# Inverse problem for $\mathrm{sl}(2)$ lattices 

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#### Abstract

We consider the inverse problem for periodic sl(2) lattices as a canonical transformation from the separation to local variables. A new concept of a factorized separation chain is introduced allowing to solve the inverse problem explicitly. The method is applied to an arbitrary representation of the corresponding Sklyanin algebra.


[^0]
## 1 Introduction

Let us denote a separating transform from the local variables ( $\mathbf{q}, \mathbf{p}$ ) to the separation variables $(\mathbf{u}, \mathbf{v})$ as $\mathcal{S}_{n}$,

$$
\begin{equation*}
\mathcal{S}_{n}: \quad(\mathbf{q}, \mathbf{p}) \mapsto(\mathbf{u}, \mathbf{v}) \tag{1.1}
\end{equation*}
$$

and the inverse separating map as $\mathcal{S}_{n}^{-1}$,

$$
\begin{equation*}
\mathcal{S}_{n}^{-1}: \quad(\mathbf{u}, \mathbf{v}) \mapsto(\mathbf{q}, \mathbf{p}) \tag{1.2}
\end{equation*}
$$

In [1], 2] and [3] Sklyanin worked out the $r$-matrix technique for the method of separation of variables. It was based on the 1976 separation of the (classical) periodic Toda lattice by van Moerbeke [7] and Flaschka-McLaughlin [5] and also on early 80's Gutzwiller's solution to the quantum periodic Toda lattice [6] and Komarov's ideas [7] on the relation between the quantum separation of variables and the quantum inverse scattering method. In those three papers, Sklyanin studied separating maps $\mathcal{S}_{n}$ for the Goryachev-Chaplygin top, periodic Toda lattice and Heisenberg magnet, respectively. He also studied the corresponding quantum separations $\hat{\mathcal{S}}_{n}$.

Further development of the separation method included: (i) the 3-particle elliptic Calogero-Moser system [8] and its trigonometric and $q$ - versions [9, 10, 11] for which both maps, $\hat{\mathcal{S}}_{3}$ and $\hat{\mathcal{S}}_{3}^{-1}$, were explicitly constructed, and (ii) recent construction of the direct map $\hat{\mathcal{S}}_{4}$ in the case of the 4 -particle Calogero-Moser system 12.

In his Nankai Lectures [3] Sklyanin announced a new method (Lecture 4) that allowed to reduce the $n$-variable spectral problem for the kernel of the inverse separating map $\hat{\mathcal{S}}_{n}^{-1}$ to (smaller) $p$ - and $(n-p)$-variable spectral problems.

Recently, Kharchev and Lebedev [13] used this method to describe the quantum separating map $\hat{\mathcal{S}}_{n}^{-1}$ for the Toda lattice.

In the present paper we show that, in the classical case, Sklyanin's approach is equivalent to explicit solution of the inverse problem for sl(2) integrable lattices. We introduce a new concept of the (inverse) factorized separation chain:

$$
\begin{equation*}
\mathcal{S}_{n}^{-1}=\mathcal{B}_{1} \circ \cdots \circ \mathcal{B}_{n-1} \circ \mathcal{B}_{n} \tag{1.3}
\end{equation*}
$$

Each of the factors $\mathcal{B}_{m}$ is constructed as a map between the separation variables of the $m$-particle lattice and those of the $(m-1)$-particle lattice, plus a pair of initial (local) variables $\left(q_{m}, p_{m}\right)$. By definition, the map $\mathcal{B}_{m}$ is the same as the map $\mathcal{B}_{n}$ with $n$ replaced by $m$, so that it acts only on $m$ degrees of freedom. This means that the whole chain (1.3) is generated by a recursive application of a single map $\mathcal{B}_{n}$.

Such factorization of the inverse separating map is explicitly performed for an arbitrary representation of the quadratic Sklyanin algebra in the considered case of the $2 \times 2$ Lax matrix. It represents an interesting algebraic structure of the intertwiner between the separation and lattice representations of the quadratic algebra. It also reveals a hidden algebraic structure of the inverse problem for integrable lattices.

In fact, the factorization (1.3) is a direct consequence of the multiplicative structure of the Lax (monodromy) matrix. Using this structure of the Lax matrix, we show that the corresponding transformation $\mathcal{B}_{n}$ : (i) exists, (ii) is unique and (iii) is a rational canonical map with a simple generating function. We derive such factorization first for the generic
case of the inhomogeneous Heisenberg magnet. The formulae for two degenerate cases, the periodic DST and Toda lattices, are also given.

The structure of the paper is following. In Section 2 we define our basic model. Sections 3 and 4 give the separation and lattice representations of the quadratic algebra, respectively. Section 5 describes the recursive procedure for solving the inverse problem. In Section 6 we find the generating function of the factorized separation map $\mathcal{B}_{n}$. In Sections 7 and 8 we apply our results to the DST and periodic Toda lattice. There are three Appendices with explicit formulae for the inverse separating maps in the case of 3 spins/particles.

## 2 Quadratic $r$-matrix algebra and the model

We study a class of finite-dimensional Liouville integrable systems described by the representations of the quadratic $r$-matrix Poisson algebra, or the Sklyanin algebra:

$$
\begin{equation*}
\{\stackrel{1}{L}(u), \stackrel{2}{L}(v)\}=r(u-v) \stackrel{1}{L}(u) \stackrel{2}{L}(v)-\stackrel{1}{L}(u) \stackrel{2}{L}(v) r(u-v) . \tag{2.1}
\end{equation*}
$$

We consider the simplest case of the $4 \times 4$ rational $r$-matrix $r(u)$ and $2 \times 2 L$-operator $L(u)$ :

$$
r(u)=\frac{\kappa}{u}\left(\begin{array}{cccc}
1 & 0 & 0 & 0  \tag{2.2}\\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{array}\right), \quad L(u)=\left(\begin{array}{cc}
A(u) & B(u) \\
C(u) & D(u)
\end{array}\right) .
$$

In (2.1) the standard notations for tensor products are used:

$$
\stackrel{1}{L}(u)=L(u) \otimes\left(\begin{array}{ll}
1 & 0  \tag{2.3}\\
0 & 1
\end{array}\right), \quad \stackrel{2}{L}(v)=\left(\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right) \otimes L(v) .
$$

In the left-hand side of (2.1) one has a $4 \times 4$ matrix with the Poisson brackets as its entries: $\left(\left\{\frac{1}{L}(u), \stackrel{2}{L}(v)\right\}\right)_{i j, k l} \equiv\left\{(L(u))_{i j},(L(v))_{k l}\right\}$. In the right-hand side of (2.1) there are $(4 \times 4)$ matrix products.

