec{\tau}_{\mathrm{elec}}]{\sim} \end{array} & \mathrm{IG}^{\Omega}_{>0}(n+1,2n) \end{array}.$$

Therefore, the composition $q_G \circ \Psi_G \circ \overleftarrow{\tau}_{\mathrm{elec}}$ solves the inverse problem. We work out the inverse map explicitly when $n = 3$ in Section 6.

We end the Introduction with some open problems. If the graph G is not well connected, then $\mathcal{R}_G$ parameterizes a smaller electroid cell in $\mathrm{IG}^{\Omega}(n+1,2n)$ which is the intersection of a positroid cell with $\mathrm{IG}^{\Omega}(n+1,2n)$ [Lam18]. Muller and Speyer defined the twist map for all postroid cells, which suggests the following problem.

Problem 1.3. Construct a stratified space whose strata are parameterized by $\mathcal{B}_G$, where G varies over the move-equivalence classes of reduced graphs with n vertices on the boundary of the disk. Define an electrical twist map that homeomorphically maps the strata to electroid cells in $\mathrm{IG}^{\Omega}(n+1,2n)$.

There is another definition of positive orthogonal Grassmannian introduced in [HWX14] which was used to parameterize the Ising model by Galashin and Pylyavskyy [GP20]. Similarly, there is a positive Lagrangian Grassmannian associated with the cluster side $\mathcal{A}$ of the Ising model, introduced by Kenyon and Pemantle [KP16, KP14] in relation to the Kashaev recurrence [Kas96]. The two definitions of positive orthogonal/Langrangian Grassmannian do not agree, but instead we expect the relationship to be as in the table below, where the two spaces in each row are related by twist.

	cluster $\mathcal{A}$ side	cluster $\mathcal{X}$ side
dimer models	positive Grassmannian [Sco06]	positive Grassmannian [Pos06]
electrical networks	positive orthogonal Grassmannian [HS10]	positive Lagrangian Grassmannian [BGKT21, CGS21]
Ising models	positive Lagrangian Grassmannian [KP16]	positive orthogonal Grassmannian [GP20]

Problem 1.4. Define a twist map for the Ising model relating the positive orthogonal Grassmannian in [GP20] with the positive Lagrangian Grassmannian in [KP14].

We mention that results relating orthogonal and Lagrangian Grassmannians also appear in [Wan22, Wan23], but the connection to the above table is unclear.

Acknowledgments. This project originated from conversations with Sunita Chepuri and David Speyer. I also thank David for many discussions on his papers [HS10] and [MS17].

2. Background on the dimer model and the positive Grassmannian

In this section, we review background on the positive Grassmannian, dimer models, and the twist map.

2.1. **Grassmannians and Plücker coordinates.** The *Grassmannian* $\mathrm{Gr}(k,n)$ is the space of k-dimensional subspaces of $\mathbb{C}^n$. Let $e_1,\dots,e_n$ denote the standard basis of $\mathbb{C}^n$. For $I=\{i_1<i_2<\cdots<i_k\}\in\binom{[n]}{k}$, let $e_I := e_{i_1}\wedge\cdots\wedge e_{i_k}$. Then, the e_I form a basis for $\bigwedge^k\mathbb{C}^n$. The *Plücker embedding* is the closed embedding $\mathrm{Pl}:\mathrm{Gr}(k,n)\hookrightarrow\mathbb{P}(\bigwedge^k\mathbb{C}^n)$ sending a subspace X spanned by $v_1,\dots,v_k$ to $[v_1\wedge\cdots\wedge v_k]$. The coefficients $\Delta_I(X)$ of e_I in $v_1\wedge\cdots\wedge v_k$ are called *Plücker coordinates*. Following [Wen21], we call $\widetilde{\mathrm{Gr}}(k,n) := \{(X,v)\mid X\in\mathrm{Gr}(k,n), v\in\bigwedge^k X\}$ the *decorated Grassmannian*. Given $(X,v)\in\widetilde{\mathrm{Gr}}(k,n)$, we denote the coefficient of e_I in v by $\Delta_I(X,v)$. Changing the basis multiplies all the Plücker coordinates by a common scalar, so they are well-defined functions on $\widetilde{\mathrm{Gr}}(k,n)$ but not on $\mathrm{Gr}(k,n)$.

Let $\mathrm{Mat}^\circ(k,n)$ denote the space of $k\times n$ matrices of rank k. $\mathrm{GL_k}$ acts on $\mathrm{Mat}^\circ(k,n)$ by left multiplication and we have identifications

$$\mathrm{GL}_k\setminus\mathrm{Mat}^\circ(k,n)\cong\mathrm{Gr}(k,n)\text{ and }\mathrm{SL}_k\setminus\mathrm{Mat}^\circ(k,n)\cong\widetilde{\mathrm{Gr}}(k,n)\tag{2.1}$$

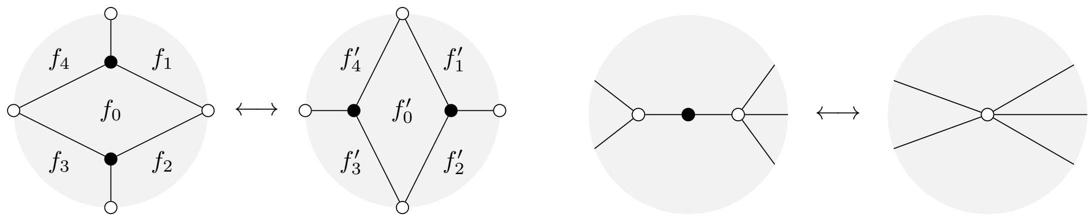
sending the matrix with rows $v_1,\dots,v_k$ to $\mathrm{span}(v_1,\dots,v_k)$ and $(\mathrm{span}(v_1,\dots,v_k),v_1\wedge\cdots\wedge v_k)$ respectively.

Let $\widetilde{\mathrm{Gr}}_{>0}(k,n)$ denote the *positive decorated Grassmannian*, the subset of $\widetilde{\mathrm{Gr}}(k,n)$ where where all Plücker coordinates are positive real numbers, and let $\mathrm{Gr}_{>0}(k,n)$ denote the *positive Grassmannian*, the subset of $\mathrm{Gr}(k,n)$ where the ratio of any two Plücker coordinates is a positive real number.

2.2. **Planar bipartite graphs in the disk.** Let $\Gamma=(B\sqcup W,E,F)$ be a bipartite graph embedded in a disk $\mathbb{D}$ with n vertices on the boundary of $\mathbb{D}$ labeled $d_1,d_2,\dots,d_n$ in clockwise cyclic order. Let $k:=\#W-\#B$. A *strand* (or *zig-zag path*) is a path in Γ that turns maximally left at the white vertices and maximally right at the black vertices, and either starts and ends at the boundary or is an internal cycle. We say that Γ is *reduced* (or *minimal*) if:

(1) Each strand starts and ends on the boundary, i.e., no strand path is an internal cycle.
(2) Each strand path of length greater than two has no self-intersections.
(3) Strands do not form "parallel bigons", i.e., there is no pair of strands that intersect twice in the same direction.

Let $d_{\pi_\Gamma(i)}$ denote the endpoint of the strand that starts at d_i ends. Then $\pi_\Gamma:[n]\to[n]$ is a permutation called the *strand permutation* of Γ. Let $\pi_{k,n}:[n]\to[n]$ be the permutation $(k+1,k+2,\dots,n,1,2,\dots,k-1)$.



(a) The spider move. (b) The contraction-uncontraction move.

Figure 1. Moves for bipartite graphs.

Remark 2.1. If $\pi_\Gamma(i) = i$, then we also have to specify a color for i, but this does not occur in $\pi_{k,n}$.

We say that two bipartite graphs Γ and Γ' are *move-equivalent* if they are related by the moves shown in Figure 1. Each move $\Gamma \rightsquigarrow \Gamma'$ induces a canonical bijection between $F(\Gamma)$ and $F(\Gamma')$; we denote the face of Γ' corresponding to the face f of Γ by f'. Postnikov [Pos06] and Thurston [Thu17] showed that two reduced bipartite graphs are move-equivalent if and only if they have the same strand permutation.

2.3. **Dimer models and boundary measurement.** Let $\mathrm{wt} : E(\Gamma) \to \mathbb{R}_{>0}$ be a function called an *edge weight*. Two edge weights wt_1 and wt_2 are said to be *gauge equivalent* if there is a function $g : B(\Gamma) \sqcup W(\Gamma) \to \mathbb{R}_{>0}$ that is equal to 1 on the boundary vertices such that for every edge $e = \mathrm{bw}$ with $\mathrm{b} \in B(\Gamma), \mathrm{w} \in W(\Gamma)$, we have $\mathrm{wt}_2(e) = g(\mathrm{b})^{-1}\mathrm{wt}_1(e)g(\mathrm{w})$. Let $\mathcal{X}_\Gamma := \mathbb{R}_{>0}^{E(\Gamma)}/\text{gauge}$ denote the space of edge weights on Γ modulo gauge equivalence. We denote the gauge equivalence class of wt by [wt]. A pair $(\Gamma, [\mathrm{wt}])$ with $[\mathrm{wt}] \in \mathcal{X}_\Gamma$ is called a *dimer model*.

For a face f of Γ with boundary $\mathrm{w}_1 \xrightarrow{e_1} \mathrm{b}_1 \xrightarrow{e_2} \mathrm{w}_2 \xrightarrow{e_3} \mathrm{b}_2 \xrightarrow{e_4} \cdots \xrightarrow{e_{2k-2}} \mathrm{w}_k \xrightarrow{e_{2k-1}} \mathrm{b}_k \xrightarrow{e_{2k}} \mathrm{w}_1$, let

$$X_f := \prod_{i=1}^{k} \frac{\mathrm{wt}(e_{2i})}{\mathrm{wt}(e_{2i-1})}$$

denote the alternating product of the edge weights around the boundary of f. The X_f's are invariant under gauge equivalence and provide coordinates on $\mathcal{X}_\Gamma$ satisfying the relation $\prod_{f\in F(\Gamma)} X_f = 1$, so $\mathcal{X}_\Gamma \cong \mathbb{R}_{>0}^{\#F(\Gamma)-1}$.

A move $\Gamma \rightsquigarrow \Gamma'$ induces a homeomorphism $\mathcal{X}_\Gamma \xrightarrow{\sim} \mathcal{X}_{\Gamma'}$ as follows:

(1) Spider move at a face f_0: The homeomorphism $\mathcal{X}_\Gamma \xrightarrow{\sim} \mathcal{X}_{\Gamma'}$ is given by

$$X_{f'_0} := \frac{1}{X_{f_0}}, X_{f'_1} := X_{f_1}(1 + X_{f_0}), X_{f'_2} := \frac{X_{f_2}}{(1 + \frac{1}{X_{f_0}})}, X_{f'_3} := X_{f'_3}(1 + X_{f_0}), X_{f_4} := \frac{X_{f'_4}}{(1 + \frac{1}{X_{f_0}})},$$

and $X_{f'} := X_f$ for $f' \in F(\Gamma') \setminus \{f'_0, f'_1, f'_2, f'_3, f'_4\}$.

(2) Contraction-uncontraction move: The homeomorphism $\mathcal{X}_\Gamma \xrightarrow{\sim} \mathcal{X}_{\Gamma'}$ is $X_{f'} := X_f$ for all $f' \in F(\Gamma')$.

Given a strand permutation π, let $\mathcal{X}_\pi := \bigsqcup_{\pi_\Gamma = \pi} \mathcal{X}_\Gamma \big/ \text{moves}$ denote the *space of dimer models*, where the union is over all reduced bipartite graphs Γ with strand permutation π.

A *dimer cover* (or *almost perfect matching*) of Γ is a subset of $E(\Gamma)$ that uses each internal vertex of Γ and a subset of the boundary vertices exactly once. The *weight* $\mathrm{wt}(M)$ of a dimer

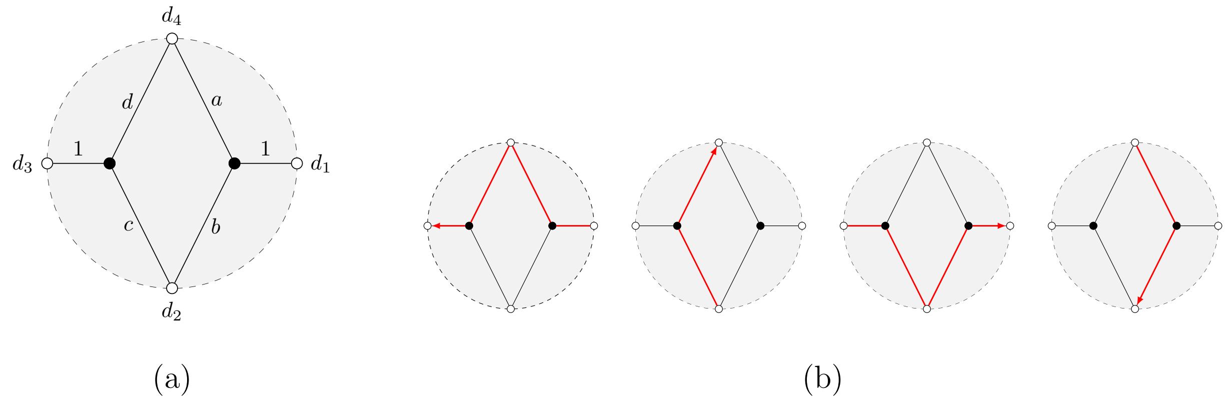


FIGURE 2. A bipartite graph (a) and its strands (b).

cover M is defined to be $\prod_{e \in M} \mathrm{wt}(e)$. For a dimer cover M, let

$$\partial M := \{i \mid d_i \text{ is black and used by } M\} \cup \{i \mid d_i \text{ is white and not used by } M\} \in \binom{[n]}{k}.$$

For $I \in \binom{[n]}{k}$, define the *dimer partition function*

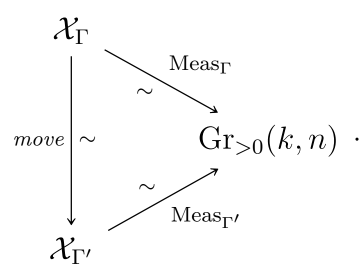
$$Z_I := \sum_{M \mid \partial M = I} \mathrm{wt}(M).$$

Postnikov [Pos06] defined the *boundary measurement map*

$$\mathrm{Meas}_\Gamma : \mathcal{X}_\Gamma \to \mathbb{P}(\bigwedge^k \mathbb{C}^n)$$

sending [wt] to $[\sum_{I \in \binom{[n]}{k}} Z_I e_I]$. Meas_Γ is well-defined, since the gauge equivalence multiplies all Z_I's by a scalar. The following theorem is due to Postnikov [Pos06] in a different language (see also [PSW09] and [Lam16, Corollary 7.14]).

Theorem 2.2. *For a reduced* Γ *with* $\pi_\Gamma = \pi_{k,n}$, $\mathrm{Meas}_\Gamma : \mathcal{X}_\Gamma \xrightarrow{\sim} \mathrm{Gr}_{>0}(k,n)$ *is a homeomorphism. If* Γ *and* Γ' *are related by a move, then the following diagram commutes:*



Therefore, the maps Meas_Γ *glue to a homeomorphism* $\mathrm{Meas} : \mathcal{X}_{\pi_{k,n}} \xrightarrow{\sim} \mathrm{Gr}_{>0}(k,n)$.

