

# Inverse problem with transmission eigenvalues for the discrete Schrödinger equation

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The discrete Schrödinger equation with the Dirichlet boundary condition is considered on a half-line lattice when the potential is real valued and compactly supported. The inverse problem of recovery of the potential from the corresponding transmission eigenvalues is analyzed. The Marchenko method and the Gel'fand-Levitan method are used to solve the inverse problem uniquely, except in one “unusual” case where the sum of the transmission eigenvalues is equal to a certain integer related to the support of the potential. It is shown that in the unusual case, there may be a unique potential corresponding to a given set of transmission eigenvalues, there may be a finite number of distinct potentials for a given set of transmission eigenvalues, or there may be infinitely many potentials for a given set of transmission eigenvalues. The theory presented is illustrated with several explicit examples. © 2015 AIP Publishing LLC. [<http://dx.doi.org/10.1063/1.4927264>]

## I. INTRODUCTION

Consider the discrete Schrödinger equation on the half-line

$$-\psi_{n+1} + 2\psi_n - \psi_{n-1} + V_n\psi_n = \lambda\psi_n, \quad n \geq 1, \quad (1.1)$$

where  $\lambda$  is the spectral parameter,  $n$  is the spacial independent variable taking positive integer values, and the subscripts are used to denote the dependence on  $n$ . Thus,  $\psi_n$  denotes the wavefunction  $\psi(\lambda, n)$ . Note that  $\psi_0$  appears in (1.1) when  $n = 1$ , and hence the value at  $n = 0$  of the wavefunction  $\psi_n$  must be compatible with (1.1). By  $V_n$ , which can also be written as  $V(n)$ , we denote the value of the potential at  $n$ . The potential  $V$  is assumed to belong to class  $\mathcal{A}_b$  defined below.

*Definition 1.1.* The potential  $V$  belongs to class  $\mathcal{A}_b$  if the  $V_n$ -values are real and the support of the potential  $V$  is confined to the finite set  $n \in \{1, 2, \dots, b\}$  for some positive integer  $b$ , i.e.,  $V_n = 0$  for  $n > b$ .

The potential  $V$  in (1.1), by belonging to class  $\mathcal{A}_b$ , is compactly supported, i.e., it vanishes for  $n \geq b + 1$ , and hence, the knowledge of  $V$  is equivalent to the knowledge of the ordered set  $\{V_1, V_2, \dots, V_b\}$ . We do not assume that all the values in this ordered set are nonzero.

Note that (1.1) is the discrete analog of the Schrödinger equation on the half-line

$$-\psi''(k, x) + V(x)\psi(k, x) = k^2\psi(k, x), \quad x > 0, \quad (1.2)$$

where  $\lambda := k^2$ , the prime denotes the  $x$ -derivative, the potential  $V$  is real valued, and the support of the potential is confined to the interval  $x \in [0, b]$ .

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We are interested in the inverse problem of recovery of the potential  $V$  in (1.1) when the available input data set consists of the so-called transmission eigenvalues. The precise meaning of transmission eigenvalues is given in Section V. The transmission eigenvalues are uniquely determined by the potential  $V$  and Dirichlet boundary condition (2.2) specified in Section II. Informally speaking, the transmission eigenvalues correspond to certain  $\lambda$ -values at which the scattering from (1.1) with Dirichlet boundary condition (2.2) agrees with the scattering when the potential is identically zero. This inverse problem was recently analyzed in Ref. 22, and it was shown that the transmission eigenvalues uniquely determine the potential with the exception of one case,<sup>22</sup> which we now call the “unusual case,” namely, when the sum of the transmission eigenvalues is equal to  $4(b - 1)$ , where  $b$  is the positive integer appearing in Definition 1.1 and related to the support of the potential.

A recursive procedure was presented in Ref. 22 to determine the potential from the transmission eigenvalues in the “usual case,” and the nonuniqueness in the unusual case was illustrated via an example in Ref. 22. However, the procedure of Ref. 22 to construct the potential is not as direct as either of the Marchenko procedure or the Gel’fand-Levitan procedure we present in this paper because it requires first the construction of the right-hand side of (2.70) in our paper, namely, the regular solution to (1.1) as a function of  $\lambda$ , and then using that regular solution in (1.1) to extract the values of the potential.

We remark that when the potential  $V$  vanishes for  $n \geq b + 1$ , while  $V_b \neq 0$ , as indicated in Theorem 5.2, the number of transmission eigenvalues (including multiplicities) is  $2b - 2$ , and hence, the inverse problem dealing with transmission eigenvalues is meaningful only when  $b \geq 2$ .

One of our main goals in this paper is to develop a comprehensive approach to solve the inverse problem in the usual case in order to determine the potential  $V$  in class  $\mathcal{A}_b$  from the set of corresponding transmission eigenvalues. This is done via the Marchenko method and also the Gel’fand-Levitan method for (1.1). Another goal we have is to elaborate on the unusual case and to show that the inverse problem in the unusual case may or may not have a unique solution. This is done by presenting various examples in Section VI and by showing that all of the following possibilities occur. There may be only one real-valued potential corresponding to a given set of transmission eigenvalues, there may be a finite number of distinct real-valued potentials corresponding to a given set of transmission eigenvalues, or there may be infinitely many distinct real-valued potentials corresponding to a given set of transmission eigenvalues. It is an open problem to explain the physical reason behind the occurrence of the unusual case.

The literature on transmission eigenvalues and related inverse problems is expanding rapidly. We refer the reader to Refs. 1, 3, 4, 7, 8, 12–15, and 19–21 and the references therein for a historical account of transmission eigenvalues and some important developments in the field. Since we are dealing with difference equation (1.1) rather than differential equation (1.2), the most relevant reference for our paper is Ref. 22. The techniques used in our paper are analogs of the techniques from Refs. 3 and 4 used in the continuous case for (1.2).

For the inverse problem for the discrete Schrödinger equation and the related Gel’fand-Levitan method, we refer the reader to Ref. 9, even though the discrete Schrödinger equation used in Ref. 9 differs from (1.1). In Ref. 9, the discrete Schrödinger equation is treated as the Jacobi equation

$$\frac{1}{2} e^{-(V_n+V_{n+1})/2} \phi_{n+1} + \frac{1}{2} e^{-(V_{n-1}+V_n)/2} \phi_{n-1} = \lambda \phi_n, \quad n \geq 1. \quad (1.3)$$

To help the reader in presenting the results for (1.1), we at times also state the corresponding results for (1.2) by illustrating the similarities and differences between the discrete and continuous cases.

Our paper is organized as follows. In Section II, the preliminaries are presented for the Marchenko method to be given in Section III, the Gel’fand-Levitan method to be given in Section IV, and the inverse problem with transmission eigenvalues to be given in Section V. This is done by introducing the so-called Jost solution  $f_n$  to (1.1), the regular solution  $\varphi_n$  to (1.1), the Jost function  $f_0$  corresponding to the value of the Jost solution at  $n = 0$ , and the scattering matrix  $S$  given in (2.45) and by presenting certain relevant properties of such quantities because those properties are needed in Secs. III–VI. Note that as in the continuous case, we call  $S$  a “matrix” even though it is scalar valued. In Section III, the linear system of Marchenko equations for (1.1) with boundary condition (2.2) is derived and it is shown how the potential can be recovered

from the solution to Marchenko system (3.15). The Marchenko system uses as input the scattering matrix and the bound-state information consisting of bound-state energies and the so-called Marchenko bound-state norming constants. In Section III, it is also shown that, as a result of the compact-support property of the potential, the bound-state information is actually contained in the scattering matrix, and hence the scattering matrix alone uniquely determines the potential. In Section IV, the Gel'fand-Levitan method for (1.1) with boundary condition (2.2) is derived and it is shown how the potential can be recovered from the solution to the linear system of Gel'fand-Levitan equations (4.17). It is also shown how the kernel of the Gel'fand-Levitan system can be constructed from the Jost function alone. In Section V, the transmission eigenvalues related to (1.1) with boundary condition (2.2) are introduced and they are shown to correspond to the zeros of the key quantity  $D$  given in (5.2) or equivalently to the zeros of the quantity  $E$  defined in (5.7). It is also shown how the Jost function  $f_0$  can be recovered from the key quantity  $E$  in the usual case, and hence, having  $f_0$  at hand, one can use either the Marchenko method or the Gel'fand-Levitan method to recover the potential. In Section V, it is further indicated what happens when the unusual case occurs, which may prevent the recovery of a unique potential from a given set of transmission eigenvalues. Finally, in Section VI, some examples are presented to illustrate the unique recovery of the potential from the transmission eigenvalues via the Marchenko method and the Gel'fand-Levitan method in the usual case. In that section, some further examples are presented in the unusual case to illustrate that there may exist a unique real-valued potential for a given set of transmission eigenvalues, there may be a finite number of distinct real-valued potentials for some other sets of transmission eigenvalues, or there may be infinitely many distinct real-valued potentials corresponding to some sets of transmission eigenvalues.

## II. PRELIMINARIES

In this section, we introduce the Jost solution  $f_n$  to (1.1), the regular solution  $\varphi_n$  to (1.1), the corresponding Jost function  $f_0$ , and scattering matrix  $S$  and we present some of their relevant properties needed later on for the solution to the inverse problem of recovery of the potential in (1.1) from the transmission eigenvalues.

Using the shift operators  $P^+$  and  $P^-$  and the identity operator  $I$ , which are given by

$$P^+\psi_n := \psi_{n+1}, \quad P^-\psi_n := \psi_{n-1}, \quad I\psi_n := \psi_n,$$

let us write (1.1) in the operator form as  $\mathcal{L}\psi_n = \lambda\psi_n$ , where we have defined

$$\mathcal{L} := -P^+ + 2I - P^- + V_n. \quad (2.1)$$

Let us associate with (1.1) the Dirichlet boundary condition

$$\psi_0 = 0, \quad (2.2)$$

which is the analog of the Dirichlet boundary condition for (1.2), namely,

$$\psi(0) = 0. \quad (2.3)$$

Note that  $\mathcal{L}$  given in (2.1) with boundary condition (2.2) is a bounded operator because the shift operators  $P^+$  and  $P^-$  appearing in (2.1) each have norm one and the multiplication by  $V_n$  is a bounded operator. Furthermore,  $\mathcal{L}$  is selfadjoint,<sup>23</sup> which can be verified by establishing the equality

$$\sum_{n=1}^{\infty} \phi_n^* (\mathcal{L} \psi_n) = \sum_{n=1}^{\infty} (\mathcal{L} \phi_n)^* \psi_n,$$

where  $\{\psi_n\}_{n=1}^{\infty}$  and  $\{\phi_n\}_{n=1}^{\infty}$  are two square-summable sequences satisfying (2.2). Note that we use an asterisk to denote complex conjugation, and we recall that the square summability of  $\psi_n$  is described by

$$\sum_{n=1}^{\infty} |\psi_n|^2 < +\infty. \quad (2.4)$$

The bound states of  $\mathcal{L}$  correspond to the discrete spectrum of  $\mathcal{L}$ , i.e., the set of those  $\lambda$ -values for which there exist corresponding square-summable solutions to (1.1) satisfying (2.2).

When the potential  $V$  belongs to class  $\mathcal{A}_b$  specified in Definition 1.1, as we will see in Theorem 2.5, the bound states can only occur when  $\lambda \in (-\infty, 0) \cup (4, +\infty)$  and the number of bound states is either zero or a finite number. We will use  $N$  to denote the number of bound states and assume that they occur at  $\lambda = \mu_j$  for  $j = 1, \dots, N$ . The continuous spectrum of the discrete Schrödinger operator  $\mathcal{L}$  with Dirichlet boundary condition (2.2) is the interval  $\lambda \in [0, 4]$  because as we will see, for each  $\lambda \in [0, 4]$ , we have a solution to (1.1) satisfying (2.2) in such a way that such a solution is either bounded in  $n$  without being square summable or it grows linearly in  $n$  as  $n \rightarrow +\infty$ .

In case  $V_n \equiv 0$  in (1.1), let us use  $\overset{\circ}{\psi}_n$  to denote the corresponding wavefunction. Thus,  $\overset{\circ}{\psi}_n$  satisfies the unperturbed discrete Schrödinger equation

$$-\overset{\circ}{\psi}_{n+1} + 2\overset{\circ}{\psi}_n - \overset{\circ}{\psi}_{n-1} = \lambda \overset{\circ}{\psi}_n, \quad n \geq 1. \quad (2.5)$$

It is convenient at times to use another spectral parameter related to  $\lambda$ , usually denoted by  $z$ , by letting

$$z := 1 - \frac{\lambda}{2} + \frac{1}{2} \sqrt{\lambda(\lambda - 4)}, \quad (2.6)$$

where the square root is used to denote the principal branch of the complex square-root function. Let us use  $\mathbf{T}$  for the unit circle  $|z| = 1$  in the complex plane  $\mathbf{C}$ ,  $\mathbf{T}^+$  for the upper portion of  $\mathbf{T}$  given by  $z = e^{i\theta}$  with  $\theta \in (0, \pi)$ , and  $\overline{\mathbf{T}^+}$  for the closure of  $\mathbf{T}^+$  given by  $z = e^{i\theta}$  with  $\theta \in [0, \pi]$ . Under the transformation  $\lambda \mapsto z$  we have  $\infty \mapsto 0$ ,  $0 \mapsto 1$ , and  $4 \mapsto -1$ . The real half-line  $\lambda \in (-\infty, 0)$  is mapped to the real interval  $z \in (0, 1)$  in the complex  $z$ -plane, the real interval  $\lambda \in (4, +\infty)$  is mapped to the real interval  $z \in (-1, 0)$ , and the real interval  $\lambda \in (0, 4)$  is mapped to  $\mathbf{T}^+$ . Hence, the real  $\lambda$ -axis is mapped to the boundary of the upper half of the unit disc in the complex  $z$ -plane. Thus, the bound states in the complex  $z$ -plane can only occur in the interval  $z \in (-1, 0)$  or  $z \in (0, 1)$ . From (2.6), it follows that

$$z^{-1} = 1 - \frac{\lambda}{2} - \frac{1}{2} \sqrt{\lambda(\lambda - 4)}. \quad (2.7)$$

We see from (2.6) and (2.7) that

$$\lambda = 2 - z - z^{-1}. \quad (2.8)$$

Letting

$$z = e^{i\theta}, \quad (2.9)$$

from (2.6), we get

$$\cos \theta = 1 - \frac{\lambda}{2}, \quad \sin \theta = \frac{1}{2} \sqrt{\lambda(4 - \lambda)}, \quad (2.10)$$

and hence, as we have already observed, as  $\theta$  takes values in the interval  $\theta \in [0, \pi]$ , the spectral parameter  $z$  traces  $\overline{\mathbf{T}^+}$  and  $\lambda$  moves in the interval  $\lambda \in [0, 4]$  from the left endpoint 0 to the right endpoint 4. Note that from (2.8) and (2.9), we have

$$\lambda = 2 - 2 \cos \theta. \quad (2.11)$$

Using (2.8), it is convenient to write (1.1) as

$$\psi_{n+1} + \psi_{n-1} = (z + z^{-1} + V_n) \psi_n, \quad n \geq 1. \quad (2.12)$$

Thus, (2.5) can be written as

$$\overset{\circ}{\psi}_{n+1} + \overset{\circ}{\psi}_{n-1} = (z + z^{-1}) \overset{\circ}{\psi}_n, \quad n \geq 1. \quad (2.13)$$

One can directly verify that  $z^n$  and  $z^{-n}$  are solutions to (2.13) and that they are linearly independent when  $z \neq \pm 1$ . Hence, the general solution to (2.13) can be written as a linear combination of  $z^n$  and

$z^{-n}$  when  $z \neq \pm 1$ . In (2.52), we display two linearly independent solutions to (2.13) when  $z = 1$ . In (2.56), we list two linearly independent solutions to (2.13) when  $z = -1$ .

There are certain relevant solutions to (2.12) and hence equivalently to (1.1). One such solution is the regular solution  $\varphi_n$  satisfying the initial conditions

$$\varphi_0 = 0, \quad \varphi_1 = 1, \tag{2.14}$$

and it is the analog of the regular solution  $\varphi(k, x)$  to Schrödinger equation (1.2) with the initial values

$$\varphi(k, 0) = 0, \quad \varphi'(k, 0) = 1. \tag{2.15}$$

Note that  $k$  and  $-k$  appear in the same way in (1.2) and (2.15), and hence, the regular solution  $\varphi(k, x)$  satisfies  $\varphi(-k, x) \equiv \varphi(k, x)$ . Similarly,  $z$  and  $z^{-1}$  appear in the same way in (2.12) and (2.14), and hence  $\varphi_n$  remains unchanged if we replace  $z$  with  $z^{-1}$  in  $\varphi_n$ .

Another relevant solution to (2.12) and hence also to (1.1) is the Jost solution  $f_n$  satisfying the asymptotic condition

$$f_n = z^n[1 + o(1)], \quad n \rightarrow +\infty. \tag{2.16}$$

It is the analog of the Jost solution  $f(k, x)$  satisfying (1.2) with the asymptotics

$$f(k, x) = e^{ikx}[1 + o(1)], \quad x \rightarrow +\infty. \tag{2.17}$$

Recall<sup>4,5,10,17,18</sup> that (1.2) has also the solution  $g(k, x)$  satisfying the asymptotics

$$g(k, x) = e^{-ikx}[1 + o(1)], \quad x \rightarrow +\infty,$$

and  $g(k, x)$  is related to the Jost solution as

$$g(k, x) = f(-k, x), \quad k \in \mathbf{R},$$

with  $\mathbf{R}$  denoting the real axis. In the discrete case, the corresponding analogous solution to (2.12) or equivalently to (1.1) is denoted by  $g_n$  and it satisfies the asymptotics

$$g_n = z^{-n}[1 + o(1)], \quad n \rightarrow +\infty. \tag{2.18}$$

From (2.12), (2.16), and (2.18), it follows that  $g_n$  is obtained from  $f_n$  by replacing  $z$  by  $z^{-1}$  in  $f_n$ .

