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INVERSE PROBLEM OF LINEAR OPTIMAL CONTROL*

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Abstract. Necessary and sufficient conditions are derived such that a multi-input, time-varying, linear state-feedback system minimizes a quadratic performance index (the inverse linear optimal control problem). A procedure for determining all such equivalent performance indices that yield the same feedback matrix is indicated.

1. Introduction. Consider a linear system given by

$$\dot{x} = Ax + Bu, \qquad x(t_0) = x_0, \tag{1.1}$$

$$u = Dx, \tag{1.2}$$

and a performance index

$$I = \frac{1}{2}x^T(t_1)Fx(t_1) + \frac{1}{2}\int_{t_0}^{t_1} (x^TQx + u^TRu)\,dt, \tag{1.3}$$

where x is the n-dimensional state, u the m-dimensional control, t_1 is a fixed terminal time, and the superscript T denotes matrix transpose. The matrices A, B, D, Q and R may be time-varying, and are assumed to be uniformly bounded and continuous on $[t_0, t_1]$.

The inverse problem of linear optimal control is to find necessary and sufficient conditions on the system matrices A, B and D so that some performance index of the type (1.3) is minimized, and to determine all such R, Q and F.

The direct problem and its solution are of course well known. The feedback matrix D such that the system (1.1), (1.2) minimizes (1.3) is given by

$$D = -R^{-1}B^TP, \tag{1.4}$$

where P is the solution of the matrix Riccati equation

$$-\dot{P} = PA + A^TP - PBR^{-1}B^TP + Q, \qquad P(t_1) = F. \tag{1.5}$$

For the existence of a unique minimizing u it is assumed that R is positive definite (denoted by $R > 0$), and as a sufficient condition for the existence of a solution $P(t)$ of (1.5) it is usually assumed that Q and F are nonnegative definite ($Q \geqq 0, F \geqq 0$). The minimal value I_* of I is then nonnegative for all x_0 and t_0, and since

$$I_* = \tfrac{1}{2}x_0^TP(t_0)x_0, \tag{1.6}$$

P is nonnegative definite. The case where $t_1 \to \infty$ is of particular interest when A and B are constant, because the performance index

$$I = \frac{1}{2}\int_0^\infty (x^TQx + u^TRu)\,dt, \tag{1.7}$$

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with Q and R constant, results in a constant feedback matrix D. If (1.1) is completely controllable and $Q = C^TC$ is such that $[A, C]$ is completely observable, then the matrix P in (1.4) is the asymptotically stable equilibrium point of the Riccati equation

$$\dot{P} = PA + A^TP - PBR^{-1}B^TP + Q, \qquad P(0) = P_0 = P_0^T \geqq 0, \tag{1.8}$$

i.e., it is the positive definite solution of the matrix equation

$$0 = PA + A^TP - PBR^{-1}B^TP + Q. \tag{1.9}$$

The present paper solves the general time-varying case with the performance index given by (1.3), as well as the time-invariant case with the performance index (1.7). The plan of attack is as follows.

In § 2 we determine necessary conditions for the existence of real symmetric matrices $R > 0$ and P (also for $P \geqq 0$ and $P > 0$), such that (1.4) is satisfied; the most restrictive of these is that DB have real eigenvalues. The sufficiency of these conditions is demonstrated in § 3 by producing general formulas for such R and P. In § 4 we give complete sets of necessary and sufficient conditions for solutions $P \geqq 0$ and $P > 0$; in view of (1.6), these conditions are necessary and sufficient for I_* to be nonnegative and positive for all x_0 and all $t_0 < t_1$, and they are necessary for construction of $Q \geqq 0$—hence their importance.

The solutions of (1.4) for R and P are pointwise in time, but P can be constructed to be differentiable, so that we have $\dot{P}$. Then $F = P(t_1)$, and Q is given by

$$Q = -\dot{P} - PA - A^TP + D^TRD. \tag{1.10}$$

We remark that Q so determined may not be nonnegative definite even if P is positive definite; this is true also in the time-invariant case. We now have the entire class of matrices $\{R, Q, F, P\}$ that satisfy (1.4) and (1.5), and we show in § 5 that each member of this class of performance indices is actually minimized by the given control law (1.2), thus solving the inverse problem (Theorem 5.1).

The inverse problem was first posed and partly solved by Kalman [1] who considered the time-invariant single-input case where R reduces to a scalar. He showed that the satisfaction of a particular inequality, the sensitivity inequality, implies that there exists a performance index (1.7) with a nonnegative definite Q and $R = 1$ which is minimized. This result was generalized by Anderson [2] to the multi-input, time-invariant case. Our approach and results, which do not necessarily produce a nonnegative definite Q, are different. Results for the case where Q is nonnegative definite, and relations with the Kalman–Anderson results, will be presented in a sequel to this paper.

The generality of the characterization of R and P (Theorems 3.1–3.4) gives them independent value; they provide new insight into the already extensively researched linear optimal control problem and are bound to find many applications, particularly in the area of equivalent loss functions [3]. We note that the results of this paper are important for the local treatment of the general nonlinear inverse problem. The fact that our results do not require $Q \geqq 0$ is then significant; this requirement, usually natural for the direct linear optimal problem, is unduly restrictive for the nonlinear case (where Q is replaced by H_{xx}, the second partial of the Hamiltonian).

We remark in conclusion that the conditions for optimality of (1.1), (1.2) derived here are in general no longer necessary when a cross-product term u^TSx is added in the integrand of (1.3) or (1.7). In fact, it is shown in [4] that *every* system (1.1), (1.2) minimizes a performance index (1.3) with a cross-product term, and that there are many ways such a performance index can be constructed. If the performance index is further generalized to include derivatives of the control, dynamic feedback and feedforward controllers can be included, and thus *every linear, finite-dimensional dynamic feedback system minimizes some sufficiently general quadratic performance index* [4].

2. Compatibility conditions. First we note that only the symmetric parts of R, Q and F appear in (1.4) and in the Riccati equation (1.5). Thus, the existence of symmetric P, R, Q and F satisfying (1.4) and (1.5) is a necessary condition for a closed-loop system (1.1), (1.2) to be optimal with respect to (1.3).

From (1.4),

$$B^TP = -RD, \tag{2.1}$$

and our objective is to solve this equation for all real, symmetric and positive definite R, $R = R^T > 0$, and all real and symmetric P. In this section we derive several necessary conditions for the existence of such solutions; in the next section we show constructively that these are also sufficient. We recall that B is $n \times m$, D is $m \times n$, and they do not necessarily have full rank. We have the following lemma.