Sklyanin's bracket (2.1) amounts to having the following Poisson brackets between the elements $A(u), B(u), C(u)$ and $D(u)$ of the $L$-operator $L(u)$ :

$$
\begin{align*}
&\{A(u), A(v)\}=\{B(u), B(v)\}=\{C(u), C(v)\}=\{D(u), D(v)\}=0,  \tag{2.4}\\
&\{B(u), A(v)\}=\frac{\kappa}{u-v}(B(u) A(v)-B(v) A(u)),  \tag{2.5}\\
&\{C(u), A(v)\}=\frac{\kappa}{u-v}(A(u) C(v)-A(v) C(u)),  \tag{2.6}\\
&\{B(u), D(v)\}=\frac{\kappa}{u-v}(D(u) B(v)-D(v) B(u))  \tag{2.7}\\
&\{C(u), D(v)\}=\frac{\kappa}{u-v}(C(u) D(v)-C(v) D(u))  \tag{2.8}\\
&\{A(u), D(v)\}=\frac{\kappa}{u-v}(C(u) B(v)-C(v) B(u)) \tag{2.9}
\end{align*}
$$

$$
\begin{equation*}
\{B(u), C(v)\}=\frac{\kappa}{u-v}(D(u) A(v)-D(v) A(u)) . \tag{2.10}
\end{equation*}
$$

We choose the entries of our $L$-operator $L_{n}(u)$ to be polynomials of degree $n$ :

$$
\begin{align*}
L_{n}(u) & =\left(\begin{array}{ll}
A_{n}(u) & B_{n}(u) \\
C_{n}(u) & D_{n}(u)
\end{array}\right)  \tag{2.11}\\
& =\left(\begin{array}{ll}
\alpha u^{n}+A_{n, 1} u^{n-1}+\ldots+A_{n, n} & \beta u^{n}+B_{n, 1} u^{n-1}+\ldots+B_{n, n} \\
\gamma u^{n}+C_{n, 1} u^{n-1}+\ldots+C_{n, n} & \delta u^{n}+D_{n, 1} u^{n-1}+\ldots+D_{n, n}
\end{array}\right) .
\end{align*}
$$

Notice that the leading coefficients, $\alpha, \beta, \gamma, \delta$, are Casimirs of the bracket (2.1). By another property of the bracket, there are $2 n$ further Casimirs $Q_{k}, k=1, \ldots, 2 n$, which are coefficients of the $\operatorname{det} L_{n}(u)$ :

$$
\begin{equation*}
\operatorname{det} L_{n}(u)=(\alpha \delta-\beta \gamma) u^{2 n}+Q_{1} u^{2 n-1}+\ldots+Q_{2 n} \tag{2.12}
\end{equation*}
$$

Therefore, we have a $4 n$-dimensional space of the coefficients $A_{n, i}, B_{n, i}, C_{n, i}, D_{n, i}$ of the matrix $L_{n}(u)$ with $2 n$ Casimir operators, leaving us with $n$ degrees of freedom. Independent, Poisson involutive integrals of motion $H_{i}, i=1, \ldots, n$, are given by the coefficients of the $\operatorname{tr} L_{n}(u)$ :

$$
\begin{equation*}
\operatorname{tr} L_{n}(u)=(\alpha+\delta) u^{n}+H_{1} u^{n-1}+\ldots+H_{n}, \quad\left\{H_{i}, H_{j}\right\}=0 . \tag{2.13}
\end{equation*}
$$

These define a Liouville integrable system which is our generic model for the whole paper. The two first Hamiltonians of the system are

$$
\begin{equation*}
H_{1}=A_{n, 1}+D_{n, 1}, \quad H_{2}=A_{n, 2}+D_{n, 2} \tag{2.14}
\end{equation*}
$$

## 3 Separation representation

Our first aim is to construct a separation representation for the quadratic algebra (2.4)(2.11). In this special representation one has $n$ canonical pairs of variables, $u_{i}, v_{i}, i=$ $1, \ldots, n$, having the standard Poisson brackets,

$$
\begin{equation*}
\left\{u_{i}, u_{j}\right\}=\left\{v_{i}, v_{j}\right\}=0, \quad\left\{v_{i}, u_{j}\right\}=\delta_{i j} \tag{3.1}
\end{equation*}
$$

with the $u$-variables being $n$ zeros of the polynomial $B_{n}(u)$ and the $\mathrm{e}^{\kappa v}$-variables being values of the polynomial $A_{n}(u)$ at those zeros,

$$
\begin{equation*}
B_{n}\left(u_{i}\right)=0, \quad \mathrm{e}^{\kappa v_{i}}=A_{n}\left(u_{i}\right), \quad i=1, \ldots, n, \tag{3.2}
\end{equation*}
$$

so that the pairs $\left(\mathrm{e}^{\kappa v_{i}}, u_{i}\right), i=1, \ldots, n$, belong to the spectral curve $\mathcal{C}_{n}$ of the $L$-operator $L_{n}(u)$ :

$$
\begin{equation*}
\mathcal{C}_{n}=\left\{\left(\mathrm{e}^{\kappa v}, u\right) \in \mathbb{C}^{2} \mid \operatorname{det}\left(L_{n}(u)-\mathrm{e}^{\kappa v}\right)=0\right\}, \quad \operatorname{det}\left(L_{n}\left(u_{i}\right)-\mathrm{e}^{\kappa v_{i}}\right)=0 . \tag{3.3}
\end{equation*}
$$

Let us parameterize the determinant of the $L$-operator as follows:

$$
\begin{equation*}
\operatorname{det} L_{n}(u)=(\alpha \delta-\beta \gamma) \prod_{i=1}^{n}\left(\left(u-c_{i}\right)^{2}+\kappa^{2} s_{i}^{2}\right) \tag{3.4}
\end{equation*}
$$

The interpolation data (3.2) plus $n+1$ identities, $A_{n}\left(u_{i}\right) D_{n}\left(u_{i}\right)=\operatorname{det} L_{n}\left(u_{i}\right)$ and $\operatorname{det} L_{n}(u)$ $=A_{n}(u) D_{n}(u)-B_{n}(u) C_{n}(u)$, allow us to construct the needed separation representation for the whole algebra:

$$
\begin{align*}
B_{n}(u) & =\beta\left(u-u_{1}\right)\left(u-u_{2}\right) \cdots\left(u-u_{n}\right)  \tag{3.5}\\
A_{n}(u) & =B_{n}(u)\left(\frac{\alpha}{\beta}+\sum_{i=1}^{n} \frac{\mathrm{e}^{\kappa v_{i}}}{\left(u-u_{i}\right) B_{n}^{\prime}\left(u_{i}\right)}\right)  \tag{3.6}\\
D_{n}(u) & =B_{n}(u)\left(\frac{\delta}{\beta}+\sum_{i=1}^{n} \frac{\operatorname{det} L_{n}\left(u_{i}\right) \mathrm{e}^{-\kappa v_{i}}}{\left(u-u_{i}\right) B_{n}^{\prime}\left(u_{i}\right)}\right)  \tag{3.7}\\
C_{n}(u) & =\frac{A_{n}(u) D_{n}(u)-\operatorname{det} L_{n}(u)}{B_{n}(u)} \tag{3.8}
\end{align*}
$$

One can easily check that the brackets (2.4)-(2.10) imply the brackets (3.1) and vice versa.

## 4 Lattice representation

Another important representation of the quadratic algebra with the generators $A_{n, i}, B_{n, i}$, $C_{n, i}$ and $D_{n, i}$ comes as a consequence of the co-multiplication property of the algebra (2.1). Essentially, it means that the $L$-operator (2.11) can be factorized into a product of elementary matrices, each containing only one degree of freedom. In this picture, our main model turns out to be an $n$-site Heisenberg magnet, which is an integrable lattice of $n \mathrm{sl}(2)$ spins with nearest neighbour interaction.