Example 2.3. Let (Γ, wt) be the weighted bipartite graph shown in Figure 2(a). The four strands are shown in Figure 2(b), from which we obtain the strand matching to be $\pi_{2,4}$. The

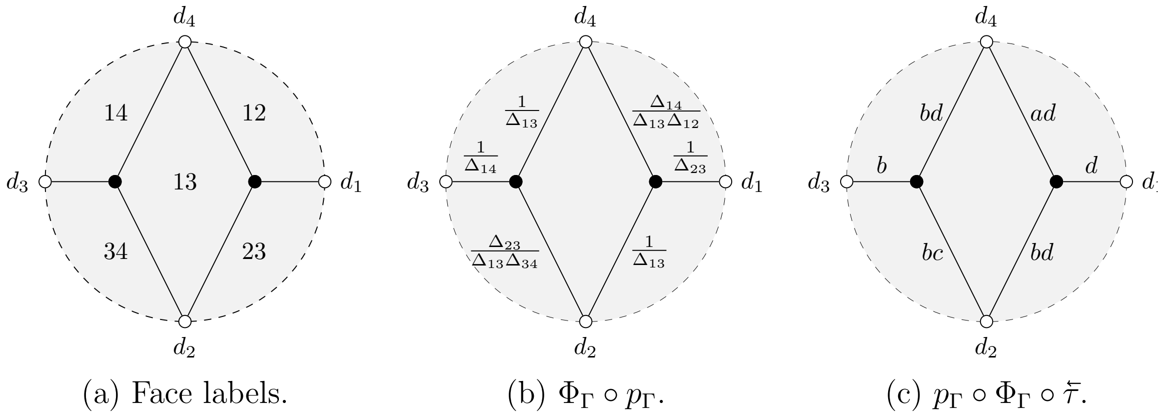


FIGURE 3. Inverting the boundary measurement map for the graph in Figure 2.

boundary measurement map sends [wt] to $[ae_{12} + (ac + bd)e_{13} + be_{14} + de_{23} + e_{24} + ce_{34}]$, which is the image under Pl of

$$X := \text{row span} \begin{bmatrix} b & 1 & c & 0 \\ -a & 0 & d & 1 \end{bmatrix}. \tag{2.2}$$

2.4. A **variables.** Let $\mathcal{A}_\Gamma := \mathbb{R}_{>0}^{F(\Gamma)}$ denote the space of functions $A : F(\Gamma) \to \mathbb{R}_{>0}$. A move $\Gamma \rightsquigarrow \Gamma'$ induces a homeomorphism $\mathcal{A}_\Gamma \xrightarrow{\sim} \mathcal{A}_{\Gamma'}$ as follows:

(1) Spider move at a face f_0: The homeomorphism $\mathcal{A}_\Gamma \xrightarrow{\sim} \mathcal{A}_{\Gamma'}$ is given by the cluster mutation formula
$$A_{f_0'} := \frac{A_{f_1}A_{f_3} + A_{f_2}A_{f_4}}{A_{f_0}}$$
and $A_{f'} := A_f$ for $f' \in F(\Gamma') \setminus \{f_0'\}$.
(2) Contraction-uncontraction move: The homeomorphism $\mathcal{A}_\Gamma \xrightarrow{\sim} \mathcal{A}_{\Gamma'}$ is $A_{f'} := A_f$ for all $f' \in F(\Gamma')$.

Let $\mathcal{A}_\pi := \bigsqcup_{\pi_\Gamma = \pi} \mathcal{A}_\Gamma \Big/ \text{moves}$.

Remark 2.4. The spaces $\mathcal{X}_\Gamma$ and $\mathcal{A}_\Gamma$ are the positive points of the $\mathcal{X}$ and $\mathcal{A}$ associated with Γ, respectively (see [FG09]), and $\mathcal{X}_\pi$ and $\mathcal{A}_\pi$ are the positive points of the $\mathcal{X}$ and $\mathcal{A}$ cluster varieties respectively. Since the cluster varieties do not appear directly in this paper, we have chosen to denote the positive points by $\mathcal{X}_\Gamma$ instead of $\mathcal{X}_\Gamma(\mathbb{R}_{>0})$, etc.

Definition 2.5. For each face f of Γ, define the *(target) face label*
$$S(f) := \{i \in [n] \mid f \text{ is on the left of the strand ending at } d_i\}.$$

For each face f, $S(f)$ is a k-element subset of $[n]$. Let $f_1^-, \ldots, f_n^-$ denote the boundary faces of Γ so that f_i^- is between d_{i-1} and d_i. If $\pi_\Gamma = \pi_{k,n}$, then $S(f_i^-) = \{i, i+1, \ldots, i+k-1\}$ are the cyclically consecutive subsets.

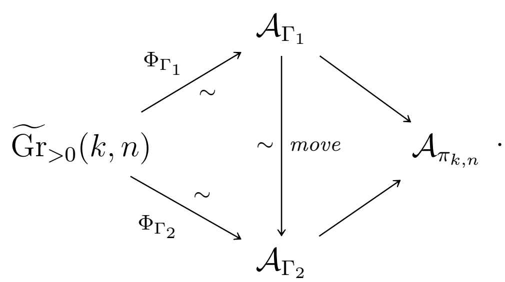
Example 2.6. For the graph in Figure 2(a), using the strands shown in Figure 2(b), we compute the face labels as shown in Figure 3.

Define the map

$$\Phi_\Gamma : \widetilde{\mathrm{Gr}}_{>0}(k,n) \to \mathcal{A}_\Gamma$$

sending a (X,v) to $(\Delta_{S(f)}(X,v))_{f\in F(\Gamma)}$.

Theorem 2.7 (Scott, [Sco06, Theorem 4])**.** *For every reduced* Γ *with* $\pi_\Gamma = \pi_{k,n}$*,* $\Phi_\Gamma : \widetilde{\mathrm{Gr}}_{>0}(k,n) \xrightarrow{\sim} \mathcal{A}_\Gamma$ *is a homeomorphism. If* Γ_1 *and* Γ_2 *are related by a move, then*



commutes, so we obtain a well-defined homeomorphism $\Phi : \widetilde{\mathrm{Gr}}_{>0}(k,n) \xrightarrow{\sim} \mathcal{A}_{\pi_{k,n}}$.

2.5. **Twist.** We introduce the twist map defined by Marsh and Scott [MS16] and generalized by Muller and Speyer [MS17]. We follow the normalization conventions of [MS17]. Let M be a $k\times n$ matrix whose $k\times k$ minors are all nonzero. For any $i \in [n]$, let M_i denote the ith column of M. We extend this definition to all $i \in \mathbb{Z}$ by defining $M_i := M_{\bar{i}}$ where $\bar{i} \in [n]$ is the reduction of $i \in \mathbb{Z}$ modulo n. Let $\langle \cdot,\cdot\rangle$ denote the standard inner product on $\mathbb{R}^k$.

Definition 2.8. The *right twist* of M is the $k \times n$ matrix $\vec{\tau}(M)$ whose column $\vec{\tau}(M)_i$ is defined by

$$\langle \vec{\tau}(M)_i, M_i\rangle = 1 \text{ and } \langle \vec{\tau}(M)_i, M_j\rangle = 0 \text{ for } i < j \le i+k-1.$$

Similarly, the *left twist* of X is the $k\times n$ matrix $\overleftarrow{\tau}(M)$ whose column $\overleftarrow{\tau}(M)_i$ is defined by

$$\langle \overleftarrow{\tau}(M)_i, M_i\rangle = 1 \text{ and } \langle \overleftarrow{\tau}(M)_i, M_j\rangle = 0 \text{ for } i-k+1 \le j < i.$$

Theorem 2.9 (Muller and Speyer, [MS17, Corollary 6.8])**.** *Under the identifications (2.1), the right and left twists descend to mutually inverse homeomorphisms of* $\widetilde{\mathrm{Gr}}_{>0}(k,n)$ *and* $\mathrm{Gr}_{>0}(k,n)$*.*

Definition 2.10. We denote the right twist of $(X,v) \in \widetilde{\mathrm{Gr}}_{>0}(k,n)$ (resp., $X \in \mathrm{Gr}_{>0}(k,n)$) by $\vec{\tau}(X,v)$ (resp., $\vec{\tau}(X)$), and similarly for the left twist.

Example 2.11. The left twist of X in (2.2) is $\overleftarrow{\tau}(X) = \text{row span}\begin{bmatrix} \frac{1}{b} & 1 & 0 & -\frac{d}{c} \\ 0 & \frac{b}{a} & \frac{1}{d} & 1\end{bmatrix}$.

Definition 2.12. Let Γ be a reduced bipartite graph with $\pi_\Gamma = \pi_{k,n}$. Define the map $p_\Gamma : \mathcal{A}_\Gamma \to \mathcal{X}_\Gamma$ sending A to $[\mathrm{wt}]$ as follows. Let $e \in E(\Gamma)$ be an edge and let $f,g \in F(\Gamma)$ be the two faces incident to e. Define

$$\mathrm{wt}(e) := \begin{cases} \frac{1}{A_f A_g} & \text{if } e \text{ is not incident to a boundary white vertex,} \\ \frac{A_{f_i^-}}{A_f A_g} & \text{if } e \text{ is incident to boundary white vertex } d_i. \end{cases}$$

where f_i^- is the boundary face of Γ between d_{i-1} and d_i.

Theorem 2.13. [MS17, Theorem 7.1 and Remark 7.2] *Let Γ be reduced bipartite graph with $\pi_\Gamma = \pi_{k,n}$. The following diagrams commute.*

$$\begin{array}{ccc} \mathcal{A}_\Gamma & \xrightarrow{p_\Gamma} & \mathcal{X}_\Gamma \\ {\scriptstyle \Phi_\Gamma}\uparrow\wr & & \wr\downarrow{\scriptstyle \mathrm{Meas}_\Gamma} \\ \widetilde{\mathrm{Gr}}_{>0}(k,n) & \xrightarrow[\vec{\tau}]{} & \mathrm{Gr}_{>0}(k,n) \end{array}, \qquad \begin{array}{ccc} \mathcal{A}_\Gamma/\mathbb{R}_{>0} & \xrightarrow[\sim]{p_\Gamma} & \mathcal{X}_\Gamma \\ {\scriptstyle \Phi_\Gamma}\uparrow\wr & & \wr\downarrow{\scriptstyle \mathrm{Meas}_\Gamma} \\ \mathrm{Gr}_{>0}(k,n) & \underset{\vec{\tau}}{\overset{\overleftarrow{\tau}}{\rightleftarrows}} & \mathrm{Gr}_{>0}(k,n) \end{array}. \tag{2.3}$$

In the diagram on the left, the quotient is by the action of $\mathbb{R}_{>0}$ on $\mathcal{A}_\Gamma$ multiplying all the A variables by a scalar.

Remark 2.14. The map p_Γ is an incarnation of the canonical map between $\mathcal{A}$ and $\mathcal{X}$ cluster varieties in Fock and Goncharov [FG09].

Example 2.15. Recall Examples 2.3 and 2.11. The Plücker coordinates of $\overleftarrow{\tau}(X)$ are

$$\Delta_{12} = \frac{1}{a}, \Delta_{13} = \frac{1}{bd}, \Delta_{14} = \frac{1}{b}, \Delta_{23} = \frac{1}{d}, \Delta_{24} = 1 + \frac{bd}{ac}, \Delta_{34} = \frac{1}{c}.$$

The compositions $p_\Gamma \circ \Phi_\Gamma$ and $p_\Gamma \circ \Phi_\Gamma \circ \overleftarrow{\tau}$ are shown in Figure 3(b) and Figure 3(c) respectively. The weights in Figure 2(a) and Figure 3(c) are easily seen to be gauge equivalent.

Definition 2.16. For $t = (t_1, \ldots, t_n) \in \mathbb{R}^n_{>0}$ and $X \in \mathrm{Gr}_{>0}(n+1, 2n)$, let $t \cdot X \in \mathrm{Gr}_{>0}(n+1, 2n)$ denote the point obtained as follows. Let M be a $k \times n$ matrix such that X is the row span of M. Then, $t \cdot X$ is the row span of the matrix $t \cdot M$ defined by $(t \cdot M)_i := t_i M_i$.

Let Γ be a reduced bipartite graph with $\pi_\Gamma = \pi_{k,n}$ and let $[\mathrm{wt}] \in \mathcal{X}_\Gamma$. Let $\mathbb{R}^n_{>0}$ act on $\mathcal{X}_\Gamma$ by multiplying the weights of all edges incident to d_i by $\frac{1}{t_i}$ if d_i is white and t_i if d_i is black. The following lemma is used in the proof of Theorem 4.2.

Lemma 2.17. *The map $\mathrm{Meas}_\Gamma : \mathcal{X}_\Gamma \to \mathrm{Gr}_{>0}(k,n)$ is $\mathbb{R}^n_{>0}$ equivariant.*

Proof. We have $\Delta_I(t \cdot X) = (\prod_{i \in I} t_i)\Delta_I(X)$. On the other hand,

$$\begin{aligned} \mathrm{Meas}_\Gamma(t \cdot [\mathrm{wt}]) &= \left[\sum_{I \in \binom{[n]}{k}} \left(\prod_{i \in I | d_i \in B(\Gamma)} t_i \right) \left(\prod_{i \notin I | d_i \in W(\Gamma)} t_i \right) Z_I e_I \right] \\ &= \left[\sum_{I \in \binom{[n]}{k}} \left(\prod_{i \in I} t_i \right) Z_I e_I \right], \end{aligned}$$

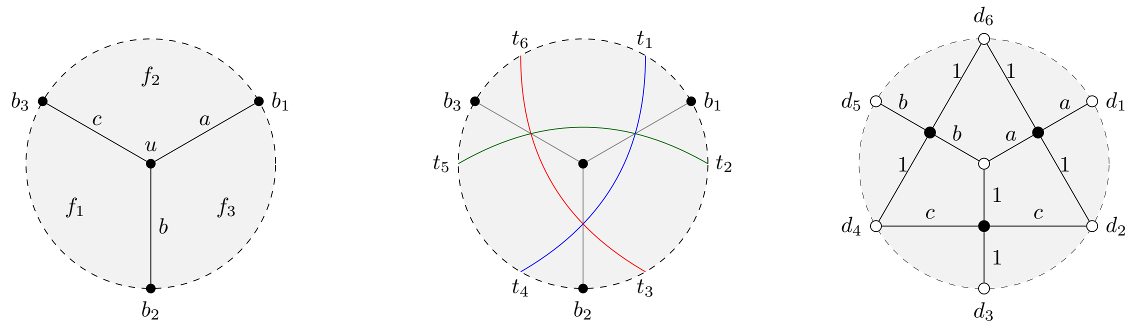
where in the second equality we rescaled by $\prod_{i \in [n] | d_i \in W(\Gamma)} t_i$. □

The following two properties of the twist will be required later.

Proposition 2.18 (Muller and Speyer, [MS17, (9) in the proof of Proposition 6.6 and Proposition 6.1])**.** *Let $X \in \mathrm{Gr}_{>0}(k,n)$.*

(1) For any boundary face f_i^-, we have $\Delta_{S(f_i^-)}(\vec{\tau}(X)) = \frac{1}{\Delta_{S(f_i^-)}(X)}$.

(2) If $t = (t_1, \ldots, t_n) \in \mathbb{R}^n_{>0}$, then $\vec{\tau}(t \cdot X) = t^{-1} \cdot \vec{\tau}(X)$, where $t^{-1} := (\frac{1}{t_1}, \ldots, \frac{1}{t_n})$.