LEMMA 2.1. *Necessary conditions for* (2.1) *to have real symmetric solutions P, and real, symmetric and positive definite solutions R, are*:

(i) *for any P, R must be such that the compatibility condition holds*:

$$B^TB^{\ddagger T}RD = RD, \tag{2.2}$$

where $B^\ddagger$ is any matrix (e.g., the Penrose generalized inverse $B^\dagger$) *such that* $BB^\ddagger B = B$;

(ii) *for P to be symmetric, the symmetry condition must hold*:

$$RDB = B^TD^TR; \tag{2.3}$$

(iii) *for R to be positive definite, a rank condition on BD must hold*:

$$\operatorname{rank} BD = \operatorname{rank} D. \tag{2.4}$$

Proof. (i) Premultiplying (2.1) by $B^TB^{\ddagger T}$ and using the identity $B^TB^{\ddagger T}B^T = B^T$, we have

$$B^TB^{\ddagger T}B^TP = B^TP = -RD = -B^TB^{\ddagger T}RD.$$

(ii) Postmultiplying (2.1) by B we have

$$B^TPB = -RDB, \tag{2.5}$$

whence the symmetry of RDB is necessary for the symmetry of P.

(iii) From (2.1),

$$PBD = -D^TRD.$$

We first observe that since R is positive definite,

$$\text{rank } D^TRD = \text{rank } D \tag{2.6}$$

because for every ρ such that $D^TRD\rho = 0$ we have

$$\rho^T D^T RD\rho = (D\rho)^T R(D\rho) = 0,$$

whence $D\rho$ must be zero. Thus, by (2.6),

$$\text{rank } BD \geqq \text{rank } PBD = \text{rank } D^TRD = \text{rank } D.$$

But since rank $BD >$ rank D is impossible, (2.4) follows. Q.E.D.

That RDB must be symmetric with a real, symmetric and positive definite R, leads to the next lemma.

LEMMA 2.2. *A real $R = R^T > 0$ such that RDB is symmetric exists only if the $m \times m$ matrix DB satisfies the eigenvector condition*:

(2.7) *DB has m linearly independent real eigenvectors.*

Proof. Let $L^TL = R > 0$ be such that RDB is symmetric. Then $(L^{-1})^T RDBL^{-1} = LDBL^{-1}$ is symmetric, and it therefore has m linearly independent real eigenvectors. But since DB is similar to $LDBL^{-1}$, (2.7) follows. Q.E.D.

The conditions (2.4) and (2.7) are necessary for the desired solution of (2.1), and it is shown constructively by Theorems 3.1, 3.2 and 3.3 in the next section that they are also sufficient. We can therefore state the following theorem.

THEOREM 2.1. *Equation* (2.1) *has solutions $R = R^T > 0$ and $P = P^T$ if and only if the rank condition* (2.4) *on BD and the eigenvector condition* (2.7) *on DB hold.*

We now proceed to derive conditions for symmetric P to be nonnegative definite and positive definite. We have the following lemma.

LEMMA 2.3. *For* (2.1) *with $R = R^T > 0$ to have solutions $P = P^T \geqq 0$ it is necessary that*

$$\text{rank } DB = \text{rank } D, \tag{2.8}$$

and that all eigenvalues λ of DB be nonpositive:

$$\lambda_i \leqq 0, \quad i = 1, 2, \cdots, m; \tag{2.9}$$

for solutions $P = P^T > 0$, it is necessary that

$$\text{rank } DB = \text{rank } D = \text{rank } B. \tag{2.10}$$

Proof. From (2.1), since R is nonsingular,

$$\text{rank } D = \text{rank } RD = \text{rank } B^TP = \text{rank } PB,$$

and from (2.5),

$$\text{rank } DB = \text{rank } RDB = \text{rank } B^TPB.$$

Thus (2.8) is equivalent to

$$\text{rank } B^TPB = \text{rank } PB. \tag{2.11}$$

Since $P \geqq 0$, we may write $P = Z^TZ$, Z real. We observe, as in the proof of (2.6), that

$$\operatorname{rank}(ZB)^T ZB = \operatorname{rank} ZB.$$

Thus

$$\operatorname{rank} B^T PB = \operatorname{rank}(ZB)^T ZB = \operatorname{rank} ZB \geqq \operatorname{rank} Z^T ZB = \operatorname{rank} PB.$$

But since rank $B^T PB >$ rank PB is impossible, (2.8) follows. If $P \geqq 0$ or $P > 0$, then from (2.5),

$$RDB \leqq 0. \tag{2.12}$$

Writing $R = L^T L$, $(L^{-1})^T RDBL^{-1} = LDBL^{-1}$ is also nonpositive definite, real, and symmetric, and therefore has nonpositive real eigenvalues. Since DB is similar to $LDBL^{-1}$, (2.9) is established. If $P > 0$, then since R is nonsingular, (2.1) shows that

$$\operatorname{rank} D = \operatorname{rank} B, \tag{2.13}$$

whence (2.10) must hold. Q.E.D.

LEMMA 2.4. *The rank condition* (2.8) *on DB together with the symmetry condition* (2.3) *on RDB imply that the compatibility condition* (2.2) *and the rank condition* (2.4) *on BD are satisfied.*

Proof. From (2.2), using the symmetry of RDB and the identity $BB^{\ddagger}B = B$, we have

$$B^T B^{\ddagger T} RDB = B^T B^{\ddagger T} B^T D^T R = B^T D^T R = RDB,$$

or

$$(B^T B^{\ddagger T} R - R)DB = 0. \tag{2.14}$$

But since rank DB = rank D, if M is a nonzero matrix such that $MDB = 0$, then also $MD = 0$. Thus, (2.14) implies (2.2). From (2.3),

$$RDBD = B^T D^T RD. \tag{2.15}$$

Let ρ be any vector such that $D\rho \neq 0$. Then in view of (2.6), $D^T RD\rho \neq 0$, and by (2.8), $B^T D^T RD\rho \neq 0$, whence by (2.15), $RDBD\rho \neq 0$. Thus $BD\rho \neq 0$ if $D\rho \neq 0$, implying (2.4). Q.E.D.

From Lemmas 2.2, 2.3 and 2.4 we see that conditions (2.7), (2.8) and (2.9) emerge as the major ones necessary for $P \geqq 0$, and in § 4 we show that they are indeed sufficient. We can therefore state the following theorem.

THEOREM 2.2. *Equation* (2.1) *has solutions* $R = R^T > 0$ *and* $P = P^T \geqq 0$ *if and only if DB has m linearly independent real eigenvectors, its rank equals that of D, and its eigenvalues are all nonpositive* (*conditions* (2.7), (2.8) *and* (2.9), *respectively*); *for* $P = P^T > 0$, *the rank of DB must also be equal to that of B* (*condition* (2.10)).

