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INVERSE PROBLEMS FOR A PERTURBED DISSIPATIVE HALF-SPACE

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ABSTRACT

This paper addresses the scattering of acoustic and electromagnetic waves from a perturbed dissipative half-space. For simplicity, the perturbation is assumed to have compact support. Section 1 discusses the application that motivated this work and explains how the scalar model used here is related to Maxwell's equations. Section 2 introduces three formulations for direct and inverse problems for the half-space geometry. Two of these formulations relate to scattering problems, and the third to a boundary value problem. Section 3 shows how the scattering problems can be related to the boundary value problem. This shows that the three inverse problems are equivalent in a certain sense. In section 4, the boundary value problem is used to outline a simple way to formulate a multidimensional layer stripping procedure. This procedure is unstable and does not constitute a practical algorithm for solving the inverse problem. The paper concludes with three appendices, the first two of which carry out a careful construction of solutions of the direct problems and the third of which contains a discussion of some properties of the scattering operator.

Short title: Inverse problems for a half-space

PACS Classification numbers: 0340K, 0380, 4320, 4110H

1. INTRODUCTION

This work is motivated by the problems encountered in using radar as a geophysical probe. For these applications, the radar antenna is positioned above the earth, often on a satellite [E], an airplane, or a tall gantry. In many cases it is reasonable to approximate the earth as an infinite half-space $x_3 < 0$. The upper half-space is assumed to be composed of dry air, whose electromagnetic characteristics we assume to be those of free space (i.e., vacuum). Electromagnetic measurements are made in the upper half-space, and from these measurements one hopes to reconstruct the electromagnetic characteristics of the lower half-space.

In this paper we consider a simplified scalar model that includes not only the variable speed of wave propagation but also the dissipation. This model is also appropriate for acoustic wave propagation.

The propagation of electromagnetic waves is governed by Maxwell's equations, which we write in the form

$$\nabla \wedge E = i\omega\mu H \tag{1.1}$$

$$\nabla \wedge H = (\sigma - i\omega\epsilon)E. \tag{1.2}$$

Here E is the electric field, H the magnetic field, ϵ the electric permittivity, μ the magnetic permeability, and σ the conductivity. These equations are obtained from the time-dependent equations by assuming a time dependence of $e^{-i\omega t}$. We will take ω to be positive throughout.

In many cases of interest, the magnetic permeability is very close to the permeability of free space; accordingly we assume $\mu = \mu_0$. If we write out the six scalar equations of (1.1) and (1.2), and assume that ϵ, σ, E, and H are independent of one of the coordinates, say x_2, then we find that the six equations decouple into two sets of equations, one set for H_1, E_2, and H_3, and the other set for E_1, H_2, and E_3. These determine independent polarizations, the former called the Transverse Electric (TE) polarization and the latter called the Transverse Magnetic (TM) polarization [J].

The equations for the TE polarization reduce to

$$(\nabla^2 + \omega^2\mu_0\epsilon + i\omega\mu_0\sigma)E_2 = 0, \tag{1.3}$$

where the Laplacian is a two-dimensional one in the x_1 and x_3 variables. We assume that the upper half-space ($x_3 > 0$) is air, which we approximate by the same electromagnetic parameters as free space, namely $\epsilon = \epsilon_0$, $\sigma = 0$. We will write $k = \omega/c_0$, where $c_0 = (\mu_0\epsilon_0)^{-1/2}$ is the speed of light in free space. We will consider only k positive. In addition, we write $n^2 = \epsilon/\epsilon_0$ and $m = \sigma\sqrt{\mu_0/\epsilon_0}$. With this notation, (1.3) becomes

$$(\nabla^2 + k^2n^2 + ikm)E = 0, \tag{1.4}$$

where we have dropped the subscript on E.