(a) An electrical network (G, c). (b) $G^\times$. (c) $(G_+, [\mathrm{wt}_+])$.

FIGURE 4. An electrical network with $n = 3$ and its associated graphs. The three medial strands in $G^\times$ are given different colors.

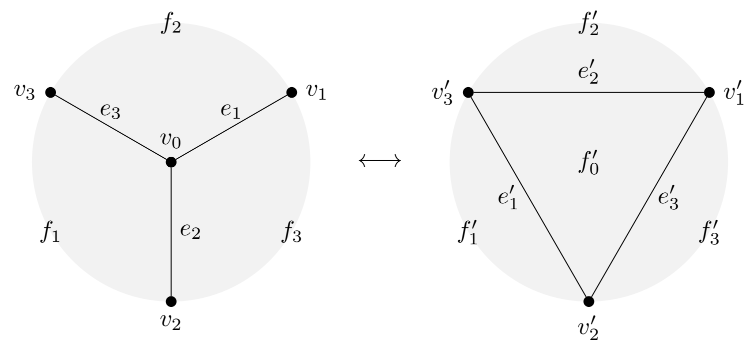


FIGURE 5. The Y-Δ move.

3. Electrical networks

3.1. **Reduced graphs in the disk.** Let $G = (V, E, F)$ be a planar graph embedded in the disk $\mathbb{D}$ with n vertices on the boundary labeled $b_1, b_2, \dots, b_n$. The *medial graph* $G^\times$ of G is the graph obtained as follows. Place $2n$ vertices of $G^\times$ labeled $t_1, t_2, \dots, t_{2n}$ on the boundary of $\mathbb{D}$ such that b_i is between t_{2i-1} and t_{2i} and a vertex v_e in the middle of each edge e of G. Connect v_e and $v_{e'}$ by an edge if they occur consecutively around a face of G. For each $i \in [n]$, connect t_{2i-1} (resp., t_{2i}) to v_e if e is the last (resp., first) edge in clockwise order incident to b_i. By construction, each t_i has degree 1 and each v_e degree 4 in $G^\times$. A *medial strand* in $G^\times$ is a maximal sequence of edges that goes straight through every v_e.

Example 3.1. Figure 4(b) shows the medial graph of the electrical network in Figure 4(a).

The graph G is called *reduced* if:

(1) Every medial strand starts and ends at a boundary vertex, i.e., no medial strand is an internal cycle.
(2) Medial strands have no self-intersections.
(3) There is no pair of medial strands that intersect twice.

The *medial pairing* of G is the matching on $[2n]$ defined by

$$\tau_G := \{\{i, j\} \mid \text{there is a medial strand between } t_i \text{ and } t_j\}.$$

Example 3.2. For the electrical network in Figure 4(a), the medial graph is shown in Figure 4(b), from which we see that G is reduced with medial pairing τ_3.

We say that G and G' are *move-equivalent* if they are related by a sequence of Y-Δ moves (Figure 5). A Y-Δ move $G \rightsquigarrow G'$ induces canonical bijections $V(G) \sqcup F(G) \xrightarrow{\sim} V(G') \sqcup F(G')$ and $E(G) \xrightarrow{\sim} E(G')$. Two graphs G and G' are move-equivalent if and only if they have the same medial pairing [CdV94]. In this paper, we only consider reduced graphs G with medial pairing $\tau_n := \{\{1, n+1\}, \{2, n+2\}, \dots, \{n, 2n\}\}$; such graphs are called *well connected*.

3.2. **The space of electrical networks and the positive Lagrangian Grassmannian.** Let $c : E(G) \to \mathbb{R}_{>0}$ be a function called *conductance*, and let $\mathcal{R}_G := \mathbb{R}_{>0}^{E(G)}$ be the space of conductances on G. A pair (G, c) with $c \in \mathcal{R}_G$ is called an *electrical network*.

A Y-Δ move $G \rightsquigarrow G'$ induces a homeomorphism $\mathcal{R}_G \xrightarrow{\sim} \mathcal{R}_{G'}$ given by

$$c(e_1') := \frac{c(e_2)c(e_3)}{C}, c(e_2') := \frac{c(e_1)c(e_3)}{C}, c(e_3') := \frac{c(e_1)c(e_2)}{C},$$

where $C := c(e_1)c(e_2) + c(e_1)c(e_3) + c(e_2)c(e_3)$ and the edges are labeled as in Figure 5. Let $\mathcal{R}_n := \bigsqcup_{\tau_G = \tau_n} \mathcal{R}_G \big/ \text{moves}$ denote the *space of electrical networks*.

We associate a dimer model $(G_+, [\mathrm{wt}_+])$ to (G, c) as follows. Place a black vertex b_e in the middle of every edge e of G, a white vertex w_v at every vertex v of G and a white vertex w_f in the middle of every face of G. If v is a vertex of G incident to edge e, draw an edge $\mathrm{b}_e\mathrm{w}_v$ and assign $\mathrm{wt}_+(\mathrm{b}_e\mathrm{w}_v) := c(e)$. If f is a face of G incident to e, draw an edge $\mathrm{b}_e\mathrm{w}_f$ and assign $\mathrm{wt}_+(\mathrm{b}_e\mathrm{w}_f) := 1$.

Example 3.3. The weighted bipartite graph associated to the electrical network in Figure 4(a) is shown in Figure 4(c).

Remark 3.4. The notation G_+ is inspired by the notation $G_\square$ for the Ising graph in [GP20], since we are replacing each edge of G with a $+$.

The map $(G, c) \mapsto (G_+, [\mathrm{wt}_+])$ defines an inclusion $j_G^+ : \mathcal{R}_G \hookrightarrow \mathcal{X}_{G_+}$.

Proposition 3.5 (Goncharov and Kenyon, [GK13, Lemma 5.11])**.** *If G and G' are related by a Y-Δ move, then there is a sequence of moves relating G_+ and G'_+ making the following diagram commute.*

$$\begin{array}{ccc} \mathcal{R}_G & \overset{j_G^+}{\hookrightarrow} & \mathcal{X}_{G_+} \\ {\scriptstyle Y\text{-}\Delta\ move}\downarrow\wr & & \wr\downarrow{\scriptstyle moves} \\ \mathcal{R}_{G'} & \overset{j_{G'}^+}{\hookrightarrow} & \mathcal{X}_{G'_+} \end{array}.$$

Therefore, the inclusions j_G^+ glue to an inclusion $j^+ : \mathcal{R}_n \to \mathcal{X}_{\pi_{n+1,2n}}$.

Let $\Omega : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}$ be the degenerate skew symmetric bilinear form

$$\Omega(x, y) = \sum_{i=1}^{n} (x_{2i-1}y_{2i} - x_{2i}y_{2i-1}) + \sum_{i=1}^{n-1} (x_{2i+1}y_{2i} - x_{2i}y_{2i+1}) + (-1)^n(x_1y_{2n} - x_{2n}y_1). \tag{3.1}$$

We say that $X \in \mathrm{Gr}(n+1, 2n)$ is *isotropic* for Ω if $\Omega(x, y) = 0$ for any $x, y \in X$. Let $\mathrm{IG}^\Omega(n+1, 2n)$ be the *Lagrangian Grassmannian* of isotropic subspaces inside $\mathrm{Gr}(n+1, 2n)$

and $\mathrm{IG}^{\Omega}_{>0}(n+1,2n) := \mathrm{IG}^{\Omega}(n+1,2n) \cap \mathrm{Gr}_{>0}(n+1,2n)$ the *positive Lagrangian Grassmannian*. The following result was independently proved by Bychkov, Gorbounov, Kazakov and Talalaev [BGKT21] and Chepuri, George and Speyer [CGS21], following earlier results of Lam [Lam18].

Theorem 3.6. *The composition* $\mathrm{Meas}_{G_+} \circ j^+_G : \mathcal{R}_G \xrightarrow{\sim} \mathrm{IG}^{\Omega}_{>0}(n+1,2n)$ *is a homeomorphism.*

Therefore, we have a commuting diagram

$$\begin{array}{ccc} \mathcal{R}_G & \xhookrightarrow{\quad j^+_G \quad} & \mathcal{X}_{G_+} \\ {\scriptstyle \mathrm{Meas}_{G_+} \circ j^+_G}\big\downarrow{\scriptstyle \sim} & & {\scriptstyle \sim}\big\downarrow{\scriptstyle \mathrm{Meas}_{G_+}} \\ \mathrm{IG}^{\Omega}_{>0}(n+1,2n) & \hookrightarrow & \mathrm{Gr}_{>0}(n+1,2n) \end{array}.$$

3.3. **A bit of representation theory of the spin group.** In this section, we give a brief background on the spin group, mostly following [FH91, Chapter 20] and [HS10, Section 5], and prove Proposition 3.8 relating Cartan and Plücker coordinates. Consider the nondegenerate symmetric bilinear form $Q : \mathbb{C}^{2n} \times \mathbb{C}^{2n} \to \mathbb{C}$ defined by

$$Q(x,y) := \frac{1}{2}\sum_{i=1}^{n}(-1)^{i-1}(x_i y_{n+i} + x_{n+i} y_i).$$

We first make a change of basis so that Q becomes the standard nondegenerate symmetric bilinear form. Let W denote the Lagrangian subspace $\mathrm{span}(e_1, e_2, \dots, e_n)$. Then, Q defines an isomorphism

$$\begin{aligned} W^{\perp} &\to W^{\vee} \\ e_{n+i} &\mapsto (-1)^{i-1} e_i^{\vee}, \end{aligned} \tag{3.2}$$

where $W^{\vee}$ denotes the dual vector space of W and $e_i^{\vee}$ is basis vector dual to e_i, i.e., $e_i^{\vee}(e_j) = \delta_{ij}$. Therefore, we have an isomorphism $\mathbb{C}^{2n} \cong W \oplus W^{\vee}$ such that the inner product Q becomes

$$Q((x,x^{\vee}),(y,y^{\vee})) = \frac{1}{2}(x^{\vee}(y) + y^{\vee}(x)) \text{ where } (x,x^{\vee}),(y,y^{\vee}) \in W \oplus W^{\vee}. \tag{3.3}$$

Note that our form Q differs from the standard form in [FH91] by a factor of $\frac{1}{2}$. Let $\mathrm{Cl}(Q) := \bigoplus_{k=0}^{\infty}(\mathbb{C}^{2n})^{\otimes k}/\langle x \otimes x - Q(x,x)\rangle$ denote the *Clifford algebra*. Since the ideal $\langle x \otimes x - Q(x,x)\rangle$ is generated by elements of even degree, the Clifford algebra has a $\mathbb{Z}/2\mathbb{Z}$ grading: $\mathrm{Cl}(Q) = \mathrm{Cl}(Q)^{\mathrm{even}} \oplus \mathrm{Cl}(Q)^{\mathrm{odd}}$.

The *Clifford group*

$$\mathrm{Cl}^*(Q) := \{x \in \mathrm{Cl}(Q) \mid \text{there exists } y \in \mathrm{Cl}(Q) \text{ such that } x \otimes y = y \otimes x = 1\}$$

is the multiplicative group of units inside $\mathrm{Cl}(Q)$. Its Lie algebra $\mathfrak{cl}^*(Q)$ is $\mathrm{Cl}(Q)$ with the Lie bracket $[x,y] := x \otimes y - y \otimes x$, and we have the exponential map $\exp : \mathfrak{cl}^*(Q) \to \mathrm{Cl}^*(Q)$ defined by

$$\exp(x) := \sum_{n \geq 0} \frac{x^{\otimes n}}{n!}. \tag{3.4}$$

The Clifford algebra has an anti-involution $u \mapsto u^*$ called *conjugation* defined by $(x_1 \otimes \cdots \otimes x_r)^* := (-1)^r x_r \otimes \cdots \otimes x_1$. The involution $\alpha : \mathrm{Cl}(Q) \to \mathrm{Cl}(Q)$ defined by $\alpha(x_1 \otimes \cdots \otimes x_r) := (-1)^r(x_1 \otimes \cdots \otimes x_r)$ is called the *main involution*. The *pin* and *spin groups* are defined as

$$\mathrm{Pin}(Q) := \{x \in \mathrm{Cl}^*(Q) : x \otimes x^* = 1 \text{ and } \alpha(x) \otimes \mathbb{C}^{2n} \otimes x^* \subseteq \mathbb{C}^{2n}\},$$
$$\mathrm{Spin}(Q) := \{x \in \mathrm{Cl}^*(Q) \cap \mathrm{Cl}(Q)^{\mathrm{even}} : x \otimes x^* = 1 \text{ and } \alpha(x) \otimes \mathbb{C}^{2n} \otimes x^* \subseteq \mathbb{C}^{2n}\}.$$

The map $\rho : \mathrm{Pin}(Q) \to \mathrm{O}(Q)$ (resp., $\rho : \mathrm{Spin}(Q) \to \mathrm{SO}(Q)$) defined by $x \mapsto \rho(x)$ where $\rho(x) : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ is the endomorphism $v \mapsto \alpha(x) \otimes v \otimes x^*$ makes $\mathrm{Pin}(Q)$ (resp., $\mathrm{Spin}(Q)$) a double cover of $\mathrm{O}(Q)$ (resp., $\mathrm{SO}(Q)$).

The Lie algebra of $\mathrm{SO}(Q)$ is

$$\mathfrak{so}(Q) := \{X \in \mathrm{End}(\mathbb{C}^{2n}) \mid Q(X(v), w) + Q(v, X(w)) = 0 \text{ for all } v, w \in \mathbb{C}^{2n}\}.$$

The map $\varphi : \bigwedge^2 \mathbb{C}^{2n} \to \mathfrak{so}(Q)$ sending $a \wedge b$ to $\varphi_{a\wedge b}$ given by

$$\varphi_{a\wedge b}(v) := 2(Q(b, v)a - Q(a, v)b) \tag{3.5}$$

is an isomorphism of Lie algebras. On the other hand, the map $\psi : \bigwedge^2 \mathbb{C}^{2n} \to \mathfrak{cl}^*(Q)$ sending $a \wedge b$ to $a \otimes b - Q(a, b)$ is a map of Lie algebras.

Lemma 3.7. [FH91, Lemma 20.7 and Exercise 20.33] *The composition $\psi \circ \varphi^{-1} : \mathfrak{so}(Q) \to \mathrm{Cl}(Q)^{\mathrm{even}}$ is an embedding of Lie algebras. The embedded image is the Lie algebra $\mathfrak{spin}(Q)$ of* $\mathrm{Spin}(Q)$.