When  $V_n \equiv 0$ , let us use  $\overset{\circ}{f}_n$  and  $\overset{\circ}{g}_n$  to denote the corresponding Jost solution  $f_n$  and its relative  $g_n$ , respectively. From (2.13), (2.16), and (2.18) we see that

$$\overset{\circ}{f}_n = z^n, \quad \overset{\circ}{g}_n = z^{-n}, \quad n \geq 1, \tag{2.19}$$

and from (2.13) and (2.19) we observe that the values of  $\overset{\circ}{f}_n$  and  $\overset{\circ}{g}_n$  at  $n = 0$  are obtained as

$$\overset{\circ}{f}_0 = 1, \quad \overset{\circ}{g}_0 = 1. \tag{2.20}$$

One can directly verify that the regular solution  $\overset{\circ}{\varphi}_n$  satisfying (2.13) and initial conditions (2.14) is given by

$$\overset{\circ}{\varphi}_n = \frac{z^n - z^{-n}}{z - z^{-1}}, \quad n \geq 0. \tag{2.21}$$

Using (2.9) in (2.21) we can write  $\overset{\circ}{\varphi}_n$  also as

$$\overset{\circ}{\varphi}_n = \frac{\sin(n\theta)}{\sin \theta}, \quad n \geq 0. \tag{2.22}$$

Equivalently, using (2.6) and (2.7) in (2.21) we can express  $\overset{\circ}{\varphi}_n$  in terms of the spectral parameter  $\lambda$  as

$$\overset{\circ}{\varphi}_n = \frac{\left(1 - \frac{\lambda}{2} + \frac{1}{2}\sqrt{\lambda(\lambda-4)}\right)^n - \left(1 - \frac{\lambda}{2} - \frac{1}{2}\sqrt{\lambda(\lambda-4)}\right)^n}{\sqrt{\lambda(\lambda-4)}}, \quad n \geq 0. \tag{2.23}$$

Let us now investigate the Jost solution  $f_n$  to (2.12) when the potential  $V$  belongs to class  $\mathcal{A}_b$ . Using  $V_n = 0$  for  $n \geq b + 1$  in (2.12), we see that

$$f_n = z^n, \quad n \geq b. \tag{2.24}$$

Let us emphasize that (2.24) holds at  $n = b$  as well. It is convenient to define

$$m_n := z^{-n} f_n, \tag{2.25}$$

which is the analog of the Faddeev function  $m(k, x)$  related to (1.2) and expressed in terms of the Jost solution  $f(k, x)$  as

$$m(k, x) := e^{-ikx} f(k, x).$$

Note that from (2.24) and (2.25), we get

$$m_n = 1, \quad n \geq b. \tag{2.26}$$

Using (2.24) and (2.25) in (2.12), we obtain

$$m_{n-1} = -z^2 m_{n+1} + (z^2 + V_n z + 1) m_n, \quad n \geq 1, \tag{2.27}$$

and hence we see that (2.27) allows us to obtain the values of  $m_n$  and  $f_n$  at  $n = 0$ .

*Proposition 2.1.* Assume that the potential  $V$  belongs to class  $\mathcal{A}_b$  for some  $b \geq 1$  and that  $V_b \neq 0$ . Then:

- (a) For  $0 \leq n \leq b - 1$ , the solution  $m_n$  to (2.27) with asymptotic condition (2.26) is a polynomial in  $z$  of degree  $(2b - 2n - 1)$  with coefficients uniquely determined by the ordered set  $\{V_{n+1}, V_{n+2}, \dots, V_b\}$  of potential values as

$$m_n = \sum_{j=0}^{2b-2n-1} K_{n(n+j)} z^j, \quad 0 \leq n \leq b - 1. \tag{2.28}$$

In particular, for  $0 \leq n \leq b - 1$ , we have

$$K_{nn} = 1, \quad K_{n(n+1)} = \sum_{j=n+1}^b V_j, \quad K_{n(n+2)} = \sum_{n+1 \leq j < l \leq b} V_j V_l, \tag{2.29}$$

$$K_{n(2b-n-2)} = V_b \sum_{j=n+1}^{b-1} V_j, \quad K_{n(2b-n-1)} = V_b. \tag{2.30}$$

- (b) For  $0 \leq n \leq b - 1$ , the Jost solution  $f_n$  to (2.12) satisfying (2.16) is a polynomial in  $z$  of degree  $(2b - n - 1)$  with the lowest-power term being  $z^n$  and the highest-power term being  $V_b z^{2b-n-1}$ , and it is given by

$$f_n = \sum_{j=0}^{2b-2n-1} K_{n(n+j)} z^{n+j}. \tag{2.31}$$

- (c) For  $0 \leq n \leq b - 1$ , the solution  $g_n$  to (2.12) satisfying (2.18) is given by

$$g_n = \sum_{j=0}^{2b-2n-1} K_{n(n+j)} z^{-n-j}. \tag{2.32}$$

- (d) The coefficients  $K_{nm}$  appearing in (2.31) are real valued, and for every non-negative pair of integers  $n$  and  $m$  we have

$$K_{nm} = 0, \quad n \geq 2b - m, \tag{2.33}$$

$$K_{nm} = 0, \quad n \geq m + 1. \tag{2.34}$$

*Proof.* We obtain (2.28) by solving (2.27) iteratively and using (2.26). We then get (2.31) with the help of (2.25) and (2.28). Finally, (2.32) is obtained by using  $z^{-1}$  instead of  $z$  in (2.31). The

coefficients  $K_{nm}$  appearing in (2.33) and (2.34) are missing in the expansions in (2.31) and (2.32), and hence, they can be chosen as zero. The real-valuedness of  $K_{nm}$  follows from the fact that  $f_n$  and  $g_n$  are complex conjugates of each other when  $z$  is on the unit circle  $\mathbb{T}$ . ■

Note that for  $1 \leq n \leq b - 1$  in going from  $f_n$  to  $f_{n-1}$ , two new powers of  $z$  are gained, which is seen by displaying  $f_n$  and  $f_{n-1}$  with powers in  $z$  in an ascending order as

$$\begin{aligned} f_n &= z^n + [V_{n+1} + \dots + V_b] z^{n+1} + \dots + V_b [V_{n+1} + \dots + V_{b-1}] z^{2b-n-2} + V_b z^{2b-n-1}, \\ f_{n-1} &= z^{n-1} + [V_n + \dots + V_b] z^n + \dots + V_b [V_n + \dots + V_{b-1}] z^{2b-n-1} + V_b z^{2b-n}. \end{aligned}$$

We remark that the coefficients  $K_{nm}$  appearing in Proposition 2.1 will be used in Section III in the Marchenko method for the inverse problem associated with (1.1).

Recall that the Wronskian of any two solutions  $\psi(k, x)$  and  $\phi(k, x)$  to (1.2) is defined with the help of a matrix determinant as

$$[\psi(k, x); \phi(k, x)] := \begin{vmatrix} \psi(k, x) & \phi(k, x) \\ \psi'(k, x) & \phi'(k, x) \end{vmatrix}. \tag{2.35}$$

It is known<sup>10,17,18</sup> that the Wronskian in (2.35) is independent of  $x$  and hence is a function of the spectral parameter  $k$  alone. The analog of (2.35) for the Wronskian of any two solutions  $\psi_n$  and  $\phi_n$  to discrete equation (1.1) is given by

$$[\psi_n; \phi_n] := \begin{vmatrix} \psi_n & \phi_n \\ \psi_{n+1} & \phi_{n+1} \end{vmatrix}. \tag{2.36}$$

Using (2.12) in (2.36), we get

$$\begin{aligned} [\psi_n; \phi_n] &= \begin{vmatrix} \psi_n & \phi_n \\ -\psi_{n-1} + (z + z^{-1} + V_n)\psi_n & -\phi_{n-1} + (z + z^{-1} + V_n)\phi_n \end{vmatrix} \\ &= \begin{vmatrix} \psi_n & \phi_n \\ -\psi_{n-1} & -\phi_{n-1} \end{vmatrix}. \end{aligned} \tag{2.37}$$

From (2.37), by interchanging the two rows in the last determinant and using (2.36), we obtain

$$[\psi_n; \phi_n] = [\psi_{n-1}; \phi_{n-1}],$$

confirming that the Wronskian of any two solutions to (1.1) is independent of  $n$ , and hence it is a function of the spectral parameter alone. Thus, the value of the Wronskian given in (2.36) can be evaluated at any  $n$ -value, e.g., at  $n = 0$  or as  $n \rightarrow +\infty$ .

For  $z \neq \pm 1$ , the regular solution  $\varphi_n$  can be expressed as a linear combination of  $f_n$  and  $g_n$  given in (2.31) and (2.32), respectively. Writing

$$\varphi_n = \alpha f_n + \beta g_n, \tag{2.38}$$

where  $\alpha$  and  $\beta$  are some coefficients depending only on the spectral parameter  $z$ , we can express  $\alpha$  and  $\beta$  in terms of Wronskians as

$$\alpha = \frac{[g_n; \varphi_n]}{[g_n; f_n]}, \quad \beta = \frac{[f_n; \varphi_n]}{[f_n; g_n]}. \tag{2.39}$$

Since the Wronskian of any two solutions to (1.1) is independent of  $n$ , we can evaluate the Wronskians appearing in the numerators in (2.39) at  $n = 0$ , and hence, with the help of (2.14) and (2.36), we obtain

$$[g_n; \varphi_n] = g_0, \quad [f_n; \varphi_n] = f_0. \tag{2.40}$$

Similarly, we can evaluate the Wronskians appearing in the denominators in (2.39) as  $n \rightarrow +\infty$ , and hence, with the help of (2.16), (2.18), and (2.36), we get

$$[f_n; g_n] = z^{-1} - z. \tag{2.41}$$

Thus, (2.38) can be written as

$$\varphi_n = \frac{1}{z - z^{-1}} (g_0 f_n - f_0 g_n). \tag{2.42}$$

Note that (2.40) is analogous to

$$[f(-k, x); \varphi(k, x)] = f(-k, 0), \quad [f(k, x); \varphi(k, x)] = f(k, 0),$$

(2.41) is analogous to

$$[f(k, x); f(-k, x)] = -2ik,$$

and (2.42) is analogous to

$$\varphi(k, x) = \frac{1}{2ik} [f(-k, 0) f(k, x) - f(k, 0) f(-k, x)]. \tag{2.43}$$

Let us remark that the property  $\varphi(-k, x) \equiv \varphi(k, x)$  readily follows from (2.43). Similarly, from (2.42), we directly observe that replacing  $z$  with  $z^{-1}$  in  $\varphi_n$  does not change the value of  $\varphi_n$ .

It is known<sup>10,17,18</sup> that  $f(k, 0)$  is the Jost function for Schrödinger equation (1.2) with Dirichlet boundary condition (2.3). Similarly, we see that  $f_0$  corresponds to the Jost function for discrete Schrödinger equation (1.1) with Dirichlet boundary condition (2.2). The scattering matrix  $S(k)$  for (1.2) with (2.3) is defined as

$$S(k) := \frac{f(-k, 0)}{f(k, 0)}, \tag{2.44}$$

and similarly, the scattering matrix associated with (1.1) and (2.2) is defined as

$$S := \frac{g_0}{f_0}. \tag{2.45}$$

Some relevant properties of the Jost function  $f_0$  are given in the following theorem.

**Theorem 2.2.** *Assume that the potential  $V$  belongs to class  $\mathcal{A}_b$  for some  $b \geq 1$  and that  $V_b \neq 0$ . Then, the Jost function  $f_0$  appearing in (2.42) is a polynomial in  $z$  of degree  $2b - 1$ , it is equal to  $m_0$  appearing in (2.28) with  $n = 0$ , and it is given by*

$$f_0 = \sum_{j=0}^{2b-1} K_{0j} z^j, \tag{2.46}$$

where the coefficients  $K_{0j}$  are uniquely determined by the ordered set  $\{V_1, \dots, V_b\}$  of potential values. In particular, we have

$$K_{00} = 1, \quad K_{01} = \sum_{j=1}^b V_j, \quad K_{02} = \sum_{1 \leq j < l \leq b} V_j V_l, \tag{2.47}$$

$$K_{0(2b-2)} = V_b \sum_{j=1}^{b-1} V_j, \quad K_{0(2b-1)} = V_b. \tag{2.48}$$

Thus,  $f_0 - 1$  is a “plus” function in the sense that it is analytic in  $z$  when  $|z| < 1$ , continuous when  $|z| \leq 1$ , and behaves like  $O(z)$  as  $z \rightarrow 0$  in  $|z| \leq 1$ . Similarly, the quantity  $g_0$  appearing in (2.42) is given by

$$g_0 = \sum_{j=0}^{2b-1} K_{0j} z^{-j}, \tag{2.49}$$

and hence  $g_0 - 1$  is a “minus” function in the sense that it is analytic in  $z$  when  $|z| > 1$ , continuous when  $|z| \geq 1$ , and  $O(1/z)$  as  $z \rightarrow \infty$  in  $|z| \geq 1$ .

*Proof.* With the help of (2.25), the results for  $f_0$  directly follow from Proposition 2.1. The expressions for the coefficients in (2.47) and (2.48) are obtained with the help of (2.29) and (2.30). The results for  $g_0$  are obtained from  $f_0$  given in (2.46) by replacing  $z$  with  $z^{-1}$ . ■



We observe from (2.46) and the first equality in (2.47) that the Jost function  $f_0$  does not vanish at  $z = 0$ , and in fact, the value of the Jost function at  $z = 0$  is given by 1. It is convenient to display  $f_0$  in powers of  $z$  in an ascending order as

$$f_0 = 1 + [V_1 + \dots + V_b]z + \dots + V_b [V_1 + \dots + V_{b-1}]z^{2b-2} + V_b z^{2b-1}. \tag{2.50}$$

**Theorem 2.3.** *Assume that the potential  $V$  belongs to class  $\mathcal{A}_b$  for some  $b \geq 1$ . Then, the ordered set  $\{V_1, V_2, \dots, V_b\}$  of potential values is uniquely determined by the ordered set  $\{K_{01}, K_{12}, K_{23}, \dots, K_{(b-1)b}\}$ , where the quantities  $K_{nm}$  are the coefficients appearing in (2.31).*

*Proof.* From the second identity in (2.29), we obtain

$$V_n = K_{(n-1)n} - K_{n(n+1)}, \quad 1 \leq n \leq b, \tag{2.51}$$

where we also use  $K_{b(b+1)} = 0$ , which follows from (2.33). Hence, the proof is complete. ■

The following theorem describes the behavior of the regular solution  $\varphi_n$  as  $n \rightarrow +\infty$  when  $\lambda = 0$  and when  $\lambda = 4$ . It shows that (1.1) with Dirichlet boundary condition (2.2) cannot have a bound state at  $\lambda = 0$  or  $\lambda = 4$ . It also shows that (1.1) and (1.2) have some similarities for the so-called generic case and also for the so-called exceptional case. Recall that<sup>5,10</sup> the exceptional case occurs when a bound-state eigenvalue has a limit occurring at the edge of the continuous spectrum. As a result, a small perturbation of the potential may result in changing the number of bound states by one. Some physicists describe the exceptional case as a “half-bound state” occurring at the limiting value at the edge of the continuous spectrum. In the exceptional case, at that limiting value, the regular solution and the Jost solution become linearly dependent.

**Theorem 2.4.** *Assume that the potential  $V$  belongs to class  $\mathcal{A}_b$  for some  $b \geq 1$ . Let  $\lambda$  and  $z$  be the spectral parameters appearing in (1.1) and (2.12), respectively, and let  $\varphi_n$  and  $f_n$  be the corresponding regular solution and the Jost solution to (1.1) appearing in (2.14) and (2.16), respectively. Let  $\overset{\circ}{f}_n$  and  $\overset{\circ}{\varphi}_n$  be the Jost and regular solutions appearing in (2.19) and (2.21), respectively, corresponding to  $V_n \equiv 0$ . Then:*

- (a) *At  $\lambda = 0$  or equivalently at  $z = 1$ , the regular solution  $\varphi_n$  either grows linearly in  $n$  as  $n \rightarrow +\infty$ , which corresponds to the generic case, or it remains bounded, which corresponds to the exceptional case. Hence,  $\lambda = 0$  never corresponds to a bound state for (1.1) with Dirichlet boundary condition (2.2).*
- (b) *At  $\lambda = 4$  or equivalently at  $z = -1$ , the regular solution  $\varphi_n$  generically grows linearly in  $n$  as  $n \rightarrow +\infty$ , and in the exceptional case it remains bounded. Hence,  $\lambda = 4$  never corresponds to a bound state for (1.1) with Dirichlet boundary condition (2.2).*
- (c) *If  $f_0$  has a zero at  $z = 1$ , then it must be a simple zero. If  $f_0$  has a zero at  $z = 1$ , then  $g_0$  also has a simple zero at  $z = 1$ .*
- (d) *If  $f_0$  has a zero at  $z = -1$ , then it must be a simple zero. If  $f_0$  has a zero at  $z = -1$ , then  $g_0$  also has a simple zero at  $z = -1$ .*
- (e) *The Jost function  $f_0$  and the quantity  $g_0$  cannot vanish simultaneously at any  $z$ -value with the possible exception of  $z = \pm 1$ .*
- (f) *The exceptional case at  $z = 1$  occurs when  $f_0 = 0$  at  $z = 1$ , and the exceptional case at  $z = -1$  occurs when  $f_0 = 0$  at  $z = -1$ . We may have  $f_0$  vanishing at  $z = 1$  but not at  $z = -1$ , we may have  $f_0$  vanishing at  $z = -1$  but not at  $z = 1$ , or we may have  $f_0$  vanishing at both  $z = 1$  and  $z = -1$ .*

*Proof.* For  $\lambda = 0$ , we can use the method of variation of parameters<sup>6</sup> and express any solution to (1.1) in terms of two linearly independent solutions to (2.5). In particular, for  $\lambda = 0$ , we can use the regular solution  $\overset{\circ}{\varphi}_n$  and the Jost solution  $\overset{\circ}{f}_n$  that are given by

$$\overset{\circ}{\varphi}_n = n, \quad \overset{\circ}{f}_n = 1, \quad n \geq 0. \tag{2.52}$$

The method of variation of parameters yields the following expression for the regular solution to (1.1):

$$\varphi_n = n \left[ 1 - \sum_{j=1}^{n-1} V_j \varphi_j \right] + \sum_{j=1}^{n-1} j V_j \varphi_j, \quad n \geq 2. \tag{2.53}$$

Using  $V_n = 0$  for  $n \geq b + 1$  in (2.53), for  $n > b$ , we obtain

$$\varphi_n = n \left[ 1 - \sum_{j=1}^b V_j \varphi_j \right] + \sum_{j=1}^b j V_j \varphi_j, \quad n \geq b + 1. \tag{2.54}$$

From (2.54), we conclude that  $\varphi_n$  at  $\lambda = 0$  grows linearly in  $n$  as  $n \rightarrow +\infty$  unless the exceptional case occurs with

$$\sum_{j=1}^b V_j \varphi_j = 1. \tag{2.55}$$

If (2.55) holds, the second summation term in (2.54) must be nonzero because otherwise we would have  $\varphi_n = 0$  for  $\lambda = 0$  and  $n \geq b + 1$ , causing  $\varphi_n = 0$  for  $\lambda = 0$  and for all  $n \geq 0$ , contradicting the second equality in (2.14). Thus, in either case, i.e., whether (2.55) holds or not, from (2.54) we conclude that  $\varphi_n$  at  $\lambda = 0$  cannot be square summable, and hence there cannot be a bound state at  $\lambda = 0$  for (1.1) with boundary condition (2.2). For  $\lambda = 4$  or equivalently  $z = -1$ , the proof is obtained similarly, in which case (2.52) is replaced with

$$\overset{\circ}{\varphi}_n = (-1)^{n-1} n, \quad \overset{\circ}{f}_n = (-1)^n, \quad n \geq 0, \tag{2.56}$$