In what follows, we will develop the theory for (1.3) and (1.4) when the Laplacian is a three-dimensional one; the theory for the two-dimensional case is similar. Both n^2 and m are assumed nonnegative. We assume that n^2 is identically one in the upper half-space, and in the lower half-space, n^2 differs from a positive constant n_-^2 only in a region of compact support. Similarly m is identically zero in the upper half-space, and in the lower half-space differs from a positive constant m_- only in a region of compact support. These assumptions are meant to include the case of an ice floe in sea water. The parameter values for sea ice in the gigahertz range are between 3 and 4 for n^2 and around 6 m^{-1} for m. For sea water, n_-^2 is 3.37, and the value of m_- is around seven thousand m^{-1} [Ca]. We will write $x = (x_1, x_2, x_3) = (x', x_3)$.

The arguments given here would need to be modified in order to apply to cases when the background medium has more layers, and for the case when the perturbation extends into the upper half-space.

The theory of scattering from a half-space for (1.4) in the non-dissipative case ($m = 0$) has been developed in [Wi, DG, We, Xu]. The layered problem has been investigated by many investigators. In particular, the papers [Chj, M, KK] show that backscattered data from a single incident plane wave suffices to determine both n^2 and m only if n^2 and m have a jump discontinuity. An abstract formulation of scattering for dissipative hyperbolic systems has been given in [LP].

2. FORMULATION OF THREE INVERSE PROBLEMS

When a layered half-space is perturbed, some thought must be given to the formulation of the direct and inverse scattering problems. For the direct problem, one commonly-used approach [TKS] is to assume an incident plane wave and then use a half-space or layered-medium Green's function to set up an integral equation. The solution of this integral equation defines the scattering solution, whose far-field asymptotics are taken to be the scattering data. Inverse problems involve using this scattering data to determine perturbations in the medium.

One problem with this approach is that in practice, the incident field is never infinite in extent. This is unimportant if energy from infinity has no effect on the scattering, but for some layered medium problems this may not be the case. One can avoid this difficulty by multiplying the incident field by a cut-off function meant to model the antenna beam pattern, but the form of this cut-off function certainly does affect the scattering, and it is difficult to find a simple way to retain the information about the incident field in the far-field pattern. One approach to this is in [GX]. Below some other methods are suggested.

THE DIRECT AND INVERSE SCATTERING PROBLEM.

For any incident field, solutions to the direct scattering problem can be constructed in the usual way by converting the differential equation (1.4) to an integral equation that builds in the boundary conditions at infinity. The kernel of this integral equation is a Green's function for the unperturbed problem with outgoing boundary conditions. The details are given in Appendix 1.

To define scattering data, we consider the field E in the upper half-space. We define the scattering operator S to be the map from the downgoing part of the wavefield to the upgoing part. We construct an explicit representation $\hat{S}$ of this map in the Fourier transform domain. In particular, we use the fact that the medium parameters are known and constant in the upper half-space. For x_3 positive, one can therefore Fourier transform (1.4) in the x_1 and x_2 coordinates. The result is an ordinary differential equation whose general solution for $x_3 > 0$ is

$$\hat{E}(\xi, x_3) = A(\xi)e^{i\lambda_+ x_3} + B(\xi)e^{-i\lambda_+ x_3}, \tag{2.1}$$

where $\lambda_+ = \sqrt{k^2 - |\xi|^2}$ and the hat denotes the two-dimensional Fourier transform

$$\hat{E}(\xi, x_3) = \int E(x', x_3)e^{-i\xi \cdot x'} dx', \tag{2.2}$$

x' denoting (x_1, x_2). When λ_+ is zero, the general solution corresponding to (2.1) is simply a constant. When $|\xi| < k$, the B term in (2.1) is a downgoing wave, whereas the A term is upgoing. The coefficient B thus determines an incident wave. This incident wave, together with continuity of E and its normal derivative at the interface $x_3 = 0$ and a radiation condition in the lower half-space, uniquely defines the scattered wave, which determines A. (See Appendix 1 for details.) Consequently, we can define $\hat{S}$ as the map from B to A. Thus $\hat{S}$ maps incident fields to scattered fields; knowledge of $\hat{S}$ is equivalent to knowledge of the scattered fields corresponding to all incident fields in some class. In particular, $\hat{S}$ can be considered as a map on the space $L^2(\mathbf{R}^2)$ of square-integrable functions. The operator S is then defined by $\widehat{Sf} = \hat{S}\hat{f}$.