Let $S := \bigwedge^\bullet W$. Define the $\mathrm{Cl}(Q)$ representation $\Gamma : \mathrm{Cl}(Q) \to \mathrm{End}(S)$ by

$$\Gamma_x(w_1 \wedge \cdots \wedge w_k) := x \wedge (w_1 \wedge \cdots \wedge w_k) \text{ for } u \in W,$$
$$\Gamma_{x^\vee}(w_1 \wedge \cdots \wedge w_k) := x^\vee \lrcorner (w_1 \wedge \cdots \wedge w_k) \text{ for } x^\vee \in W^\vee.$$

This is an isomorphism $\mathrm{Cl}(Q) \cong \mathrm{End}(S)$. Let $S_+ := \bigwedge^{\mathrm{even}} W$ and $S_- := \bigwedge^{\mathrm{odd}} W$. Restricting Γ, we obtain an isomorphism

$$\Gamma : \mathrm{Cl}(Q)^{\mathrm{even}} \xrightarrow{\cong} \mathrm{End}(S_+) \oplus \mathrm{End}(S_-).$$

The embedding $\mathrm{Spin}(Q) \subset \mathrm{Cl}(Q)^{\mathrm{even}}$ makes $S_\pm$ into $\mathrm{Spin}(Q)$ representations, called *half-spin representations.*

For $j \in [n]$, let $c_j(t) := \frac{1}{2}(te_j \otimes e_{n+j} + t^{-1}e_{n+j} \otimes e_j)$, and for $t = (t_1, \ldots, t_n) \in (\mathbb{C}^\times)^n$, let

$$c(t) := \prod_{j=1}^{n} c_j(t_j). \tag{3.6}$$

The image of c is the *maximal torus* inside $\mathrm{Spin}(Q)$ and under the covering $\rho : \mathrm{Spin}(Q) \to \mathrm{SO}(Q)$, we get $\rho(c(t)) = \mathrm{diag}(t_1^2, \ldots, t_n^2, t_1^{-2}, \ldots, t_n^{-2})$.

1 (resp., e_1) is a highest weight vector of S_+ (resp. S_-) with weight $(-1, -1, \ldots, -1)$ (resp., $(1, -1, \ldots, -1)$). For $I, I^\vee \subseteq [n]$ such that $\#I + \#I^\vee = n + 1$, let $e_{I,I^\vee}$ denote wedge product indexed by $I \sqcup I^\vee$: If $I = \{i_1 < i_2 < \cdots < i_k\}$ and $I^\vee = \{j_1 < j_2 < \cdots < j_{n+1-k}\}$, then $e_{I,I^\vee} := e_{i_1} \wedge \cdots \wedge e_{i_k} \wedge e^\vee_{j_1} \wedge \cdots \wedge e^\vee_{j_{n+1-k}}$. Since $\bigwedge^{n+1} \mathbb{C}^{2n}$ is an irreducible $\mathrm{Spin}(Q)$ representation with highest weight vector $e_{\{1\},[n]}$ with weight $(0, -2, \ldots, -2)$, $\bigwedge^{n+1} \mathbb{C}^{2n}$ is a direct summand of $S_+ \otimes S_-$. Let $p : \bigwedge^{n+1} \mathbb{C}^{2n} \hookrightarrow S_+ \otimes S_-$ denote the morphism of $\mathrm{Spin}(Q)$ representations, sending $e_{1,[n]}$ to $(-1)^{\sum_{j\in[n]}(j-1)} 1 \otimes e_1$. Let $\sigma(I)$ be 1 if $\#I \equiv 2$ modulo 4 and 1 otherwise.

Proposition 3.8. *Suppose $I, I^\vee \subseteq [n]$ are such that $\#I + \#I^\vee = n+1$ and $I \cap I^\vee = \{l\}$. Then,*

$$(-1)^{\sum_{j\in I^\vee}(j-1)} p(e_{I,I^\vee}) = \sigma(I)\sigma(I\setminus\{l\}) e_I \otimes e_{I\setminus\{l\}} \quad \textit{if } \#I \textit{ is even, and}$$
$$(-1)^{\sum_{j\in I^\vee}(j-1)} p(e_{I,I^\vee}) = \sigma(I)\sigma(I\setminus\{l\}) e_{I\setminus\{l\}} \otimes e_I \quad \textit{if } \#I \textit{ is odd.} \tag{3.7}$$

Proof. We will use the action of $\mathrm{Spin}(Q)$ to send $e_{1,[n]}$ to $e_{I,I^\vee}$ and use $\mathrm{Spin}(Q)$ equivariance of p. The main difficulty will be in keeping track of the signs.

We start by defining the required elements of $\mathrm{Spin}(Q)$. By the Cartan–Dieudonné theorem [LM89, Theorem 2.7], any element of $\mathrm{O}(Q)$ can be written as a product of reflections, so we look for appropriate reflections. If $w \in V$ with $Q(w,w) = -1$ and R_w is the reflection in the hyperplane orthogonal to w, then $w \in \mathrm{Pin}(Q)$ and $\rho(w) = R_w$. Let $u_{jk} := \frac{i}{\sqrt{2}}(e_j - e_k + e_j^\vee - e_k^\vee)$ and $v_{jk} := \frac{1}{\sqrt{2}}(e_j - e_k - e_j^\vee + e_k^\vee)$ so that $Q(u_{jk}, u_{jk}) = Q(v_{jk}, v_{jk}) = -1$. A computation shows that the composition $R_{v_{jk}} \circ R_{u_{jk}}$ is in $\mathrm{SO}(Q)$ and is the transformation $e_j \longleftrightarrow e_k, e_j^\vee \longleftrightarrow e_k^\vee$. Let $w_j := e_j - e_j^\vee$, so that $Q(w_j, w_j) = -1$. The composition $R_{w_j} \circ R_{w_k} \in \mathrm{SO}(Q)$ is the transformation $e_j \longleftrightarrow e_j^\vee, e_k \longleftrightarrow e_k^\vee$. The transformations $R_{v_{jk}} \circ R_{u_{jk}}$ and $R_{w_j} \circ R_{w_k}$ have lifts $v_{jk} \otimes u_{jk}$ and $w_j \otimes w_k$ to $\mathrm{Spin}(Q)$ respectively.

Now, we proceed by induction on $m := \#I$. When $m = 1$, we have $I = \{l\}$ and $I^\vee = [n]$ for some $l \in [n]$. Suppose $l \neq 1$. We have

$$\begin{aligned} v_{1l} \otimes u_{1l} \cdot e_{1,[n]} &= R_{v_{1l}} \circ R_{u_{1l}}(e_{1,[n]}) \\ &= e_l \wedge e_l^\vee \wedge e_2 \wedge \cdots \wedge e_{l-1}^\vee \wedge e_1^\vee \wedge e_{l+1}^\vee \wedge \cdots \wedge e_n^\vee \\ &= -e_{l,[n]}, \end{aligned}$$

where the -1 arises when we reorder the alternating tensor. On the other hand,

$$\begin{aligned} v_{1l} \otimes u_{1l} \cdot 1 &= v_{1l} \cdot \frac{i}{\sqrt{2}}(e_1 - e_l) \\ &= \frac{i}{2}(e_1 \wedge - e_l \wedge - e_1^\vee \lrcorner + e_l^\vee \lrcorner)(e_1 - e_l) \\ &= -i1, \end{aligned}$$

and

$$\begin{aligned} v_{1l} \otimes u_{1l} \cdot e_1 &= v_{1l} \cdot \frac{i}{\sqrt{2}}(1 - e_l \wedge e_1) \\ &= -ie_l. \end{aligned}$$

Therefore, $v_{1l} \otimes u_{1l} \cdot (1 \otimes e_1) = -1 \otimes e_l$. By $\mathrm{Spin}(Q)$ equivariance of p, for all $l \in [n]$, we have

$$\begin{aligned} p(e_{l,[n]}) &= p(-v_{1l} \otimes u_{1l} \cdot e_{1,[n]}) \\ &= -v_{1l} \otimes u_{1l} \cdot p(e_{1,[n]}) \\ &= -v_{1l} \otimes u_{1l} \cdot (-1)^{\sum_{j\in[n]}(j-1)} v_{1l} \otimes u_{1l} \cdot 1 \otimes e_1 \\ &= (-1)^{\sum_{j\in[n]}(j-1)} 1 \otimes e_l. \end{aligned}$$

Since $\sigma(I \setminus \{l\}) = \sigma(\varnothing) = 1$ and $\sigma(I) = \sigma(\{l\}) = 1$, we get (3.7).

Now suppose $m = \#I > 1$. Let k be the largest element of $I \setminus \{l\}$ and let $I_0 := I \setminus \{k\}$. We compute

$$\begin{aligned} R_{w_k} \otimes R_{w_l}(e_{I_0,I_0^\vee}) &= (-1)^{(k-1)+m} e_{I,I^\vee}, \\ w_k \otimes w_l \cdot e_{I_0\setminus\{l\}} &= e_k \wedge e_l \wedge e_{I_0\setminus\{l\}} \\ &= (-1)^{m-1+\#\{j\in I_0 | j<l\}} e_I, \text{ and} \\ w_k \otimes w_l \cdot e_{I_0} &= e_k \wedge e_l \lrcorner e_{I_0} \\ &= (-1)^{m-2+\#\{j\in I_0 | j<l\}} e_{I\setminus\{l\}}. \end{aligned}$$

Assume $\#I$ is even so $\#I_0$ is odd. By the induction hypothesis,

$$(-1)^{\sum_{j\in I_0^\vee}(j-1)} p(e_{I_0,I_0^\vee}) = \sigma(I_0)\sigma(I_0 \setminus \{l\}) e_{I_0\setminus\{l\}} \otimes e_{I_0}.$$

By $\mathrm{Spin}(Q)$ equivariance of p,

$$(-1)^{\sum_{j\in I_0^\vee}(j-1)} p((-1)^{(k-1)+m} e_{I,I^\vee}) = -\sigma(I_0)\sigma(I_0 \setminus \{l\}) e_I \otimes e_{I\setminus\{l\}}.$$

Since $I_0 = I \setminus \{k\}$, $I^\vee = I_0^\vee \cup \{k\}$. Therefore, $(-1)^{\sum_{j\in I_0^\vee}(j-1)+(k-1)} = (-1)^{\sum_{j\in I^\vee}(j-1)}$. Note that

$$\{\#I_0 \text{ modulo } 4, \#(I \cup \{l\}) \text{ modulo } 4\} = \begin{cases} \{0,2\} & \text{if } m \text{ is odd, and} \\ \{1,3\} & \text{if } m \text{ is even.} \end{cases}$$

Therefore, $\sigma(I_0)\sigma(I \cup \{l\}) = (-1)^{m+1}$, using which we get $(-1)^{\sum_{j\in I^\vee}(j-1)} p(e_{I,I^\vee}) = \sigma(I)\sigma(I \setminus \{l\}) e_I \otimes e_{I\setminus\{l\}}$. The case when $\#I$ is odd is identical. □

3.4. **The decorated positive orthogonal Grassmannian.** In this section, we define the orthogonal Grassmannian and its Cartan embedding; for further background, see [Che97, BHH21, HS10].

For a subspace U of V, let $U^\perp := \{x \in V \mid Q(x,y) = 0 \text{ for every } y \in U\}$ denote its orthogonal complement. A subspace U is said to be *isotropic* (resp., *coisotropic*) for Q if $U \subseteq U^\perp$ (resp., $U^\perp \subseteq U$). Let $\mathrm{OG}(n, 2n)$ denote the orthogonal Grassmannian of isotropic n dimensional subspaces. Then $\mathrm{OG}(n, 2n) = \mathrm{OG}_+(n, 2n) \sqcup \mathrm{OG}_-(n, 2n)$ has two irreducible components, where $\mathrm{OG}_+(n, 2n)$ (resp., $\mathrm{OG}_-(n, 2n)$) is the $\mathrm{Spin}(Q)$ orbit of $\mathrm{span}(e_{n+1}, \ldots, e_{2n})$ (resp., $\mathrm{span}(e_1, e_{n+2}, e_{n+3}, \ldots, e_{2n})$). We have $\mathrm{Spin}(Q)$ equivariant embeddings $\mathrm{Ca}_\pm : \mathrm{OG}_\pm(n, 2n) \hookrightarrow \mathbb{P}(S_\pm)$, called *Cartan embeddings*, defined by

$$\mathrm{span}(e_{n+1}, \ldots, e_{2n}) \mapsto 1 \text{ and } \mathrm{span}(e_1, e_{n+2}, e_{n+3}, \ldots, e_{2n}) \mapsto e_1.$$

Let $\mathrm{OG}(n+1, 2n)$ denote the orthogonal Grassmannian of coisotropic $(n+1)$-dimensional subspaces. Given $X \in \mathrm{OG}(n+1, 2n)$, there are two maximal isotropic subspaces $X_\pm \in \mathrm{OG}_\pm(n, 2n)$ contained in X. The composition

$$\begin{array}{ccccc} \mathrm{OG}(n+1,2n) & \hookrightarrow & \mathrm{OG}_+(n,2n) \times \mathrm{OG}_-(n,2n) & \xrightarrow{\mathrm{Ca}_+ \times \mathrm{Ca}_-} & \mathbb{P}(S_+) \times \mathbb{P}(S_-) \\ \Psi & & \Psi & & \Psi \\ X & \longmapsto & (X_+, X_-) & \longmapsto & (\mathrm{Ca}_+(X_+), \mathrm{Ca}_-(X_-)) \end{array}$$

defines a $\mathrm{Spin}(Q)$ equivariant embedding $\mathrm{Ca} : \mathrm{OG}(n+1, 2n) \hookrightarrow \mathbb{P}(S_+) \times \mathbb{P}(S_-)$. Let

$$\widetilde{\mathrm{OG}}(n+1, 2n) := \{(X, s_+, s_-) \mid X \in \mathrm{OG}(n+1, 2n), s_\pm \in \mathrm{Ca}(X_\pm)\}$$

denote the *decorated orthogonal Grassmannian*. Then, we have an embedding $\widetilde{\mathrm{OG}}(n+1, 2n) \hookrightarrow S_+ \times S_-$ sending (X, s_+, s_-) to (s_+, s_-).

Recall that $\sigma(I)$ is defined to be -1 if $\#I \equiv 2$ modulo 4 and 1 otherwise. The coefficients $\Sigma_I(X, s_+, s_-)$ of $\sigma(I)e_I$ in (s_+, s_-) are called *Cartan coordinates*. Consider the bihomogeneous equations

$$\Sigma_{I\cup\{j,l\}}\Sigma_{I\cup\{k\}} = \Sigma_I\Sigma_{I\cup\{j,k,l\}} + \Sigma_{I\cup\{j,k\}}\Sigma_{I\cup\{l\}} + \Sigma_{I\cup\{k,l\}}\Sigma_{I\cup\{j\}}, \tag{3.8}$$

for $j < k < l$.

Theorem 3.9 (Henriques and Speyer, [HS10, Theorem 5.3])**.** *The image of $\widetilde{\mathrm{OG}}(n+1, 2n)$ in $S_+ \times S_-$ is the subvariety cut out by all the equations (3.8).*

Consider the $\mathrm{Spin}(Q)$ equivariant map $\eta : \widetilde{\mathrm{OG}}(n+1, 2n) \to \widetilde{\mathrm{Gr}}(n+1, 2n)$ defined by $(\mathrm{span}(e_1, e_1^\vee, \ldots, e_n^\vee), 1, e_1) \mapsto (\mathrm{span}(e_1, e_1^\vee, \ldots, e_n^\vee), e_{1,[n]})$.

Remark 3.10. The maps $\eta : \widetilde{\mathrm{OG}}(n+1, 2n) \to \widetilde{\mathrm{Gr}}(n+1, 2n)$ and $S_+ \times S_- \to S_+ \otimes S_-$ are not embeddings, but they become embeddings upon projectivization.

Let $I \in \binom{[2n]}{n+1}$, let

$$J := I \cap [n] \text{ and } J^\vee := \{i - n \mid i \in I \cap [n+1, 2n]\}. \tag{3.9}$$

Under the change of basis (3.2), e_I becomes $(-1)^{\sum_{j\in J^\vee}(j-1)}e_{J,J^\vee}$. The following proposition relates Plücker and Cartan coordinates.