(2.53) is replaced with

$$\varphi_n = (-1)^{n-1} n \left[ 1 - \sum_{j=1}^{n-1} (-1)^j V_j \varphi_j \right] + (-1)^{n-1} \sum_{j=1}^{n-1} (-1)^j j V_j \varphi_j, \quad n \geq 2, \tag{2.57}$$

(2.54) is replaced with

$$\varphi_n = (-1)^{n-1} n \left[ 1 - \sum_{j=1}^b (-1)^j V_j \varphi_j \right] + (-1)^{n-1} \sum_{j=1}^b (-1)^j j V_j \varphi_j, \quad n \geq b + 1, \tag{2.58}$$

and (2.55) is replaced with

$$\sum_{j=1}^b (-1)^j V_j \varphi_j = 1.$$

Thus, the proofs of (a) and (b) are complete. Let us now prove (c), (d), and (e). Since  $g_0$  is obtained from  $f_0$  by replacing  $z$  by  $z^{-1}$ , it follows that  $g_0$  vanishes at  $z = 1$  if  $f_0$  vanishes there. Similarly,  $g_0$  vanishes at  $z = -1$  if  $f_0$  vanishes there. However,  $f_0$  and  $g_0$  cannot vanish simultaneously at any other  $z$ -value because otherwise, as a result of (2.42), the regular solution  $\varphi_n$  would be zero at that  $z$ -value for all  $n \geq 1$ , contradicting  $\varphi_1 = 1$  stated in (2.14). Thus, in order to complete the proofs of (c), (d), and (e), we only need to show that a possible zero of  $f_0$  at  $z = 1$  must be a simple zero and that a possible zero of  $f_0$  at  $z = -1$  must be a simple zero. By Proposition 2.1(b), we know that  $f_n$  is a polynomial in  $z$  and this is true also when  $V_b = 0$ . Hence, we can expand  $f_n$  around  $z = 1$  as

$$f_n(z) = f_n(1) + (z - 1) \dot{f}_n(1) + O((z - 1)^2), \quad z \rightarrow 1 \text{ in } \mathbf{C}, \tag{2.59}$$

where  $f_n(z)$  denotes the value of  $f_n$  at  $z$  and  $\dot{f}_n(1)$  denotes the derivative of  $f_n$  with respect to  $z$  evaluated at  $z = 1$ . Since  $g_n$  is obtained by replacing  $z$  with  $z^{-1}$  in  $f_n$ , from (2.59), we get

$$g_n(z) = f_n(1) + \left( \frac{1}{z} - 1 \right) \dot{f}_n(1) + O\left( \left( \frac{1}{z} - 1 \right)^2 \right), \quad z \rightarrow 1 \text{ in } \mathbf{C}. \tag{2.60}$$

Using

$$\frac{1}{z} - 1 = -(z - 1) + O((z - 1)^2), \quad z \rightarrow 1 \text{ in } \mathbf{C},$$

we can write (2.60) as

$$g_n(z) = f_n(1) - (z - 1) \dot{f}_n(1) + O((z - 1)^2), \quad z \rightarrow 1 \text{ in } \mathbf{C}. \tag{2.61}$$

Using (2.36), (2.59), and (2.61), we write the left-hand side of (2.41) as

$$[f_n(z); g_n(z)] = 2(z - 1) \begin{vmatrix} \dot{f}_n(1) & f_n(1) \\ \dot{f}_{n+1}(1) & f_{n+1}(1) \end{vmatrix} + O((z - 1)^2), \quad z \rightarrow 1 \text{ in } \mathbf{C}. \tag{2.62}$$

Since we have

$$z - z^{-1} = 2(z - 1) + O((z - 1)^2), \quad z \rightarrow 1 \text{ in } \mathbf{C}, \tag{2.63}$$

with the help of (2.41) and (2.63), we obtain

$$[f_n(z); g_n(z)] = -2(z - 1) + O((z - 1)^2), \quad z \rightarrow 1 \text{ in } \mathbf{C}. \tag{2.64}$$

Comparing (2.62) and (2.64), we see that

$$\begin{vmatrix} f_n(1) & \dot{f}_n(1) \\ f_{n+1}(1) & \dot{f}_{n+1}(1) \end{vmatrix} = 1. \tag{2.65}$$

From (2.65), we conclude that  $f_n(1)$  and  $\dot{f}_n(1)$  cannot vanish simultaneously. Thus, if  $f_n(1) = 0$ , then we must have  $\dot{f}_n(1) \neq 0$ . Therefore,  $z = 1$  can only be a simple zero of  $f_n(z)$ . Proceeding similarly, we can prove that if  $f_n$  has a zero at  $z = -1$ , then such a zero must be a simple zero. Thus, the proof of (c)-(e) is complete. Finally, let us prove (f). For this, let us first show that  $\varphi_n$  at  $z = 1$  remains bounded as  $n \rightarrow +\infty$  if and only if  $f_0(1) = 0$ . Using (2.59) and (2.61) with  $n = 0$  as well as using (2.12), (2.13) and (2.62), we expand the right-hand side of (2.42) around  $z = 1$  as  $n \rightarrow +\infty$ , and we obtain

$$\varphi_n = n f_0(1) - \dot{f}_0(1) + O(z - 1), \quad z \rightarrow 1 \text{ in } \mathbf{C}, \tag{2.66}$$

where the terms represented with  $O(z - 1)$  remain bounded as  $n \rightarrow +\infty$ . From (2.66), we conclude that  $\varphi_n$  at  $z = 1$  remains bounded if and only if  $f_0(1) = 0$ , proving that the exceptional case with  $z = 1$  occurs if and only if  $f_0(1) = 0$ . A similar argument proves that the exceptional case with  $z = -1$  occurs if and only if  $f_0(-1) = 0$ . We can show with some explicit examples that  $f_0$  can vanish at  $z = 1$  but not at  $z = -1$ , and vice versa. For example, in Example 6.1 in Section VI, from (6.1), we see that we have  $f_0(1) = 0$  and  $f_0(-1) = 2$  if we choose  $V_1 = -1$ . In that example, we get  $f_0(-1) = 0$  and  $f_0(1) = 2$  if we choose  $V_1 = 1$ . In Example 6.3, choosing  $V_1 = -\sqrt{2}$  and  $V_2 = 1/\sqrt{2}$ , from the second equality in (6.12), we see that  $f_0(1) = 0$  and  $f_0(-1) = 0$ . ■

We remark that the summation terms appearing in (2.57) and (2.58) are the discrete analogs of the integrals appearing in (5.5) and (5.6) of Ref. 2. In the next theorem, we summarize the facts relevant to the bound states of (1.1) with boundary condition (2.2). Recall that the bound states correspond to the  $\lambda$ -values at which (1.1) has square-summable solutions satisfying boundary condition (2.2).

**Theorem 2.5.** *Assume that the potential  $V$  belongs to class  $A_b$  for some  $b \geq 1$ . Let  $\lambda$  and  $z$  be the spectral parameters appearing in (1.1) and (2.12), respectively, and let  $f_n, \varphi_n$ , and  $f_0$  be the corresponding Jost solution appearing in (2.16), the regular solution appearing in (2.42), and the Jost function appearing in (2.46). Then:*

- (a) *A bound state can only occur when  $\lambda \in (-\infty, 0)$  or  $\lambda \in (4, +\infty)$ . Equivalently, a bound state can only occur when  $z \in (-1, 0)$  or  $z \in (0, 1)$ .*
- (b) *The regular solution evaluated at a bound state, i.e., the value of  $\varphi_n$  evaluated at a  $\lambda$ -value corresponding to a bound state must be real. Similarly, the Jost solution evaluated at a bound state must be real.*

- (c) At a bound state, the Jost function  $f_0$  has a simple zero in  $\lambda$  and in  $z$ . At a bound state, the value of the Jost solution at  $n = 1$  cannot vanish, i.e.,  $f_1 \neq 0$  at a bound state.
- (d) The number of bound states, denoted by  $N$ , must be finite, and we have  $0 \leq N \leq 2b - 1$ , where  $b$  is the positive number related to the support of  $V$  and appearing in Definition 1.1. In particular, we have  $N = 0$  when  $V_n \equiv 0$ .
- (e) At a bound state, the quantity  $g_0$  appearing in (2.49) cannot vanish. Thus, the scattering matrix appearing in (2.45) has a simple pole in  $\lambda$  and in  $z$  at each bound state.

*Proof.* By its definition, at a bound state, (1.1) must have a square-summable solution and that solution must satisfy (2.2). Since the regular solution  $\varphi_n$  satisfies (2.2), any bound-state solution must be linearly dependent on the regular solution. Since the corresponding operator  $\mathcal{L}$  is self-adjoint, a bound state can only occur when  $\lambda$  is real valued. Recall that, as seen from (2.6), the real  $\lambda$ -values in the interval  $(0, 4)$  correspond to the  $z$ -values on  $\mathbf{T}^+$ , the upper portion of the unit circle. On the other hand, the  $z$ -value corresponding to a bound state cannot occur on  $\mathbf{T}^+$ , because, as a consequence of (2.16) and (2.18), neither of the two linearly independent solutions  $f_n$  and  $g_n$  to (1.1) vanish at those  $z$ -values as  $n \rightarrow +\infty$ . Furthermore, from Theorem 2.4, we know that a bound state cannot occur at  $z = 1$  or at  $z = -1$ . We also know from (2.6) that  $z = 0$  corresponds to  $\lambda = \infty$  and hence, a bound state cannot occur at  $z = 0$ . Thus, we have proved (a). Let us now prove (b). Since the bound states can only occur at some real  $z$ -values, from Proposition 2.1(d), we conclude that  $f_n$  given in (2.31) and  $g_n$  given in (2.32) are both real valued at a bound state for any  $n \geq 0$ . As a result, from (2.42) we conclude that the regular solution  $\varphi_n$  also takes real values at a bound state. Thus, we have proved (b). Let us now turn to the proof of (c). At a bound state, we conclude that the coefficient of  $g_n$  on the right-hand side of (2.42) must be zero because otherwise (2.18) implies that  $\varphi_n$  cannot be square summable at the bound state occurring at some  $z$ -value lying in the interval  $z \in (-1, 0)$  or  $z \in (0, 1)$ . Thus, at a bound state, the Jost function  $f_0$  must be zero. At a bound state, we cannot have  $f_1 = 0$  because then the only solution to (1.1) with  $f_0 = 0$  and  $f_1 = 0$  would have to be  $f_n \equiv 0$ , which is not compatible with (2.16). Let us now prove that the zero of  $f_0$  at a bound state must be simple. From (1.1), we obtain

$$\begin{cases} f_{n+1} + f_{n-1} = (2 - \lambda + V_n) f_n, \\ \frac{df_{n+1}}{d\lambda} + \frac{df_{n-1}}{d\lambda} = -f_n + (2 - \lambda + V_n) \frac{df_n}{d\lambda}, \end{cases} \tag{2.67}$$

where the second line is obtained by taking the  $\lambda$ -derivative of the first line. Multiplying the first line in (2.67) by  $df_n/d\lambda$  and the second line by  $f_n$  and taking the difference of the resulting equations, we get

$$f_{n+1} \frac{df_n}{d\lambda} - f_n \frac{df_{n+1}}{d\lambda} + f_{n-1} \frac{df_n}{d\lambda} - f_n \frac{df_{n-1}}{d\lambda} = f_n^2. \tag{2.68}$$

Let us evaluate (2.68) at a  $\lambda$ -value corresponding to a bound state and take the summation of both sides starting with  $n = 1$ . Many cancellations on the left-hand side in the summation yield

$$f_0 \frac{df_1}{d\lambda} - f_1 \frac{df_0}{d\lambda} = \sum_{n=1}^{\infty} f_n^2. \tag{2.69}$$

We already know that at a bound state  $f_0 = 0, f_1 \neq 0$ , and the  $f_n$ -values are real. Thus, (2.69) implies that at a bound state, we have

$$\frac{df_0}{d\lambda} = -\frac{1}{f_1} \sum_{n=1}^{\infty} f_n^2,$$

with the right-hand side being nonzero. Thus, the zero of  $f_0$  as a function of  $\lambda$  at the bound state must be simple. From (2.8) and Theorem 2.5(a), we conclude that  $d\lambda/dz$  at a bound state is strictly positive and hence, a simple zero of  $f_0$  in  $\lambda$  corresponds to a simple zero of  $f_0$  in  $z$ . Thus, the proof of (c) is complete. By Theorem 2.2, we know that  $f_0$  is a polynomial in  $z$  of degree  $2b - 1$ . Since the bound states correspond to the zeros of  $f_0$ , we conclude that the number of bound states cannot

exceed  $2b - 1$ . Furthermore, from the first equality in (2.20), we know that the Jost function  $f_0^\circ$  corresponding to the zero potential does not have any zeros, and hence the number of bound states corresponding to the zero potential is zero. Thus, we have proved (d). At a bound state  $z$ -value, we already know that  $f_0$  vanishes and that  $z$ -value cannot be equal to 1 or  $-1$ . Thus, from (2.42) we see that the vanishing of  $g_0$  at a bound state would imply  $\varphi_n \equiv 0$ , contradicting the second condition in (2.14). Then, from (2.45) and the simplicity of the zeros of  $f_0$  at the bound states, we conclude that the scattering matrix  $S$  must have a simple pole at each bound state. Thus, the proof of (e) is complete. ■

In (2.42), the regular solution  $\varphi_n$  to (1.1) with initial values (2.14) is expressed in terms of the spectral parameter  $z$ . It is possible to express  $\varphi_n$  in terms of the spectral parameter  $\lambda$  appearing in (1.1). That result is stated in the following theorem and will be useful in the formulation of the Gel'fand-Levitan procedure for (1.1).

**Theorem 2.6.** *Assume that the potential  $V$  belongs to class  $\mathcal{A}_b$  for some  $b \geq 1$ . Then, for  $n \geq 1$ , the regular solution  $\varphi_n$  to (1.1) with initial values (2.14) is a polynomial in  $\lambda$  of degree  $n - 1$  and is given by*

$$\varphi_n = \sum_{j=0}^{n-1} B_{nj} \lambda^j, \tag{2.70}$$

where the coefficients  $B_{nj}$  are real valued and are uniquely determined by the ordered set  $\{V_1, V_2, \dots, V_{n-1}\}$  of potential values. In particular, we have

$$B_{n(n-1)} = (-1)^{n-1}, \quad B_{n(n-2)} = (-1)^{n-2} \left[ 2(n-1) + \sum_{j=1}^{n-1} V_j \right], \tag{2.71}$$

$$B_{n(n-3)} = (-1)^{n-3} \left[ (n-2)(2n-3) + 2(n-2) \sum_{j=1}^{n-1} V_j + \sum_{1 \leq k < j \leq n-1} V_j V_k \right]. \tag{2.72}$$

*Proof.* As seen from (1.1),  $\varphi_n$  satisfies

$$\varphi_{n+1} + \varphi_{n-1} = (2 - \lambda + V_n) \varphi_n, \quad n \geq 1. \tag{2.73}$$

Solving (2.73) iteratively and using the initial values  $\varphi_0 = 0$  and  $\varphi_1 = 1$  as stated in (2.14), we get (2.70) with the coefficients given in (2.71) and (2.72). Since the initial values are real valued and the coefficients in (2.73) for  $\varphi_{n-1}$  and  $\varphi_n$  are real valued for real  $\lambda$ -values, the coefficients  $B_{nj}$  appearing in (2.70) are all real valued. ■

With the help of Theorem 2.6, we see that, for  $n \geq 1$ , the regular solution  $\varphi_n^\circ$  appearing in (2.23) is a polynomial in  $\lambda$  of degree  $n - 1$ . The same result can also be obtained directly from (2.23) by using the expansions

$$\left( 1 - \frac{\lambda}{2} \pm \frac{1}{2} \sqrt{\lambda(\lambda - 4)} \right)^n = \sum_{j=0}^n \binom{n}{j} \left( 1 - \frac{\lambda}{2} \right)^{n-j} \left( \pm \frac{1}{2} \sqrt{\lambda(\lambda - 4)} \right)^j, \quad n \geq 1, \tag{2.74}$$

with  $\binom{n}{j}$  denoting the binomial coefficient  $n!/(j!(n-j)!)$ . With the help of (2.74), from (2.23), we obtain

$$\varphi_n^\circ = \sum_{s=0}^{\lfloor (n-1)/2 \rfloor} \binom{n}{2s+1} \left( 1 - \frac{\lambda}{2} \right)^{n-1-2s} \lambda^s \left( \frac{\lambda}{4} - 1 \right)^s, \quad n \geq 1, \tag{2.75}$$

with  $\lfloor x \rfloor$  denoting the floor function, i.e., the greatest integer less than or equal to  $x$ .

Using either (2.75) or Theorem 2.6, we directly obtain the following corollary.