We close this section with the remark that the rank condition rank BD = rank D can be seen as a direct consequence of optimality. There is no loss of generality (see Appendix) in assuming that B is in canonical form and u is partitioned accordingly:

$$B = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}. \tag{2.16}$$

Since u_1 does not affect $x(t)$, it must minimize at all times the quadratic form u^TRu:

$$[u_1^T \quad u_2^T]\begin{bmatrix} R_{11} & R_{12} \\ R_{12}^T & R_{22} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = u_1^TR_{11}u_1 + 2u_1^TR_{12}u_2 + u_2^TR_{22}u_2,$$

whence the minimizing u_1 is given by

$$u_1 = -R_{11}^{-1}R_{12}u_2,$$

i.e., u_1 is proportional to u_2. Thus, if we partition D as $D^T = [D_1^T \quad D_2^T]$, then D_1 must satisfy

$$D_1 = KD_2, \tag{2.17}$$

or, rank D = rank D_2. But

$$BD = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} D_1 \\ D_2 \end{bmatrix} = \begin{bmatrix} 0 \\ D_2 \end{bmatrix},$$

whence rank BD = rank D_2 = rank D.

3. General representations of R **and** P. We first construct $R = R^T > 0$ so that RDB is symmetric. By Lemma 2.2 we must assume that the eigenvectors of DB, and hence of B^TD^T, are real and linearly independent. Thus the matrix V whose columns are the eigenvectors v of B^TD^T is real and nonsingular. We may write the set of equations $B^TD^Tv_i = \lambda_iv_i$, $i = 1, 2, \cdots, m$, as

$$B^TD^TV = V\Lambda, \tag{3.1}$$

where Λ is the diagonal matrix of eigenvalues λ of B^TD^T. Then for any real, nonsingular matrix Π that commutes with Λ we have

$$B^TD^TV\Pi = V\Lambda\Pi = V\Pi\Lambda, \tag{3.2}$$

whence the columns of $V\Pi$ are seen as a new, linearly independent set of real eigenvectors; in fact, all such sets of eigenvectors are generated by all such Π. (If all eigenvalues λ are distinct, then Π must be diagonal and it simply scales the eigenvectors v; if $\lambda_i = \lambda_j$, then any linear combination of v_i and v_j is also an eigenvector and all such combinations are generated by the now permissible off-diagonal elements π_{ij} and π_{ji} of Π.) We have the following theorem.

THEOREM 3.1. *Let the necessary eigenvector condition* (2.7) *hold. Then every given real* $R = R^T > 0$ *such that* RDB *is symmetric, is necessarily given by*

$$R = VV^T, \tag{3.3}$$

where the columns v *of* V *are suitably chosen eigenvectors of* B^TD^T. *Let* V *be any such given matrix, and let* Γ *be a real matrix such that*

$$\Gamma = \Gamma^T > 0 \quad \textit{and} \quad \Gamma\Lambda = \Lambda\Gamma, \tag{3.4}$$

where Λ *is defined by* (3.1). *Then all real* $R = R^T > 0$ *such that* RDB *is symmetric are generated by all real* Γ *in*

$$R = V\Gamma V^T, \tag{3.5}$$

where Γ *satisfies* (3.4).

Proof. Let $R = MM^T$. Then, since B^TD^TR is symmetric,

$$M^{-1}(B^TD^TR)(M^{-1})^T = M^{-1}B^TD^TM$$

is a symmetric matrix whose eigenvalues are those of B^TD^T. There exists therefore an orthogonal matrix H such that

$$H^TM^{-1}B^TD^TMH = \Lambda, \qquad HH^T = I,$$

whence the columns of MH are seen to be eigenvectors of B^TD^T. If we define $V = MH$, then

$$VV^T = MHH^TM^T = MM^T = R,$$

as claimed in (3.3). Conversely, for any V in (3.3), RDB is symmetric:

$$VV^TDB = V\Lambda V^T = B^TD^TVV^T.$$

Recalling the comment that follows (3.2), we let $\Pi\Pi^T = \Gamma$, whence (3.5) follows from (3.3). Q.E.D.

Not every $R = R^T > 0$ that satisfies the symmetry condition (2.3) automatically satisfies also the compatibility condition (2.2). For example, let

$$B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \qquad R = \begin{bmatrix} r_{11} & r_{12} \\ r_{12} & r_{22} \end{bmatrix}.$$

Then

$$BD = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \qquad DB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

whence rank BD = rank D and RDB is symmetric for any R. But since

$$RD = \begin{bmatrix} r_{11} + r_{12} & 0 \\ r_{12} + r_{22} & 0 \end{bmatrix} \quad \text{and} \quad B^TB^{\ddagger T}RD = \begin{bmatrix} 0 & 0 \\ r_{12} + r_{22} & 0 \end{bmatrix},$$

the compatibility condition holds only if $r_{11} + r_{12} = 0$. Nevertheless, the following theorem holds.

THEOREM 3.2. *If the rank condition* (2.4) *on BD and the eigenvector condition* (2.7) *hold, there exist* $R = R^T > 0$ *such that both the compatibility condition* (2.2) *and the symmetry condition* (2.3) *on RDB are satisfied, and all such R can be constructed by suitably choosing* Γ *in* (3.5).

Proof. This is shown in the Appendix, where B is first transformed to canonical form (2.16). For a set of eigenvectors of B^TD^T, all Γ in (3.5) are found such that the compatibility condition is satisfied.

Remark 3.1. The rule for structuring R in (3.3), or (3.5), so as to satisfy the compatibility condition (see (A.12)–(A.15) in the Appendix) rarely need be invoked because the compatibility condition always holds in the most common case when rank $B = m$ (see § 6). The compatibility condition also always holds if rank DB = rank D, as needed for the usual case $P \geqq 0$ (see Lemmas 2.3 and 2.4). Thus, in most cases of interest, the formula (3.5) for R provides all the requisite matrices R of the inverse problem.

Given a real $R = R^T > 0$ satisfying the compatibility condition (2.2) and the symmetry condition (2.3) on RDB, we next solve (2.1) for a real symmetric P. Let U be any real $n \times m$ matrix such that

$$B^T U^T R D = RD; \tag{3.6}$$

in view of (2.2), one such matrix is $U = B^{\ddagger}$. By inspection of (3.6), $-U^T R D$ is a solution of (2.1) for P, which, however, is not necessarily symmetric. To obtain a symmetric solution set

$$P_0 = -U^T RD - D^T RU + U^T RDBU. \tag{3.7}$$

Now by (3.6) and the symmetry of RDB,

$$B^T P_0 = -RD - B^T D^T RU + RDBU = -RD.$$

Further, if P is any real symmetric solution of (2.1), then

$$B^T(P - P_0) = 0,$$

whence the general solution of (2.1) for a real symmetric P is

$$P = -U^T RD - D^T RU + U^T RDBU + Y, \tag{3.8}$$

where Y is any real matrix such that

$$B^T Y = 0, \qquad Y = Y^T. \tag{3.9}$$

In summary, we have the following theorem.