In this lattice representation the $L$-operator (2.11) acquires the following form:

$$
\begin{gather*}
L_{n}(u)=\ell_{n}\left(u-c_{n}\right) \ell_{n-1}\left(u-c_{n-1}\right) \cdots \ell_{1}\left(u-c_{1}\right)\left(\begin{array}{ll}
\alpha & \beta \\
\gamma & \delta
\end{array}\right),  \tag{4.1}\\
\ell_{i}(u):=\left(\begin{array}{cc}
u-\mathrm{i} \kappa s_{i}^{(3)} & -\mathrm{i} \kappa s_{i}^{(-)} \\
-\mathrm{i} \kappa s_{i}^{(+)} & u+\mathrm{i} \kappa s_{i}^{(3)}
\end{array}\right), \quad i=1, \ldots, n . \tag{4.2}
\end{gather*}
$$

The local variables $\mathbf{s}_{i}, i=1, \ldots, n$, are generators of $n$ copies of the $\mathrm{sl}(2)$ Poisson algebra:

$$
\begin{equation*}
\left\{s^{(3)}, s^{( \pm)}\right\}=\mp \mathrm{i} s^{( \pm)}, \quad\left\{s^{(-)}, s^{(+)}\right\}=2 \mathrm{i} s^{(3)} \tag{4.3}
\end{equation*}
$$

The (non-local) Hamiltonians $H_{1}$ and $H_{2}(2.14)$ are as follows:

$$
\begin{align*}
H_{1}= & -\mathrm{i} \kappa \sum_{i=1}^{n}\left((\alpha-\delta) s_{i}^{(3)}+\beta s_{i}^{(+)}+\gamma s_{i}^{(-)}\right)-(\alpha+\delta) \sum_{i=1}^{n} c_{i}  \tag{4.4}\\
H_{2}= & -\kappa^{2} \sum_{i>j}\left[\alpha\left(s_{i}^{(3)} s_{j}^{(3)}+s_{i}^{(-)} s_{j}^{(+)}\right)+\beta\left(s_{i}^{(+)} s_{j}^{(3)}-s_{i}^{(3)} s_{j}^{(+)}\right)\right.  \tag{4.5}\\
& \left.+\gamma\left(s_{i}^{(3)} s_{j}^{(-)}-s_{i}^{(-)} s_{j}^{(3)}\right)+\delta\left(s_{i}^{(3)} s_{j}^{(3)}+s_{i}^{(+)} s_{j}^{(-)}\right)\right] \\
& +\sum_{j \neq i} c_{j}\left(\frac{\alpha+\delta}{2} c_{i}+\mathrm{i} \kappa(\alpha-\delta) s_{i}^{(3)}+\mathrm{i} \kappa \beta s_{i}^{(+)}+\mathrm{i} \kappa \gamma s_{i}^{(-)}\right)
\end{align*}
$$

We use the following 'holomorphic' representation for the local spins $\mathbf{s}_{i}$, and correspondingly for the local $L$-operators $\ell_{i}(u)$, in terms of $n$ pairs of canonical variables $\mathbf{q}$ and
p :

$$
\begin{gather*}
\ell_{i}(u)=\left(\begin{array}{cc}
u+\mathrm{i} \kappa\left(\mathrm{i} q_{i} p_{i}-s_{i}\right) & -\mathrm{i} \kappa p_{i} \\
-\mathrm{i} \kappa\left(q_{i}^{2} p_{i}+2 \mathrm{i} s_{i} q_{i}\right) & u-\mathrm{i} \kappa\left(\mathrm{i} q_{i} p_{i}-s_{i}\right)
\end{array}\right), \quad i=1, \ldots, n,  \tag{4.6}\\
\left\{q_{i}, q_{j}\right\}=\left\{p_{i}, p_{j}\right\}=0, \quad\left\{p_{i}, q_{j}\right\}=\delta_{i j} . \tag{4.7}
\end{gather*}
$$

Here, the variables $s_{i}$ are the spin values: $\mathrm{s}_{i}^{2}=\left(s_{i}^{(3)}\right)^{2}+s_{i}^{(+)} s_{i}^{(-)}=s_{i}^{2}, i=1, \ldots, n$. One can easily check that the brackets (2.4)-(2.10) imply the brackets (4.7) and vice versa.

Our aim is to describe in explicit terms the transformation between the two representations outlined in this and in the previous Section, notably in the direction from the variables $\mathbf{u}$ and $\mathbf{v}$ to the local spins $\mathbf{s}_{1}, \ldots, \mathbf{s}_{n}$ or to the variables $\mathbf{q}$ and $\mathbf{p}$. In other words, we present below a constructive solution to the inverse problem which provides the formulae for the local variables in terms of separation variables for a large class of integrable lattices.

## 5 Inverse problem

The formulae (3.2) are the defining equations for the symplectic separating map $\mathcal{S}_{n}$ from the initial (local) variables to the separation variables. A natural question arises whether this map can be inverted in a sensible way. The answer is affirmative.

The map $\mathcal{S}_{n}$ and its inverse are rather complicated canonical transforms, meaning that it is difficult to write down their corresponding generating functions or to quantize these maps starting directly from the defining equations (3.2). Our prime motivation therefore is to solve the inverse problem explicitly in such a way that consequent quantization is straightforward.

In order to construct our solution to the inverse problem we shall introduce a new concept of a factorized separation chain. It means that the inverse separating transform $\mathcal{S}_{n}^{-1}$ will be factorized into a composition (a chain) of elementary canonical transforms $\mathbb{\square}$. The meaning of the factors can be understood from considering the first map in the chain, $\mathcal{B}_{n}$, which, in a sense, represents a transformation between the model with $n$ degrees of freedom and the model with $n-1$ degrees of freedom. Consequent iteration of the found map will then result in the needed inverse separating transform $\mathcal{S}_{n}^{-1}$ :

$$
\begin{equation*}
\mathcal{S}_{n}^{-1}=\mathcal{B}_{1} \circ \cdots \circ \mathcal{B}_{n-1} \circ \mathcal{B}_{n} . \tag{5.1}
\end{equation*}
$$

Let us construct a canonical transformation that factorizes out the $n$th local $L$-operator $\ell_{n}\left(u-c_{n}\right)$ from the $n$-spin $L$-operator $L_{n}(u)$ (cf. (4.1)). This is the transformation $\mathcal{B}_{n}$. That is to say that we have the following matrix equation:

$$
\begin{align*}
L_{n}(u)= & \left(\begin{array}{cc}
u-c_{n}+\mathrm{i} \kappa\left(\mathrm{i} q_{n} p_{n}-s_{n}\right) & -\mathrm{i} \kappa p_{n} \\
-\mathrm{i} \kappa\left(q_{n}^{2} p_{n}+2 \mathrm{i} s_{n} q_{n}\right) & u-c_{n}-\mathrm{i} \kappa\left(\mathrm{i} q_{n} p_{n}-s_{n}\right)
\end{array}\right) \times  \tag{5.2}\\
& \times L_{n-1}(u),
\end{align*}
$$

where the $n$-spin matrix on the left is given in terms of separation variables $u_{i}, v_{i}, i=$ $1, \ldots, n$, by interpolation formulae (3.5)-(3.8) and the $(n-1)$-spin matrix on the right is