**Corollary 2.7.** *For  $n \geq 1$ , the regular solution to (2.5) with initial values (2.14), namely,  $\varphi_n^\circ$  appearing in (2.23), is a polynomial in  $\lambda$  of degree  $n - 1$  and is given by*

$$\overset{\circ}{\varphi}_n = \sum_{j=0}^{n-1} \overset{\circ}{B}_{nj} \lambda^j, \quad n \geq 1, \tag{2.76}$$

where the coefficients  $\overset{\circ}{B}_{nj}$  are real valued and obtained from the expansion in (2.75). In particular, we have

$$\overset{\circ}{B}_{n(n-1)} = (-1)^{n-1}, \quad \overset{\circ}{B}_{n(n-2)} = 2(-1)^{n-2}(n-1), \tag{2.77}$$

$$\overset{\circ}{B}_{n(n-3)} = (-1)^{n-3}(n-2)(2n-3). \tag{2.78}$$

Let us now consider expressing the regular solution  $\varphi_n$  to (1.1) in terms of the elements in the set  $\{\overset{\circ}{\varphi}_j\}_{j=1}^n$ . This problem arises in the formulation of the Gel'fand-Levitan method, and it is also closely related to the theory of orthogonal polynomials.<sup>9</sup>

**Theorem 2.8.** *Assume that the potential  $V$  belongs to class  $\mathcal{A}_b$  for some  $b \geq 1$ . Let  $\overset{\circ}{\varphi}_n$  be the regular solution to (2.5) satisfying initial conditions (2.14). Then, for  $n \geq 1$ , the regular solution  $\varphi_n$  to (1.1) with initial values (2.14) can be written as a linear combination of the elements in the set  $\{\overset{\circ}{\varphi}_j\}_{j=1}^n$ , and we have*

$$\varphi_n = \overset{\circ}{\varphi}_n + \sum_{j=1}^{n-1} A_{nj} \overset{\circ}{\varphi}_j, \quad n \geq 1, \tag{2.79}$$

with the coefficients  $A_{nj}$  for  $j = 1, \dots, n-1$  uniquely determined by the ordered set  $\{V_1, \dots, V_{n-1}\}$ . In particular, we have

$$A_{n(n-1)} = \sum_{j=1}^{n-1} V_j, \quad n \geq 2, \tag{2.80}$$

with the convention that

$$\begin{aligned} A_{n0} &= 0, & A_{(n+1)(n+1)} &= 1, & n &\geq 0, \\ A_{nm} &= 0, & 0 &\leq n < m. \end{aligned} \tag{2.81}$$

*Proof.* From Theorem 2.6, we see that  $\varphi_n$  is a polynomial in  $\lambda$  of degree  $n-1$ . From Corollary 2.7, we know that, for  $j \geq 1$ , the quantity  $\overset{\circ}{\varphi}_j$  is a polynomial in  $\lambda$  of degree  $j-1$ . Thus, we can use the set  $\{\overset{\circ}{\varphi}_j\}_{j=1}^n$  as a basis for polynomials in  $\lambda$  of degree  $n-1$ , and we express the polynomial  $\varphi_n$  as

$$\varphi_n = A_{nn} \overset{\circ}{\varphi}_n + \sum_{j=1}^{n-1} A_{nj} \overset{\circ}{\varphi}_j, \quad n \geq 1. \tag{2.82}$$

With the help of (2.70) and (2.76), we can compare the coefficients of  $\lambda^{n-1}$  on both sides of (2.82). From the value of  $B_{n(n-1)}$  in (2.71) and the value of  $\overset{\circ}{B}_{n(n-1)}$  in (2.77), we know that those coefficients are both equal to  $(-1)^{n-1}$ , and hence, we must have  $A_{nn} = 1$  in (2.82). Let us remark that we can use  $A_{n0}$  for  $n \geq 1$  as the coefficient of  $\overset{\circ}{\varphi}_0$ , by recalling that  $\overset{\circ}{\varphi}_0 = 0$  as a result of the first initial condition given in (2.14). Thus, the choices in (2.81) are appropriate. Using (2.70), (2.76), and an analog of (2.76) but with  $n-1$  instead of  $n$  there, we compare the coefficients of  $\lambda^{n-2}$  on both sides of (2.82). This leads us to the equation

$$B_{n(n-2)} = A_{nn} \overset{\circ}{B}_{n(n-2)} + A_{n(n-1)} \overset{\circ}{B}_{(n-1)(n-2)}, \quad n \geq 2. \tag{2.83}$$

Inserting in (2.83) the value  $A_{nn} = 1$  and the value of  $B_{n(n-2)}$  from (2.71) and the values of  $\overset{\circ}{B}_{n(n-2)}$  and  $\overset{\circ}{B}_{(n-1)(n-2)}$  from (2.77), we get (2.80). ■

We remark that the coefficients  $A_{nm}$  will appear in Section IV as the unknown quantities in the Gel'fand-Levitan system for (1.1) with Dirichlet boundary condition (2.2). From (2.80) and the first equality in (2.81), we obtain the following corollary.

*Corollary 2.9. Let  $V$  be a potential in class  $\mathcal{A}_b$  for some  $b \geq 1$ . Then,  $V$  is uniquely determined by the ordered set  $\{A_{21}, A_{32}, \dots, A_{(b+1)b}\}$  of the corresponding coefficients appearing in (2.79), and we have*

$$V_n = A_{(n+1)n} - A_{n(n-1)}, \quad n \geq 1, \tag{2.84}$$

with the understanding that  $A_{10} = 0$ .

### III. THE MARCHENKO METHOD

In this section, we outline the Marchenko method to recover the potential  $V$  from the corresponding scattering data for (1.1) with Dirichlet boundary condition (2.2). We also show that the bound-state information is contained in the scattering matrix  $S$  appearing in (2.45) as a result of the compact support of  $V$ . Thus, the corresponding scattering matrix alone determines the potential  $V$  in class  $\mathcal{A}_b$  uniquely without any need to specify the bound-state information independently.

The Marchenko equation for (1.2) is given<sup>5,10,17,18</sup> by

$$K(x, y) + M(x + y) + \int_x^\infty dt K(x, t) M(t + y), \quad y > x, \tag{3.1}$$

in which the unknown function  $K(x, y)$  is related to the Jost solution  $f(k, x)$  appearing in (2.17) as

$$K(x, y) := \frac{1}{2\pi} \int_{-\infty}^\infty dk [f(k, x) - e^{ikx}] e^{-iky},$$

or equivalently as

$$K(x, y) := \frac{1}{2\pi} \int_{-\infty}^\infty dk [f(-k, x) - e^{-ikx}] e^{iky}.$$

Note that  $K(x, y) = 0$  for  $x > y$  as a result of the facts<sup>5,10,17,18</sup> that if the potential  $V$  in (1.2) is real valued and integrable on  $x \in (0, b)$  and vanishes for  $x > b$ , then for each fixed  $x \geq 0$ , the Jost solution  $f(k, x)$  is analytic in  $k$  in the open upper-half complex plane  $\mathbf{C}^+$ , continuous in  $k$  in the closed upper-half complex plane  $\overline{\mathbf{C}^+}$ , and  $e^{-ikx} f(k, x) = 1 + O(1/k)$  as  $k \rightarrow \infty$  in  $\overline{\mathbf{C}^+}$ . The kernel of the Marchenko equation is expressed in terms of the scattering matrix  $S(k)$  given in (2.44) and the bound-state information as

$$M(y) := \frac{1}{2\pi} \int_{-\infty}^\infty dk [1 - S(k)] e^{iky} + \sum_{s=1}^N c_s^2 e^{-\kappa_s y}, \tag{3.2}$$

where the values  $k = i\kappa_s$  for  $s = 1, \dots, N$  with distinct positive  $\kappa_s$ -values correspond to the zeros of  $f(k, 0)$ , and the constants  $c_s$  denote the Marchenko bound-state norming constants given by

$$c_s := \frac{1}{\sqrt{\int_0^\infty dx f(i\kappa_s, x)^2}}. \tag{3.3}$$

Because the potential  $V$  is assumed to be supported within the finite interval  $x \in [0, b]$ , it turns out that the value of the norming constant  $c_s$  is uniquely determined by the scattering matrix  $S(k)$  as<sup>3,4</sup>

$$c_s = \sqrt{i \operatorname{Res}[S(k), i\kappa_s]}, \tag{3.4}$$

with  $\operatorname{Res}[S(k), i\kappa_s]$  denoting the residue of  $S(k)$  at the pole  $k = i\kappa_s$ . We note that the poles of  $S(k)$  in  $\mathbf{C}^+$  are all simple and that the number of bound states denoted by  $N$  is either 0 or a positive integer.

One can derive (3.1) from (2.43) as follows. From (2.43), we get

$$f(k, 0) f(-k, x) - f(-k, 0) f(k, x) = -2ik \varphi(k, x), \quad k \in \mathbf{R}. \tag{3.5}$$

Dividing (3.5) by  $f(k, 0)$  and using (2.44), we obtain

$$f(-k, x) - S(k) f(k, x) = -\frac{2ik}{f(k, 0)} \varphi(k, x), \quad k \in \mathbf{R}. \tag{3.6}$$

We can write (3.6) as

$$[f(-k, x) - e^{-ikx}] + [1 - S(k)] e^{ikx} + [1 - S(k)][f(k, x) - e^{ikx}] = H(k, x), \quad k \in \mathbf{R}, \tag{3.7}$$

where we have defined

$$H(k, x) := f(k, x) - e^{-ikx} - \frac{2ik}{f(k, 0)} \varphi(k, x). \tag{3.8}$$

With the help of the Fourier transform of (3.7) with  $(2\pi)^{-1} \int_{-\infty}^{\infty} dk e^{iky}$  for  $y > x$ , we obtain (3.1). When the potential  $V$  in (1.2) is real valued and compactly supported, the bound-state information can be obtained<sup>3,4</sup> from the scattering matrix  $S$  given in (2.44) because in that case  $S(k)$  has a meromorphic extension from  $k \in \mathbf{R}$  to  $k \in \mathbf{C}^+$  with simple poles at  $k = i\kappa_s$  for  $s = 1, \dots, N$  and the residues of  $S(k)$  at  $k = i\kappa_s$  are related to the bound-state norming constants as in (3.4). Note that the contribution of the Fourier transform of (3.8) from each bound state at  $k = i\kappa_s$  is evaluated with the help of (3.5) and the fact that  $f(i\kappa_s, 0) = 0$ , and the total contribution from all the bound states is given by

$$-\frac{1}{2\pi} \int_{-\infty}^{\infty} dk \frac{2ik}{f(k, 0)} \varphi(k, x) e^{iky} = -i \sum_{s=1}^N \frac{f(-i\kappa_s, 0)}{\dot{f}(i\kappa_s, 0)} f(i\kappa_s, x) e^{-\kappa_s y}, \tag{3.9}$$

where an overdot denotes the derivative with respect to  $k$ . From (2.44), we see that

$$\text{Res}[S(k), i\kappa_s] = \frac{f(-i\kappa_s, 0)}{\dot{f}(i\kappa_s, 0)}, \tag{3.10}$$

and hence  $c_s$  appearing in (3.2) is given by (3.4). We remark that the right-hand side of (3.8) does not have a pole at  $k = 0$ , and hence there is no residue contribution in (3.9) from  $k = 0$ . This is because  $f(k, 0)$  is nonzero for  $k \in \mathbf{R} \setminus \{0\}$ , and either  $f(k, 0)$  does not vanish at  $k = 0$  or it has a simple zero at  $k = 0$ .

The analog of (3.1) for (1.1) is derived in a similar way. We can write (2.42) as

$$f_0 g_n - g_0 f_n = -(z - z^{-1}) \varphi_n, \quad z \in \mathbf{T}, \tag{3.11}$$

where we recall that  $\mathbf{T}$  denotes the unit circle  $|z| = 1$  in the complex  $z$ -plane. Dividing (3.11) by the Jost function  $f_0$  and using (2.45), we get

$$g_n - S f_n = -\frac{z - z^{-1}}{f_0} \varphi_n, \quad z \in \mathbf{T}. \tag{3.12}$$

We can write (3.12) as

$$(g_n - z^{-n}) + (1 - S) z^n + (1 - S)(f_n - z^n) = H_n, \quad z \in \mathbf{T}, \tag{3.13}$$

where we have defined

$$H_n := f_n - z^{-n} - \frac{z - z^{-1}}{f_0} \varphi_n. \tag{3.14}$$

Let us take the Fourier transform of (3.13) with  $(2\pi i)^{-1} \oint dz z^{m-1}$  for  $m \geq n + 1$ , where the integral is along the unit circle  $\mathbf{T}$  in the counterclockwise direction traversed once. This yields the linear system of Marchenko equations

$$K_{nm} + M_{n+m} + \sum_{j=n+1}^{2b-n-1} K_{nj} M_{j+m} = 0, \quad 0 \leq n < m, \tag{3.15}$$



where we have defined

$$K_{nm} := \frac{1}{2\pi i} \oint dz (g_n - z^{-n}) z^{m-1}, \tag{3.16}$$

$$M_n := \frac{1}{2\pi i} \oint dz (1 - S) z^{n-1} + \sum_{s=1}^N c_s^2 z_s^n. \tag{3.17}$$

Because of (2.31), we may consider (3.15) only for  $m \leq 2b - 2n - 1$  for each  $n = 0, 1, \dots, b - 1$ . In (3.17), the values  $z = z_s$  for  $s = 1, \dots, N$  correspond to the zeros of  $f_0$  inside the unit circle  $\mathbf{T}$ , and in fact by Theorem 2.5(a) we know that  $z_s \in (-1, 0) \cup (0, 1)$ . The constants  $c_s$  appearing in (3.17) are the Marchenko bound-state norming constants, and they are defined as

$$c_s := \frac{1}{\sqrt{\sum_{n=1}^{\infty} f_n^2|_{z=z_s}}}, \tag{3.18}$$

where we have used the fact that the value of  $f_n$  at a bound state is real as asserted in Theorem 2.5(b), and hence we have replaced the absolute square in the denominator in (3.18) with the ordinary square. Since the potential  $V$  is compactly supported, it turns out that the Marchenko norming constant  $c_s$  defined in (3.18) can be evaluated from the scattering matrix  $S$  appearing in (2.45) via the residue of  $S/z$  at the bound-state pole  $z = z_s$ . We have

$$c_s = \sqrt{\text{Res} \left[ \frac{S}{z}, z_s \right]}, \quad s = 1, \dots, N. \tag{3.19}$$

We recall that each bound state  $z = z_s$  occurs when either  $z_s \in (-1, 0)$  or  $z_s \in (0, 1)$ . Let us remark that (3.16) can also be written as

$$K_{nm} = \frac{1}{2\pi i} \oint dz (f_n - z^n) z^{-m-1}, \tag{3.20}$$

and that the contribution to the Fourier transform from the right-hand side of (3.13) at the zeros  $z = z_s$  for  $s = 1, \dots, N$  of the Jost function  $f_0$  is evaluated by using

$$\begin{aligned} \frac{1}{2\pi i} \oint dz H_n z^{m-1} &= -\frac{1}{2\pi i} \oint dz \frac{z - z^{-1}}{z f_0} \varphi_n z^m \\ &= -\sum_{s=1}^N \frac{g_0}{z_s \dot{f}_0} z_s^m f_n \Big|_{z=z_s}, \end{aligned} \tag{3.21}$$

where we have used (3.11) and the fact that  $f_0$  vanishes at  $z = z_s$ . The overdot in (3.21) indicates the derivative with respect to  $z$ . The value of  $c_s$  appearing in (3.17) is given by (3.19) by noting that from (2.45) and (3.21) we have

$$\text{Res} \left[ \frac{S}{z}, z_s \right] = \frac{g_0}{z_s \dot{f}_0} \Big|_{z=z_s}, \tag{3.22}$$

which is the analog of (3.10) in the discrete case. We remark that there cannot be any residue contribution from  $z = 0$  to the integral in (3.21) as a result of the fact that  $H_n$  given in (3.14), when expanded in  $z$ , contains only terms of  $z^k$  for  $k \geq -n$  and that we have  $m > n$ . This is seen as follows. From (2.31), we know that  $f_n$  contains terms of  $z^k$  only for  $k \geq n$ ; from (2.46) and (2.47), we know that  $f_0$  contains terms of  $z^k$  only for  $k \geq 0$  and that  $f_0 = 1$  when  $z = 0$ ; furthermore, from (2.8) and (2.70), it follows that  $\varphi_n$  contains terms of  $z^k$  only for  $-n + 1 \leq k \leq n - 1$ . Hence, the integrand  $H_n z^{m-1}$ , with  $m > n$  in (3.21), when expanded in  $z$ , does not contain any negative powers of  $z$ .

Let us note that the quantities  $K_{nm}$  given in (3.16) and (3.20) coincide with those appearing in (2.28)-(2.32) and (2.46)-(2.49). The following theorem states that the ordered set  $\{V_1, \dots, V_b\}$  of potential values is uniquely determined by the scattering matrix  $S$  appearing in (2.45) and it is also uniquely determined by the Jost function  $f_0$  appearing in (2.46).

**Theorem 3.1.** Assume that the potential  $V$  belongs to class  $\mathcal{A}_b$  specified in Definition 1.1 for some  $b \geq 1$ . Then, the Jost function  $f_0$  appearing in (2.46) uniquely determines the potential. Similarly, the scattering matrix  $S$  given in (2.45) uniquely determines the potential. The recovery of the potential can be accomplished by solving Marchenko system (3.15) for  $K_{nm}$  with  $0 \leq n < m \leq 2b - 2n - 1$  for each  $n = 0, 1, \dots, b - 1$  and then obtaining the potential  $V$  via (2.51).

*Proof.* Given the Jost function  $f_0$  as a function of  $z$ , we also have  $g_0$  given in (2.49) because  $g_0$  is obtained by replacing  $z$  with  $z^{-1}$  in  $f_0$ . Thus, as seen from (2.45), we have at hand the scattering matrix  $S$  as a function of the spectral parameter  $z$ . On the other hand, if we are given the scattering matrix  $S$ , using (2.45) and (3.19), we then construct the Marchenko kernel  $M_n$  given in (3.17). Next, we solve Marchenko system (3.15) and uniquely obtain  $K_{nm}$  for  $0 \leq n < m \leq 2b - 2n - 1$  for each  $n = 0, 1, \dots, b - 1$ . Finally, we use (2.51) to obtain the ordered set  $\{V_1, \dots, V_b\}$  of potential values. ■

#### IV. SPECTRAL FUNCTIONS AND THE GEL'FAND-LEVITAN METHOD

Let us use  $\overset{\circ}{\rho}$  to denote the spectral function<sup>9</sup> associated with the unperturbed discrete Schrödinger operator corresponding to (2.5) with Dirichlet boundary condition (2.2). Similarly, let us use  $\rho$  to denote the spectral function associated with perturbed discrete Schrödinger operator (1.1) with Dirichlet boundary condition (2.2). Our goal in this section is to derive the corresponding Gel'fand-Levitan system and establish the recovery of the potential  $V$  in class  $\mathcal{A}_b$  from the spectral functions  $\rho$  and  $\overset{\circ}{\rho}$ . Recall that as the spectral parameter, we can use  $\lambda$  appearing in (1.1), or use  $z$  appearing in (2.6), or even use  $\theta$  appearing in (2.9). Thus,  $\overset{\circ}{\rho}$  and  $\rho$  can be viewed as functions of  $\lambda$  or functions of  $z$  or functions of  $\theta$ . We will refer to  $d\overset{\circ}{\rho}$  and  $d\rho$  as the spectral measures associated with (2.5) and (1.1), respectively, with Dirichlet boundary condition (2.2).