THEOREM 3.3. *Let R be a real, symmetric and positive definite matrix satisfying the compatibility condition* (2.2) *and the symmetry condition* (2.3). *Then a real symmetric P satisfying* (2.1) *exists, and all such P are represented by* (3.8), *where U is any real matrix satisfying* (3.6) *and Y is any real matrix satisfying* (3.9).

Theorems 3.1, 3.2 and 3.3 provide the sufficiency part of Theorem 2.1.

Under the rank condition (2.8) on DB, additional representations of P are available. Furthermore, by Lemma 2.4, the compatibility condition (2.2) is then redundant. We then have the following.

THEOREM 3.4. *Let R be a real, symmetric and positive definite matrix such that RDB is symmetric. If*

$$\operatorname{rank} DB = \operatorname{rank} D, \tag{3.10}$$

then all real symmetric P satisfying (2.1) *are represented in terms of the given R by*

$$P = -D^T R(RDB)^{\dagger} RD + Y, \tag{3.11}$$

where † denotes the Penrose generalized inverse; in terms of the eigenvectors of $B^T D^T$, P is given by

$$P = -D^T V \Gamma \Lambda^{\dagger} V^T D + Y, \tag{3.12}$$

where Y are all real matrices that satisfy (3.9), *V and Λ are defined by* (3.1), *and Γ is defined by* (3.4).

Proof. Consider (3.6) and (3.7), and let

$$U = (RDB)^{\dagger} RD. \tag{3.13}$$

If DB is nonsingular, this U satisfies (3.6). Consider the case where DB is singular. By the identity $XX^\dagger X = X$,

$$[B^TD^TR(RDB)^\dagger R - R]DB = 0. \tag{3.14}$$

We note that (3.10) implies that $MDB = 0$ only if $MD = 0$. Hence (3.14) gives

$$B^TD^TR(RDB)^\dagger RD = RD.$$

Thus U given by (3.13) is seen to satisfy (3.6). Substituting U into the last term of (3.7), we have

$$\begin{aligned} U^TRDBU &= D^TR(RDB)^\dagger RDB(RDB)^\dagger RD \\ &= D^TR(RDB)^\dagger RD = D^TRU, \end{aligned}$$

so that (3.8) yields (3.11). To prove (3.12), we note that $\Lambda^\dagger$ is a diagonal matrix with elements $1/\lambda_i$ if $\lambda_i \neq 0$, and 0 if $\lambda_j = 0$. Thus $\Gamma\Lambda^\dagger = \Lambda^\dagger\Gamma$, and (3.12) is seen to be symmetric. Also, (3.10) implies that all eigenvectors v of B^TD^T that correspond to zero eigenvalues are such that $D^Tv = 0$. Considering all such eigenvectors we have that

$$D^TV = D^TV\Lambda^\dagger\Lambda,$$

whence P given by (3.12) is seen to satisfy (2.1):

$$B^TP = -B^TD^TV\Gamma\Lambda^\dagger V^TD = -V\Lambda\Gamma\Lambda^\dagger V^TD = -V\Gamma V^TD = -RD.$$

Q.E.D.

The rank condition (3.10) is, by Lemma 2.3, necessary for $P \geqq 0$ and $P > 0$. Formulas (3.11) and (3.12) are therefore useful for, but by no means restricted to, these cases. A general representation of P, equivalent to (3.8), or under the rank condition (3.10) equivalent to (3.12), and based on transformation of B to canonical form (2.16), is given in the Appendix (see (A.17)–(A.19)).

Remark 3.2. We used the generalized inverse in the sense of Moore–Penrose because it is unique and perhaps best known of the various pseudoinverses. However, of the identities

$$XX^\dagger X = X, \qquad X^\dagger XX^\dagger = X^\dagger, \qquad (XX^\dagger)^T = XX^\dagger, \qquad (X^\dagger X)^T = X^\dagger X,$$

defining the Penrose inverse, we need only the first two. We may define $X^\#$ by $XX^\#X = X$ and $X^\#XX^\# = X^\#$, a pseudoinverse that is no longer unique. Let $(RDB)^\#$ be

$$(RDB)^\# = (V\Gamma\Lambda V^T)^\# \triangleq V^{T^{-1}}\Lambda^\#\Gamma^{-1}V^{-1}; \tag{3.15}$$

this is a symmetric matrix that satisfies the two identities for $^\#$. Then, replacing † in (3.11) by $^\#$, using (3.15), and $\Lambda^\# = \Lambda^\dagger$ and $\Lambda^\dagger\Gamma = \Gamma\Lambda^\dagger$, (3.11) reduces to (3.12). We remark that in passing from one general representation for P to another, the matrix Y in the second representation, while still satisfying $B^TY = 0$, may not be the same as in the first representation.

Since the symmetry of RDB and the compatibility condition (2.2) are necessary, R given by Theorems 3.1 and 3.2 is the general, real, symmetric, positive definite solution of (2.1). Given R, Theorem 3.3 provides the general, real, symmetric solution P. We thus have all solutions $R = R^T > 0$ and $P = P^T$ of (2.1); in the next section we obtain all such solutions where in addition P is nonnegative definite and positive definite.

4. Conditions for $P \geqq 0$ and for $P > 0$. In view of (1.6), conditions for $P \geqq 0$ (for $P > 0$) are necessary and sufficient for I_* to be nonnegative (positive) for all x_0 and all $t_0 < t_1$. They are important also for the solution of the inverse problem with positiveness conditions on Q in (1.3), as will be discussed in the sequel to this paper.

Recalling Lemma 2.3, we use the representations (3.11) and (3.12) for P, and seek conditions on Y for P to be nonnegative definite and positive definite. We have the following theorem.