[^1]given by similar formulae in terms of its own separation variables $\int \tilde{u}_{j}, \tilde{v}_{j}, j=1, \ldots, n-1$ :
\[

$$
\begin{align*}
& B_{n-1}(u)=\beta\left(u-\tilde{u}_{1}\right)\left(u-\tilde{u}_{2}\right) \cdots\left(u-\tilde{u}_{n-1}\right)  \tag{5.3}\\
& A_{n-1}(u)=B_{n-1}(u)\left(\frac{\alpha}{\beta}+\sum_{j=1}^{n-1} \frac{\mathrm{e}^{\kappa \tilde{v}_{j}}}{\left(u-\tilde{u}_{j}\right) B_{n-1}^{\prime}\left(\tilde{u}_{j}\right)}\right),  \tag{5.4}\\
& D_{n-1}(u)=B_{n-1}(u)\left(\frac{\delta}{\beta}+\sum_{j=1}^{n-1} \frac{\operatorname{det} L_{n-1}\left(\tilde{u}_{j}\right) \mathrm{e}^{-\kappa \tilde{v}_{j}}}{\left(u-\tilde{u}_{j}\right) B_{n-1}^{\prime}\left(\tilde{u}_{j}\right)}\right),  \tag{5.5}\\
& C_{n-1}(u)=\frac{A_{n-1}(u) D_{n-1}(u)-\operatorname{det} L_{n-1}(u)}{B_{n-1}(u)} . \tag{5.6}
\end{align*}
$$
\]

Transformation $\mathcal{B}_{n}$ maps separation variables $u_{i}, v_{i}, i=1, \ldots, n$, into new separation variables $\tilde{u}_{j}, \tilde{v}_{j}, j=1, \ldots, n-1$, which parameterize the matrix $L_{n-1}(u)$, and into a pair of local variables, $q_{n}$ and $p_{n}$. It is a single-valued rational map. Indeed, rewrite the matrix equation (5.2) in the equivalent form,

$$
\begin{equation*}
L_{n-1}(u)=\frac{1}{\operatorname{det} \ell_{n}\left(u-c_{n}\right)} \ell_{n}^{\wedge}\left(u-c_{n}\right) L_{n}(u) \tag{5.7}
\end{equation*}
$$

where $\ell_{n}^{\wedge}\left(u-c_{n}\right)$ stands for the adjoint matrix. Equating to zero the residues of the righthand side at two zeros $c_{n} \pm \mathrm{i} \kappa s_{n}$ of the $\operatorname{det} \ell_{n}\left(u-c_{n}\right)$ and solving the resulting equations in $q_{n}$ and $p_{n}$, we obtain

$$
\begin{gather*}
q_{n}=-\mathrm{i} \frac{D_{n}\left(c_{n}-\mathrm{i} \kappa s_{n}\right)}{B_{n}\left(c_{n}-\mathrm{i} \kappa s_{n}\right)},  \tag{5.8}\\
p_{n}=\frac{2 s_{n} B_{n}\left(c_{n}+\mathrm{i} \kappa s_{n}\right) B_{n}\left(c_{n}-\mathrm{i} \kappa s_{n}\right)}{B_{n}\left(c_{n}+\mathrm{i} \kappa s_{n}\right) D_{n}\left(c_{n}-\mathrm{i} \kappa s_{n}\right)-B_{n}\left(c_{n}-\mathrm{i} \kappa s_{n}\right) D_{n}\left(c_{n}+\mathrm{i} \kappa s_{n}\right)} . \tag{5.9}
\end{gather*}
$$

This gives the answer for the $n$th pair of local variables. Now, the formula (5.7) uniquely defines the matrix $L_{n-1}(u)$ in terms of the separation variables $\mathbf{u}$ and $\mathbf{v}$. The procedure can be recursively repeated, leading to reconstruction of all local variables $\mathbf{q}$ and $\mathbf{p}$. For instance, explicit formulae for the inverse separating map $\mathcal{S}_{3}^{-1}$ for the homogeneous chain of three 0 -spins, i.e. when all parameters vanish, $s_{i}=c_{i}=0$, are given in the Appendix A. The corresponding formulae for the case of non-zero parameters $s_{i}$ and $c_{i}$ are too long.

## 6 Generating function

In the previous Section we introduced the factorized separation chain $\mathcal{S}_{n}^{-1}=\mathcal{B}_{1} \circ \cdots \circ$ $\mathcal{B}_{n-1} \circ \mathcal{B}_{n}$ that maps the canonical variables in the following order:

$$
\begin{equation*}
\mathcal{S}_{n}^{-1}:(\mathbf{u}, \mathbf{v}) \stackrel{\mathcal{B}_{n}}{\mapsto}\left(\tilde{\mathbf{u}}, \tilde{\mathbf{v}} \mid q_{n}, p_{n}\right) \stackrel{\mathcal{B}_{n-1}}{\mapsto}\left(\tilde{\tilde{\mathbf{u}}}, \tilde{\tilde{\mathbf{v}}} \mid q_{n-1}, p_{n-1} ; q_{n}, p_{n}\right) \cdots \stackrel{\mathcal{B}_{7}}{\mapsto}(\mathbf{q}, \mathbf{p}) . \tag{6.1}
\end{equation*}
$$

In this inverse separating chain, the initial variables $(\mathbf{u}, \mathbf{v})$ are the separation variables for the starting $n$-spin integrable system and the terminal variables ( $\mathbf{q}, \mathbf{p}$ ) are the local variables. The intermediate variables ( $\tilde{\mathbf{u}}, \tilde{\mathbf{v}})$ have the meaning of being the separation variables for the $(n-1)$-spin system, the variables $(\tilde{\tilde{\mathbf{u}}}, \tilde{\tilde{\mathbf{v}}})$ being those for the $(n-2)$-spin

[^2]system, and so on. Notice that every next transformation in the chain acts only on the new separation pairs from the outcome of a previous transform, without touching the pairs $\left(q_{i}, p_{i}\right)$ of local variables that have already been factorized out, so that the transform $\mathcal{B}_{m}$ acts on less and less variables (degrees of freedom) as its index $m$ decreases.

In this Section we show that the factors $\mathcal{B}_{m}, m=1, \ldots, n$, of the composition (6.1) are very simple canonical transforms and we describe them in completely explicit terms by presenting their generating functions. The complexity of the compound transform $\mathcal{S}_{n}^{-1}$ is therefore being explained as a composition of these elementary transforms. Such factorization of a complex separating transform must be a universal feature of all separating maps and it must provide a unified approach to their description. The application of this approach is not limited by the model chosen in this paper.

Let us start from the first map $\mathcal{B}_{n}$ and let us fix the $2 n$ variables of its generating function $F_{n}$ to be the coordinates $\mathbf{u}$ and, respectively, $\tilde{\mathbf{u}}$ and $q_{n}$, so that one has the following $2 n$ equations defining the map $\mathcal{B}_{n}$,

$$
\begin{align*}
\mathcal{B}_{n}: \quad v_{i} & =\frac{\partial F_{n}\left(\tilde{\mathbf{u}}, q_{n} \mid \mathbf{u}\right)}{\partial u_{i}}, \quad i=1, \ldots, n,  \tag{6.2}\\
\tilde{v}_{j} & =-\frac{\partial F_{n}\left(\tilde{\mathbf{u}}, q_{n} \mid \mathbf{u}\right)}{\partial \tilde{u}_{j}}, \quad j=1, \ldots, n-1,  \tag{6.3}\\
p_{n} & =-\frac{\partial F_{n}\left(\tilde{\mathbf{u}}, q_{n} \mid \mathbf{u}\right)}{\partial q_{n}} \tag{6.4}
\end{align*}
$$