As seen from (1.1) and (1.3), we observe that the spectral parameter used in Ref. 9 is not the same as the spectral parameter  $\lambda$  used in (1.1) but the two are closely related to each other. Furthermore, the discrete Schrödinger equation used in Ref. 9 is not the same as (1.1) we use. Nevertheless, by proceeding as in Ref. 9, we obtain our spectral measure  $d\overset{\circ}{\rho}$  in terms of  $\lambda$  as

$$d\overset{\circ}{\rho} = \begin{cases} 0, & \lambda < 0, \\ \frac{1}{2\pi} \sqrt{\lambda(4-\lambda)} d\lambda, & 0 \leq \lambda \leq 4, \\ 0, & \lambda > 4. \end{cases} \tag{4.1}$$

As a function of  $\theta$ , the nonzero portion of  $d\overset{\circ}{\rho}$  corresponding to  $0 \leq \lambda \leq 4$  can be expressed as

$$d\overset{\circ}{\rho} = \frac{2}{\pi} \sin^2 \theta d\theta, \quad 0 \leq \theta \leq \pi. \tag{4.2}$$

As a function of  $z$ , the nonzero portion of  $d\overset{\circ}{\rho}$  corresponding to  $0 \leq \lambda \leq 4$  is given by

$$d\overset{\circ}{\rho} = -\frac{1}{2\pi i} (z - z^{-1})^2 \frac{dz}{z}, \quad z \in \overline{\mathbf{T}}^+, \tag{4.3}$$

where we recall that  $\overline{\mathbf{T}}^+$  denotes the closure of the upper portion of the unit circle  $\mathbf{T}$ . We already know from (2.20) and Theorem 2.5(c) that (2.5) with Dirichlet boundary condition (2.2) does not have any bound states and hence, the expressions for  $d\overset{\circ}{\rho}$  do not contain any terms related to bound states.

The idea<sup>9,11,16,18</sup> behind the spectral function  $\overset{\circ}{\rho}$  is the following. The regular solution  $\overset{\circ}{\varphi}_n$  appearing in (2.21)-(2.23) as a function of  $n$  forms the sequence  $\{\overset{\circ}{\varphi}_n\}_{n=1}^\infty$ , where each term is a function of the spectral parameter  $\lambda$ . In fact, by Theorem 2.6, we know that, for  $n \geq 1$ , the quantity  $\overset{\circ}{\varphi}_n$  is a polynomial in  $\lambda$  of degree  $n - 1$ . The terms in the sequence  $\{\overset{\circ}{\varphi}_n\}_{n=1}^\infty$  form an orthonormal set with respect to the spectral measure  $d\overset{\circ}{\rho}$ , i.e., we have

$$\int d\overset{\circ}{\rho} (\overset{\circ}{\varphi}_j; \overset{\circ}{\varphi}_l) = \delta_{jl}, \tag{4.4}$$

where  $\delta_{jl}$  is the Kronecker delta and we omit the integration limits for notational simplicity with the understanding that the integral is over  $\lambda \in \mathbf{R}$ .

The orthonormality condition given in (4.4) can be stated in  $\lambda$ , in  $z$ , or in  $\theta$ . For example, in terms of  $\theta$ , as seen from (2.22) and (4.2), we can write the left-hand side of (4.4) as

$$\int d\overset{\circ}{\rho} (\overset{\circ}{\varphi}_j; \overset{\circ}{\varphi}_l) = \frac{2}{\pi} \int_0^\pi d\theta \sin^2\theta \left[ \frac{\sin(j\theta)}{\sin\theta} \right] \left[ \frac{\sin(l\theta)}{\sin\theta} \right].$$

Since we already know that

$$\int_0^\pi d\theta \sin(j\theta) \sin(l\theta) = \frac{\pi}{2} \delta_{jl},$$

we see that (4.4) holds when it is expressed in  $\theta$ .

Let us now consider the main idea behind the spectral function  $\rho$  associated with (1.1) with Dirichlet boundary condition (2.2). The regular solution  $\varphi_n$  to (1.1) with initial values (2.14) as a function of  $n$  forms the sequence  $\{\varphi_n\}_{n=1}^\infty$ , where each term in the sequence is a function of the spectral parameter  $\lambda$ . In fact, by Theorem 2.6, we know that for  $n \geq 1$ , the quantity  $\varphi_n$  is a polynomial in  $\lambda$  of degree  $n - 1$ . The terms in the sequence  $\{\varphi_n\}_{n=1}^\infty$  are orthonormal with respect to the spectral measure  $d\rho$ , i.e., we have

$$\int d\rho (\varphi_j; \varphi_l) = \delta_{jl}, \tag{4.5}$$

where, again for simplicity, we suppress the integration limits in our notation.

The spectral function  $\rho$  contains all the relevant information about the potential  $V$  and boundary condition (2.2). In the presence of the potential  $V$ , assuming that the bound states occur at  $\lambda = \mu_s$  for  $s = 1, \dots, N$ , we have

$$d\rho = \begin{cases} \sum_{s=1}^N C_s^2 \delta(\lambda - \mu_s) d\lambda, & \lambda < 0 \text{ or } \lambda > 4, \\ \frac{1}{2\pi} \sqrt{\lambda(4-\lambda)} \frac{d\lambda}{|f_0|^2}, & 0 \leq \lambda \leq 4, \end{cases} \tag{4.6}$$

where  $f_0$  is the Jost function appearing in (2.46). Notice that the continuous part of  $d\rho$  in (4.6) is obtained from  $d\overset{\circ}{\rho}$  given in (4.1) via a division by  $|f_0|^2$ .

Recall that, as stated in Theorem 2.5(c), the bound states correspond to the zeros of the Jost function  $f_0$ . The constant  $C_s$  appearing in (4.6) is the Gel'fand-Levitan bound-state norming constant at the bound state  $\lambda = \mu_s$ , i.e.,

$$C_s := \frac{1}{\sqrt{\sum_{n=1}^\infty \varphi_n^2|_{\lambda=\mu_s}}}, \tag{4.7}$$

where we have used the fact that the value of  $\varphi_n$  at a bound state is real, as stated by Theorem 2.5(b).

The contribution to the spectral function  $\rho$  from each eigenvalue is given by the square of the norming constant  $C_s$  appearing in (4.7). We can obtain the spectral measure given in (4.6), and in particular its continuous part, as the limiting case  $p \rightarrow +\infty$  from the spectral density  $\rho_p$  corresponding to the discrete Schrödinger equation on the finite lattice  $n \in \{1, 2, \dots, p\}$  with boundary condition (2.2) at  $n = 0$  and an additional boundary condition at  $n = p + 1$ , e.g., the Dirichlet condition  $\psi_{p+1} = 0$ . The discrete Schrödinger operator corresponding to such a problem on the finite lattice has only the discrete spectrum and we can explicitly evaluate the corresponding spectral function  $\rho_p$ . We then get the continuous part of the expression in (4.6) by taking the limit of  $\rho_p$  as  $p \rightarrow +\infty$ .

By comparing the Marchenko norming constant  $c_s$  defined in (3.18) with the Gel'fand-Levitan norming constant  $C_s$  defined in (4.7), we observe that they differ from each other in the sense that

the Marchenko norming constant  $c_s$  is obtained by normalizing the Jost solution  $f_n$  at the bound state  $\lambda = \mu_s$ , whereas the Gel'fand-Levitan norming constant is obtained by normalizing the regular solution  $\varphi_n$  at the same bound state. On the other hand, we already know that the regular solution  $\varphi_n$  and the Jost solution  $f_n$  are linearly dependent at a bound state. Thus, evaluating (2.42) at the bound state  $\lambda = \mu_s$  and using the fact that  $f_0$  vanishes at  $\lambda = \mu_s$ , we obtain

$$\varphi_n|_{\lambda=\mu_s} = \left( \frac{g_0}{z - z^{-1}} \Big|_{\lambda=\mu_s} \right) f_n|_{\lambda=\mu_s}. \tag{4.8}$$

Since the quantity inside the parentheses in (4.8) is independent of  $n$ , by squaring both sides of (4.8) and then using a summation over  $n$ , we obtain

$$\sum_{n=1}^{\infty} \varphi_n^2|_{\lambda=\mu_s} = \left( \frac{g_0}{z - z^{-1}} \Big|_{\lambda=\mu_s} \right)^2 \sum_{n=1}^{\infty} f_n^2|_{\lambda=\mu_s}. \tag{4.9}$$

Using (3.18) and (4.7) in (4.9), we conclude that the Marchenko and Gel'fand-Levitan bound-state norming constants are related to each other as

$$c_s = \left| \frac{g_0}{z - z^{-1}} \Big|_{\lambda=\mu_s} \right| C_s. \tag{4.10}$$

If we know  $|f_0|$ , because  $g_0$  is obtained from  $f_0$  by replacing  $z$  with  $z^{-1}$ , we can evaluate the factor in (4.10) relating the Marchenko norming constant  $c_s$  to the Gel'fand-Levitan norming constant  $C_s$ . We remind the reader that  $z$  and  $z^{-1}$  are related to  $\lambda$  as in (2.6) and (2.7), respectively.

As already mentioned, the continuous spectrum of the discrete Schrödinger operator associated with (1.1) and (2.2) is  $\lambda \in [0, 4]$  and the discrete spectrum consists of at most a finite number points  $\lambda = \mu_s$ , where each  $\mu_s$  is located either in the interval  $\lambda \in (-\infty, 0)$  or  $\lambda \in (4, +\infty)$ . Viewed as a function of the parameter  $\theta$ , the continuous part of the spectral measure  $d\rho$  is given by

$$d\rho = \frac{2}{\pi} \sin^2\theta \frac{d\theta}{|f_0|^2}, \quad 0 \leq \theta \leq \pi, \tag{4.11}$$

which is obtained from (4.6) with the help of the transformation  $\theta \mapsto \lambda$  given in (2.11). Similarly, we obtain the same quantity as a function of the spectral parameter  $z$  by using (2.8) in (4.6) and have

$$d\rho = -\frac{1}{2\pi i} (z - z^{-1})^2 \frac{dz}{z |f_0|^2}, \quad z \in \overline{\mathbf{T}^+}. \tag{4.12}$$

Note that in (4.6), (4.11), and (4.12), instead of  $|f_0|^2$ , we can equally use the product  $f_0 g_0$  because  $|f_0|^2 = f_0 g_0$  when  $z$  is on the upper unit semicircle  $\mathbf{T}^+$  because  $g_0$  is obtained from  $f_0$  by replacing  $z$  by  $z^{-1}$ , which is equivalent to taking the complex conjugate of  $f_0$  when  $z$  is on  $\mathbf{T}^+$ .

We derive the linear system of Gel'fand-Levitan equations as follows. Consider the regular solution  $\varphi_n$  to (1.1) with initial values (2.14). The set  $\{\varphi_j\}_{j=1}^n$  forms an orthonormal basis for polynomials in  $\lambda$  of degree  $n - 1$  with respect to the spectral measure  $d\rho$  given in (4.6). By Theorem 2.6, each element  $\varphi_j$  in the set is a polynomial in  $\lambda$  of degree  $j - 1$ . Note that we can write any polynomial in  $\lambda$  of degree  $n - 2$  or less as a linear combination of the elements in the set  $\{\varphi_j\}_{j=1}^{n-1}$ . Thus, with respect to the spectral measure  $d\rho$ , the quantity  $\varphi_n$  is orthogonal to any polynomial with degree  $n - 2$  or less. Each element  $\mathring{\varphi}_m$  in the set  $\{\mathring{\varphi}_m\}_{m=1}^{n-1}$  is a polynomial in  $\lambda$  of degree  $m - 1$ , and because  $m < n$ , we can conclude that  $\varphi_n$  is orthogonal to  $\mathring{\varphi}_m$  with respect to the measure  $d\rho$ , i.e., we have

$$\int d\rho \left( \varphi_n \mathring{\varphi}_m \right) = 0, \quad 1 \leq m < n, \tag{4.13}$$

where we again suppress the integration limits for notational simplicity. Let us replace  $d\rho$  in (4.13) with  $(d\rho - d\mathring{\rho}) + d\mathring{\rho}$  and let us replace  $\varphi_n$  in (4.13) with the right-hand side of (2.79). For  $1 \leq m < n$ , we get

$$\begin{aligned}
 0 &= \int (d\rho - d\overset{\circ}{\rho}) \overset{\circ}{\varphi}_n \overset{\circ}{\varphi}_m + \int d\overset{\circ}{\rho} \left( \overset{\circ}{\varphi}_n \overset{\circ}{\varphi}_m \right) \\
 &+ \sum_{j=1}^{n-1} \int (d\rho - d\overset{\circ}{\rho}) A_{nj} \overset{\circ}{\varphi}_j \overset{\circ}{\varphi}_m + \sum_{j=1}^{n-1} \int d\overset{\circ}{\rho} \left( A_{nj} \overset{\circ}{\varphi}_j \overset{\circ}{\varphi}_m \right).
 \end{aligned}
 \tag{4.14}$$

Let us define

$$G_{nm} := \int (d\rho - d\overset{\circ}{\rho}) \overset{\circ}{\varphi}_n \overset{\circ}{\varphi}_m.
 \tag{4.15}$$

We apply the orthonormality given in (4.4) to the second and fourth terms on the right-hand side in (4.14) and use (4.15) in the first and third terms on the right-hand side in (4.14). This yields

$$G_{nm} + 0 + \sum_{j=1}^{n-1} A_{nj} G_{jm} + \sum_{j=1}^{n-1} A_{nj} \delta_{jm} = 0, \quad 1 \leq m < n.
 \tag{4.16}$$

Simplifying (4.16), we obtain the linear system of Gel'fand-Levitan equations

$$A_{nm} + G_{nm} + \sum_{j=1}^{n-1} A_{nj} G_{jm} = 0, \quad 1 \leq m < n.
 \tag{4.17}$$

Let us remark that it is sufficient to consider (4.17) for only  $n = 2, 3, \dots, b$  as a result of the compact support property of the potential, i.e.,  $V_n = 0$  for  $n \geq b + 1$ . The solution to the inverse problem of recovery of the potential  $V$  via the Gel'fand-Levitan method is achieved as follows. Given the absolute value of the Jost function  $f_0$  and the bound-state data, i.e., the zeros of the Jost function  $f_0$  and the corresponding Gel'fand-Levitan bound-state norming constants  $C_s$  appearing in (4.7), with the help of (4.1) and (4.6) we form the spectral measures  $d\overset{\circ}{\rho}$  and  $d\rho$ . We then evaluate  $G_{nm}$  given in (4.15). Next, we solve Gel'fand-Levitan system (4.17) and recover  $A_{nm}$ . Then, we use the values of  $A_{n(n-1)}$  for  $n = 2, 3, \dots, b$  and recover the ordered set  $\{V_1, V_2, \dots, V_b\}$  of potential values via (2.84).

Note that the derivation of discrete Gel'fand-Levitan system (4.17) is similar to the following derivation of the continuous version of the Gel'fand-Levitan equation for (1.2) with Dirichlet boundary condition (2.3). Let  $\varphi(k, x)$  be the regular solution to (1.2) satisfying (2.15). Let us use  $\overset{\circ}{\varphi}(k, x)$  to denote the regular solution to (1.2) with boundary condition (2.15) when  $V(x) \equiv 0$ . In fact, we have

$$\overset{\circ}{\varphi}(k, x) = \frac{\sin(kx)}{k}, \quad x \geq 0,$$

where we recall that  $\lambda$  and  $k$  are related to each other as  $k = \sqrt{\lambda}$ . In the continuous case, the spectral measures  $d\overset{\circ}{\rho}$  and  $d\rho$  are related to the orthonormality relations given by

$$\int d\overset{\circ}{\rho} \left[ \overset{\circ}{\varphi}(k, x) \overset{\circ}{\varphi}(k, y) \right] = \delta(x - y), \quad x, y \in [0, +\infty),
 \tag{4.18}$$

$$\int d\rho \left[ \varphi(k, x) \varphi(k, y) \right] = \delta(x - y), \quad x, y \in [0, +\infty),
 \tag{4.19}$$

which are the analogs of (4.4) and (4.5), respectively. Note that we use  $\delta(x)$  to denote the Dirac delta distribution. As an analog of (4.15), let us define

$$G(x, y) := \int (d\rho - d\overset{\circ}{\rho}) \left[ \overset{\circ}{\varphi}(k, x) \overset{\circ}{\varphi}(k, y) \right].
 \tag{4.20}$$

We have the analog of the expansion in (2.79), namely,

$$\varphi(k, x) = \overset{\circ}{\varphi}(k, x) + \int_0^x dz A(x, z) \overset{\circ}{\varphi}(k, z).
 \tag{4.21}$$

From (4.19) and (4.21), we conclude that, for  $y < x$ , the regular solutions  $\varphi(k, x)$  and  $\overset{\circ}{\varphi}(k, y)$  are orthogonal with respect to the spectral measure  $d\rho$ , i.e., we have the analog of (4.13) given by

$$\int d\rho \left[ \varphi(k, x) \mathring{\varphi}(k, y) \right] = 0, \quad y < x. \tag{4.22}$$

Using (4.21) in (4.22), we obtain the analog of (4.14), namely, for  $y < x$ , we get

$$\begin{aligned} 0 = & \int (d\rho - d\mathring{\rho}) \mathring{\varphi}(k, x) \mathring{\varphi}(k, y) + \int d\mathring{\rho} \left[ \mathring{\varphi}(k, x) \mathring{\varphi}(k, y) \right] \\ & + \int_0^x dz \int (d\rho - d\mathring{\rho}) A(x, z) \mathring{\varphi}(k, z) \mathring{\varphi}(k, y) + \int_0^x dz \int d\mathring{\rho} A(x, z) \mathring{\varphi}(k, z) \mathring{\varphi}(k, y). \end{aligned} \tag{4.23}$$

Since we use  $x > y$ , the second of the four terms on the right-hand side in (4.23) vanishes as a result of (4.18). The fourth term there, with the help of (4.18), is seen to be equal to  $A(x, y)$ . Furthermore, using (4.20) in the first and third terms, from (4.23), we get the Gel'fand-Levitan equation<sup>5,10,17,18</sup>

$$A(x, y) + G(x, y) + \int_0^x dz A(x, z) G(z, y) = 0, \quad x > y, \tag{4.24}$$

which is the analog of (4.17). The potential  $V$  is recovered from the solution to (4.24) as

$$V(x) = 2 \frac{dA(x, x)}{dx},$$

which is the analog of (2.84). Let us finally remark that the analog of (4.6) in the continuous case is<sup>5,10,17,18</sup>

$$d\rho = \begin{cases} \sum_{s=1}^N C_s^2 \delta(\lambda - \mu_s), & \lambda < 0, \\ \frac{1}{\pi} \frac{\sqrt{\lambda}}{|f(\sqrt{\lambda}, 0)|^2} d\lambda, & \lambda \geq 0. \end{cases} \tag{4.25}$$

where  $f(\sqrt{\lambda}, 0)$  with  $k := \sqrt{\lambda}$  is the Jost function  $f(k, 0)$  and is obtained from the Jost solution  $f(k, x)$  appearing in (2.17) by putting  $x = 0$ . In (4.25), we assume that there are  $N$  bound states occurring at  $\lambda = \mu_s$  for  $s = 1, \dots, N$ . The Gel'fand-Levitan norming constant  $C_s$  appearing in (4.25) is defined as

$$C_s := \frac{1}{\sqrt{\int_0^\infty dx \varphi(\sqrt{\mu_s}, x)^2}}. \tag{4.26}$$

The analog of (4.1) in the continuous case with the zero potential is then given by<sup>5,10,17,18</sup>

$$d\mathring{\rho} = \begin{cases} 0, & \lambda < 0, \\ \frac{\sqrt{\lambda}}{\sqrt{\pi}} d\lambda, & \lambda \geq 0. \end{cases}$$

When the potential in (1.2) is compactly supported, the analog of (4.10) in the continuous case can be obtained from (2.43) with the help of (3.3) and (4.26) as

$$c_s = \frac{|f(-i\sqrt{|\mu_s|}, 0)|}{2\sqrt{|\mu_s|}} C_s.$$

### V. TRANSMISSION EIGENVALUES

In this section, we introduce the transmission eigenvalues for discrete Schrödinger equation (1.1) with Dirichlet boundary condition (2.2). We relate the transmission eigenvalues to the zeros of the key quantity  $D$  defined in (5.2) and also equivalently to the zeros of the quantity  $E$  given in

(5.7). We express  $D$  in terms of the spectral parameter  $z$  and relate it as in (5.8) to the Jost function  $f_0$  and its relative  $g_0$ , which allows us to obtain relationship (5.10) relating  $E$  to  $f_0$  and  $g_0$ . We introduce the unusual case and characterize it in several equivalent ways. We recognize that (5.10) is a discrete analog of a Riemann-Hilbert problem, which enables us to solve it uniquely in the usual case and recover  $f_0$  from  $E$ . Thus, we uniquely recover the Jost function  $f_0$  from the set of transmission eigenvalues in the usual case. We are then able to use either of the Marchenko method and the Gel'fand-Levitan method to uniquely recover the potential in class  $\mathcal{A}_b$  when the transmission eigenvalues are given as input in the usual case. In the unusual case, the unique recovery may or may not be possible. This is illustrated with some examples in Section VI.