Proposition 3.11. *Let $(X, s_+, s_-) \in \widetilde{\mathrm{OG}}(n+1, 2n)$, let $(X, v) = \eta(X, s_+, s_-)$, and let $J, J^\vee$ be defined as in (3.9). If $\#(J \cap J^\vee) = 1$, then*

$$\Delta_I(X, v) = \Sigma_J(X, s_+, s_-)\Sigma_{[n]\setminus J^\vee}(X, s_+, s_-).$$

Proof. Consider the following commutative diagram

$$\begin{array}{ccccc} \widetilde{\mathrm{OG}}(n+1, 2n) & & \hookrightarrow & & S_+ \times S_- \\ \eta\downarrow & & & & \downarrow \\ \widetilde{\mathrm{Gr}}(n+1, 2n) & \hookrightarrow & \bigwedge^{n+1}\mathbb{C}^{2n} & \overset{p}{\hookrightarrow} & S_+ \otimes S_- \end{array}.$$

Let $(X, s_+, s_-) \in \widetilde{\mathrm{OG}}(n+1, 2n)$ and let $(X, v) = \eta(X, s_+, s_-)$. The coefficient of $\sigma(J)\sigma([n] \setminus J^\vee)e_J \otimes e_{[n]\setminus J^\vee}$ in $s_+ \otimes s_-$ is $\Sigma_J(X, s_+, s_-)\Sigma_{[n]\setminus J^\vee}(X, s_+, s_-)$. Using Proposition 3.8, and commutativity of the diagram, we get that this coefficient is also equal to $\Delta(X, v)$. $\square$

Definition 3.12. Let $\widetilde{\mathrm{OG}}_{>0}(n+1, 2n)$ denote the subset of $\widetilde{\mathrm{OG}}(n+1, 2n)$ where all the Cartan coordinates are positive, which we call the *decorated positive orthogonal Grassmannian*. Let $\mathrm{OG}_{>0}(n+1, 2n)$ denote the *positive orthogonal Grassmannian*, the image of $\widetilde{\mathrm{OG}}_{>0}(n+1, 2n)$ under the projection $\widetilde{\mathrm{OG}}(n+1, 2n) \to \mathrm{OG}(n+1, 2n)$, or equivalently, the subset of $\mathrm{OG}(n+1, 2n)$ where the ratio of any two Cartan coordinates of the same parity is positive.

Example 3.13. Given $(X, s_+, s_-) \in \widetilde{\mathrm{OG}}_{>0}(n+1, 2n)$, Proposition 3.11 lets us write down a matrix whose row span is X. For example, let $n = 3$ and let $(X, s_+, s_-) \in \widetilde{\mathrm{OG}}_{>0}(4, 6)$ be such that $(\Sigma_J(X, s_+, s_-))_{J\subseteq[3]} = (\Sigma_J)_{J\subseteq[3]}$. Then,

$$X = \text{row span}\begin{bmatrix} \Sigma_\varnothing\Sigma_1 & \Sigma_\varnothing\Sigma_2 & \Sigma_\varnothing\Sigma_3 & 0 & 0 & 0 \\ 0 & \frac{\Sigma_{12}}{\Sigma_\varnothing} & \frac{\Sigma_{13}}{\Sigma_\varnothing} & 1 & 0 & 0 \\ 0 & -\frac{\Sigma_{12}\Sigma_2}{\Sigma_\varnothing\Sigma_1} & -\frac{\Sigma_{12}\Sigma_{23}+\Sigma_{12}\Sigma_3}{\Sigma_\varnothing\Sigma_1} & 0 & 1 & 0 \\ 0 & \frac{\Sigma_\varnothing\Sigma_{13}+\Sigma_{12}\Sigma_3}{\Sigma_\varnothing\Sigma_1} & \frac{\Sigma_{13}\Sigma_3}{\Sigma_\varnothing\Sigma_1} & 0 & 0 & 1 \end{bmatrix},$$

where $\Sigma_2 = \frac{\Sigma_\varnothing\Sigma_{123}+\Sigma_1\Sigma_{23}+\Sigma_{12}\Sigma_3}{\Sigma_{13}}$.

3.5. **Pfaffian formulas for Cartan coordinates.** The main result of this section is Proposition 3.15 expressing each Cartan coordinate as the Pfaffian of a certain matrix. Let $A = (a_{ij})$ be a $2n \times 2n$ skew symmetric matrix. Let $\omega_A := \sum_{1\le i<j\le 2n} a_{ij}e_i \wedge e_j$ denote the associated alternating form. The pfaffian $\mathrm{pf}(A)$ of A is defined by the formula

$$\frac{1}{n!}\omega_A^n = \mathrm{pf}(A)e_1 \wedge \cdots \wedge e_{2n}.$$

For $I \subseteq [n]$, let A_I^I denote the principal submatrix of A with rows and columns indexed by I.

Lemma 3.14 ([Pro06, Chapter 5, Equation (3.6.3)])**.** *We have*

$$\exp(\omega_A) = \sum_{I\subseteq[n]} \mathrm{pf}(A_I^I)e_I = \sum_{I\subseteq[n]|\#I \text{ even}} \mathrm{pf}(A_I^I)e_I.$$

Recall from Section 3.4 that the orthogonal Grassmannian $\mathrm{OG}(n, 2n) = \mathrm{OG}_+(n, 2n) \sqcup \mathrm{OG}_-(n, 2n)$ is the union of two components. If $X_+ \in \mathrm{OG}_+(n, 2n)$ and $\Delta_{[n+1,2n]}(X_+) \neq 0$, then in the coordinates (3.2), X_+ is the row span of a matrix of the form $\begin{bmatrix} M_+ & I_n \end{bmatrix}$, where M_+ is a skew symmetric $n \times n$ matrix. Similarly, if $X_- \in \mathrm{OG}_-(n, 2n)$ is such that $\Delta_{\{1,n+2,n+3,\dots,2n\}}(X_-) \neq 0$, then X_- is the row span of an $n \times 2n$ matrix with I_n in columns $1, n+2, n+3, \dots, 2n$ and a matrix $\tilde{M}_-$ in columns $2, 3, \dots, n+1$ such that the matrix M_- obtained from $\tilde{M}_-$ by cyclically rotating the columns by one step to the right is skew symmetric. For $J \subseteq [n]$, let $J\Delta\{1\}$ denote the symmetric difference, i.e.,

$$J\Delta\{1\} := \begin{cases} J \setminus \{1\} & \text{if } 1 \in J; \\ J \cup \{1\} & \text{if } 1 \notin J. \end{cases}$$

Proposition 3.15. *Let* $(X, s_+, s_-) \in \widetilde{\mathrm{OG}}_{>0}(n+1, 2n)$ *and let* (X_+, X_-) *denote the maximal isotropic subspaces in* X*. Let* M_+, M_- *be as above. Then,*

$$\Sigma_J(X, s_+, s_-) = \begin{cases} \Sigma_\varnothing(X, s_+, s_-)\,\mathrm{pf}((M_+)_J^J) & \text{if } \#J \text{ is even;} \\ (-1)^{\frac{\#J-1}{2}}\Sigma_{\{1\}}(X, s_+, s_-)\,\mathrm{pf}((M_-)_{J\Delta\{1\}}^{J\Delta\{1\}}) & \text{if } \#J \text{ is odd.} \end{cases}$$

Proof. We use $\mathrm{Spin}(Q)$ equivariance of the Cartan map and the following commutative diagram of exponential maps:

$$\begin{array}{ccc} \mathfrak{so}(Q) & \xrightarrow[\sim]{\psi\circ\varphi^{-1}} & \mathfrak{spin}(Q) \\ \downarrow{\scriptstyle\exp} & & \downarrow{\scriptstyle\exp} \\ \mathrm{SO}(Q) & \xleftarrow{\rho} & \mathrm{Spin}(Q) \end{array},$$

where the exp on the left is the matrix exponential map and on the right is (3.4).

Consider the element $m_+ := \begin{bmatrix} 0 & -M_+ \\ 0 & 0 \end{bmatrix} \in \mathfrak{so}(Q)$. Exponentiating m_+, we get $\begin{bmatrix} I_n & -M_+ \\ 0 & I_n \end{bmatrix} \in$ SO(Q), so that we have $\begin{bmatrix} 0 & I_n \end{bmatrix} (\exp(m_+))^T = \begin{bmatrix} M_+ & I_n \end{bmatrix}$. On the other hand, under the isomorphism $\psi \circ \varphi^{-1} : \mathfrak{so}(Q) \xrightarrow{\sim} \mathfrak{spin}(Q)$, m_+ goes to $-\sum_{1 \leq i < j \leq n} (M_+)_{ij} e_i \otimes e_j$. Exponentiating, and using Lemma 3.14 along with Spin(Q) equivariance of the Cartan map, we get

$$[s_+] = [\exp(m_+) \cdot 1] = \left[\sum_{J \text{ even}} \operatorname{pf}((-M^+)^J_J) e_J \right]. \tag{3.10}$$

Since $\Sigma_J(X, s_+, s_-)$ is the coefficient of $\sigma(J) e_J$ in s_+, we get

$$\frac{\Sigma_J(X, s_+, s_-)}{\Sigma_\varnothing(X, s_+, s_-)} = \sigma(J) \operatorname{pf}((-M^+)^J_J) = \operatorname{pf}((M^+)^J_J),$$

where we used $\operatorname{pf}((-M^+)^J_J) = (-1)^{\frac{\#J}{2}} \operatorname{pf}((M^+)^J_J)$ and $(-1)^{\frac{\#J}{2}} = \sigma(J)$.

The orthogonal transformation R_w for $w = e_1 - e_1^\vee$ is given by $e_1 \leftrightarrow e_{n+1}$, and it interchanges the two connected components OG$_\pm(n, 2n)$ and also interchanges M_- and M_+. Its lift w to Pin(Q) acts on S by

$$w \cdot e_J := \begin{cases} -e_{J\Delta\{1\}} & \text{if } 1 \in J; \\ e_{J\Delta\{1\}} & \text{if } 1 \notin J. \end{cases}$$

Using (3.10), we get

$$\left[-\sum_{J \text{ odd} | 1 \in J} \Sigma_J(X, s_+, s_-) e_{J\Delta\{1\}} + \sum_{J \text{ odd} | 1 \notin J} \Sigma_J(X, s_+, s_-) e_{J\Delta\{1\}} \right] = \left[\sum_{J \text{ odd}} \operatorname{pf}((-M_-)^{J\Delta\{1\}}_{J\Delta\{1\}}) e_{J\Delta\{1\}} \right].$$

Now, we have to check two cases. If $1 \in J$, then

$$\frac{\Sigma_J(X, s_+, s_-)}{\Sigma_{\{1\}}(X, s_+, s_-)} = (-1)^{\frac{\#J-1}{2}} \operatorname{pf}((M_-)^{J\Delta\{1\}}_{J\Delta\{1\}}),$$

and if $1 \notin J$, then

$$\frac{\Sigma_J(X, s_+, s_-)}{\Sigma_{\{1\}}(X, s_+, s_-)} = -(-1)^{\frac{\#J+1}{2}} \operatorname{pf}((M_-)^{J\Delta\{1\}}_{J\Delta\{1\}}) = (-1)^{\frac{\#J-1}{2}} \operatorname{pf}((M_-)^{J\Delta\{1\}}_{J\Delta\{1\}}).$$

□

Example 3.16. Recall Example 3.13. After making the change of basis (3.2), the two maximal isotropic subspaces X_+ and X_- are the row spans of

$$\begin{bmatrix} 0 & \frac{\Sigma_{12}}{\Sigma_\varnothing} & \frac{\Sigma_{13}}{\Sigma_\varnothing} & 1 & 0 & 0 \\ -\frac{\Sigma_{12}}{\Sigma_\varnothing} & 0 & \frac{\Sigma_{23}}{\Sigma_\varnothing} & 0 & 1 & 0 \\ -\frac{\Sigma_{13}}{\Sigma_\varnothing} & -\frac{\Sigma_{23}}{\Sigma_\varnothing} & 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & \frac{\Sigma_2}{\Sigma_1} & \frac{\Sigma_3}{\Sigma_1} & 0 & 0 & 0 \\ 0 & 0 & -\frac{\Sigma_{123}}{\Sigma_1} & -\frac{\Sigma_2}{\Sigma_1} & 1 & 0 \\ 0 & \frac{\Sigma_{123}}{\Sigma_1} & 0 & -\frac{\Sigma_3}{\Sigma_1} & 0 & 1 \end{bmatrix} \text{ respectively.}$$

Therefore,

$$M_+ = \begin{bmatrix} 0 & \frac{\Sigma_{12}}{\Sigma_\varnothing} & \frac{\Sigma_{13}}{\Sigma_\varnothing} \\ -\frac{\Sigma_{12}}{\Sigma_\varnothing} & 0 & \frac{\Sigma_{23}}{\Sigma_\varnothing} \\ -\frac{\Sigma_{13}}{\Sigma_\varnothing} & -\frac{\Sigma_{23}}{\Sigma_\varnothing} & 0 \end{bmatrix} \text{ and } M_- = \begin{bmatrix} 0 & \frac{\Sigma_2}{\Sigma_1} & \frac{\Sigma_3}{\Sigma_1} \\ -\frac{\Sigma_2}{\Sigma_1} & 0 & -\frac{\Sigma_{123}}{\Sigma_1} \\ -\frac{\Sigma_3}{\Sigma_1} & \frac{\Sigma_{123}}{\Sigma_1} & 0 \end{bmatrix},$$

using which we verify Proposition 3.15. For example,

$$\Sigma_\varnothing \operatorname{pf}((M_+)^{12}_{12}) = \Sigma_\varnothing \operatorname{pf}\begin{bmatrix} 0 & \frac{\Sigma_{12}}{\Sigma_\varnothing} \\ -\frac{\Sigma_{12}}{\Sigma_\varnothing} & 0 \end{bmatrix} = \Sigma_{12} \text{ when } J = \{1,2\},$$

$$(-1)^{\frac{1-1}{2}}\Sigma_1 \operatorname{pf}((M_-)^{12}_{12}) = \Sigma_1 \operatorname{pf}\begin{bmatrix} 0 & \frac{\Sigma_2}{\Sigma_1} \\ -\frac{\Sigma_2}{\Sigma_1} & 0 \end{bmatrix} = \Sigma_2 \text{ when } J = \{2\}, \text{ and}$$

$$(-1)^{\frac{2-1}{2}}\Sigma_1 \operatorname{pf}((M_-)^{23}_{23}) = -\Sigma_1 \operatorname{pf}\begin{bmatrix} 0 & \frac{-\Sigma_{123}}{\Sigma_1} \\ \frac{\Sigma_{123}}{\Sigma_1} & 0 \end{bmatrix} = \Sigma_{123} \text{ when } J = \{1,2,3\}.$$

3.6. B **variables.** Consider the space $\mathcal{B}_G := \mathbb{R}_{>0}^{V(G)\sqcup F(G)}$ of functions $B : V(G) \sqcup F(G) \to \mathbb{R}_{>0}$. We call a pair (G, B) a *B-network*. Since there is a bijection $V(G) \sqcup F(G) \xrightarrow{\sim} W(G_+)$, we will sometimes write B_{w_u} instead of B_u for $u \in V(G) \sqcup F(G)$.