We will first find these equations and then find the function $F_{n}\left(\tilde{\mathbf{u}}, q_{n} \mid \mathbf{u}\right)$.
Equation for the polynomial $B_{n}(u)$ from the equality (5.2) reads

$$
\begin{equation*}
B_{n}(u)=\left(u-c_{n}-\mathrm{i} \kappa s_{n}-\kappa q_{n} p_{n}\right) B_{n-1}(u)-\mathrm{i} \kappa p_{n} D_{n-1}(u) \tag{6.5}
\end{equation*}
$$

Substituting $u=\tilde{u}_{j}$, one obtains

$$
\begin{equation*}
B_{n}\left(\tilde{u}_{j}\right)=-\mathrm{i} \kappa p_{n} D_{n-1}\left(\tilde{u}_{j}\right), \quad j=1, \ldots, n-1 \tag{6.6}
\end{equation*}
$$

Equation for the polynomial $B_{n-1}(u)$ from the equality (5.7) reads

$$
\begin{equation*}
\operatorname{det}\left(\ell_{n}\left(u-c_{n}\right)\right) B_{n-1}(u)=\left(u-c_{n}+\mathrm{i} \kappa s_{n}+\kappa q_{n} p_{n}\right) B_{n}(u)+\mathrm{i} \kappa p_{n} D_{n}(u) \tag{6.7}
\end{equation*}
$$

Substituting $u=u_{i}$, one obtains

$$
\begin{equation*}
\operatorname{det}\left(\ell_{n}\left(u_{i}-c_{n}\right)\right) B_{n-1}\left(u_{i}\right)=\mathrm{i} \kappa p_{n} D_{n}\left(u_{i}\right), \quad i=1, \ldots, n . \tag{6.8}
\end{equation*}
$$

Equating the leading coefficients in $u$ in (6.5) and unwrapping the formulae (6.6) and (6.8), one finally obtains $2 n$ needed equations for the map $\mathcal{B}_{n}$ in the following form:

$$
\begin{align*}
p_{n} & =\frac{\beta}{\kappa} \frac{\sum_{i=1}^{n} u_{i}-\sum_{j=1}^{n-1} \tilde{u}_{j}-c_{n}-\mathrm{i} \kappa s_{n}}{\beta q_{n}+\mathrm{i} \delta},  \tag{6.9}\\
\mathrm{e}^{\kappa v_{i}} & =\frac{\mathrm{i} \kappa p_{n}(\alpha \delta-\beta \gamma)}{\beta} \prod_{m=1}^{n-1} \frac{\left(u_{i}-c_{m}\right)^{2}+\kappa^{2} s_{m}^{2}}{u_{i}-\tilde{u}_{m}}, \quad i=1, \ldots, n,  \tag{6.10}\\
\mathrm{e}^{\kappa \tilde{v}_{j}} & =\frac{-\mathrm{i} \kappa p_{n}(\alpha \delta-\beta \gamma)}{\beta\left(\tilde{u}_{j}-u_{n}\right)} \prod_{m=1}^{n-1} \frac{\left(\tilde{u}_{j}-c_{m}\right)^{2}+\kappa^{2} s_{m}^{2}}{\tilde{u}_{j}-u_{m}}, j=1, \ldots, n-1 . \tag{6.11}
\end{align*}
$$

These equations can be easily integrated, resulting in explicit formula for the generating function $F_{n}\left(\tilde{\mathbf{u}}, q_{n} \mid \mathbf{u}\right)$ of the map $\mathcal{B}_{n}$ :

$$
\begin{align*}
F_{n}\left(\tilde{\mathbf{u}}, q_{n} \mid \mathbf{u}\right)= & \frac{1}{\kappa} \sum_{i=1}^{n} \sum_{j=1}^{n-1} \Omega\left(\tilde{u}_{j}-u_{i}\right)  \tag{6.12}\\
& -\frac{1}{\kappa} \Omega\left(\sum_{j=1}^{n-1} \tilde{u}_{j}-\sum_{i=1}^{n} u_{i}+c_{n}+\mathrm{i} \kappa s_{n}\right) \\
& -\frac{1}{\kappa}\left(\sum_{j=1}^{n-1} \tilde{u}_{j}-\sum_{i=1}^{n} u_{i}\right) \log \frac{\mathrm{i}(\alpha \delta-\beta \gamma)}{\beta q_{n}+\mathrm{i} \delta} \\
& +\frac{1}{\kappa}\left(c_{n}+\mathrm{i} \kappa s_{n}\right) \log \left(\beta q_{n}+\mathrm{i} \delta\right)-\frac{\mathrm{i} \pi n}{\kappa} \sum_{i=1}^{n} u_{i} \\
& +\frac{1}{\kappa} \sum_{i=1}^{n} \sum_{j=1}^{n-1}\left(\Omega\left(u_{i}-c_{j}+\mathrm{i} \kappa s_{j}\right)+\Omega\left(u_{i}-c_{j}-\mathrm{i} \kappa s_{j}\right)\right) \\
& -\frac{1}{\kappa} \sum_{j=1}^{n-1} \sum_{j^{\prime}=1}^{n-1}\left(\Omega\left(\tilde{u}_{j}-c_{j^{\prime}}+\mathrm{i} \kappa s_{j^{\prime}}\right)+\Omega\left(\tilde{u}_{j}-c_{j^{\prime}}-\mathrm{i} \kappa s_{j^{\prime}}\right)\right) .
\end{align*}
$$

Here, the function $\Omega(u)$ is the anti-derivative of $\log (u)$ :

$$
\begin{equation*}
\Omega(u)=\int^{u} \log \left(u^{\prime}\right) d u^{\prime}=u(\log (u)-1)+C, \quad \frac{d \Omega(u)}{d u}=\log (u) . \tag{6.13}
\end{equation*}
$$

In order to obtain the corresponding formulae for the maps $\mathcal{B}_{m}, m=1, \ldots, n-1$, one must replace $n$ by $m$ in the formulae (6.9), (6.10), (6.11) and (6.12). It was already mentioned that a reduction in the number of variables happens along with decreasing of map's index.

The formulae in this Section completely describe the inverse separating map $\mathcal{S}_{n}^{-1}$ as a canonical map, through the explicit representation for its factors. In the next two Sections we consider two degenerate cases of the Heisenberg magnet: the integrable DST model and the periodic Toda lattice. The main reason for inserting these Sections is to exemplify the new concept of the factorized separation chain that has been introduced above. For these two degenerate cases we give definitions of the models, equations and generating functions for the basic map $\mathcal{B}_{n}$ and, finally, analogues of the rational map $\mathcal{S}_{3}^{-1}$ in the 3-particle case.

## $7 \quad$ Integrable DST model

The integrable case of the DST (discrete self-trapping) model with $n$ degrees of freedom was introduced in [14] and studied in (15]. It appears as a specialization of our basic model when several parameters vanish:

$$
\begin{equation*}
\beta=\gamma=\delta=0 \quad \text { and } \quad Q_{j}=0, \quad j=1, \ldots, n-1 \tag{7.1}
\end{equation*}
$$

We also put $\alpha=1$, leading to the $L$-operator

$$
L_{n}(u)=\left(\begin{array}{cc}
u^{n}+A_{n, 1} u^{n-1}+\ldots+A_{n, n} & B_{n, 1} u^{n-1}+\ldots+B_{n, n}  \tag{7.2}\\
C_{n, 1} u^{n-1}+\ldots+C_{n, n} & D_{n, 2} u^{n-2}+\ldots+D_{n, n}
\end{array}\right)
$$

Notice that $D_{n, 1} \equiv Q_{1}=0$. Set $Q_{n}=b^{n}$ and parameterize $\operatorname{det} L_{n}(u)$ as follows:

$$
\begin{equation*}
\operatorname{det} L_{n}(u)=b^{n}\left(u-c_{1}\right)\left(u-c_{2}\right) \cdots\left(u-c_{n}\right) \tag{7.3}
\end{equation*}
$$

Let us also choose $\kappa=-1$. By definition, the $L$-operator (7.2) obeys the quadratic relations of the Poisson algebra (2.4)-(2.10).