The transmission eigenvalues<sup>22</sup> for (1.1) with the potential  $V$  in class  $\mathcal{A}_b$  and with Dirichlet boundary condition (2.2) are the  $\lambda$ -values for which we have nontrivial solutions  $\psi_n$  and  $\overset{\circ}{\psi}_n$  to the system

$$\begin{cases} -\psi_{n+1} + 2\psi_n - \psi_{n-1} + V_n\psi_n = \lambda\psi_n, & n \geq 1, \\ -\overset{\circ}{\psi}_{n+1} + 2\overset{\circ}{\psi}_n - \overset{\circ}{\psi}_{n-1} = \lambda\overset{\circ}{\psi}_n, & n \geq 1, \\ \psi_0 = 0, \quad \overset{\circ}{\psi}_0 = 0, \\ \psi_b = \overset{\circ}{\psi}_b, \quad \psi_{b+1} = \overset{\circ}{\psi}_{b+1}. \end{cases} \tag{5.1}$$

In other words, at a transmission eigenvalue  $\lambda$ , we must have (1.1) satisfied for a nontrivial wavefunction  $\psi_n$ , the unperturbed problem corresponding to (1.1) with  $V_n \equiv 0$  must have a nontrivial solution  $\overset{\circ}{\psi}_n$ , the solutions  $\psi_n$  and  $\overset{\circ}{\psi}_n$  both must vanish at  $n = 0$ , and the solutions  $\psi_n$  and  $\overset{\circ}{\psi}_n$  must agree at  $n = b$  and also at  $n = b + 1$ , where  $b$  is the positive integer appearing in Definition 1.1 and related to the support of the potential  $V$ . We note that the discrete transmission-eigenvalue problem described in (5.1) is the analog<sup>3</sup> of the transmission-eigenvalue problem for the Schrödinger equation with Dirichlet boundary condition (2.3).

From the first line of (5.1) and the first equation in the third line, it follows that  $\psi_n$  must be linearly dependent on the regular solution  $\varphi_n$  appearing in (2.42). Similarly, from the second line of (5.1) and the second equation in the third line, it follows that  $\overset{\circ}{\psi}_n$  must be linearly dependent on the regular solution  $\overset{\circ}{\varphi}_n$  appearing in (2.23). The fourth line of (5.1) is equivalent to the statement that the column vectors  $\begin{bmatrix} \overset{\circ}{\psi}_b \\ \overset{\circ}{\psi}_{b+1} \end{bmatrix}$  and  $\begin{bmatrix} \psi_b \\ \psi_{b+1} \end{bmatrix}$  are linearly dependent and hence, the matrix formed by using those two vectors as columns must have zero determinant. Consequently, the transmission eigenvalues are exactly the  $\lambda$ -values for which the two column vectors  $\begin{bmatrix} \overset{\circ}{\varphi}_b \\ \overset{\circ}{\varphi}_{b+1} \end{bmatrix}$  and  $\begin{bmatrix} \varphi_b \\ \varphi_{b+1} \end{bmatrix}$  are linearly dependent. Equivalently, the transmission eigenvalues correspond to the zeros of the quantity  $D$  defined as a matrix determinant via

$$D := \begin{vmatrix} \overset{\circ}{\varphi}_b & \varphi_b \\ \overset{\circ}{\varphi}_{b+1} & \varphi_{b+1} \end{vmatrix}. \tag{5.2}$$

When  $V_n \equiv 0$ , from (5.1), it follows that any  $\lambda$  is a transmission eigenvalue, and in this case from (5.2) we get  $D \equiv 0$ . When  $V_n = 0$  for  $n \geq 2$  and  $V_1 \neq 0$ , from (5.1) it follows that no  $\lambda$ -value can be a transmission eigenvalue, and in this case from (5.2) we get  $D = V_1$ . Thus, in the analysis of the inverse problem with transmission eigenvalues, we can assume that  $b \geq 2$ , where  $b$  is the positive integer related to the support of  $V$  given in Definition 1.1.

**Theorem 5.1.** *Assume that the potential  $V$  belongs to class  $\mathcal{A}_b$  with  $V_b \neq 0$  and further suppose that  $b \geq 2$ , where  $b$  denotes the positive integer related to the support of the potential. Then, the transmission eigenvalues for (5.1) correspond to the zeros of the quantity  $D$  defined in (5.2). The quantity  $D$  is a polynomial in  $\lambda$  of degree  $2b - 2$ , and we have*

$$D = \sum_{s=0}^{2b-2} D_s \lambda^s, \tag{5.3}$$

where the coefficients  $D_s$  for  $s = 0, 1, \dots, 2b - 2$  are uniquely determined by the potential  $V$ . In particular, we have

$$D_{2b-2} = V_b, \quad D_{2b-3} = -V_b \left( 4(b-1) + \sum_{j=1}^{b-1} V_j \right). \quad (5.4)$$

*Proof.* The fact that the transmission eigenvalues correspond to the zeros of  $D$  has already been established in the paragraph containing (5.2). Using (2.70)-(2.72) and (2.76)-(2.78) in (5.2), we establish the remaining results stated. ■

With the help of Theorem 5.1, we obtain the following result.

**Theorem 5.2.** *Assume that the potential  $V$  belongs to class  $\mathcal{A}_b$  with  $V_b \neq 0$  and further suppose that  $b \geq 2$ , where  $b$  denotes the positive integer related to the support of the potential. Then, the quantity  $D$  defined in (5.2), as a function of  $\lambda$ , has exactly  $2b - 2$  zeros, and hence the number of transmission eigenvalues for (5.1) is  $2b - 2$ . In terms of the transmission eigenvalues  $\lambda_1, \dots, \lambda_{2b-2}$ , we have*

$$D = V_b \prod_{j=1}^{2b-2} (\lambda - \lambda_j), \quad (5.5)$$

$$\sum_{j=1}^{2b-2} \lambda_j = 4(b-1) + \sum_{j=1}^{b-1} V_j. \quad (5.6)$$

Hence, the knowledge of the transmission eigenvalues is equivalent to the knowledge of the quantity  $E$  defined as

$$E := \frac{D}{V_b}, \quad (5.7)$$

and the transmission eigenvalues correspond to the zeros of  $E$ .

*Proof.* By Theorem 5.1, we know that the zeros of  $D$  correspond to the transmission eigenvalues for (5.1) and that  $D$  is a polynomial in  $\lambda$  of degree  $2(b-1)$  with the leading term  $V_b \lambda^{2b-2}$ . Thus, we have established (5.5). Comparing the coefficients of  $\lambda^{2b-3}$  in (5.3) and (5.5), with the help of (5.4) we get (5.6). From (5.5), it directly follows that knowing  $E$  is equivalent to knowing the transmission eigenvalues. ■

Recall that we can also use  $z$  as the spectral parameter instead of  $\lambda$ , where  $z$  is related to  $\lambda$  as in (2.6). Next, we express the key quantity  $D$  given in (5.2) in terms of the spectral parameter  $z$ . This will help us to solve the relevant inverse problem to recover the potential with the help of the Marchenko method or the Gel'fand-Levitan method.

**Theorem 5.3.** *Assume that the potential  $V$  belongs to class  $\mathcal{A}_b$  with  $V_b \neq 0$  and further suppose that  $b \geq 2$ , where  $b$  denotes the positive integer related to the support of the potential. Then:*

- (a) *The key quantity  $D$  given in (5.2) can be expressed in terms of the spectral parameter  $z$  via*

$$D = \frac{1}{z - z^{-1}} (f_0 - g_0), \quad (5.8)$$

where  $f_0$  is the Jost function appearing in (2.46) and  $g_0$  is the quantity in (2.49). Expression (5.8) holds for  $b = 1$  as well.

- (b) *The quantity  $(f_0 - 1)/V_b$  is uniquely determined by the transmission eigenvalues for (5.1). The quantity  $K_{01}/V_b$  is also uniquely determined by the transmission eigenvalues, where  $K_{01}$  is the coefficient of  $z$  in polynomial expansion (2.46) in terms of  $z$  for the Jost function  $f_0$ . In fact, the value of  $K_{01}/V_b$  is equal to the coefficient of  $z$  when  $(f_0 - 1)/V_b$  is expressed in  $z$  as a polynomial.*



(c) Identity (5.6) is equivalent to

$$\sum_{j=1}^{2b-2} \lambda_j = 4(b-1) + \left( \frac{K_{01}}{V_b} - 1 \right) V_b. \tag{5.9}$$

(d) Unless the quantity  $K_{01}/V_b$  is equal to 1, where the case  $K_{01}/V_b = 1$  corresponds to the unusual case,  $V_b$  is uniquely determined by the transmission eigenvalues by solving (5.9) for  $V_b$ .

(e) Unless we are in the unusual case, the transmission eigenvalues uniquely determine the Jost function  $f_0$ . Thus, unless we are in the unusual case, the transmission eigenvalues uniquely determine the ordered set  $\{V_1, \dots, V_b\}$  of potential values.

*Proof.* Using (2.21) and (2.42) in the first and second columns, respectively, of (5.2), we establish (a). Using (5.7) in (5.8), we get

$$(z - z^{-1})E = \frac{f_0 - 1}{V_b} - \frac{g_0 - 1}{V_b}. \tag{5.10}$$

From Theorem 5.2, we know that the left-hand side of (5.10) is uniquely determined by the transmission eigenvalues of (5.1). From Theorem 2.2, we know that  $f_0 - 1$  is a polynomial in  $z$  of degree  $2b - 1$  and the quantity  $g_0 - 1$  is a polynomial in  $z^{-1}$  of degree  $2b - 1$ . Thus, given the left-hand side of (5.10), we can uniquely determine  $(f_0 - 1)/V_b$  as follows. From Theorem 2.2, it follows that  $(f_0 - 1)/V_b$  is a “plus” function and  $(g_0 - 1)/V_b$  is a “minus” function, and hence, writing the left-hand side of (5.10) in the spectral parameter  $z$ , we see that the terms containing the positive powers of  $z$  make up  $(f_0 - 1)/V_b$  and the terms containing the negative powers of  $z$  make up  $(g_0 - 1)/V_b$ . Having  $(f_0 - 1)/V_b$  at hand, we can use (3.20) with  $n = 0$  and  $m = 1$  to obtain the value of  $K_{01}/V_b$ . We remark that  $K_{01}/V_b$  is the coefficient of  $z$  when  $(f_0 - 1)/V_b$  is written in terms of  $z$  as a polynomial in  $z$ . Thus, we have proved (b). From the second equality in (2.47), we get

$$\frac{K_{01}}{V_b} = 1 + \frac{1}{V_b} \sum_{j=1}^{b-1} V_j, \tag{5.11}$$

and hence, using (5.11) in (5.6), we obtain (5.9), which establishes (c). By (b), we know that  $K_{01}/V_b$  is uniquely determined by the transmission eigenvalues, and hence we can solve (5.9) for  $V_b$  and determine  $V_b$  uniquely in terms of the transmission eigenvalues, provided  $K_{01}/V_b \neq 1$ . Thus, we have also established (d). From (b), we already know that the transmission eigenvalues uniquely determine  $(f_0 - 1)/V_b$  and from (d), we know that in the usual case, the transmission eigenvalues uniquely determine  $V_b$ . Thus, in the usual case, the transmission eigenvalues uniquely determine  $f_0$ . By Theorem 3.1, we know that  $f_0$  uniquely determines the potential  $V$ . Thus, the proof of (e) is complete. ■

The following result is needed to prove Theorem 5.5. It shows that a transmission eigenvalue for (5.1) cannot occur at a zero of the Jost function  $f_0$ .

*Proposition 5.4.* Assume that the potential  $V$  belongs to class  $\mathcal{A}_b$  specified in Definition 1.1 for some  $b \geq 2$ . Let  $f_0$  be the Jost function appearing in (2.46),  $g_0$  the quantity in (2.49), and  $D$  the quantity defined in (5.2). Let  $z$  be the spectral parameter appearing in (2.12). Then,  $D$  and  $f_0$  cannot vanish simultaneously at any  $z$ -value.

*Proof.* From (5.8), we see that if  $D$  and  $f_0$  vanished at the same  $z$ -value with  $z \neq \pm 1$ , then we would also have  $g_0$  vanishing at that  $z$ -value, which would contradict Theorem 2.4(e). Thus,  $D$  and  $f_0$  cannot simultaneously vanish at any  $z$ -value with  $z \neq \pm 1$ . On the other hand, using (2.59), (2.61), and (2.63) in (5.8), we get the expansion for  $D$  around  $z = 1$  as

$$D(z) = \dot{f}_0(1) + O(z - 1), \quad z \rightarrow 1 \text{ in } \mathbb{C}, \tag{5.12}$$

where we recall that  $\dot{f}_0(1)$  denotes the derivative of  $f_0$  with respect to  $z$  evaluated at  $z = 1$ . Because of (2.65), we know that  $\dot{f}_0(1) \neq 0$  if  $f_0(1) = 0$ . Thus,  $D$  and  $f_0$  cannot simultaneously vanish at  $z = 1$ . A similar argument shows that  $D$  and  $f_0$  cannot simultaneously vanish at  $z = -1$  either. ■

From (2.20) and (2.45), it follows that the scattering matrix for the unperturbed system is given by

$$\overset{\circ}{S} \equiv 1.$$

The next theorem explores the connection between the transmission eigenvalues and the energies at which the scattering from perturbed system (1.1) and from unperturbed system (2.5) coincides. It shows that at a transmission eigenvalue the scattering matrix  $S$  of the perturbed system takes the value of 1. With the possible exception of  $\lambda = 0$  and  $\lambda = 4$ , it shows that any  $\lambda$ -value at which  $S$  becomes equal to 1 is a transmission eigenvalue. It also gives some necessary and sufficient conditions for each of  $\lambda = 0$  and  $\lambda = 4$  to be a transmission eigenvalue for system (5.1).

**Theorem 5.5.** *Assume that the potential  $V$  belongs to class  $\mathcal{A}_b$  with  $V_b \neq 0$  and further suppose that  $b \geq 2$ , where  $b$  denotes the positive integer related to the support of the potential. Then:*

- (a) *At a transmission eigenvalue  $\lambda$  for (5.1), the scattering matrix  $S$  given in (2.45) takes the value 1. Thus, at a transmission eigenvalue  $\lambda$ , the scattering from perturbed system (1.1) with Dirichlet boundary condition (2.2) agrees with the scattering from unperturbed system (2.5) with Dirichlet boundary condition (2.2).*
- (b) *With the possible exception of  $z = 1$  and  $z = -1$ , corresponding to  $\lambda = 0$  and  $\lambda = 4$ , respectively, each  $\lambda$ -value at which the scattering matrix  $S$  takes the value of 1 is a transmission eigenvalue for (5.1).*
- (c) *If the scattering matrix  $S$  takes the value 1 at  $\lambda = 0$ , then  $\lambda = 0$  is a transmission eigenvalue if and only if  $f_0$  is nonzero and  $\dot{f}_0$  is zero at  $\lambda = 0$ .*
- (d) *If the scattering matrix  $S$  takes the value 1 at  $\lambda = 4$ , then  $\lambda = 4$  is a transmission eigenvalue if and only if  $f_0$  is nonzero and  $\dot{f}_0$  is zero at  $\lambda = 4$ .*
- (e) *If the scattering matrix  $S$  takes the value 1 at  $\lambda = 0$ , then  $\lambda = 0$  is a transmission eigenvalue if and only if  $S - 1$  has a double zero at  $\lambda = 0$ .*
- (f) *If the scattering matrix  $S$  takes the value 1 at  $\lambda = 4$ , then  $\lambda = 4$  is a transmission eigenvalue if and only if  $S - 1$  has a double zero at  $\lambda = 4$ .*

*Proof.* Recall that  $\lambda = 0$ , as a result of (2.6), corresponds to  $z = 1$  in a one-to-one manner and that  $\lambda = 4$  corresponds to  $z = -1$  in a one-to-one manner. With the help of (2.45), let us write (5.8) as

$$(z - z^{-1}) \frac{D}{f_0} = 1 - S. \tag{5.13}$$

By Theorem 5.1, we know that the transmission eigenvalues correspond to the zeros of  $D$ . First, consider a zero of  $D$  occurring at a  $z$ -value with  $z \neq \pm 1$ . From Proposition 5.4, we know that  $f_0$  must be nonzero at such a  $z$ -value. Hence, the left-hand side of (5.13) is zero at that  $z$ -value, causing  $S$  to take the value of 1 at that  $z$ -value. If the zero of  $D$  occurs at  $z = 1$ , then from Proposition 5.4 we know that  $f_0|_{z=1} \neq 0$  and hence the left-hand side of (5.13) is zero, which results in  $S$  taking the value of 1 at  $z = 1$ . A similar argument shows that if  $z = -1$  corresponds to a zero of  $D$ , then we have  $S$  taking the value of 1 at  $z = -1$ . Thus, the proof of (a) is complete. The proof of (b) is given as follows. If  $S = 1$  at some  $z$ -value other than  $z = \pm 1$ , then the left-hand side of (5.13) must be equal to zero at that  $z$ -value, and hence (5.13) implies that  $D$  must vanish at that same  $z$ -value. Thus, that  $z$ -value being a zero of  $D$  must be a transmission eigenvalue, establishing (b). To prove (c), we proceed as follows. Using (2.59) and (2.61) in (2.45), we can expand the scattering matrix  $S$  around  $z = 1$  as

$$S(z) = \frac{f_0(1) - (z - 1)\dot{f}_0(1) + O((z - 1)^2)}{f_0(1) + (z - 1)\dot{f}_0(1) + O((z - 1)^2)}, \quad z \rightarrow 1 \text{ in } \mathbf{C},$$

which yields, for  $z \rightarrow 1$  in  $\mathbf{C}$ ,

$$S(z) = \begin{cases} 1 - 2(z - 1)\frac{\dot{f}_0(1)}{f_0(1)} + O((z - 1)^2), & f_0(1) \neq 0, \\ -1 + O(z - 1), & f_0(1) = 0. \end{cases} \tag{5.14}$$

Thus, we have  $S(1) = 1$  if  $f_0(1) \neq 0$  and we get  $S(1) = -1$  if  $f_0(1) = 0$ . On the other hand, from (5.12) we know that  $D$  vanishes at  $z = 1$  if and only if  $\dot{f}_0(1) = 0$ . Therefore, at  $z = 1$  we have  $S = 1$  and  $D = 0$  if and only if  $f_0(1) \neq 0$  and  $\dot{f}_0(1) = 0$ . Hence, the proof of (c) is complete. The proof of (d) is obtained in a similar way as in the proof of (c). From (5.14), we see that  $S - 1$  has a double zero at  $z = 1$  if and only if we have  $f_0 \neq 0$  and  $\dot{f}_0 = 0$  at  $z = 1$ . Thus, (e) is equivalent to (c). Similarly, one can show that (f) is equivalent to (d), completing the proof of our theorem. ■

Let us now investigate the unusual case in the transmission-eigenvalue problem for (5.1). Recall from Theorem 5.3(d) that the unusual case occurs for (5.1) when

$$\sum_{j=1}^{2b-2} \lambda_j = 4(b-1), \quad (5.15)$$

where the  $\lambda_j$ -values denote the transmission eigenvalues for  $j = 1, 2, \dots, 2b-2$  and  $b$  is the integer related to the support of the potential. From (5.9) and (5.11), or equivalently from (5.6) and (5.15), we see that the unusual case can also be identified as the case with

$$\sum_{j=1}^{b-1} V_j = 0. \quad (5.16)$$

In the following theorem, we characterize the unusual case in several different ways.