If $|\xi| > k$, then the second term on the right side of (2.1) grows exponentially as x_3 becomes large. Because it is not physically reasonable for the incident wave in a scattering experiment to be exponentially large at infinity, in the scattering case we take B to be zero for $|\xi| > k$. For these values of ξ, the scattered wave also decays exponentially as x_3 goes to infinity; thus for a scattering experiment in which measurements are made in the far field, the relevant scattering operator is $P\hat{S}P$, where P denotes the projection operator of multiplication by the function that is one for $|\xi| \leq k$ and zero for $|\xi| > k$. Appendix 3 contains a proof that $P\hat{S}P$, as a map on a certain L^2 space, has norm less than one.

In the case when the incident wave is a plane wave independent of x_2, making an angle θ with the vertical, B is a delta function supported at $\xi = (k \sin\theta, 0)$. If the lower half-space varies only in the depth coordinate x_3, then $\hat{S} = P\hat{S}P$ is simply multiplication by the usual reflection coefficient [To, TKS]. Thus data from a single angle of incidence θ defines the action of $\hat{S}$ on $\delta(\xi - k(sin\theta, 0))$. As θ varies and the incident beam rotates around the vertical axis, the set $\{|\xi| < k\}$ is swept out. Thus knowledge of $\hat{S}$ incorporates knowledge of scattering for all angles of incidence.

This definition of scattering data differs from that in [TKS, Xu, We] in that no far-field asymptotic expansion is needed. The present definition may thus be useful in cases when measurements are made close to the surface. The present definition can handle any antenna beam pattern. However, this definition has the disadvantage that measurements are needed everywhere on a horizontal surface to completely determine $\hat{S}$. This makes it unsuitable for use with satellite-borne radar. If n^2 and m are assumed to depend only on x_3, then $\hat{S}$ can be determined for all ξ with magnitude less than k by measuring the reflection coefficient for all angles of incidence.

The inverse scattering problem is to determine n^2 and m in the lower half-space from knowledge of $\hat{S}$. In the three-dimensional case, if $\hat{S}$ is thought of as an integral operator mapping functions of two variables to functions of two variables, it is clear that $\hat{S}$ depends on four variables. The unknowns n^2 and m depend on only three variables, so this inverse problem is overdetermined in the three-dimensional case. In the two-dimensional case, $\hat{S}$, n^2, and m are all functions of two variables.

THE POINT SOURCE INVERSE PROBLEM.

Another way to define scattering data is to assume that the incident field is due to a point source located either on the surface $x_3 = 0$ or in the upper half-space. In this case, (1.4) becomes

$$(\nabla^2 + k^2 n^2 + ikm)G(x,y) = -\delta(x-y), \tag{2.3}$$

where y is the location of the source. To define G uniquely, one needs an outgoing radiation condition at infinity. (See Appendix 1 for details.) Scattering data in this case can be taken to be knowledge of $G(x,y)$ for all x with $x_3 =$ constant and all y with $y_3 =$ constant.

The point source inverse problem is to determine n^2 and m in the lower half-space from the scattering data. In the three-dimensional case, the scattering data depend on four variables; in the two-dimensional case, on two.

THE INVERSE BOUNDARY VALUE PROBLEM.