THEOREM 4.1. *A real symmetric P given by* (3.11) *or* (3.12) *satisfies* (2.1) *with $R = R^T > 0$ and is nonnegative definite if and only if the following conditions hold*:

$$\text{rank } DB = \text{rank } D, \tag{4.1}$$

the eigenvalues λ of DB (*which by* (2.7) *must be real*) *are nonpositive*:

$$\lambda_i \leqq 0, \quad i = 1, 2, \cdots, m, \tag{4.2}$$

and in (3.11) *and* (3.12),

$$Y = Y^T \geqq 0; \tag{4.3}$$

P is positive definite if and only if

$$\text{rank } DB = \text{rank } D = \text{rank } B, \tag{4.4}$$

(4.2) *holds, and Y is of the form*

$$Y = WW^T, \tag{4.5}$$

where W has $n - r_B$ linearly independent columns ($r_B = $ rank B) *and*

$$B^T W = 0. \tag{4.6}$$

Proof. The necessity of (4.1), (4.2) and (4.4) is established in Lemma 2.3 and we now show the necessity of the conditions on Y. Consider formula (3.11) for P. Let

$$x = (I - B(RDB)^\dagger RD)\eta, \tag{4.7}$$

where η is any nonzero vector. Then

$$\begin{aligned} x^T P x = & -\eta^T(I - D^T R(RDB)^\dagger B^T)D^T R(RDB)^\dagger RD(I - B(RDB)^\dagger RD)\eta \\ & + \eta^T Y \eta = \eta^T Y \eta, \end{aligned} \tag{4.8}$$

whence (4.3) is seen to be necessary. To prove the necessity of (4.5), assume that the columns of W do not span $B^\perp$, the $(n - r_B)$-dimensional orthogonal complement of the range space of B. There is then a nonzero η in $B^\perp$ such that $\eta^T Y \eta = 0$, and x given by (4.7) is nonzero (being the sum of a vector in $B^\perp$ and a vector in the range space of B). Then (4.8) shows that $x^T P x = 0, x \neq 0$, proving the necessity of (4.5). To establish sufficiency, we first note that as in the proof of Lemma 2.3, (4.2) implies that $RDB \leqq 0$; whence $(RDB)^\dagger \leqq 0$ (since RDB is symmetric, $RDB = H\Omega H^T$, $HH^T = I$, and $(RDB)^\dagger = H\Omega^\dagger H^T$, where $\Omega = \operatorname{diag}(\omega_i)$ and ω_i are eigenvalues of RDB). Thus the first term in (3.11) is nonnegative and $P \geqq 0$ under (4.3). For the case $P > 0$, let $x = \eta + B\rho$, where η is in the orthogonal complement $B^\perp$ of the range space of B. Then

$$x^T P x = -x^T D^T R (RDB)^\dagger R D x + \eta^T W W^T \eta,$$

where the first term is nonnegative and the second is, under (4.5), positive for all nonzero η. Thus $x^T P x > 0$ for $\eta \neq 0$. If $\eta = 0$, then $x = B\rho$, and for $x = B\rho \neq 0$ we have

$$x^T P x = -\rho^T B^T D^T R (RDB)^\dagger RDB\rho = -\rho^T RDB\rho.$$

Since $-RDB$ is nonnegative, $-RDB = Z^T Z$ for some real Z and then $DB\rho = -R^{-1} Z^T Z\rho$. But since $B\rho \neq 0$, then under (4.4), $DB\rho \neq 0$ and $Z^T Z\rho \neq 0$. Thus $x^T P x = -\rho^T RDB\rho = \rho^T Z^T Z\rho > 0$. Hence $P > 0$. For the representation (3.12) of P we can proceed in a similar manner, or simply note that (3.12) follows from (3.11) if we replace the Penrose inverse † by the pseudoinverse $^\#$ defined in Remark 3.2. Q.E.D.

Theorem 4.1 together with the theorems of § 3 establish the sufficiency part of Theorem 2.2. The equivalent of Theorem 4.1, for a partitioned P that results when B is in canonical form (2.16), is given in the Appendix (see (A.21)–(A.25)).

Since (3.10) is necessary for $P \geqq 0$, it is clear that (3.11) or (3.12), together with the conditions (4.3) and (4.5) on Y, are the general, real, symmetric solutions $P \geqq 0$ and $P > 0$ of (2.1). We thus have all the desired solutions R and P of (2.1), needed for solving the inverse problem.

5. The inverse problem. If matrices B and D of (1.1), (1.2) satisfy the conditions of Theorem 2.1, we can construct an $R = R^T > 0$ as in Theorems 3.1 and 3.2, and a $P = P^T$ as in Theorem 3.3, so that (1.4) is satisfied. To construct a bounded Q from (1.10) we require that P be differentiable. We therefore make the following assumption.

ASSUMPTION 5.1. In (1.1), (1.2), $B(t)$ and $D(t)$ are differentiable on $[t_0, t_1]$ and are of constant rank.

The latter part prevents an apparent discontinuity in P due to a change in the rank of $Y = WW^T$, mandated by Theorem 4.1 for $P > 0$, at the instant B changes rank. However, since the direct problem does not require Assumption 5.1, and our representations for P are general, it is clear that the assumption is for convenience only.

We now show that a performance index (1.3) with the weighting matrices R, Q and $F = P(t_1)$ constructed in §§ 3 and 4 is minimized by the control (1.2).

LEMMA 5.1. *Consider a closed-loop linear system* (1.1), (1.2). *Let $R > 0$, Q, P, and $F = P(t_1)$ be arbitrary uniformly bounded symmetric matrices satisfying* (1.4) *and the Riccati equation* (1.5). *Then the performance index* (1.3) *attains its absolute minimum I_* given by* (1.6), *over all square-integrable controls, for all x_0 and all $t_0 < t_1 \leqq \infty$. The optimal control is uniquely given by* (1.2).

Proof. Substituting (1.4) into (1.5) we have

$$-\dot{P} = PA + A^T P - D^T R D + Q. \tag{5.1}$$

Multiplying both sides by x and using $Ax = \dot{x} - Bu$, we have

$$-\frac{d}{dt}(x^T P x) = -(u - Dx)^T R(u - Dx) + x^T Q x + u^T R u. \tag{5.2}$$

By integrating (5.2), setting $P(t_1) = F$, and multiplying by $\frac{1}{2}$, we have

$$\begin{aligned} &\frac{1}{2}x_0^T P(t_0)x_0 + \frac{1}{2}\int_{t_0}^{t_1} (u - Dx)^T R(u - Dx)\,dt \\ &\qquad = \frac{1}{2}x(t_1)^T F x(t_1) + \frac{1}{2}\int_{t_0}^{t_1} (x^T Q x + u^T R u)\,dt. \end{aligned} \tag{5.3}$$

The right side of (5.3) is the performance index I of (1.3). Since $R > 0$, the integral on the left side is nonnegative. Thus I attains its absolute minimum if and only if $u = Dx$, as stated. Q.E.D.