A Y-Δ move $G \rightsquigarrow G'$ induces a homeomorphism $\mathcal{B}_G \xrightarrow{\sim} \mathcal{B}_{G'}$ given by the *cube recurrence*

$$B_{f_0'} := \frac{B_{v_1}B_{f_1} + B_{v_2}B_{f_2} + B_{v_3}B_{f_3}}{B_{v_0}}$$

and $B_{v'} := B_v$ for all other $v \in V(G')$ and $B_{f'} := B_f$ for all other $f \in F(G')$, where vertices and faces are labeled as in Figure 5, and let $\mathcal{B}_n := \bigsqcup_{\tau_G = \tau_n} \mathcal{B}_G \big/ \text{moves}$.

Each face g of G_+ has degree 4 and is incident to two white vertices w_v and w_f, where $v \in V(G)$ and $f \in F(G)$. Define the inclusion $i_G^+ : \mathcal{B}_G \hookrightarrow \mathcal{A}_{G_+}$ by $A_g := B_v B_f$.

Proposition 3.17 (Goncharov and Kenyon, [GK13, Lemma 5.11])**.** *If G and G' are related by a Y-Δ move, then there is a sequence of moves relating G_+ and G'_+ such that the following diagram commutes.*

$$\begin{array}{ccc} \mathcal{B}_G & \xhookrightarrow{i_G^+} & \mathcal{A}_{G_+} \\ {\scriptstyle Y\text{-}\Delta\ move}\downarrow\wr & & \wr\downarrow{\scriptstyle moves} \\ \mathcal{B}_{G'} & \xhookrightarrow{i_{G'}^+} & \mathcal{A}_{G'_+} \end{array}$$

Therefore, the inclusions i_G^+ glue to an inclusion $i^+ : \mathcal{B}_n \hookrightarrow \mathcal{A}_{\pi_{n+1,2n}}$.

Definition 3.18. Given G a reduced graph with $\tau_G = \tau_n$, we assign to each vertex and face of G a subset of $[n]$ as follows. For $i \in [n]$, let β_i denote the strand in G from t_{n+i} to t_i. If $u \in V(G) \sqcup F(G)$, define

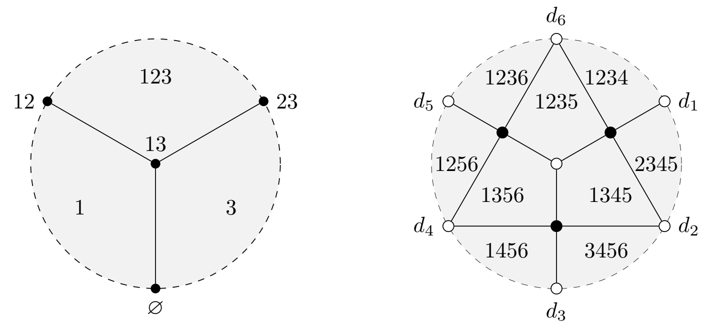
$$J(u) := \{j \in [n] \mid u \text{ is to the left of } \beta_i\}.$$

Lemma 3.19. *Let g be a face of G_+ incident to white vertices $\mathrm{w}_v, \mathrm{w}_f$ where $v \in V(G)$ and $f \in F(G)$. If $I := S(g)$, then $\{J, [n] \setminus J^\vee\} = \{J(v), J(f)\}$, where J and $J^\vee$ are as in (3.9).*

Proof. Let α_i denote the strand in G_+ ending at d_i. By construction of G_+, the strand α_i (resp., α_{n+i}) is parallel (resp., antiparallel) to β_i. Faces of G_+ are in bijection with edges of $G^\times$. Let $l \in [n]$ be such that β_l is the strand containing the edge of $G^\times$ corresponding to g. Without loss of generality, assume that v is to the left and f to the right of β_l. Then,

$$J(v) = J(f) \cup \{l\} \text{ and } S(g) = \{l, n+l\} \cup \{i \mid i \in J(f)\} \cup \{n+i \mid i \notin J(v)\}.$$

Therefore, $J = S(g) \cap [n] = J(f) \cup \{l\} = J(v)$ and $J^\vee = ([n] \setminus J(v)) \cup \{l\} = [n] \setminus J(f)$. $\square$



(a) (b)

FIGURE 6. (a) Labeling the vertices and faces of the electrical network from Figure 4(a) and (b) the face labels of G_+.

Example 3.20. Figures 6(a) and (b) show the labels J and S for G and G_+ from Figure 4.

Define the map

$$\Psi_G : \widetilde{\mathrm{OG}}_{>0}(n+1,2n) \to \mathcal{B}_G$$

sending (X, s_+, s_-) to $(\Sigma_{J(u)}(X, s_+, s_-))_{u \in V(G) \sqcup F(G)}$.

Theorem 3.21 (Henriques and Speyer, [HS10]). *For every reduced G with $\tau_G = \tau_n$, $\Psi_G : \widetilde{\mathrm{OG}}_{>0}(n+1,2n) \xrightarrow{\sim} \mathcal{B}_G$ is a homeomorphism.*

Suppose G and G' are related by a Y-Δ move with vertices and faces labeled as in Figure 5. Then, up to cyclic rotation of the tuple $(v_1, f_1, v_2, f_2, v_3, f_3)$ and swapping v and f, we have

$$\begin{aligned} &J(v_0) = I \cup \{j,l\}, && J(v_1) = I, && J(v_2) = I \cup \{j,k\}, && J(v_3) = I \cup \{k,l\}, \\ &J(f_0) = I \cup \{k\}, && J(f_1) = I \cup \{j,k,l\}, && J(f_2) = I \cup \{l\}, && J(f_3) = I \cup \{j\}. \end{aligned}$$

for some $I \subset [n]$ and $j < k < l$. By Theorem 3.9, the following diagram commutes

$$\begin{array}{ccccc} & & \mathcal{B}_G & & \\ & \overset{\Psi_G}{\underset{\sim}{\nearrow}} & & \searrow & \\ \widetilde{\mathrm{OG}}_{>0}(n+1,2n) & & \sim \big\downarrow \text{Y-}\Delta\text{ move} & & \mathcal{B}_n \\ & \overset{\sim}{\underset{\Psi_{G'}}{\searrow}} & & \nearrow & \\ & & \mathcal{B}_{G'} & & \end{array},$$

so we obtain a well-defined homeomorphism $\Psi : \widetilde{\mathrm{OG}}_{>0}(n+1,2n) \xrightarrow{\sim} \mathcal{B}_n$.

Proposition 3.22. *The following diagram commutes.*

$$\begin{array}{ccc} \mathcal{B}_G & \xrightarrow{i_G^+} & \mathcal{A}_{G_+} \\ \Psi_G \big\uparrow \sim & & \sim \big\uparrow \Phi_{G_+} \\ \widetilde{\mathrm{OG}}_{>0}(n+1,2n) & \longrightarrow & \widetilde{\mathrm{Gr}}_{>0}(n+1,2n) \end{array} \quad (3.11)$$

Proof. Let g be a face of G_+ incident to white vertices $\mathrm{w}_v, \mathrm{w}_f$ where $v \in V(G)$ and $f \in F(G)$. Using Lemma 3.19 and Proposition 3.11, we get

$$\Delta_{S(g)}(X, v) = \Sigma_{J(v)}(X, s_+, s_-)\Sigma_{J(f)}(X, s_+, s_-).,\tag{3.12}$$

which implies that (3.11) commutes. □

4. The electrical right twist

In this section, we define the electrical right twist and prove Theorem 1.1. Let $e = uv$ be an edge of G and let f, g denote the faces of G incident to e. Following equation (56) in [GK13, Section 5.3.1], define $q_G : \mathcal{B}_G \to \mathcal{R}_G$ by $c(e) := \frac{B_u B_v}{B_f B_g}$. If G and G' are related by a Y-Δ move, then the following diagram commutes

$$\begin{array}{ccc} \mathcal{B}_G & \xrightarrow{q_G} & \mathcal{R}_G \\ {\scriptstyle \text{Y-}\Delta\text{ move}}\big\downarrow{\scriptstyle \sim} & & {\scriptstyle \sim}\big\downarrow{\scriptstyle \text{Y-}\Delta\text{ move}} \\ \mathcal{B}_{G'} & \xrightarrow{q_{G'}} & \mathcal{R}_{G'} \end{array},$$

and therefore, the q_G glue to a map $q : \mathcal{B}_n \to \mathcal{R}_n$.

Recall the action (2.16) of $\mathbb{R}^{2n}_{>0}$ on $\mathrm{Gr}_{>0}(n+1, 2n)$ by rescaling columns.

Definition 4.1. Let $(X, s_+, s_-) \in \widetilde{\mathrm{OG}}_{>0}(n+1, 2n)$, and let $t_i := \frac{\Sigma_{J(d_{i-1})}(X, s_+, s_-)}{\Sigma_{J(d_i)}(X, s_+, s_-)}$ for $i \in [2n]$. The *electrical right twist* of (X, s_+, s_-), denoted $\vec{\tau}_{\mathrm{elec}}(X, s_+, s_-)$, is defined to be $t \cdot \vec{\tau}(X) \in \mathrm{Gr}_{>0}(n+1, 2n)$.

Theorem 4.2. *Let G be a reduced graph with $\tau_G = \tau_n$. The image of $\vec{\tau}_{\mathrm{elec}}$ is contained in $\mathrm{IG}^{\Omega}_{>0}(n+1, 2n)$, and the following diagrams commute:*

$$\begin{array}{ccc} \mathcal{B}_G & \xrightarrow{\quad q_G \quad} & \mathcal{R}_G \\ {\scriptstyle \Psi_G}\big\uparrow{\scriptstyle \sim} & & {\scriptstyle \sim}\big\downarrow{\scriptstyle \mathrm{Meas}_{G_+} \circ j^+_G} \\ \widetilde{\mathrm{OG}}_{>0}(n+1, 2n) & \xrightarrow{\vec{\tau}_{\mathrm{elec}}} & \mathrm{IG}^{\Omega}_{>0}(n+1, 2n) \end{array}, \qquad \begin{array}{ccc} \mathcal{B}_n & \xrightarrow{\quad q \quad} & \mathcal{R}_n \\ {\scriptstyle \Psi}\big\uparrow{\scriptstyle \sim} & & {\scriptstyle \sim}\big\downarrow{\scriptstyle \mathrm{Meas} \circ j^+} \\ \widetilde{\mathrm{OG}}_{>0}(n+1, 2n) & \xrightarrow{\vec{\tau}_{\mathrm{elec}}} & \mathrm{IG}^{\Omega}_{>0}(n+1, 2n) \end{array}.$$

Proof. We will show commutativity of the left diagram by showing that $\mathrm{Meas}^{-1}_{G_+} \circ \vec{\tau}_{\mathrm{elec}} = j^+_G \circ q_G \circ \Psi_G$. The right diagram is then obtained by gluing.

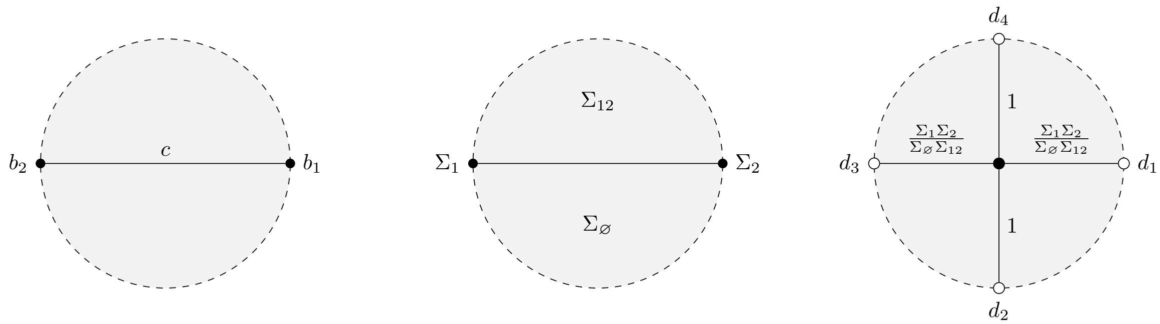
Define $B := \Psi_G(X, s_+, s_-)$. Let $e = uv$ be an edge of G and let f, g denote the faces of G incident to e. From definitions, if $[\mathrm{wt}] := j_G \circ q_G(B)$, then

$$\mathrm{wt}(\mathrm{b}_e \mathrm{w}_x) = \begin{cases} \frac{B_u B_v}{B_f B_g} & \text{if } x \in \{u, v\} \\ 1 & \text{if } x \in \{f, g\} \end{cases}.$$

We define a gauge transformation $\tilde{g}$ by $\tilde{g}(\mathrm{b}_e) := \frac{1}{B_u B_v}$ and for each internal white vertex w_u, $\tilde{g}(\mathrm{w}_u) := B_u^2$. We have the following cases for edges $\mathrm{b}_e \mathrm{w}_x$ in G_+.

(1) $x = u$. Let h and h' be the two faces of G_+ incident to $\mathrm{b}_e \mathrm{w}_u$, where h is between u and f and h' is between u and g.
   (a) w_u is an internal vertex of G_+. Then, $\mathrm{Meas}^{-1}_{G_+} \circ \vec{\tau}_{\mathrm{elec}}$ assigns weight $\frac{1}{A_h A_{h'}} = \frac{1}{B_u^2 B_f B_g}$ to $\mathrm{b}_e \mathrm{w}_u$. Applying the gauge transformation $\tilde{g}$, we get

$$B_u B_v \frac{1}{B_e^2 B_f B_g} B_u^2 = \frac{B_u B_v}{B_f B_g}.$$



(a) An electrical network (G, c). (b) $\Psi_G(X, s_+, s_-)$. (c) $j_G^+ \circ q_G \circ \Psi_G(X, s_+, s_-)$.

FIGURE 7. Commutativity of the diagram in Theorem 4.2 when $n = 2$.

(b) w_u is a boundary vertex d_{2i-1} of G_+. $\mathrm{Meas}_{G_+}^{-1} \circ \vec{\tau}_{\mathrm{elec}}$ assigns weight $\frac{B_{d_{2i-1}}}{B_{d_{2i-2}}} \frac{A_{f_{2i-1}^-}}{A_h A_{h'}} = \frac{1}{B_f B_g}$ to $\mathrm{b}_e \mathrm{w}_u$. Applying the gauge transformation $\tilde{g}$, we get $\frac{B_u B_v}{B_f B_g}$.

(2) $x = f$. Let h and h' be the two faces of G_+ incident to $\mathrm{b}_e \mathrm{w}_f$, where h is between u and f and h' is between f and v.

(a) If w_f is an internal vertex of G_+, then $\mathrm{Meas}_{G_+}^{-1} \circ \vec{\tau}_{\mathrm{elec}}$ assigns weight $\frac{1}{A_h A_{h'}} = \frac{1}{B_e B_f^2 B_v}$ to $\mathrm{b}_e \mathrm{w}_f$. Applying the gauge transformation $\tilde{g}$, we get

$$B_u B_v \frac{1}{B_u B_f^2 B_v} B_g^2 = 1.$$

(b) If w_f is the boundary vertex d_{2i}, then $\mathrm{Meas}_{G_+}^{-1} \circ \vec{\tau}_{\mathrm{elec}}$ assigns weight $\frac{B_{d_{2i}}}{B_{d_{2i-1}}} \frac{A_{f_{2i}^-}}{A_h A_{h'}} = \frac{1}{B_u B_v}$ to $\mathrm{b}_e \mathrm{w}_f$. Applying the gauge transformation $\tilde{g}$, we get 1.