Separation variables $\left(\mathrm{e}^{-v_{i}}, u_{i}\right)$ are introduced as before:

$$
\begin{equation*}
B_{n}\left(u_{i}\right)=0, \quad \mathrm{e}^{-v_{i}}=A_{n}\left(u_{i}\right), \quad i=1, \ldots, n-1, \tag{7.4}
\end{equation*}
$$

the only difference now is that this gives only $n-1$ instead of $n$ separation pairs. The missing pair of canonical variables is defined as follows:

$$
\begin{equation*}
v_{n}:=B_{n, 1}, \quad u_{n}:=\frac{A_{n, 1}}{B_{n, 1}} . \tag{7.5}
\end{equation*}
$$

We remind that the $n$ pairs introduced by (7.4) and (7.5) are canonical variables, i.e.

$$
\begin{equation*}
\left\{u_{i}, u_{j}\right\}=\left\{v_{i}, v_{j}\right\}=0, \quad\left\{v_{i}, u_{j}\right\}=\delta_{i j}, \quad i, j=1, \ldots, n \tag{7.6}
\end{equation*}
$$

The separation representation of the algebra in this special case has the form

$$
\begin{align*}
B_{n}(u) & =v_{n}\left(u-u_{1}\right)\left(u-u_{2}\right) \cdots\left(u-u_{n-1}\right)  \tag{7.7}\\
A_{n}(u) & =B_{n}(u)\left(\frac{u+u_{n} v_{n}+\sum_{i=1}^{n-1} u_{i}}{v_{n}}+\sum_{i=1}^{n-1} \frac{\mathrm{e}^{-v_{i}}}{\left(u-u_{i}\right) B_{n}^{\prime}\left(u_{i}\right)}\right)  \tag{7.8}\\
D_{n}(u) & =B_{n}(u) \sum_{i=1}^{n-1} \frac{\operatorname{det} L_{n}\left(u_{i}\right) \mathrm{e}^{v_{i}}}{\left(u-u_{i}\right) B_{n}^{\prime}\left(u_{i}\right)}  \tag{7.9}\\
C_{n}(u) & =\frac{A_{n}(u) D_{n}(u)-\operatorname{det} L_{n}(u)}{B_{n}(u)} \tag{7.10}
\end{align*}
$$

In the lattice representation, the $L$-operator (7.2) acquires the form

$$
\begin{equation*}
L_{n}(u)=\ell_{n}\left(u-c_{n}\right) \ell_{n-1}\left(u-c_{n-1}\right) \cdots \ell_{1}\left(u-c_{1}\right), \tag{7.11}
\end{equation*}
$$

with the local $L$-operators

$$
\ell_{i}(u):=\left(\begin{array}{cc}
u-q_{i} p_{i} & b q_{i}  \tag{7.12}\\
-p_{i} & b
\end{array}\right), \quad i=1, \ldots, n .
$$

The local variables $\left(q_{i}, p_{i}\right)$ are canonical,

$$
\begin{equation*}
\left\{q_{i}, q_{j}\right\}=\left\{p_{i}, p_{j}\right\}=0, \quad\left\{p_{i}, q_{j}\right\}=\delta_{i j}, \quad i, j=1, \ldots, n \tag{7.13}
\end{equation*}
$$

The Hamiltonians $H_{1}$ and $H_{2}$ from the $\operatorname{tr} L_{n}(u)=u^{n}+H_{1} u^{n-1}+\ldots+H_{n}$ are

$$
\begin{align*}
H_{1} & =-\sum_{i=1}^{n}\left(q_{i} p_{i}+c_{i}\right)  \tag{7.14}\\
H_{2} & =\sum_{i>j}\left(q_{i} p_{i}+c_{i}\right)\left(q_{j} p_{j}+c_{j}\right)-b \sum_{i=1}^{n} q_{i+1} p_{i}, \quad\left(q_{n+1} \equiv q_{1}\right)
\end{align*}
$$

The (inverse) factorized separation chain $\mathcal{S}_{n}^{-1}=\mathcal{B}_{1} \circ \cdots \circ \mathcal{B}_{n-1} \circ \mathcal{B}_{n}$ is produced by a recursive application of the map $\mathcal{B}_{n}$, which is defined by the following equations:

$$
\begin{array}{rlr}
v_{n} & =\tilde{v}_{n-1}=\frac{\sum_{j=1}^{n-2} \tilde{u}_{j}-\sum_{i=1}^{n-1} u_{i}}{u_{n}-\tilde{u}_{n-1}}, \quad p_{n}=\frac{\sum_{i=1}^{n-1} u_{i}-\sum_{j=1}^{n-2} \tilde{u}_{j}-c_{n}}{q_{n}}, \\
\mathrm{e}^{-v_{i}} & =-\frac{b^{n} q_{n}\left(u_{i}-c_{n-1}\right)}{v_{n}} \prod_{m=1}^{n-2} \frac{u_{i}-c_{m}}{u_{i}-\tilde{u}_{m}}, \quad i=1, \ldots, n-1, \\
\mathrm{e}^{-\tilde{v}_{j}} & =\frac{b^{n} q_{n}}{v_{n}} \prod_{m=1}^{n-1} \frac{\tilde{u}_{j}-c_{m}}{\tilde{u}_{j}-u_{m}}, & j=1, \ldots, n-2 . \tag{7.17}
\end{array}
$$

These equations define a canonical map with the generating function

$$
\begin{align*}
F_{n}\left(\tilde{\mathbf{u}}, q_{n} \mid \mathbf{u}\right)= & \sum_{i=1}^{n-1} \sum_{j=1}^{n-2} \Omega\left(u_{i}-\tilde{u}_{j}\right)+\Omega\left(\sum_{i=1}^{n-1} u_{i}-\sum_{j=1}^{n-2} \tilde{u}_{j}\right)  \tag{7.18}\\
& -\left(\sum_{i=1}^{n-1} u_{i}-\sum_{j=1}^{n-2} \tilde{u}_{j}\right) \log \left(b^{n}\left(u_{n}-\tilde{u}_{n-1}\right)\right)-\mathrm{i} \pi n \sum_{j=1}^{n-2} \tilde{u}_{j} \\
& -\sum_{i=1}^{n-1} \sum_{i^{\prime}=1}^{n-1} \Omega\left(u_{i}-c_{i^{\prime}}\right)+\sum_{i=1}^{n-1} \sum_{j=1}^{n-2} \Omega\left(\tilde{u}_{j}-c_{i}\right) \\
& +\left(\sum_{j=1}^{n-2} \tilde{u}_{j}-\sum_{i=1}^{n-1} u_{i}+c_{n}\right) \log q_{n} .
\end{align*}
$$

Explicit expressions for the $n$th pair of local variables are

$$
\begin{equation*}
q_{n}=\frac{B_{n}\left(c_{n}\right)}{D_{n}\left(c_{n}\right)}, \quad p_{n}=-\frac{D_{n, 2}}{v_{n}}=-\sum_{i=1}^{n-1} \frac{\operatorname{det} L_{n}\left(u_{i}\right) \mathrm{e}^{v_{i}}}{B_{n}^{\prime}\left(u_{i}\right)} . \tag{7.19}
\end{equation*}
$$

This process can be iterated, leading to explicit rational formulae for the inverse separating map $\mathcal{S}_{n}^{-1}$. The corresponding formulae in the homogeneous $\left(c_{i}=0\right) 3$-particle case are given in the Appendix B.