When $t_1 = \infty$ in (1.3), the lemma is valid even if the closed-loop system (1.1), (1.2) is not asymptotically stable, since the integral on the left side of (5.3) must approach $+\infty$ if it is not finite. The time-invariant case, however, must be treated distinctly. We then arrive at (5.3) by starting with the constant quadratic matrix equation (1.9) rather than with the matrix Riccati equation (1.5), and since the performance index (1.7) does not have the terminal term $\frac{1}{2}x^T(t_1)Fx(t_1)$, (5.3) is, for $u = Dx$, replaced by

$$\tfrac{1}{2}x^T(0)P(0)x(0) - \tfrac{1}{2}x^T(\infty)P(\infty)x(\infty) = I.$$

The term $\frac{1}{2}x^T(\infty)P(\infty)x(\infty)$ can be positive and finite, raising the possibility, pointed out to us by B. P. Molinari, of an optimal control law that is unstable. Thus, to draw conclusions from (5.3), we restrict consideration to *stabilizing controls*, i.e., such that $x(t_1) \to 0$ as $t_1 \to \infty$. We have the following lemma.

LEMMA 5.2. *Consider a time-invariant asymptotically stable system* (1.1), (1.2), *and symmetric constant matrices* $R > 0$, P_∞, *and* Q *satisfying* (1.4) *and the quadratic matrix equation* (1.9). *Then the performance index* (1.7) *attains its absolute minimum over all square-integrable stabilizing controls for all* x_0, (1.2) *is the unique minimizing control, and* P_∞ *is the unique asymptotically stable equilibrium point of the Riccati equation* (1.8).

Proof. Only the last assertion remains to be proved. To prove it, we shift the origin of the Riccati equation (1.8) to P_∞ by considering

$$\bar{P}(t) = P(t) - P_\infty. \tag{5.4}$$

We find that

$$\dot{\bar{P}} = \bar{P}A_c + A_c^T\bar{P} - \bar{P}BR^{-1}B^T\bar{P}, \tag{5.5}$$

where

$$A_c = A + BD. \tag{5.6}$$

Since $\operatorname{Re}\{\lambda_i\} < 0$ for any eigenvalue λ_i of A_c, we have

$$\operatorname{Re}\{\lambda_i + \lambda_j\} < 0, \quad \text{all } i, j. \tag{5.7}$$

By (5.7), the linear part of (5.5) is asymptotically stable and hence, by Lyapunov's first method, $\bar{P}_\infty = 0$ is a locally asymptotically stable equilibrium point of (5.5); the same holds for P_∞ with respect to the Riccati equation (1.8). It remains to show that P_∞ is the only asymptotically stable equilibrium point of (1.8). First we rewrite (1.9) as

$$PA_c + A_c^T P = -D^T R D - Q. \tag{5.8}$$

For given A, B, D, R and Q, this is a linear matrix equation in P, and it has a unique solution P_∞ because by (5.7), $\lambda_i + \lambda_j \neq 0$ for all i, j. Thus any other solution of (1.9), say P'_∞, must yield D':

$$D' = -R^{-1}B^T P'_\infty \neq D = -R^{-1}B^T P_\infty.$$

Now suppose P'_∞ is an asymptotically stable equilibrium point of (1.8). Then, by reversing the previous arguments,

$$A'_c = A + BD'$$

is asymptotically stable, and $u = D'x$ provides the minimizing control for I. Hence

$$I_* = \tfrac{1}{2}x_0^T P_\infty x_0 = \tfrac{1}{2}x_0^T P'_\infty x_0, \quad \text{all } x_0,$$

whence $P'_\infty = P_\infty$ and $D' = D$.

Remark 5.1. Lemma 5.2 extends, to the case where Q is not necessarily nonnegative definite, the well-known facts (for $Q \geqq 0$) (i) that there is a one-to-one relation between the stability of the Riccati equation and that of the corresponding closed-loop optimal system, and (ii) that the Riccati equation (1.8) has at most one asymptotically stable equilibrium point. In contrast with the case $Q \geqq 0$, however, the equilibrium point P_∞ may not be positive definite and its domain of attraction is not generally known. See also recent results in [6] and [7].

We now have all the elements needed for solution of the inverse problem. Theorems 2.1, 2.2, 3.1, 3.2, 3.4, 3.5 and 4.1, and Lemmas 5.1 and 5.2, together imply the next theorem.

THEOREM 5.1. *Consider a closed-loop linear system* (1.1), (1.2) *satisfying Assumption* 5.1. *It is possible to construct a performance index* (1.3) *with*

$$F = F^T, \quad Q = Q^T, \quad R = R^T > 0, \tag{5.9}$$

that attains its absolute minimum I_* *over all square-integrable controls, for all* x_0 *and all* $t_0 < t_1 \leqq \infty$, *if and only if for all* $t, t_0 \leqq t \leqq t_1$, *the following conditions hold*:

(5.10) DB *has m linearly independent real eigenvectors*

and

$$\operatorname{rank} BD = \operatorname{rank} D. \tag{5.11}$$

The minimal value I_* *can be negative. An index* (1.3) *such that* $I_* \geqq 0$ *for all* x_0 *and all* $t_0 < t_1 \leqq \infty$ *can be constructed if and only if in addition to* (5.10), *for all* t, $t_0 \leqq t \leqq t_1$:

(5.12) *all eigenvalues of DB are nonpositive*, *and* (5.11) *is strengthened to*

$$\text{rank}\, DB = \text{rank}\, D. \tag{5.13}$$

An index (1.3), *such that* $I_* > 0$ *for all* x_0 *and all* $t_0 < t_1 \leqq \infty$, *can be constructed if and only if, in addition to* (5.10) *and* (5.12), *the rank condition* (5.13) *is strengthened to*

$$\text{rank}\, DB = \text{rank}\, D = \text{rank}\, B. \tag{5.14}$$

If the system (1.1), (1.2) *is constant and asymptotically stable, then it is possible to construct a performance index* (1.7) *with constant symmetric* Q *and* $R > 0$, *that attains its absolute minimum* I_* *over all square-integrable stabilizing controls, for all* x_0, *if and only if the above conditions hold. All performance indices corresponding to these conditions can be constructed by the general formulas of* § 3.

We observe that all conditions for the inverse problem are on the system matrices B and D, while A is arbitrary (aside from stability in the constant case). This is so because the inverse problem obviates the stability and existence problems of the direct problem. Conditions on A, as well as on B and D emerge when $Q \geqq 0$ is desired (see [1], [2], and the sequel to this paper); the conditions on B, D, BD, and DB discovered here remain of course necessary properties of a linear optimal system.

6. Consequences of B having full rank. Normally the $n \times m$ system matrix B has full rank and $m \leqq n$; in particular, this is so in a single-input system. It is therefore of interest to record the resulting simplifications in our previous results.