**Theorem 5.6.** *Assume that the potential  $V$  belongs to class  $\mathcal{A}_b$  with  $V_b \neq 0$  and further suppose that  $b \geq 2$ , where  $b$  denotes the positive integer related to the support of the potential. Then, the unusual case for the transmission eigenvalue problem for (5.1) occurs if and only if any one of the following equivalent conditions is satisfied:*

- (a) *The transmission eigenvalues  $\lambda_j$  satisfy (5.15).*
- (b) *The potential  $V$  satisfies (5.16).*
- (c) *The coefficient  $K_{01}$  of  $z$  in the polynomial expression in  $z$  given in (2.46) for the Jost function  $f_0$  is equal to  $V_b$ .*
- (d) *The coefficient  $K_{0(2b-2)}$  of  $z^{2b-2}$  in the polynomial expression in  $z$  given in (2.46) for the Jost function  $f_0$  is zero.*
- (e) *The coefficients of  $z$  and  $z^{2b-1}$  in the polynomial expansion in  $z$  for  $(f_0 - 1)/V_b$  are both equal to 1.*
- (f) *The coefficient of  $z^{2b-2}$  in the polynomial expansion in  $z$  for  $(f_0 - 1)/V_b$  is zero.*
- (g) *The coefficient  $D_{2b-3}$  of  $\lambda^{2b-3}$  in the polynomial expression in  $\lambda$  given in (5.3) for the key quantity  $D$  is  $-4(b-1)V_b$ .*
- (h) *The coefficient of  $\lambda^{2b-3}$  in the polynomial expression in  $\lambda$  for the quantity  $E$  defined in (5.7) is  $-4(b-1)$ .*

*Proof.* Note that (a) can be used as the definition of the unusual case. From (5.9), (5.15), and (5.16), we get the equivalence of (a) and (b). Using (5.16) in the second equality in (2.47), we see that the unusual case occurs if and only if  $K_{01} = V_b$ , establishing (c). Using (5.16) in the second equation in (2.48), we observe that the unusual case is equivalent to having  $K_{0(2b-2)} = 0$ , which establishes (d). From (2.46)-(2.48), it follows that the coefficients of  $z$  and  $z^{2b-1}$  in the polynomial expansion in  $z$  for  $(f_0 - 1)/V_b$  are given by  $K_{01}/V_b$  and  $K_{0(2b-1)}/V_b$ , respectively, and hence, with the help of (5.11) and (5.16), we see that  $K_{01}/V_b = 1$  in the unusual case and  $K_{0(2b-1)}/V_b = 1$  always, which establishes (e). From (2.46) and the first equality in (2.48), or equivalently from (2.50), we see that the coefficient of  $z^{2b-2}$  in the polynomial expansion in  $z$  for  $(f_0 - 1)/V_b$  is zero if and only if (5.16) holds, establishing (f). We get (g) directly from (5.3)-(5.6) and (5.15). Finally, (h) follows from (g) and (5.7). ■

In the usual case, we already know that the transmission eigenvalues uniquely determine the potential. By presenting some explicit examples, we will show in Section VI that the transmission eigenvalues for (5.1) may or may not uniquely determine the potential  $V$  in the unusual case. We

will show that in the unusual case, we may have a one-parameter family of real-valued potentials corresponding to a given set of transmission eigenvalues, we may have a finite number of distinct real-valued potentials corresponding to a given set of transmission eigenvalues, or we may have a unique real-valued potential corresponding to a given set of transmission eigenvalues.

## VI. EXAMPLES

In this section, we present various explicit examples to illustrate the results of Secs. I–V. In particular, we illustrate the recovery of the potential from the transmission eigenvalues, the Marchenko method, the Gel'fand-Levitan method, the unusual case, and the bound states.

In the first example below, we illustrate the fact that the bound-state information is contained in the scattering matrix  $S$  when the corresponding potential  $V$  belongs to class  $\mathcal{A}_b$  described in Definition 1.1. As stated in Theorem 2.5(e), the bound-state energies correspond to the poles of the scattering matrix, and as indicated in (3.22), the corresponding Marchenko bound-state norming constants are obtained from  $S$  via a residue evaluation.

*Example 6.1.* Assume that the potential is supported at  $n = 1$  with  $V_1 \neq 0$  and hence  $V_n = 0$  for  $n \geq 2$ . From (2.24), we have  $f_n = z^n$  for  $n \geq 1$ . Using (2.29) and (2.46), we obtain the Jost function as

$$f_0 = V_1 z + 1. \quad (6.1)$$

Thus, the only zero of  $f_0$  occurs at  $z = -1/V_1$ . From Theorem 2.5(a), we know that a bound state exists if the zero of  $f_0$  is in one of the real line segments  $z \in (0, 1)$  and  $z \in (-1, 0)$ . Thus, there is exactly one bound state if  $V_1 < -1$  or  $V_1 > 1$ . Assume that one of these conditions is satisfied so that we have a bound state at  $z = -1/V_1$ . From (6.1), we then get

$$f_0|_{z=-1/V_1} = \left(-\frac{1}{V_1}\right)^n. \quad (6.2)$$

With the help of (6.2), using in (3.18) the fact that  $f_n = z^n$  for  $n \geq 1$ , we evaluate the Marchenko bound-state norming constant as

$$c_1 = \frac{1}{\sqrt{\sum_{n=1}^{\infty} \frac{1}{V_1^{2n}}}},$$

which simplifies to

$$c_1 = \sqrt{V_1^2 - 1}. \quad (6.3)$$

From (6.1), replacing  $z$  by  $z^{-1}$ , we get the quantity  $g_0$  appearing in (2.49) as

$$g_0 = V_1 z^{-1} + 1. \quad (6.4)$$

Using (6.1) and (6.4) in (2.45), we obtain the scattering matrix  $S$  as

$$S = \frac{V_1 z^{-1} + 1}{V_1 z + 1}. \quad (6.5)$$

Let us verify that (3.19) yields the Marchenko bound-state norming constant given in (6.3). From (6.5), we evaluate the residue of  $S/z$  at  $z = -1/V_1$ , and we have

$$\text{Res} \left[ \frac{S}{z}, -\frac{1}{V_1} \right] = V_1^2 - 1,$$

and a comparison with (6.3) shows that (3.19) is verified. Regarding the transmission eigenvalues, from (5.3) we get  $D \equiv V_1$ , and hence there are no transmission eigenvalues.

In the second example given below, we demonstrate that the transmission eigenvalues may have multiplicities greater than one. We also illustrate that in the unusual case, there may be infinitely many distinct potentials corresponding to a given set of transmission eigenvalues.

*Example 6.2.* Let us assume that  $V_n = 0$  for  $n \geq 1$  except when  $n = b$  for some fixed positive integer  $b$ , and suppose that  $V_b$  is a nonzero real parameter. By Theorem 5.6(b), we see that this corresponds to a special case of the unusual case. Proceeding as in the proof of Theorem 2.6, we can determine all the values of the regular solution  $\varphi_n$ . We obtain

$$\varphi_b = \overset{\circ}{\varphi}_b, \quad \varphi_{b+1} = \overset{\circ}{\varphi}_{b+1} + V_b \overset{\circ}{\varphi}_b, \tag{6.6}$$

where we recall that  $\overset{\circ}{\varphi}_n$  is the regular solution when  $V_n \equiv 0$  and it appears in (2.21) and (2.22). Using (6.6) in (5.2), we simplify the determinant defining  $D$  and get

$$D = \begin{vmatrix} \overset{\circ}{\varphi}_b & 0 \\ \overset{\circ}{\varphi}_{b+1} & V_b \overset{\circ}{\varphi}_b \end{vmatrix}. \tag{6.7}$$

From (5.7) and (6.7), we observe that the quantity  $E$  is given by

$$E = \left( \overset{\circ}{\varphi}_b \right)^2. \tag{6.8}$$

Using (2.22) in (6.8), we have

$$E = \frac{\sin^2(b\theta)}{\sin^2\theta}. \tag{6.9}$$

Comparing (5.5) and (6.9), we see that there are  $2(b - 1)$  transmission eigenvalues  $\lambda_j$  including multiplicities, and in fact, the transmission eigenvalues occur when  $\theta = j\pi/b$  for  $j = 1, 2, \dots, b - 1$  and each has multiplicity two. Using (2.11), we obtain the  $\lambda$ -values for the transmission eigenvalues as

$$\lambda_j = 2 - 2 \cos\left(\frac{j\pi}{b}\right), \quad j = 1, 2, \dots, b - 1. \tag{6.10}$$

Thus, with the help of (5.5), (5.7), and (6.10), we can write (6.9) as

$$E = \prod_{j=1}^{b-1} \left[ \lambda - 2 + 2 \cos\left(\frac{j\pi}{b}\right) \right]^2. \tag{6.11}$$

From (6.11), we observe that there are no transmission eigenvalues if  $b = 1$ , there is a transmission eigenvalue at  $\lambda = 2$  with multiplicity two if  $b = 2$ , and there are the transmission eigenvalues at  $\lambda = 1$  and  $\lambda = 3$  with multiplicities two if  $b = 3$ . It is clear from (6.11) that the value of  $V_b$  cannot be determined from the transmission eigenvalues, and hence there is a one-parameter family of potentials corresponding to the transmission eigenvalues for  $b \geq 2$ , with  $V_b$  being the nonzero real parameter.

In the third example given below, we illustrate that the transmission eigenvalues may be all real, may be all complex, or may have multiplicities greater than one.

*Example 6.3.* Assume that  $b = 2$  in Definition 1.1 and hence  $V_n = 0$  for  $n \geq 3$ . Further assume that  $V_2 \neq 0$ . Thus, the potential is known by specifying  $V_1$  and  $V_2$ . From (2.24) and (2.29)-(2.31), we get  $f_n = z^n$  for  $n \geq 2$  and

$$f_1 = V_2 z^2 + z, \quad f_0 = V_2 z^3 + V_1 V_2 z^2 + (V_1 + V_2) z + 1. \tag{6.12}$$

Note that we can get  $f_0$  given in (6.12) also from (2.50). Thus, the quantity  $g_0$  appearing in (2.49) is obtained from  $f_0$  given in (6.12) by replacing  $z$  with  $z^{-1}$ . We have

$$g_0 = V_2 z^{-3} + V_1 V_2 z^{-2} + (V_1 + V_2) z^{-1} + 1. \tag{6.13}$$

Using (6.12) and (6.13) in (5.8), we obtain

$$D = \frac{V_2(z^3 - z^{-3}) + V_1 V_2(z^2 - z^{-2}) + (V_1 + V_2)(z - z^{-1})}{z - z^{-1}},$$

or equivalently,

$$D = V_2(z^2 + 1 + z^{-2}) + V_1V_2(z + z^{-1}) + (V_1 + V_2). \quad (6.14)$$

Using (2.8) in (6.14), we express  $D$  as a function of  $\lambda$  as

$$D = V_2\lambda^2 - V_2(4 + V_1)\lambda + (V_1 + 2V_1V_2 + 4V_2). \quad (6.15)$$

Thus, the quantity  $E$  is obtained by using (6.15) in (5.7) as

$$E = \lambda^2 - (4 + V_1)\lambda + \left(4 + 2V_1 + \frac{V_1}{V_2}\right). \quad (6.16)$$

We know by Theorem 5.2 that the transmission eigenvalues correspond to the zeros of  $E$  given in (6.16). Thus, from (6.16), we see that the two transmission eigenvalues are given by

$$\lambda = 2 + \frac{V_1}{2} \pm \sqrt{\frac{V_1^2}{4} - \frac{V_1}{V_2}}. \quad (6.17)$$

Hence, depending on the potential, we may have two real distinct transmission eigenvalues, a double real transmission eigenvalue, or a pair of complex conjugate transmission eigenvalues. For example, we have a double transmission eigenvalue at  $\lambda = 0$  if  $(V_1, V_2) = (-4, -1)$ , we have  $(\lambda_1, \lambda_2) = (i, -i)$  when  $(V_1, V_2) = (-4, -4/5)$ , we have  $(\lambda_1, \lambda_2) = (3, -3)$  when  $(V_1, V_2) = (-4, 4/5)$ , and we have  $(\lambda_1, \lambda_2) = (1 + i, 1 - i)$  when  $(V_1, V_2) = (-2, -1)$ . From Theorem 5.6(b), it follows that the unusual case occurs when  $V_1 = 0$ , in which case we have

$$E = (\lambda - 2)^2.$$

As already explained in Example 6.2, in the unusual case, corresponding to the transmission eigenvalues  $(\lambda_1, \lambda_2) = (2, 2)$  we have a one-parameter family of potentials with  $V_1 = 0$  and  $V_2$  being the nonzero real parameter.

In the fourth example given below, we illustrate Theorems 5.5(c) and 5.5(d).

*Example 6.4.* As in Example 6.3, let us assume that  $b = 2$  with  $V_2 \neq 0$  and hence  $V_n = 0$  for  $n \geq 3$ . Let us first illustrate Theorem 5.5(c). Using  $f_0$  given in (6.12), at  $z = 1$  we obtain

$$f_0(1) = 2V_2 + V_1 + V_1V_2 + 1, \quad \dot{f}_0(1) = 4V_2 + V_1 + 2V_1V_2. \quad (6.18)$$

From (6.18), we see that  $\dot{f}_0(1) = 0$  is equivalent to

$$V_2 = -\frac{V_1}{4 + 2V_1}, \quad (6.19)$$

in which case from (5.14) we get  $S|_{z=1} = 1$ . Using (6.19) in the first equality in (6.18), we see that  $\dot{f}_0(1) \neq 0$  is equivalent to  $V_1 \neq -2$ . From the denominator on the right-hand side in (6.19), we already see that we must have  $V_1 \neq -2$  for (6.19) to make sense. Let us now determine when  $\lambda = 0$  becomes a transmission eigenvalue. From (6.14), we see that  $\lambda = 0$  is a transmission eigenvalue if and only if the last of the three terms on the right-hand side in (6.16) is zero, i.e.,

$$4 + 2V_1 + \frac{V_1}{V_2} = 0,$$

which is equivalent to having (6.19). Thus,  $\lambda = 0$  is a transmission eigenvalue if and only if we have  $f_0(1) \neq 0$  and  $\dot{f}_0(1) = 0$ , confirming Theorem 5.5(c). Let us now illustrate Theorem 5.5(d). Evaluating  $f_0$  of (6.12) at  $z = -1$ , we get

$$f_0(-1) = -2V_2 - V_1 + V_1V_2 + 1, \quad \dot{f}_0(-1) = 4V_2 + V_1 - 2V_1V_2. \quad (6.20)$$

From (6.20), we see that  $\dot{f}_0(-1) = 0$  is equivalent to

$$V_2 = -\frac{V_1}{4 - 2V_1}. \quad (6.21)$$

Using (6.21) in the first equality in (6.20), we see that  $f_0(-1) \neq 0$  is equivalent to  $V_1 \neq 2$ . From the denominator on the right-hand side in (6.21), we already see that we must have  $V_1 \neq 2$  for (6.21) to

make sense. Let us now determine when  $\lambda = 4$  becomes a transmission eigenvalue. From (6.17), we see that  $\lambda = 4$  is a transmission eigenvalue if and only if

$$4 = 2 + \frac{V_1}{2} \pm \sqrt{\frac{V_1^2}{4} - \frac{V_1}{V_2}},$$

or equivalently, if and only if

$$\left(-2 + \frac{V_1}{2}\right)^2 = \frac{V_1^2}{4} - \frac{V_1}{V_2}. \tag{6.22}$$

Simplifying (6.22), we see that (6.22) is equivalent to (6.21). Thus,  $\lambda = 4$  is a transmission eigenvalue if and only if we have  $f_0(-1) \neq 0$  and  $\dot{f}_0(-1) = 0$ , confirming Theorem 5.5(d).

In the fifth example given below, we illustrate that, in the unusual case, we may have a unique real-valued potential  $V$  in class  $\mathcal{A}_b$  corresponding to a given set of transmission eigenvalues, we may have a finite number of distinct real-valued potentials corresponding to a given set of transmission eigenvalues, or we may have infinitely many distinct real-valued potentials corresponding to a given set of transmission eigenvalues.