A boundary value problem can be defined by

$$(\nabla^2 + k^2 n^2 + ikm)u = 0 \qquad \text{for } x_3 < 0 \tag{2.4}$$

$$u|_{x_3=0} = f, \tag{2.5}$$

together with an outgoing radiation condition in the lower half-space. If f is in the Sobolev space $H^{1/2}$ and $m > 0$, the Lax-Milgram theorem can be used [T] to show that the boundary value problem (2.4), (2.5) has a unique H^1 solution in the lower half-space. (A more explicit construction, involving Green's functions, is given in Appendix 2.) Thus the normal derivative $\partial u/\partial\nu$ on the surface $x_3 = 0$ is uniquely determined. The mapping from $H^{1/2}$ to $H^{-1/2}$

$$\Lambda : u|_{x_3=0} \mapsto \frac{\partial u}{\partial \nu}\Big|_{x_3=0} \tag{2.6}$$

is called the Dirichlet-to-Neumann map. Such maps have been used a great deal recently in the study of inverse problems [SU, SCII, Sy].

The inverse boundary value problem is to determine n^2 and m in the lower half-space from knowledge of Λ. In the three-dimensional case, Λ depends on four variables; in the two-dimensional case, it depends on two.

For some purposes, it is more convenient to work with the inverse of Λ; this inverse can be defined directly in a similar way.

3. CONNECTIONS BETWEEN THE SCATTERING PROBLEMS AND THE BOUNDARY VALUE PROBLEM

In this section, we discuss the sense in which the above inverse problems are equivalent.

The scattering problem and the boundary value problem. To see how the scattering problem is related to the boundary value problem, we recall that E and its normal derivative are continuous at the interface $x_3 = 0$. In the upper half-space, however, E is given by (2.1). If $E = f$ on $x_3 = 0$, then we have

$$\hat{f} = (\hat{S} + I)B, \tag{3.1}$$

where I denotes the identity operator, and, differentiating (2.1) with respect to x_3,

$$\widehat{\Lambda f} = i\lambda_+(\hat{S} - I)B. \tag{3.2}$$

Eliminating B from (3.1) and (3.2) and defining $\hat{\Lambda}\hat{f} = \widehat{\Lambda f}$, we have

$$\hat{\Lambda}(\hat{S} + I) = i\lambda_+(\hat{S} - I). \tag{3.3}$$

This is an operator equation that holds on a certain function space that is discussed in Appendix 3.

To recover Λ from $\hat{S}$, it appears that we need only invert the operator $\hat{S} + I$ appearing on the left side of (3.3). To find the inverse, we solve the system (3.1), (3.2) of linear equations for B, obtaining $B = \frac{1}{2}(I - (i\lambda_+)^{-1}\hat{\Lambda})\hat{f}$. This shows that

$$(\hat{S} + I)^{-1} = \frac{1}{2}(I - (i\lambda_+)^{-1}\hat{\Lambda}). \tag{3.4}$$

This expression itself can be used to recover Λ from $\hat{S}$.

A similar argument shows that

$$(I - (i\lambda_+)^{-1}\hat{\Lambda})^{-1} = \frac{1}{2}(\hat{S} + I);$$

this expression can be used to obtain $\hat{S}$ from Λ. Note that this formula and (3.4) each contain terms with a singularity at $\lambda_+ = 0$. This is to be expected because $\hat{S}$ is not defined at $\lambda_+ = 0$.

The point source problem and the boundary value problem. To see how the point source problem is connected to the boundary value problem, we follow [N], where this connection was worked out for the case of a bounded body. We write $q = k^2n^2 + ikm$ and $q_0 = k^2$ so that the perturbed and unperturbed point source problems can be written

$$(\nabla^2 + q)G = -\delta \tag{3.5a}$$

and

$$(\nabla^2 + q_0)G_0 = -\delta, \tag{3.5b}$$

respectively. The scattering solutions G and G_0 satisfy radiation conditions at infinity.

We write Λ_q and Λ_0 for the Dirichlet-to-Neumann maps for the operators $\nabla^2 + q$ and $\nabla^2 + q_0$, respectively.

We next use the scattering solutions G and G_0 to define two integral operators,

$$\Gamma f(x') = \lim_{y_3 \to 0^-} \int G((x',0),(y',y_3))f(y')dy' \tag{3.6a}$$

and

$$\Gamma_0 f(x') = \lim_{y_3 \to 0^-} \int G_0((x',0),(y',y_3))f(y')dy'. \tag{3.6b}$$

Theorem. G is related to Λ_q by the "boundary resolvent equation"

$$\Gamma - \Gamma_0 = \Gamma_0(\Lambda_q - \Lambda_0)\Gamma. \tag{3.7}$$

For the proof of this theorem, we need the following notation and lemma.