Case 1. rank $B = m$, $m < n$. We observe that the rank condition (5.11) always holds, because by Sylvester's inequality,

$$\text{rank}\, BD \geqq \text{rank}\, B + \text{rank}\, D - m = \text{rank}\, D,$$

which implies (5.11). Further, the compatibility condition (2.2), which somewhat complicates the construction of R (see Theorem 3.2 and Remark 3.1), is always satisfied because now a $B^\ddagger$ such that $B^T B^{\ddagger^T} = I$ always exists (e.g., $B^\ddagger = B^\dagger = (B^T B)^{-1} B^T$).

If DB is nonsingular, then formula (3.11) for P reduces to

$$P = -D^T (B^T D^T)^{-1} R D + Y, \tag{6.1}$$

the rank conditions (5.13) and (5.14) always hold, and the eigenvalue condition (5.12) becomes simply

$$DB < 0. \tag{6.2}$$

Case 2. rank $B = m = 1$. Here R reduces to a scalar $r > 0$, B to a column vector b, and D to a row vector d^T. All the conditions of Lemma 2.1 are now satisfied and P can always be represented by (3.8), where now U is a row vector, say, $U = b^\dagger = b^T / b^T b$. In particular, if b is in canonical form, then since

$$b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad b^{\dagger^T} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad d = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}, \quad (3.9) \Rightarrow Y = \begin{bmatrix} Y_{11} & 0 \\ 0 & 0 \end{bmatrix},$$

we find that (3.8) becomes

$$P = \begin{bmatrix} Y_{11} & -rd_1 \\ -rd_1^T & -rd_2 \end{bmatrix}. \tag{6.3}$$

For any b, rank DB = rank D reduces to

$$d^T b \neq 0; \tag{6.4}$$

then (6.1) becomes

$$P = -(r/d^T b) d d^T + Y, \tag{6.5}$$

and (6.2) reduces to

$$d^T b < 0. \tag{6.6}$$

Since r can be any positive scalar and P can always be constructed as in (3.8) or (6.3), we have a corollary.

COROLLARY 6.1. *Every single-input linear feedback system* (1.1), (1.2), *such that Assumption* 5.1 *holds, minimizes a performance index of the type* (1.3). *A performance index* (1.3) *such that* $I_* > 0$ *for all* x_0 *and all* $t_0 < t_1$ *exists if and only if for all* t, $t_0 \leqq t \leqq t_1$, $d^T b < 0$ *holds.*

Case 3. rank $B = m = n$. The case where B is nonsingular is rather trivial: P is simply $-B^{T^{-1}}RD$ and rank conditions (2.4) and (2.8) are automatically satisfied.

The case of $m > n$ is unusual, but will be discussed for completeness.

Case 4. rank $B = n < m$. In contrast with the case of $m < n$, the compatibility condition is not automatically satisfied, nor is the rank condition (5.11). However, (5.11) is implied by the symmetry condition

$$RDB = B^T D^T R$$

because, since BB^T is now nonsingular, it yields

$$(BB^T)^{-1} BRDBD = D^T RD,$$

whence, by (2.6),

$$\text{rank } BD \geqq \text{rank } (BB^T)^{-1} BRDBD = \text{rank } D^T RD = \text{rank } D,$$

which implies (5.11). Also in contrast with the case $m < n$, the rank condition (5.13) is now automatically satisfied as can be verified by Sylvester's inequality. Finally, (2.1) is now readily solvable for P, yielding

$$P = -(BB^T)^{-1} BRD, \tag{6.7}$$

which by postmultiplying by $BB^T(BB^T)^{-1}$ is seen to be symmetric under the symmetry of RDB,

$$P = -(BB^T)^{-1} BRD = (BB^T)^{-1} B(RDB) B^T (BB^T)^{-1}.$$

Appendix: Proof of Theorem 3.2. We first reduce B to canonical form by means of an equivalence transformation

$$\bar{B} = NBM = \begin{bmatrix} 0 & 0 \\ 0 & I_{r_B} \end{bmatrix}, \tag{A.1}$$

where N and M are suitable nonsingular matrices and r_B is the rank of B. If we define

(A.2) $$\bar{R} = M^T R M, \qquad \bar{P} = (N^{-1})^T P N^{-1}, \qquad \bar{D} = M^{-1} D N^{-1},$$

we find that (2.1)–(2.7) remain valid in terms of the new matrices. We may therefore assume with no loss of generality that B is initially in canonical form.

According to the hypotheses of Theorem 3.2, the rank condition (2.4) and the eigenvector condition (2.7) hold. We have

$$BD = \begin{bmatrix} 0 & 0 \\ 0 & I_{r_B} \end{bmatrix} \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ D_{21} & D_{22} \end{bmatrix},$$

whence rank BD = rank D requires

(A.3) $$D = \begin{bmatrix} KD_{21} & KD_{22} \\ D_{21} & D_{22} \end{bmatrix}$$

for some matrix K. Now

$$B^T D^T = \begin{bmatrix} 0 & 0 \\ 0 & I_{r_B} \end{bmatrix} \begin{bmatrix} D_{21}^T K^T & D_{21}^T \\ D_{22}^T K^T & D_{22}^T \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ D_{22}^T K^T & D_{22}^T \end{bmatrix}.$$

Thus the eigenvector equation $B^T D^T v = \lambda v$ has $m - r_B$ solutions

$$v = \begin{bmatrix} v_1 \\ -K^T v_1 \end{bmatrix}, \qquad \lambda = 0,$$

where the $m - r_B$ vectors v_1 are any set of linearly independent real $(m - r_B)$-vectors. The remaining r_B solutions are

$$v = \begin{bmatrix} 0 \\ v_2 \end{bmatrix},$$

where the v_2 are eigenvectors of D_{22}^T, and by the eigenvector condition (2.7) they are real and linearly independent. Thus in (3.1),

(A.4) $$V = \begin{bmatrix} V_{11} & 0 \\ -K^T V_{11} & V_{22} \end{bmatrix}, \quad \Lambda = \begin{bmatrix} 0 & 0 \\ 0 & \Lambda_2 \end{bmatrix},$$

where V_{11} is any nonsingular real matrix, and V_{22}, which is given by

(A.5) $$D_{22}^T V_{22} = V_{22} \Lambda_2,$$

is nonsingular and real. By Theorem 3.1, all R given by

(A.6) $$R = \begin{bmatrix} V_{11} & 0 \\ -K^T V_{11} & V_{22} \end{bmatrix} \begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{12}^T & \Gamma_{22} \end{bmatrix} \begin{bmatrix} V_{11}^T & -V_{11}^T K \\ 0 & V_{22}^T \end{bmatrix}$$

satisfy the symmetry condition on RDB and we only have to satisfy the compatibility condition (2.2) by choice of Γ. Expanding (A.6) gives