□

Corollary 4.3. *The electrical right twist* $\vec{\tau}_{\mathrm{elec}} : \widetilde{\mathrm{OG}}_{>0}(n+1, 2n) \to \mathrm{IG}^{\Omega}_{>0}(n+1, 2n)$ *is surjective.*

Proof. [KW17, Proposition 4] shows that q_G is surjective. By Theorem 4.2, $\vec{\tau}_{\mathrm{elec}}$ is surjective. □

Example 4.4. Let $n = 2$ and let $(X, s_+, s_-) \in \widetilde{\mathrm{OG}}_{>0}(3, 4)$ be such that $(\Sigma_J(X, s_+, s_-))_{J \subseteq [2]} = (\Sigma_J)_{J \subseteq [2]}$. Then, X is the row span of the matrix

$$\begin{bmatrix} \Sigma_\varnothing \Sigma_1 & \Sigma_\varnothing \Sigma_2 & 0 & 0 \\ 0 & \frac{\Sigma_{12}}{\Sigma_\varnothing} & 1 & 0 \\ 0 & -\frac{\Sigma_{12}\Sigma_2}{\Sigma_\varnothing \Sigma_1} & 0 & 1 \end{bmatrix}, \text{ so we compute } \vec{\tau}_{\mathrm{elec}}(X, s_+, s_-) = \begin{bmatrix} \frac{\Sigma_{12}}{\Sigma_\varnothing \Sigma_1 \Sigma_2} & \frac{1}{\Sigma_\varnothing^2} & 0 & 0 \\ 0 & 0 & \frac{\Sigma_\varnothing}{\Sigma_1} & \frac{\Sigma_2}{\Sigma_{12}} \\ \frac{\Sigma_\varnothing}{\Sigma_2} & 0 & 0 & \frac{\Sigma_1}{\Sigma_{12}} \end{bmatrix},$$

whose Plücker coordinates are

$$\Delta_{123} = \frac{1}{\Sigma_1 \Sigma_2}, \Delta_{124} = \frac{1}{\Sigma_\varnothing \Sigma_{12}}, \Delta_{134} = \frac{1}{\Sigma_1 \Sigma_2}, \Delta_{234} = \frac{1}{\Sigma_\varnothing \Sigma_{12}}. \tag{4.1}$$

Consider the electrical network in Figure 7(a). Using Figure 7(c), we compute

$$\mathrm{Meas}_{G_+} \circ j_G^+ \circ q_G \circ \Psi_G(X, s_+, s_-) = \left[e_{123} + \frac{\Sigma_1\Sigma_2}{\Sigma_\varnothing \Sigma_{12}} e_{124} + e_{134} + \frac{\Sigma_1\Sigma_2}{\Sigma_\varnothing \Sigma_{12}} e_{234} \right],$$

which agrees with (4.1) upon multiplying by $\Sigma_1\Sigma_2$.

5. The electrical left twist

In this section, we define the electrical left twist and prove Theorem 1.2. By Theorem 2.9, the right twist is a homeomorphism $\vec{\tau} : \widetilde{\mathrm{Gr}}_{>0}(n+1, 2n)/\mathbb{R}_{>0} \cong \mathrm{Gr}_{>0}(n+1, 2n) \xrightarrow{\sim} \mathrm{Gr}_{>0}(n+1, 2n)$ whose inverse is the left twist. We look for a similar statement for the electrical right twist. The dimension of $\widetilde{\mathrm{OG}}_{>0}(n+1, 2n)$ is $\binom{n+1}{2} + 1$ [HS10, Lemma 5.7], whereas the dimension of $\mathrm{IG}^{\Omega}(n+1, 2n)$ is $\binom{n}{2}$ (since this is the number of edges in G, hence the dimension of $\mathcal{R}_G$), so

$$\dim \widetilde{\mathrm{OG}}_{>0}(n+1, 2n) - \dim \mathrm{IG}^{\Omega}_{>0}(n+1, 2n) = n + 1.$$

We will see that there is an action of $\mathbb{R}^{n+1}_{>0}$ on $\widetilde{\mathrm{OG}}_{>0}(n+1, 2n)$ preserving $\vec{\tau}_{\mathrm{elec}}$.

Consider the action of $\mathbb{R}_{>0} \times \mathbb{R}^n_{>0}$ on $\mathcal{B}_G$ defined as follows. For $s \in \mathbb{R}_{>0}$ and $t = (t_1, \dots, t_n) \in \mathbb{R}^n_{>0}$,

$$((s,t) \cdot B)_v := s \left(\frac{\prod_{i \in J(v)} t_i}{\prod_{i \notin J(v)} t_i} \right) B_v. \tag{5.1}$$

Under Ψ_G, this action has the following description.

1. $\mathbb{R}_{>0}$ acts on $\widetilde{\mathrm{OG}}_{>0}(n+1, 2n)$ by $s \cdot (X, s_+, s_-) := (X, ss_+, ss_-)$.
2. Recall from (3.6) the maximal torus $(\mathbb{C}^\times)^n$ inside $\mathrm{Spin}(Q)$ which has the parameterization $c : (\mathbb{C}^\times)^n \to \mathrm{Spin}(Q)$. Restricting to $\mathbb{R}^n_{>0} \subset (\mathbb{C}^\times)^n$, we get a copy of $\mathbb{R}^n_{>0}$ inside $\mathrm{Spin}(Q)$ parameterized by $c : \mathbb{R}^n_{>0} \to \mathrm{Spin}(Q)$. We have the action $t \cdot (X, s_+, s_-) = (X\rho(c(t))^T, c(t)s_+, c(t)s_-)$, where $\rho(c(t)) \in \mathrm{SO}(Q)$ is $\mathrm{diag}(t_1^2, \dots, t_n^2, t_1^{-2}, \dots, t_n^{-2})$.

Lemma 5.1. *The map q_G is invariant under the action (5.1).*

Proof. Let $e = uv$ be an edge of G with incident faces f, g. The map q_G assigns to e the conductance $\frac{B_u B_v}{B_f B_g}$. The four labels $(J(u), J(f), J(v), J(g))$ are some cyclic rotation of $(I, I \cup \{j\}, I \cup \{j, k\}, I \cup \{k\})$, so the factors coming from the action of (s, t) in the numerator and denominator cancel. □

By Theorem 4.2 and Lemma 5.1, q_G and $\vec{\tau}_{\mathrm{elec}}$ descend to the quotients to yield the commuting diagram

$$\begin{array}{ccc} \mathcal{B}_G/\mathbb{R}^{n+1}_{>0} & \xrightarrow{\quad q_G \quad} & \mathcal{R}_G \\ \Psi_G \big\uparrow \wr & & \wr \big\downarrow \mathrm{Meas}_{G_+} \circ j_G^+ \\ \widetilde{\mathrm{OG}}_{>0}(n+1, 2n)/\mathbb{R}^{n+1}_{>0} & \xrightarrow{\vec{\tau}_{\mathrm{elec}}} & \mathrm{IG}^{\Omega}_{>0}(n+1, 2n) \end{array},$$

where each of the spaces has dimension $\binom{n}{2}$. We will show in Theorem 5.6 that the two horizontal maps are also homeomorphisms.

Let α_i denote the strand in G_+ from d_{n+i-1} to d_i. Let $[\mathrm{wt}](\alpha_i)$ denote the alternating product of edge weights along α_i, where the weights of edges oriented from black to white in α_i appear in the numerator and the weights of edges oriented from white to black in the denominator.

Lemma 5.2. *If* $[\mathrm{wt}] = q_G(B)$ *and* $X = \mathrm{Meas}_\Gamma([\mathrm{wt}])$*, then* $[\mathrm{wt}](\alpha_i) = \frac{\Delta_{S(f_{n+i-1}^-)}(X)}{\Delta_{S(f_{n+i}^-)}(X)} = \frac{B_{d_i}B_{d_{n+i}}}{B_{d_{i-1}}B_{d_{n+i-1}}}$.

Proof. If $A = \Phi_{G_+} \circ \overleftarrow{\tau}(X)$, then by Theorem 2.13, $[\mathrm{wt}] = p_{G_+}(A)$. Let $d_{n+i-1} = \mathrm{w}_1 \xrightarrow{e_1} \mathrm{b}_1 \xrightarrow{e_2} \mathrm{w}_2 \xrightarrow{e_3} \mathrm{b}_2 \xrightarrow{e_4} \cdots \xrightarrow{e_{2k-2}} \mathrm{w}_k \xrightarrow{e_{2k-1}} \mathrm{b}_k \xrightarrow{e_{2k}} \mathrm{w}_{k+1} = d_i$ denote the sequence of vertices and edges in α_i. For each edge e_i, let g_i^- (resp. g_i^+) denote the face of G_+ on the right (resp., left) of e_i. Notice that $g_{2j-1}^- = g_{2j}^-$ for $j \in [k]$ and $g_{2j}^+ = g_{2j+1}^+$ for $j \in [k-1]$. Moreover, $g_1^+ = f_{n+i}^-$ and $g_{2k}^- = f_i^-$. Therefore,

$$[\mathrm{wt}](\alpha_i) = \left(\frac{A_{g_1}^+ A_{g_1}^-}{A_{f_{n+i-1}^-}}\right)\left(\frac{1}{A_{g_2}^+ A_{g_2}^-}\right)\left(\frac{A_{g_3}^+ A_{g_3}^-}{1}\right)\cdots\left(\frac{A_{f_i^-}}{A_{g_{2k}}^+ A_{g_{2k}}^-}\right) = \frac{A_{f_{n+i}^-}}{A_{f_{n+i-1}^-}}. \tag{5.2}$$

Using Proposition 2.18(1), we get $[\mathrm{wt}](\alpha_i) = \frac{\Delta_{S(f_{n+i-1}^-)}(X)}{\Delta_{S(f_{n+i}^-)}(X)}$.

Let $[\mathrm{wt}'] := p_{G^+} \circ i_G^+(B)$. Using (5.2) for $[\mathrm{wt}']$, we get $[\mathrm{wt}'](\alpha_i) = \frac{i_G^+(B)_{f_{n+i}^-}}{i_G^+(B)_{f_{n+i-1}^-}} = \frac{B_{d_{n+i}}}{B_{d_{n+i-2}}}$. By definition of the electrical right twist and Proposition 2.18(2), we have $[\mathrm{wt}] = t \cdot [\mathrm{wt}']$, where $t \in \mathbb{R}_{>0}^{2n}$ is given by $t_j = \frac{B_{d_j}}{B_{d_{j-1}}}$ for all $j \in [2n]$. Therefore, $\mathrm{wt}(e_1) = t_{n+i-1}\mathrm{wt}'(e_1)$ and $\mathrm{wt}(e_{2k}) = t_i \mathrm{wt}'(e_{2k})$, so $[\mathrm{wt}](\alpha_i) = \frac{t_i}{t_{n+i-1}}[\mathrm{wt}'](\alpha_i) = \frac{B_{d_i}B_{d_{n+i}}}{B_{d_{i-1}}B_{d_{n+i-1}}}$.

□

Lemma 5.3. *Given* $X \in \mathrm{IG}_{>0}^\Omega(n+1, 2n)$*, let* $t \in \mathbb{R}_{>0}^{2n}$ *be such that* $t_i t_{n+i} = \frac{\Delta_{S(f_{n+i}^-)}(X)}{\Delta_{S(f_{n+i-1}^-)}(X)}$*. Then,* $t \cdot \overleftarrow{\tau}(X) \in \mathrm{OG}_{>0}(n+1, 2n)$.

Proof. By Corollary 4.3, there exists $(Y, s_+, s_-) \in \widetilde{\mathrm{OG}}_{>0}(n+1, 2n)$ such that $\vec{\tau}_{\mathrm{elec}}(Y, s_+, s_-) = X$. If $B := \Psi_G(Y, s_+, s_-)$, then by Lemma 5.2, $t_i t_{n+i} = \frac{B_{d_{i-1}}B_{d_{n+i-1}}}{B_{d_i}B_{d_{n+i}}}$. Therefore, there exists $\lambda \in \mathbb{R}_{>0}^{2n}$ such that $t_i = \lambda_i \frac{B_{d_{i-1}}}{B_{d_i}}$ and $\lambda_{i+n} = \frac{1}{\lambda_i}$. Let $\mu \in \mathbb{R}_{>0}^{2n}$ be given by $\mu_i := \frac{B_{d_{i-1}}}{B_{d_i}}$. By definition, $\vec{\tau}_{\mathrm{elec}}(Y, s_+, s_-) = \mu \cdot \vec{\tau}(Y)$, so by Proposition 2.18(2) and Theorem 2.13, we have

$$t \cdot \overleftarrow{\tau}(X) = t \cdot \overleftarrow{\tau}(\mu \cdot \vec{\tau}(Y)) = t \cdot \mu^{-1} \cdot \overleftarrow{\tau}(\vec{\tau}(Y)) = \lambda \cdot Y.$$

Since $Y \in \mathrm{OG}_{>0}(n+1, 2n)$ and λ preserves Q, $\lambda \cdot Y \in \mathrm{OG}_{>0}(n+1, 2n)$. □

Example 5.4. Consider the electrical network (G, c) in Figure 7(a). We compute

$$\mathrm{Meas}_{G_+} \circ j_G^+(c) = [e_{123} + ce_{124} + e_{134} + ce_{234}] \in \mathrm{IG}_{>0}^\Omega(3, 4),$$

which is $\mathrm{Pl}(X)$, where $X = \text{row span}\begin{bmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & 0 & c \\ 0 & 0 & -1 & -c \end{bmatrix}$. We have

$$S(f_1^-) = 123, S(f_2^-) = 234, S(f_3^-) = 134 \text{ and } S(f_4^-) = 124,$$

so we need to choose $t \in \mathbb{R}^4_{>0}$ such that

$$t_1 t_3 = \frac{\Delta_{134}(X)}{\Delta_{234}(X)} = \frac{1}{c} \text{ and } t_2 t_4 = \frac{\Delta_{124}(X)}{\Delta_{134}(X)} = c,$$

so $t_1 = \frac{1}{ct_3}$ and $t_2 = \frac{c}{t_4}$. Then, we compute

$$t \cdot \overleftarrow{\tau}(X) = \text{row span} \begin{bmatrix} \frac{1}{t_3} & \frac{c}{t_4} & 0 & 0 \\ \frac{1}{t_3 c} & 0 & 0 & \frac{t_4}{c} \\ 0 & -\frac{1}{t_4} & -t_3 & 0 \end{bmatrix}.$$

To check that $t \cdot \overleftarrow{\tau}(X) \in \mathrm{OG}_{>0}(3,4)$, we compute the orthogonal complement $(t \cdot \overleftarrow{\tau}(X))^\perp = \mathrm{span}(v)$, where $v = \left(\frac{1}{t_3 t_4}, \frac{c}{t_4^2}, \frac{ct_3}{t_4}, 1\right)$, and check that $Q(v,v) = \frac{1}{t_3 t_4} \cdot \frac{ct_3}{t_4} - \frac{c}{t_4^2} \cdot 1 = 0$.