Given any f defined on the surface $x_3 = 0$, we use Γf to define the solutions u and v of the following boundary value problems:

$$(\nabla^2 + q)u = 0 \tag{3.8}$$

$$u|_{x_3=0} = \Gamma f \tag{3.9}$$

and

$$(\nabla^2 + q_0)v = 0 \tag{3.10}$$

$$v|_{x_3=0} = \Gamma f. \tag{3.11}$$

Both u and v are outgoing at infinity in the lower half-space.

Lemma. The solution u to (3.8), (3.9) is actually given by

$$u(x) = \int_{y_3=0} G(x,y)f(y)dy. \tag{3.12}$$

Proof. The function defined by equation (3.12) is an outgoing solution to (3.8), which on the boundary $x_3 = 0$ is equal to Γf.

QED

Proof of Theorem. Here we carry out the argument of [N] for the half-space case. Relation (3.7) is obtained by using two different methods to compute the integral

$$I(x) = \int_{y_3<0} \left(G_0(x-y)\nabla^2(u-v)(y) - (u-v)(y)\nabla^2 G_0(x-y)\right) dy. \tag{3.13}$$

This integral is the limit as h goes to infinity of the integral I_h, in which the integrand is the same but the region of integration is C_h, a large cylindrical region with radius h^2 whose top is a disk in the plane $y_3 = 0$ and whose bottom is a disk in the plane $y_3 = -h$.

First, by Green's theorem,

$$I_h = \int_{\partial C_h} (G_0\partial_\nu(u-v) - (u-v)\partial_\nu G_0)dA, \tag{3.14}$$

where ∂_ν denotes differentiation with respect to the outward unit normal and ∂C_h denotes the boundary of C_h. The boundary of C_h has three parts: the disk of radius h^2 on the surface $y_3 = 0$, the disk of radius h^2 on the surface $y_3 = -h$, and the side of the cylinder. We denote the corresponding integrals by I_h^1, I_h^2, and I_h^3, respectively.

First we consider I_h^1. Because $u = v$ on $y_3 = 0$, the second term of I_h^1 vanishes. Taking into account definitions of Γ_0 and the Dirichlet-to-Neumann maps, we find that for x with $x_3 = 0$, the first term is equal to $\Gamma_0(\Lambda_q u - \Lambda_0 v) = \Gamma_0(\Lambda_q - \Lambda_0)\Gamma f$.

The integrals I_h^2 and I_h^3 vanish as h goes to infinity because of the asymptotics of G_0. (See Lemma A1.2 in Appendix 1.)

Thus we have shown that

$$I|_{x_3=0} = \Gamma_0(\Lambda_q - \Lambda_0)\Gamma. \tag{3.15}$$

On the other hand, we can compute I without using Green's theorem. We see from (3.5), (3.8), and (3.10) that

$$I(x) = \int_{y_3<0} (G_0(q_0 v - qu) + (u-v)q_0 G_0)\, dy - (u-v)(x). \tag{3.16}$$

The terms in (3.16) involving $q_0 G_0 v$ cancel. Moreover, as x approaches the surface $x_3 = 0$, the term $(u-v)(x)$ vanishes. Thus (3.16) becomes

$$I = \int_{y_3<0} G_0(q - q_0)u dy. \tag{3.17}$$

The solution u of the boundary value problem, however, is given by $u = \int Gf$. Using this in (3.17), interchanging integrals, and using the resolvent equation $G - G_0 = \int G_0(q-q_0)G$, we obtain

$$I|_{x_3=0} = \int_{y_3=0} (G - G_0) f dy' = \Gamma f - \Gamma_0 f. \tag{3.18}$$

QED

In order to use (3.7) to obtain the Dirichlet-to-Neumann map from knowledge of the point source data, we need to be able to invert the integral operators Γ and Γ_0. This is discussed in Appendix 1.