*Example 6.5.* Assume that  $b = 3$  with  $V_3 \neq 0$  and hence  $V_n = 0$  for  $n \geq 4$ . By Theorem 5.2, system (5.1) has four transmission eigenvalues uniquely determined by the ordered set  $\{V_1, V_2, V_3\}$ . By Theorem 5.6(b), we know that the unusual case occurs when  $V_2 = -V_1$ . Let us see to what extent the transmission eigenvalues determine the potential  $V$  in the unusual case. Proceeding as in the proof of Theorem 2.6, we can evaluate the values of the regular solution  $\varphi_n$ . We get

$$\varphi_3 = \lambda^2 - 4\lambda + 3 - V_1^2, \tag{6.23}$$

$$\varphi_4 = -\lambda^3 + (6 + V_3)\lambda^2 + (V_1^2 - 4V_3 - 10)\lambda + (4 - V_1 - 2V_1^2 + 3V_3 - V_3V_1^2), \tag{6.24}$$

and hence we also have the values of the regular solution  $\overset{\circ}{\varphi}_n$  corresponding to  $V_n \equiv 0$ , and in particular, from (2.75) we obtain

$$\overset{\circ}{\varphi}_3 = \lambda^2 - 4\lambda + 3, \quad \overset{\circ}{\varphi}_4 = -\lambda^3 + 6\lambda^2 - 10\lambda + 4. \tag{6.25}$$

Since  $b = 3$  in our example, using (6.23)-(6.25) in (5.2), with the help of (5.7) we get

$$E = \lambda^4 - 8\lambda^3 + (22 - \gamma)\lambda^2 + (4\gamma + \epsilon - 24)\lambda + (9 - 3\gamma - 2\epsilon), \tag{6.26}$$

where we have let

$$\gamma := V_1^2 + \frac{V_1}{V_3}, \quad \epsilon := \frac{V_1^2}{V_3}. \tag{6.27}$$

Recall that we assume  $V_2 = -V_1$ , which signifies the unusual case. Since the inverse problem of determining the potential  $V$  from the transmission eigenvalues is equivalent to the recovery of  $V$  from  $E$ , our inverse problem is equivalent to the determination of  $V_1$  and  $V_3$  from the ordered set  $\{\gamma, \epsilon\}$ . Let us analyze this problem algebraically and see to what extent we can determine the real parameter  $V_1$  and the nonzero real parameter  $V_3$  from the ordered set  $\{\gamma, \epsilon\}$ . Eliminating  $V_3$  in (6.27) we obtain

$$V_1^3 - \gamma V_1 + \epsilon = 0. \tag{6.28}$$

Given the ordered set  $\{\gamma, \epsilon\}$ , the solution to the algebraic equation in (6.28) may yield one, two, or three distinct real values for  $V_1$ . Once we obtain  $V_1$  by solving (6.28), we can recover  $V_3$  from the second identity in (6.27). If  $\gamma = 0$  and  $\epsilon = 0$ , then  $V_1 = 0$  is the solution with multiplicity three. This special case is analyzed in Example 6.2 already, and it corresponds to the unusual case. We already know that, in this unusual case,  $V_3$  cannot be determined from the knowledge of transmission eigenvalues. In this particular case, the four transmission eigenvalues are given by

$$\lambda_1 = 1, \quad \lambda_2 = 1, \quad \lambda_3 = 3, \quad \lambda_4 = 3,$$

corresponding to the one-parameter family of potentials with  $V_n = 0$  for  $n \neq 3$  and  $V_3$  being any nonzero real constant. Let us next consider the case  $\gamma = 0$  and  $\epsilon \neq 0$ . In this case from (6.27) and (6.28), we get

$$V_1 = -\sqrt[3]{\epsilon}, \quad V_3 = \frac{1}{\sqrt[3]{\epsilon}},$$

indicating that the transmission eigenvalues uniquely determine the potential  $V$  in this particular case. For example, the choice  $\gamma = 0$  and  $\epsilon = 1$  yields the unique potential  $V$  with

$$(V_1, V_2, V_3) = (-1, 1, 1),$$

corresponding to the transmission eigenvalues

$$\lambda_1 = 1.4751\bar{1}, \quad \lambda_2 = 0.50978\bar{4}, \quad \lambda_3 = 3.0075\bar{5} + 0.51311\bar{6}i, \quad \lambda_4 = 3.0075\bar{5} - 0.51311\bar{6}i,$$

which are obtained as the zeros of  $E$  given in (6.26). Note that we use an overline on a digit to indicate a round off. Let us next consider the case where (6.28) has three distinct real solutions for  $V_1$ . For example, when  $\gamma = 7$  and  $\epsilon = 6$ , we have three distinct solutions for  $V_1$  in (6.28), namely, 1, 2, and  $-3$ . In this case, the transmission eigenvalues  $\lambda_j$  are obtained as the zeros of  $E$  from (6.26) as

$$\lambda_1 = 4, \quad \lambda_2 = 3.8557\bar{7}, \quad \lambda_3 = 1.3216\bar{4}, \quad \lambda_4 = -1.1774\bar{1},$$

corresponding to exactly the three distinct potentials with the values of  $(V_1, V_2, V_3)$  given by

$$\left(1, -1, \frac{1}{6}\right), \quad \left(2, -2, \frac{2}{3}\right), \quad \left(-3, 3, \frac{3}{2}\right).$$

When  $\gamma = 3$  and  $\epsilon = 2$ , we get exactly two distinct real  $V_1$ -values as solutions to (6.28), namely,  $V_1 = 1$  and  $V_1 = -2$ . In this particular case, the transmission eigenvalues  $\lambda_j$  are obtained as the zeros of  $E$  from (6.26) as

$$\lambda_1 = 1.249\bar{5}, \quad \lambda_2 = -0.25862\bar{7}, \quad \lambda_3 = 3.5045\bar{6} + 0.30989\bar{2}i, \quad \lambda_4 = 3.5045\bar{6} - 0.30989\bar{2}i,$$

corresponding to exactly the two distinct potentials with the values for  $(V_1, V_2, V_3)$  given by

$$\left(1, -1, \frac{1}{2}\right), \quad (-2, 2, 2).$$

As a summary, in this example, corresponding to the four transmission eigenvalues, we have demonstrated that we may have a unique potential in class  $\mathcal{A}_b$ , two distinct potentials, three distinct potentials, or a one-parameter family of infinitely many potentials.

In the sixth example below, we illustrate the use of the Marchenko method to obtain the potential from the transmission eigenvalues.

*Example 6.6.* Assume that  $b = 2$  with  $V_2 \neq 0$  and hence  $V_n = 0$  for  $n \geq 3$ . We have  $2(b - 1)$  equal to 2, and hence there are two transmission eigenvalues, as indicated in Theorem 5.2. Let us assume that they are given by

$$\lambda_1 = \frac{11 + \sqrt{57}}{4}, \quad \lambda_2 = \frac{11 - \sqrt{57}}{4}. \tag{6.29}$$

Using (5.5), (5.7), and (6.29), we see that the quantity  $E$  defined in (5.7) is given by

$$E = \left(\lambda - \frac{11 + \sqrt{57}}{11}\right) \left(\lambda - \frac{11 - \sqrt{57}}{11}\right). \tag{6.30}$$

Using (2.8) in (6.30), we express  $E$  in terms of the spectral parameter  $z$  and obtain

$$E = z^2 + \frac{3}{2}z - 1 + \frac{3}{2}z^{-1} + z^{-2}.$$

Thus, we have

$$(z - z^{-1})E = z^3 + \frac{3}{2}z^2 - 2z + 2z^{-1} - \frac{3}{2}z^{-2} + z^{-3}. \tag{6.31}$$



From (6.31), with the help of (5.10) we extract  $(f_0 - 1)/V_2$  by taking the positive powers of  $z$  and get

$$\frac{f_0 - 1}{V_2} = z^3 + \frac{3}{2} z^2 - 2z. \quad (6.32)$$

As seen from Theorem 5.3(b), the coefficient of  $z$  on the right-hand side in (6.32) gives us  $K_{01}/V_2$ , and hence

$$\frac{K_{01}}{V_2} = -2. \quad (6.33)$$

Using (6.29) and (6.33) in (5.9), we write (5.9) as

$$\frac{11}{2} = 4 + (-2 - 1) V_2,$$

and hence recover  $V_2$  as

$$V_2 = -\frac{1}{2}. \quad (6.34)$$

Note that the first equality in (2.48) indicates that the coefficient of  $z^2$  in (6.32) must be  $V_1$ , i.e., we have

$$V_1 = \frac{3}{2}.$$

However, if  $b$  were a large integer, we would not be able to obtain the ordered set  $\{V_1, V_2, \dots, V_{b-1}\}$  of potential values in such a straightforward manner. Thus, we will use the Marchenko procedure to illustrate the recovery of  $V_1$  because it is a more systematic way of recovering the potential. Using (6.34) in (6.32), we obtain  $f_0$  as

$$f_0 = -\frac{1}{2} z^3 - \frac{3}{4} z^2 + z + 1. \quad (6.35)$$

From (2.46) and (2.49), we see that the quantity  $g_0$  is obtained by replacing  $z$  with  $z^{-1}$  in (6.35) and we have

$$g_0 = -\frac{1}{2} z^{-3} - \frac{3}{4} z^{-2} + z^{-1} + 1. \quad (6.36)$$

Using (6.35) and (6.36) in (2.45), we obtain the scattering matrix  $S$  as

$$S = \frac{-\frac{1}{2} z^{-3} - \frac{3}{4} z^{-2} + z^{-1} + 1}{-\frac{1}{2} z^3 - \frac{3}{4} z^2 + z + 1}. \quad (6.37)$$

Let us illustrate the determination of the potential  $V$  from  $S$  given in (6.37). We determine the zeros of  $f_0$  inside the unit circle  $|z| = 1$ , which is equivalent to finding the poles of  $S$ , as indicated in Theorem 2.5(e). Note that  $f_0$  has three zeros at  $z = z_j$  for  $j = 1, 2, 3$ , where

$$z_1 = \frac{1 - \sqrt{17}}{4}, \quad z_2 = \frac{1 + \sqrt{17}}{4}, \quad z_3 = -2.$$

We have

$$z_1 = -0.780776\bar{i}, \quad z_2 = 1.28078\bar{i}, \quad z_3 = -2,$$

and hence, only  $z_1$  corresponds to a bound state based on the criteria given in Theorem 2.5(a), i.e., among the three  $z_j$ -values, only  $z_1$  lies in the union of the intervals  $(-1, 0)$  and  $(0, 1)$ . Next, for  $n \geq 1$ , we determine  $M_n$  defined in (3.17). The integral in (3.17) can be evaluated in terms of the residues of  $S$  at the two poles  $z = 0$  and  $z = z_1$  inside the unit circle  $\mathbf{T}$ . From (3.17) and (3.19), we see that the summation term in (3.17) consists of one term only and it cancels the contribution from the integral due to the pole at  $z = z_1$ . Thus, the only contribution to  $M_n$  comes from the residues at  $z = 0$  and we have

$$M_1 = -\text{Res}[S, 0], \quad M_2 = -\text{Res}[zS, 0], \quad M_3 = -\text{Res}[z^2S, 0]. \quad (6.38)$$

Using (6.37) in (6.38), we get

$$M_1 = -\frac{7}{8}, \quad M_2 = \frac{1}{4}, \quad M_3 = \frac{1}{2}, \quad (6.39)$$

$$M_n = 0, \quad n \geq 4. \quad (6.40)$$

Using (6.39) and (6.40), let us write Marchenko system (3.15) by retaining only the nonzero terms. We get

$$\begin{cases} K_{01} + M_1 + K_{01}M_2 + K_{02}M_3 = 0, \\ K_{02} + M_2 + K_{01}M_3 = 0, \\ K_{03} + M_3 = 0, \\ K_{12} + M_3 = 0. \end{cases} \quad (6.41)$$

Thus, from Marchenko system (3.15), we get  $K_{nm} = 0$  for  $0 \leq n < m$ , with the exception of the four nonzero terms  $K_{01}, K_{02}, K_{03}$ , and  $K_{12}$ . By solving linear system (6.41), we obtain

$$K_{01} = 1, \quad K_{02} = -\frac{3}{4}, \quad K_{03} = -\frac{1}{2}, \quad K_{12} = -\frac{1}{2}.$$

Finally, we recover the potential via (2.51) and obtain

$$V_1 = K_{01} - K_{12}, \quad V_2 = K_{12} - K_{23},$$

which yields

$$V_1 = \frac{3}{2}, \quad V_2 = -\frac{1}{2}.$$

In the next example, we illustrate the Gel'fand-Levitan method presented in Section IV.

*Example 6.7.* In this example, we illustrate the use of the Gel'fand-Levitan method to recover the potential from the transmission eigenvalues. Assume that  $b = 2$  and  $V_2 \neq 0$ . Hence,  $V_n = 0$  for  $n \geq 3$ . We have  $2(b - 1)$  equal to 2, and hence there are two transmission eigenvalues as asserted in Theorem 5.2. Let us assume that they are given by

$$\lambda_1 = -1, \quad \lambda_2 = 4. \quad (6.42)$$

From (5.5) and (6.42), we see that the quantity  $E$  defined in (5.7) is given by

$$E = \lambda^2 - 3\lambda - 4. \quad (6.43)$$

Using (2.8) in (6.43), we express  $E$  in terms of the spectral parameter  $z$  and obtain

$$(z - z^{-1})E = z^3 - z^2 - 5z + 5z^{-1} + z^{-2} - z^{-3}. \quad (6.44)$$

From (6.44), with the help of (5.10) we extract  $(f_0 - 1)/V_2$  by taking only the positive powers of  $z$  and get

$$\frac{f_0 - 1}{V_2} = z^3 - z^2 - 5z. \quad (6.45)$$

As seen from Theorem 5.3(b), the coefficient of  $z$  on the right-hand side in (6.45) gives us  $K_{01}/V_2$ , and hence we have

$$\frac{K_{01}}{V_2} = -5. \quad (6.46)$$

Using (6.42) and (6.46) in (5.9), we write (5.9) as

$$3 = 4 + (-5 - 1) V_2,$$

and hence recover  $V_2$  as

$$V_2 = \frac{1}{6}. \quad (6.47)$$

Note that the first equality in (2.48) indicates that the coefficient of  $z^2$  in (6.45) must be  $V_1$ , i.e., we have

$$V_1 = -1.$$

However, if  $b$  were a large integer, we would not be able to obtain the ordered set  $\{V_1, V_2, \dots, V_{b-1}\}$  of potential values in such a straightforward manner. Instead, we will use the Gel'fand-Levitan procedure to illustrate the recovery of  $V_1$  because that is a fundamental procedure to obtain the potential. Using (6.47) in (6.45), we obtain  $f_0$  as

$$f_0 = \frac{1}{6} z^3 - \frac{1}{6} z^2 + \frac{5}{6} z + 1. \tag{6.48}$$

From (2.46) and (2.49), we see that the quantity  $g_0$  is obtained by replacing  $z$  with  $z^{-1}$  in (6.48), and hence we have

$$g_0 = \frac{1}{6} z^{-3} - \frac{1}{6} z^{-2} + \frac{5}{6} z^{-1} + 1. \tag{6.49}$$

We determine the zeros of  $f_0$  inside the unit circle  $|z| = 1$ . Note that  $f_0$  has three zeros at  $z = z_j$  for  $j = 1, 2, 3$ , where

$$z_1 = \frac{-1 - \sqrt{13}}{2}, \quad z_2 = \frac{-1 + \sqrt{13}}{2}, \quad z_3 = 2. \tag{6.50}$$

We have

$$z_1 = -2.3027\bar{8}, \quad z_2 = 1.3027\bar{8}, \quad z_3 = 2,$$

and hence, among the three  $z_j$ -values, none lies in the union of the real intervals  $(-1, 0)$  and  $(0, 1)$ . Thus, by Theorem 2.5(a) there are no bound states. Let us now evaluate  $G_{nm}$  given in (4.15). Since there are no bound states, from (4.3) and (4.12) we obtain

$$d(\rho - \overset{\circ}{\rho}) = \frac{1}{2\pi i} (z - z^{-1})^2 \left( 1 - \frac{1}{f_0 g_0} \right) \frac{dz}{z}, \quad z \in \bar{\mathbf{T}}^+, \tag{6.51}$$

which is extended in a straightforward manner from  $\mathbf{T}^+$  to the whole circle  $\mathbf{T}$ . Using (2.21) and (6.51) in (4.15), we get

$$G_{nm} = \frac{1}{2} \frac{1}{2\pi i} \oint \frac{dz}{z} \left( 1 - \frac{1}{f_0 g_0} \right) (z^n - z^{-n}) (z^m - z^{-m}), \tag{6.52}$$

where the factor  $1/2$  in front of the integral in (6.52) is due to the fact that we now integrate along the entire unit circle  $\mathbf{T}$  in the positive direction instead of integrating along the upper semicircle  $\mathbf{T}^+$ . We remark the symmetry  $G_{nm} = G_{mn}$ , as seen from (6.52). The integral in (6.52) can be evaluated by using the residues at the poles of the integrand inside the unit circle. Such poles occur at  $z = 0$  and  $z = 1/z_j$  for  $j = 1, 2, 3$ , where the  $z_j$ -values are given in (6.50). We only need to determine  $V_1$  and  $V_2$  via the Gel'fand-Levitan method, and hence it is sufficient to determine  $G_{nm}$  only for some small values of  $n$  and  $m$ . Since we will use (2.84) for  $n = 1$  and  $n = 2$ , from (2.84) we see that it is enough for us to set up the Gel'fand-Levitan system only for  $1 \leq m < n \leq 3$ . Using (6.48) and (6.49) in (6.52), we obtain

$$G_{11} = 0, \quad G_{21} = 1, \quad G_{22} = 1, \quad G_{31} = 1, \quad G_{32} = \frac{11}{6}. \tag{6.53}$$

We write Gel'fand-Levitan system (4.17) for  $1 \leq m < n \leq 3$  as

$$\begin{cases} A_{21} + G_{21} + A_{21} G_{11} = 0, \\ A_{31} + G_{31} + A_{31} G_{11} + A_{32} G_{21} = 0, \\ A_{32} + G_{32} + A_{31} G_{12} + A_{32} G_{22} = 0. \end{cases} \tag{6.54}$$

Using (6.53) in (6.54), we get

$$\begin{cases} A_{21} + 1 = 0, \\ A_{31} + 1 + A_{32} = 0, \\ A_{32} + \frac{11}{6} + A_{31} + A_{32} = 0, \end{cases}$$

which is uniquely solvable and yielding

$$A_{21} = -1, \quad A_{31} = -\frac{1}{6}, \quad A_{32} = -\frac{5}{6}. \quad (6.55)$$

We recall that  $A_{10} = 0$  as a result of (2.81). From (2.84), we have

$$V_1 = A_{21} - A_{10}, \quad V_2 = A_{32} - A_{21}, \quad (6.56)$$

and hence, using (6.55) and (6.56), we get

$$V_1 = -1, \quad V_2 = \frac{1}{6}.$$

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