Definition 5.5. Given $X \in \mathrm{IG}^\Omega_{>0}(n+1, 2n)$, let $t \in \mathbb{R}^{2n}_{>0}$ be such that $t_i t_{n+i} = \frac{\Delta_{S(f^-_{n+i})}(X)}{\Delta_{S(f^-_{n+i-1})}(X)}$ and $t_{n+1} = 1$, and let $Y := t \cdot \overleftarrow{\tau}(X)$. By Lemma 5.3, $Y \in \mathrm{OG}_{>0}(n+1, 2n)$. Let (Y, s_+, s_-) be the lift of Y to $\widetilde{\mathrm{OG}}_{>0}(n+1, 2n)$ such that $\Sigma_\varnothing(Y, s_+, s_-) = \Sigma_{\{1\}}(Y, s_+, s_-) = 1$. The *electrical left twist* $\overleftarrow{\tau}_{\mathrm{elec}} : \mathrm{IG}^\Omega_{>0}(n+1, 2n) \to \widetilde{\mathrm{OG}}_{>0}(n+1, 2n)/\mathbb{R}^{n+1}_{>0}$ is defined as $\overleftarrow{\tau}_{\mathrm{elec}}(X) := (Y, s_+, s_-)$.

Theorem 5.6. *The electrical left twist is well-defined in the sense that it is independent of the choice of $t \in \mathbb{R}^{2n}_{>0}$. The electrical right and left twists are mutually inverse homeomorphisms between $\widetilde{\mathrm{OG}}_{>0}(n+1, 2n)/\mathbb{R}^{n+1}_{>0}$ and $\mathrm{IG}^\Omega_{>0}(n+1, 2n)$ sitting in the commuting diagram*

$$\begin{array}{ccc} \mathcal{B}_G/\mathbb{R}^{n+1}_{>0} & \xrightarrow[\sim]{q_G} & \mathcal{R}_G \\ \Psi_G \uparrow \wr & & \wr \downarrow \mathrm{Meas}_{G_+} \circ j^+_G \\ \widetilde{\mathrm{OG}}_{>0}(n+1, 2n)/\mathbb{R}^{n+1}_{>0} & \underset{\overrightarrow{\tau}_{\mathrm{elec}},\ \sim}{\overset{\overleftarrow{\tau}_{\mathrm{elec}},\ \sim}{\rightleftarrows}} & \mathrm{IG}^\Omega_{>0}(n+1, 2n) \end{array},$$

gluing which we get

$$\begin{array}{ccc} \mathcal{B}_n/\mathbb{R}^{n+1}_{>0} & \xrightarrow[\sim]{q} & \mathcal{R}_n \\ \Psi \uparrow \wr & & \wr \downarrow \mathrm{Meas} \circ j^+ \\ \widetilde{\mathrm{OG}}_{>0}(n+1, 2n)/\mathbb{R}^{n+1}_{>0} & \underset{\overrightarrow{\tau}_{\mathrm{elec}},\ \sim}{\overset{\overleftarrow{\tau}_{\mathrm{elec}},\ \sim}{\rightleftarrows}} & \mathrm{IG}^\Omega_{>0}(n+1, 2n) \end{array}.$$

Proof. If $t' \in \mathbb{R}^{2n}_{>0}$ is another choice, then there exists $\lambda \in \mathbb{R}^{2n}_{>0}$ such that $\lambda_1 = \lambda_{n+1} = 1$ and $t'_i = t_i \lambda_i$ for all $i \in [2n]$. Let $Y' := t' \cdot \overleftarrow{\tau}(X) = \lambda \cdot Y$ and let (Y', s'_+, s'_-) denote its lift to $\widetilde{\mathrm{OG}}_{>0}(n+1, 2n)$ such that $\Sigma_\varnothing(Y', s'_+, s'_-) = \Sigma_{\{1\}}(Y', s'_+, s'_-) = 1$. Then,

$$\Sigma_J(Y', s'_+, s'_-) = \begin{cases} \frac{1}{\sqrt{\lambda_1 \cdots \lambda_n}} \frac{\prod_{i \in J} \sqrt{\lambda_i}}{\prod_{i \notin J} \sqrt{\lambda_i}} \Sigma_J(Y, s_+, s_-) & \text{if } J \text{ is even, and} \\ \frac{\sqrt{\lambda_1}}{\sqrt{\lambda_2 \cdots \lambda_n}} \frac{\prod_{i \in J} \sqrt{\lambda_i}}{\prod_{i \notin J} \sqrt{\lambda_i}} \Sigma_J(Y, s_+, s_-) & \text{if } J \text{ is odd.} \end{cases}$$

Since $t_{n+1} = t'_{n+1} = 1$, we have $\lambda_1 = \lambda_{n+1} = 1$. Under the action of $\mathbb{R}^{n+1}_{>0}$ on $\widetilde{\mathrm{OG}}_{>0}(n+1, 2n)$, $(\sqrt{\lambda_2 \cdots \lambda_n}, (1, \sqrt{\lambda_2}, \ldots, \sqrt{\lambda_n})) \cdot (Y, s_+, s_-) = (Y', s'_+, s'_-)$. Therefore, $\overleftarrow{\tau}_{\mathrm{elec}}(X)$ is well-defined.

Given $(Y, s_+, s_-) \in \widetilde{\mathrm{OG}}_{>0}(n+1, 2n)$, we can use the action of $\mathbb{R}^{n+1}_{>0}$ to make $\Sigma_\varnothing(Y, s_+, s_-) = \Sigma_{\{1\}}(Y, s_+, s_-) = 1$. If we choose $t_i := \frac{\Sigma_{J(d_{i-1})}(Y,s_+,s_-)}{\Sigma_{J(d_i)}(Y,s_+,s_-)}$ to define the electrical left twist, then $\overleftarrow{\tau}_{\text{elec}} \circ \vec{\tau}_{\text{elec}}(Y, s_+, s_-) = (Y, s_+, s_-)$, so $\vec{\tau}_{\text{elec}}$ is injective with left inverse $\overleftarrow{\tau}_{\text{elec}}$. By Corollary 4.3, $\vec{\tau}_{\text{elec}}$ is also surjective, so $\overleftarrow{\tau}_{\text{elec}}$ is the two-sided inverse.

□

Example 5.7. Recall Example 5.4 and set $t_3 = 1$. Using row operations, we can write

$$Y := t \cdot \overleftarrow{\tau}(X) = \text{row span} \begin{bmatrix} 1 & \frac{c}{t_4} & 0 & 0 \\ 0 & \frac{1}{t_4} & 1 & 0 \\ 0 & -\frac{c}{t_4^2} & 0 & 1 \end{bmatrix}.$$

Letting $\Sigma_\varnothing(Y, s_+, s_-) = \Sigma_1(Y, s_+, s_-) = 1$ and comparing with the matrix in Example 4.4, we see that $\Sigma_2(Y, s_+, s_-) = \frac{c}{t_4}$ and $\Sigma_{12}(Y, s_+, s_-) = \frac{1}{t_4}$. Therefore, $q_G \circ \Psi_G(Y, s_+, s_-)$ assigns to the edge the conductance

$$\frac{\Sigma_1(Y, s_+, s_-)\Sigma_2(Y, s_+, s_-)}{\Sigma_\varnothing(Y, s_+, s_-)\Sigma_{12}(Y, s_+, s_-)} = \frac{1 \cdot \frac{c}{t_4}}{1 \cdot \frac{1}{t_4}} = c,$$

verifying commutativity of the diagram in Theorem 5.6.

6. An example of the inverse map

In this section, we work out in detail the inverse map when $n = 3$. For background on electrical networks, the Laplacian and the response matrix, see [Ken12]. Let (G, c) denote the electrical network in Figure 4(a). The Laplacian is

$$\Delta = \begin{array}{c} \begin{matrix} b_1 & b_2 & b_3 & u \end{matrix} \\ \begin{bmatrix} a & 0 & 0 & -a \\ 0 & b & 0 & -b \\ 0 & 0 & c & -c \\ -a & -b & -c & a+b+c \end{bmatrix} \end{array} \begin{matrix} b_1 \\ b_2 \\ b_3 \\ u \end{matrix},$$

from which the response matrix is obtained as the Schur complement

(6.1)
$$L = -\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} + \begin{bmatrix} -a \\ -b \\ -c \end{bmatrix} \begin{bmatrix} a+b+c \end{bmatrix}^{-1} \begin{bmatrix} -a & -b & -c \end{bmatrix} = \begin{bmatrix} -\frac{a(b+c)}{a+b+c} & \frac{ab}{a+b+c} & \frac{ac}{a+b+c} \\ \frac{ab}{a+b+c} & -\frac{b(a+c)}{a+b+c} & \frac{bc}{a+b+c} \\ \frac{ac}{a+b+c} & \frac{bc}{a+b+c} & -\frac{c(a+b)}{a+b+c} \end{bmatrix}.$$

By [CGS21, Theorem 1.8], the point $X := \mathrm{Pl}^{-1} \circ \mathrm{Meas}_{G_+} \circ j_G^+(c) \in \mathrm{IG}^\Omega_{>0}(4, 6)$ is

$$\text{row span} \begin{bmatrix} 0 & 1 & 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & L_{12} & 0 & -L_{12} - L_{13} \\ 0 & 0 & -1 & -L_{12} - L_{23} & 0 & L_{12} \\ 0 & 0 & 0 & L_{23} & 1 & L_{13} \end{bmatrix}.$$

Using the face labels that have been computed in Figure 6(b), to define the electrical left twist, we need to choose $t \in \mathbb{R}^6_{>0}$ such that

$$t_1 t_4 = \frac{\Delta_{1456}(X)}{\Delta_{3456}(X)} = \frac{L_{23}}{L_{13}}, t_2 t_5 = \frac{\Delta_{1256}(X)}{\Delta_{1456}(X)} = \frac{L_{12}}{L_{23}}, t_3 t_6 = \frac{\Delta_{1236}(X)}{\Delta_{1256}(X)} = \frac{L_{13}}{L_{12}} \text{ and } t_4 = 1,$$

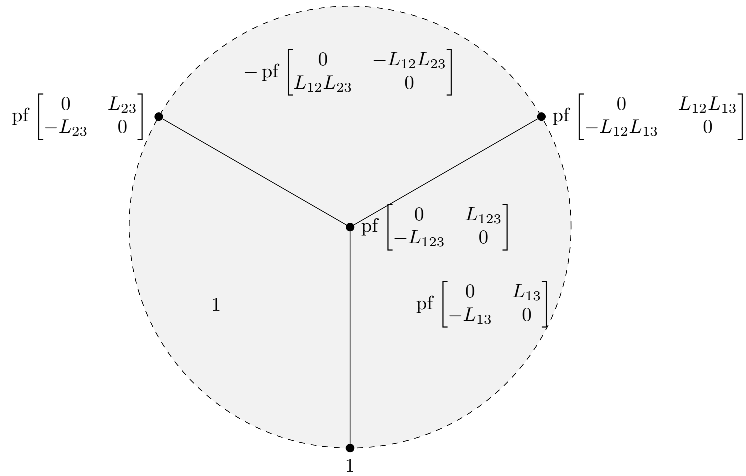


FIGURE 8. $\Psi_G \circ \overleftarrow{\tau}_{\mathrm{elec}}(X)$, where $L_{123} := L_{12}L_{13} + L_{12}L_{23} + L_{13}L_{23}$.

so let us take $t_1 = \frac{L_{23}}{L_{13}}, t_2 = L_{12}, t_3 = L_{13}, t_4 = 1, t_5 = \frac{1}{L_{23}}$ and $t_6 = \frac{1}{L_{12}}$. We compute

$$Y := t \cdot \overleftarrow{\tau}(X) = \text{row span} \begin{bmatrix} \frac{L_{12}L_{13}+L_{12}L_{23}+L_{13}L_{23}}{L_{13}} & L_{12} & 0 & 0 & 0 & -\frac{1}{L_{13}} \\ \frac{L_{23}}{L_{13}} & 0 & 0 & 0 & -\frac{1}{L_{12}} & -\frac{1}{L_{12}L_{13}} \\ -1 & -1 & -L_{13} & 0 & 0 & 0 \\ 0 & 0 & L_{12} & \frac{1}{L_{23}} & \frac{1}{L_{23}} & 0 \end{bmatrix}.$$

The skew symmetric matrices M_+ and M_- as in Section 3.5 are

$$M_+ = \begin{bmatrix} 0 & L_{23} & L_{12}L_{13} + L_{12}L_{23} + L_{13}L_{23} \\ -L_{23} & 0 & L_{12}L_{13} \\ -(L_{12}L_{13} + L_{12}L_{23} + L_{13}L_{23}) & -L_{12}L_{13} & 0 \end{bmatrix} \text{ and}$$

$$M_- = \begin{bmatrix} 0 & 1 & L_{13} \\ -1 & 0 & -L_{12}L_{23} \\ -L_{13} & L_{12}L_{23} & 0 \end{bmatrix}.$$

Using the labels in Figure 6(a) and Proposition 3.15, we get that $\Psi_G \circ \overleftarrow{\tau}_{\text{elec}}(X)$ is as shown in Figure 8, so $q_G \circ \Psi_G \circ \overleftarrow{\tau}_{\text{elec}}(X)$ is given by

$$c(ub_1) = -\frac{\operatorname{pf}\begin{bmatrix} 0 & L_{12}L_{13}+L_{12}L_{23}+L_{13}L_{23} \\ -(L_{12}L_{13}+L_{12}L_{23}+L_{13}L_{23}) & 0 \end{bmatrix} \operatorname{pf}\begin{bmatrix} 0 & L_{12}L_{13} \\ -L_{12}L_{13} & 0 \end{bmatrix}}{\operatorname{pf}\begin{bmatrix} 0 & L_{13} \\ -L_{13} & 0 \end{bmatrix} \operatorname{pf}\begin{bmatrix} 0 & -L_{12}L_{23} \\ L_{12}L_{23} & 0 \end{bmatrix}}$$
$$= \frac{L_{12}L_{13}+L_{12}L_{23}+L_{13}L_{23}}{L_{23}},$$
$$c(ub_2) = \frac{\operatorname{pf}\begin{bmatrix} 0 & L_{12}L_{13}+L_{12}L_{23}+L_{13}L_{23} \\ -(L_{12}L_{13}+L_{12}L_{23}+L_{13}L_{23}) & 0 \end{bmatrix}}{\operatorname{pf}\begin{bmatrix} 0 & L_{13} \\ -L_{13} & 0 \end{bmatrix}}$$
$$= \frac{L_{12}L_{13}+L_{12}L_{23}+L_{13}L_{23}}{L_{13}},$$
$$c(ub_3) = -\frac{\operatorname{pf}\begin{bmatrix} 0 & L_{12}L_{13}+L_{12}L_{23}+L_{13}L_{23} \\ -(L_{12}L_{13}+L_{12}L_{23}+L_{13}L_{23}) & 0 \end{bmatrix} \operatorname{pf}\begin{bmatrix} 0 & L_{23} \\ -L_{23} & 0 \end{bmatrix}}{\operatorname{pf}\begin{bmatrix} 0 & -L_{12}L_{23} \\ L_{12}L_{23} & 0 \end{bmatrix}}$$
$$= \frac{L_{12}L_{13}+L_{12}L_{23}+L_{13}L_{23}}{L_{12}}.$$

From (6.1), we have $L_{12} = \frac{ab}{a+b+c}, L_{13} = \frac{ac}{a+b+c}, L_{23} = \frac{bc}{a+b+c}$, so $L_{12}L_{13}+L_{12}L_{23}+L_{13}L_{23} = \frac{abc}{a+b+c}$. Plugging in these formulas, we get $c(ub_1) = a, c(ub_2) = b, c(ub_3) = c$.

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