## 8 Periodic Toda lattice

The periodic Toda lattice appears as a further specialization of our basic model when the parameters are fixed as follows:

$$
\begin{equation*}
\beta=\gamma=\delta=0 \quad \text { and } \quad Q_{k}=0, \quad k=1, \ldots, 2 n-1 \tag{8.1}
\end{equation*}
$$

We also put $\alpha=1$ (and $\kappa=-1$ ), leading to the $L$-operator which is similar to the one for the DST lattice:

$$
L_{n}(u)=\left(\begin{array}{cc}
u^{n}+A_{n, 1} u^{n-1}+\ldots+A_{n, n} & B_{n, 1} u^{n-1}+\ldots+B_{n, n}  \tag{8.2}\\
C_{n, 1} u^{n-1}+\ldots+C_{n, n} & D_{n, 2} u^{n-2}+\ldots+D_{n, n}
\end{array}\right)
$$

Set $Q_{2 n}=1$, so that

$$
\begin{equation*}
\operatorname{det} L_{n}(u)=1 \tag{8.3}
\end{equation*}
$$

Separation variables $\left(\mathrm{e}^{-v_{i}}, u_{i}\right), i=1, \ldots, n$, are introduced by the same formulae as for the DST lattice, cf. (7.4) and (7.5). The separation representation of the quadratic algebra in this case reads

$$
\begin{align*}
B_{n}(u) & =v_{n}\left(u-u_{1}\right)\left(u-u_{2}\right) \cdots\left(u-u_{n-1}\right)  \tag{8.4}\\
A_{n}(u) & =B_{n}(u)\left(\frac{u+u_{n} v_{n}+\sum_{i=1}^{n-1} u_{i}}{v_{n}}+\sum_{i=1}^{n-1} \frac{\mathrm{e}^{-v_{i}}}{\left(u-u_{i}\right) B_{n}^{\prime}\left(u_{i}\right)}\right)  \tag{8.5}\\
D_{n}(u) & =B_{n}(u) \sum_{i=1}^{n-1} \frac{\mathrm{e}^{v_{i}}}{\left(u-u_{i}\right) B_{n}^{\prime}\left(u_{i}\right)},  \tag{8.6}\\
C_{n}(u) & =\frac{A_{n}(u) D_{n}(u)-1}{B_{n}(u)} . \tag{8.7}
\end{align*}
$$

In the lattice representation, the $L$-operator (8.2) acquires the form

$$
L_{n}(u)=\ell_{n}(u) \ell_{n-1}(u) \cdots \ell_{1}(u)=\left(\begin{array}{cc}
A_{n}(u) & B_{n}(u)  \tag{8.8}\\
C_{n}(u) & D_{n}(u)
\end{array}\right)
$$

with the local $L$-operators

$$
\ell_{i}(u):=\left(\begin{array}{cc}
u-p_{i} & \mathrm{e}^{q_{i}}  \tag{8.9}\\
-\mathrm{e}^{-q_{i}} & 0
\end{array}\right), \quad i=1, \ldots, n .
$$

The Hamiltonians $H_{1}$ and $H_{2}$ from the $\operatorname{tr} L_{n}(u)=u^{n}+H_{1} u^{n-1}+\ldots+H_{n}$ are

$$
\begin{equation*}
H_{1}=-\sum_{i=1}^{n} p_{i}, \quad H_{2}=\sum_{i>j} p_{i} p_{j}-\sum_{i=1}^{n} \mathrm{e}^{q_{i+1}-q_{i}}, \quad\left(q_{n+1} \equiv q_{1}\right) \tag{8.10}
\end{equation*}
$$

The (inverse) factorized separation chain $\mathcal{S}_{n}^{-1}=\mathcal{B}_{1} \circ \cdots \circ \mathcal{B}_{n-1} \circ \mathcal{B}_{n}$ is produced by a recursive application of the map $\mathcal{B}_{n}$, which is defined by the following equations:

$$
\begin{array}{rlrl}
v_{n}=\tilde{v}_{n-1}=\frac{\sum_{j=1}^{n-2} \tilde{u}_{j}-\sum_{i=1}^{n-1} u_{i}}{u_{n}-\tilde{u}_{n-1}}, & p_{n}=\sum_{i=1}^{n-1} u_{i}-\sum_{j=1}^{n-2} \tilde{u}_{j}, \\
\mathrm{e}^{v_{i}}=-v_{n} \mathrm{e}^{-q_{n}} \prod_{m=1}^{n-2}\left(u_{i}-\tilde{u}_{m}\right), & & i=1, \ldots, n-1, \\
\mathrm{e}^{\tilde{v}_{j}}=v_{n} \mathrm{e}^{-q_{n}} \prod_{m=1}^{n-1}\left(\tilde{u}_{j}-u_{m}\right), & & j=1, \ldots, n-2 . \tag{8.13}
\end{array}
$$

These equations define a canonical map with the generating function

$$
\begin{align*}
F_{n}\left(\tilde{\mathbf{u}}, q_{n} \mid \mathbf{u}\right)= & \sum_{i=1}^{n-1} \sum_{j=1}^{n-2} \Omega\left(u_{i}-\tilde{u}_{j}\right)+\Omega\left(\sum_{i=1}^{n-1} u_{i}-\sum_{j=1}^{n-2} \tilde{u}_{j}\right)  \tag{8.14}\\
& -\left(\sum_{i=1}^{n-1} u_{i}-\sum_{j=1}^{n-2} \tilde{u}_{j}\right)\left(q_{n}+\log \left(u_{n}-\tilde{u}_{n-1}\right)\right)-\mathrm{i} \pi n \sum_{j=1}^{n-2} \tilde{u}_{j} .
\end{align*}
$$

Explicit expressions for the $n$th pair of local variables are

$$
\begin{equation*}
\mathrm{e}^{-q_{n}}=-\frac{D_{n, 2}}{v_{n}}=-\sum_{i=1}^{n-1} \frac{\mathrm{e}^{v_{i}}}{B_{n}^{\prime}\left(u_{i}\right)}, p_{n}=\sum_{i=1}^{n-1} u_{i}+\frac{D_{n, 3}}{D_{n, 2}}=\frac{\sum_{i=1}^{n-1} \frac{u_{i} \mathrm{e}^{v_{i}}}{B_{n}\left(u_{i}\right)}}{\sum_{i=1}^{n-1} \frac{\mathrm{e}^{i} i}{B_{n}^{\prime}\left(u_{i}\right)}} . \tag{8.15}
\end{equation*}
$$

This process can be iterated. The corresponding formulae for the 3-particle periodic Toda lattice are given in the Appendix C.