(A.7) $$\begin{aligned} R_{11} &= V_{11} \Gamma_{11} V_{11}^T, \qquad R_{12} = -R_{11} K + V_{11} \Gamma_{12} V_{22}^T, \\ R_{22} &= K^T R_{11} K + V_{22} \Gamma_{22} V_{22}^T - V_{22} \Gamma_{12}^T V_{11} K - K^T V_{11} \Gamma_{12} V_{22}^T. \end{aligned}$$

Letting $B^{\ddagger}$ be the $m \times n$ matrix:

$$B^{\ddagger} = \begin{bmatrix} 0 & 0 \\ 0 & I_{r_B} \end{bmatrix},$$

and using (A.3), (2.2) yields the conditions

$$(R_{11}K + R_{12})D_{21} = 0, \qquad (R_{11}K + R_{12})D_{22} = 0. \tag{A.8}$$

From the expression for R_{12} in (A.7), $R_{11}K + R_{12} = V_{11}\Gamma_{12}V_{22}^T$, and (A.8) becomes

$$V_{11}\Gamma_{12}V_{22}^T D_{21} = 0, \qquad V_{11}\Gamma_{12}V_{22}^T D_{22} = 0. \tag{A.9}$$

In view of (A.5) and the nonsingularity of V_{11} and V_{22}, these conditions reduce to

$$\Gamma_{12}V_{22}^T D_{21} = 0 \tag{A.10}$$

and

$$\Gamma_{12}\Lambda_2 = 0, \tag{A.11}$$

both of which are satisfied by the choice $\Gamma_{12} = 0$.

Thus, *all* $R = R^T > 0$ *such that both the symmetry and compatibility conditions hold, are given by all* Γ *in* (A.6) *such that* $\Gamma = \Gamma^T > 0$, $\Gamma\Lambda = \Lambda\Gamma$ *and* Γ_{12} *satisfies* (A.10) *and* (A.11). *One such* Γ_{12} *is* $\Gamma_{12} = 0$ *which is also necessary when* D_{22} *(or equivalently* Λ_2*) is nonsingular.*

This proves Theorem 3.2. The rule for Γ can be broken down further:

Γ_{22} is any real $r_B \times r_B$ matrix such that

$$\Gamma_{22} = \Gamma_{22}^T > 0, \qquad \Gamma_{22}\Lambda_2 = \Lambda_2\Gamma_{22}; \tag{A.12}$$

$\Gamma_{12} = \Gamma_{21}^T$ is any real $(m - r_B) \times r_B$ matrix such that

$$\Gamma_{12}V_{22}^T D_{21} = 0, \qquad \Gamma_{12}\Lambda_2 = 0; \tag{A.13}$$

Γ_{11} is any real $(m - r_B) \times (m - r_B)$ matrix such that

$$\Gamma_{11} = \Gamma_{11}^T > \Gamma_{12}\Gamma_{22}^{-1}\Gamma_{12}^T. \tag{A.14}$$

Analysis of (A.13) shows that

$$\operatorname{rank} \Gamma_{12} \leqq \operatorname{rank} B - \operatorname{rank} D. \tag{A.15}$$

We conclude this Appendix by deriving a general formula for P when B is in canonical form (A.1). Using B given by (A.1), D given by (A.3), and $U = B^{\dagger}$, (3.7) yields

$$P_0 = -\begin{bmatrix} 0 & D_{21}^T(K^T R_{12} + R_{22}) \\ (R_{12}^T K + R_{22})D_{21} & (R_{12}^T K + R_{22})D_{22} \end{bmatrix}, \tag{A.16}$$

and (3.9) yields

$$Y = \begin{bmatrix} Y_{11} & 0 \\ 0 & 0 \end{bmatrix}, \tag{A.17}$$

where Y_{11} is any real symmetric matrix. From the expression for R_{12} in (A.7),

$$K = -R_{11}^{-1}R_{12} + R_{11}^{-1}V_{11}\Gamma_{12}V_{22}^T.$$

Postmultiplying K by D_{21} and D_{22}, and using (A.9) gives

$$KD_{21} = -R_{11}^{-1}R_{12}D_{21}, \qquad KD_{22} = R_{11}^{-1}R_{12}D_{22}.$$

Thus (A.16) becomes

$$P_0 = -\begin{bmatrix} 0 & D_{21}^T(R_{22} - R_{12}^TR_{11}^{-1}R_{12}) \\ (R_{22} - R_{12}^TR_{11}^{-1}R_{12})D_{21} & (R_{22} - R_{12}^TR_{11}^{-1}R_{12})D_{22} \end{bmatrix}.$$

By defining R_0 as

(A.18) $$R_0 = R_{22} - R_{12}^TR_{11}^{-1}R_{12},$$

the general solution $P = P_0 + Y$, with Y given by (A.17), is

(A.19) $$P = \begin{bmatrix} P_{11} & -D_{21}^TR_0 \\ -R_0D_{21} & -R_0D_{22} \end{bmatrix},$$

where P_{11} is any real symmetric matrix. The term $-R_0D_{22}$ in (A.19) is symmetric (as expected), because we find that

(A.20) $$R_0 = V_{22}(\Gamma_{22} - \Gamma_{12}^T\Gamma_{11}^{-1}\Gamma_{12})V_{22}^T,$$

whence, using (A.13), we have

$$R_0D_{22} = V_{22}\Gamma_{22}V_{22}^TD_{22} = V_{22}\Gamma_{22}\Lambda_2V_{22}^T.$$

The conditions for $P \geqq 0$ and $P > 0$ can be obtained in terms of the partitioned blocks of P, from Theorem 4.1 or directly from (A.19) by the results in [5]. Corresponding to conditions (4.1), (4.2) and (4.3), we find that *P given by* (A.19) *is nonnegative definite if and only if*

(A.21) $$\operatorname{rank}[D_{21} \quad D_{22}] = \operatorname{rank} D_{22},$$

(A.22) *all eigenvalues of D_{22} are nonpositive*,

and

(A.23) $$P_{11} \geqq -D_{21}^TR_0D_{22}^{\#}D_{21},$$

where $D_{22}^{\#}$ is any matrix such that $D_{22}D_{22}^{\#}D_{22} = D_{22}$ and $D_{22}^{\#}D_{22}D_{22}^{\#} = D_{22}^{\#}$. For $P > 0$ it is necessary and sufficient that:

(A.24) *all eigenvalues of D_{22} are negative*,

and

(A.25) $$P_{11} > -D_{21}^TR_0D_{22}^{-1}D_{21}.$$

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