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## Appendix A

Here we give explicit formulae for the inverse map $\mathcal{S}_{3}^{-1}$ for the chain of three spins when $s_{i}=c_{i}=0, i=1,2,3$. For notation, see the end of Section 5. Denote $w_{i}=\mathrm{e}^{\kappa v_{i}}, i=1,2,3$, then

$$
\begin{align*}
& p_{3}=\frac{\mathrm{i} \beta w_{1} w_{2} w_{3}}{(\alpha \delta-\beta \gamma) \kappa} \frac{\left|\begin{array}{ccc}
1 & 1 & 1 \\
u_{1} & u_{2} & u_{3} \\
u_{1}^{2} & u_{2}^{2} & u_{3}^{2}
\end{array}\right|}{\left|\begin{array}{ccc}
u_{1} w_{1} & u_{2} w_{2} & u_{3} w_{3} \\
w_{1} & w_{2} & w_{3} \\
u_{1}^{4} & u_{2}^{4} & u_{3}^{4}
\end{array}\right|},  \tag{A.3}\\
& q_{1}=-\mathrm{i} \frac{\delta}{\beta}+\frac{\mathrm{i}(\alpha \delta-\beta \gamma) u_{1} u_{2} u_{3}}{\beta} \frac{\left|\begin{array}{ccc}
1 & 1 & 1 \\
u_{1} & u_{2} & u_{3} \\
u_{1}^{2} & u_{2}^{2} & u_{3}^{2}
\end{array}\right|}{\left|\begin{array}{ccc}
w_{1} & w_{2} & w_{3} \\
u_{1} & u_{2} & u_{3} \\
u_{1}^{2} & u_{2}^{2} & u_{3}^{2}
\end{array}\right|},  \tag{A.4}\\
& q_{2}=-\mathrm{i} \frac{\delta}{\beta}+\frac{\mathrm{i}(\alpha \delta-\beta \gamma)}{\beta} \frac{\left|\begin{array}{ccc}
w_{1} & w_{2} & w_{3} \\
u_{1}^{3} & u_{2}^{3} & u_{3}^{3} \\
u_{1}^{4} & u_{2}^{4} & u_{3}^{4}
\end{array}\right|}{\left|\begin{array}{ccc}
u_{1} w_{1} & u_{2} w_{2} & u_{3} w_{3} \\
w_{1} & w_{2} & w_{3} \\
u_{1}^{3} & u_{2}^{3} & u_{3}^{3}
\end{array}\right|}, \tag{A.5}
\end{align*}
$$

$$
q_{3}=-\mathrm{i} \frac{\delta}{\beta}-\frac{\mathrm{i}(\alpha \delta-\beta \gamma)}{\beta w_{1} w_{2} w_{3}} \frac{\left|\begin{array}{ccc}
u_{1} w_{1} & u_{2} w_{2} & u_{3} w_{3}  \tag{A.6}\\
w_{1} & w_{2} & w_{3} \\
u_{1}^{5} & u_{2}^{5} & u_{3}^{5}
\end{array}\right|}{\left|\begin{array}{ccc}
1 & 1 & 1 \\
u_{1} & u_{2} & u_{3} \\
u_{1}^{2} & u_{2}^{2} & u_{3}^{2}
\end{array}\right|}
$$

## Appendix B

Here we give explicit formulae for the inverse map $\mathcal{S}_{3}^{-1}$ for the 3-particle DST lattice when $c_{i}=0, i=1,2,3$ (cf. Section 7). Denote $w_{i}=\mathrm{e}^{-v_{i}}, i=1,2$, then

$$
\begin{align*}
& p_{1}=-b u_{3}-\frac{b}{u_{1} u_{2} v_{3}} \frac{\left|\begin{array}{cc}
w_{1}-u_{1}^{3} & w_{2}-u_{2}^{3} \\
u_{1} & u_{2}
\end{array}\right|}{\left|\begin{array}{cc}
1 & 1 \\
u_{1} & u_{2}
\end{array}\right|}, \quad q_{1}=\frac{v_{3}}{b},  \tag{B.1}\\
& p_{2}=\frac{b^{2}}{v_{3}}+\frac{b^{2} u_{1}^{2} u_{2}^{2}}{v_{3}} \frac{\left|\begin{array}{cc}
1 & 1 \\
u_{1} & u_{2}
\end{array}\right|}{\left|\begin{array}{ll}
w_{1} & w_{2} \\
u_{1}^{2} & u_{2}^{2}
\end{array}\right|}, \quad q_{2}=\frac{v_{3}}{b^{2} u_{1} u_{2}} \frac{\left|\begin{array}{cc}
w_{1} & w_{2} \\
u_{1} & u_{2}
\end{array}\right|}{\left|\begin{array}{cc}
1 & 1 \\
u_{1} & u_{2}
\end{array}\right|},  \tag{B.2}\\
& p_{3}=-\frac{b^{3}}{v_{3} w_{1} w_{2}} \frac{\left|\begin{array}{cc}
w_{1} & w_{2} \\
u_{1}^{3} & u_{2}^{3}
\end{array}\right|}{\left|\begin{array}{cc}
1 & 1 \\
u_{1} & u_{2}
\end{array}\right|}, \quad q_{3}=-\frac{v_{3} w_{1} w_{2}}{b^{3}} \frac{\left|\begin{array}{cc}
1 & 1 \\
u_{1} & u_{2}
\end{array}\right|}{\left|\begin{array}{cc}
w_{1} & w_{2} \\
u_{1}^{2} & u_{2}^{2}
\end{array}\right|} . \tag{B.3}
\end{align*}
$$

## Appendix C

Here we give explicit formulae for the inverse map $\mathcal{S}_{3}^{-1}$ for the 3-particle periodic Toda lattice (cf. Section 8). Denote $w_{i}=\mathrm{e}^{-v_{i}}, i=1,2$, then

$$
\begin{array}{lll}
p_{1}= & -u_{3} v_{3}-u_{1}-u_{2}, & \mathrm{e}^{q_{1}}=v_{3}, \\
p_{2}= & \frac{\left|\begin{array}{cc}
u_{1} w_{1} & u_{2} w_{2} \\
1 & 1
\end{array}\right|}{\left|\begin{array}{cc}
w_{1} & w_{2} \\
1 & 1
\end{array}\right|}, & \mathrm{e}^{q_{2}}=v_{3} \frac{\left|\begin{array}{cc}
w_{1} & w_{2} \\
1 & 1
\end{array}\right|}{\left|\begin{array}{cc}
1 & 1 \\
u_{1} & u_{2}
\end{array}\right|} \\
p_{3}=\frac{\left|\begin{array}{cc}
w_{1} & w_{2} \\
u_{1} & u_{2}
\end{array}\right|}{\left|\begin{array}{cc}
w_{1} & w_{2} \\
1 & 1
\end{array}\right|,} & \mathrm{e}^{q_{3}}=-v_{3} w_{1} w_{2} \frac{\left|\begin{array}{cc}
1 & 1 \\
u_{1} & u_{2}
\end{array}\right|}{\left|\begin{array}{cc}
w_{1} & w_{2} \\
1 & 1
\end{array}\right|} \tag{C.3}
\end{array}
$$

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[^1]:    ${ }^{1}$ an analogous factorization chain exists for the direct separating map $\mathcal{S}_{n}$ but this is beyond the scope of this paper

[^2]:    ${ }^{2}$ N.B.: $n-1$ canonical pairs/degrees of freedom