Similarly, to obtain the point source data from the Dirichlet-to-Neumann map, one needs invertibility of the map $I - \Gamma_0(\Lambda_q - \Lambda_0) = \Gamma_0\Gamma^{-1}$, which follows from invertibility of Γ and Γ_0.

4. THE INVERSE BOUNDARY VALUE PROBLEM

Because the inverse scattering and inverse point source problems can be converted into the inverse boundary value problem, it is this problem we address here. We outline a possible approach, one based on the idea of layer-stripping. Roughly, the idea is first to use the measured data to find the medium parameters on the boundary, then to use that information to synthesize data on a nearby inner subsurface. The process is then repeated. In this manner, the medium is mathematically stripped away, layer by layer, and the medium parameters are found in the process.

For one-dimensional problems, this is an old idea; we make no attempt to trace its history here. For multidimensional problems it has not been so clear how to proceed; various multidimensional layer-stripping algorithms have been suggested in [CK, SCII, W, Sm, DH, Y]. We outline here a simple way to formulate a multidimensional layer-stripping procedure.

Most of the layer-stripping schemes involve some sort of Riccati equation to remove a known layer of the medium. A Riccati equation, moreover, can be useful as a theoretical tool in working with inverse problems [LU]. As we see below, using the Dirichlet-to-Neumann map makes the appearance of a Riccati equation especially easy to understand.

Synthesizing the subsurface data. To synthesize the subsurface data, we obtain a differential equation for the boundary data in the depth variable. This requires that we extend the definition of the Dirichlet-to-Neumann map to any $z < 0$:

$$\Lambda(z)u|_{x_3=z} = \frac{\partial u}{\partial x_3}\Big|_{x_3=z}. \tag{4.1}$$

This map satisfies the following Riccati equation:

$$\frac{d\Lambda}{dz} = -\Lambda^2 - (\partial_{x_1}^2 + \partial_{x_2}^2) - q. \tag{4.2}$$

This equation is obtained by differentiating (4.1) with respect to z, using (2.4) to eliminate $\partial^2 u/\partial z^2$, and using (4.1) to eliminate $\partial u/\partial z$.

We note that equation (4.2) together with (3.3) or (3.4) can also be used to obtain a differential equation for the scattering operator $\hat{S}$. In the case when $d\hat{S}/dz$ commutes with $\hat{S}$ (such as in the layered case when $\hat{S}$ is a multiplication operator), this differential equation has the form

$$2i\lambda_+ \frac{d\hat{S}}{dz} = \lambda_+^2(\hat{S} - I)^2 + (\hat{S} + I)^2|\xi|^2 - (\hat{S} + I)^2 Q, \tag{4.3}$$

where $Q\hat{f} = \widehat{qf}$.

Finding the medium parameters on the boundary. To solve the inverse problem, we also need to use the boundary data to find the medium parameters on that same boundary. One approach to doing this is to use the idea of [KV, SCII] that is based on the principle that highly oscillatory boundary data corresponds to waves that penetrate only a short distance into the body. The difficulty with this approach, however, lies in the practical problem of creating such a field on the boundary: even in a nondissipative homogeneous medium such as air, fields with rapid spatial oscillations decay exponentially. This can be seen by writing a solution of

$$(\nabla^2 + q_0)u = 0 \tag{4.4}$$

as $u(x) = v(\xi, x_3)\exp(i\xi \cdot x')$, so that v satisfies the ordinary differential equation

$$(\partial_{x_3}^2 + q_0 - \xi^2)v = 0. \tag{4.5}$$

Even when q_0 is real, for large ξ the solution v decays exponentially. This suggests that conventional radar experiments, in which the antenna is far from the sample, could not supply highly oscillatory boundary data.

Accordingly, we consider an alternate method for obtaining the medium parameters on the boundary, namely geometrical optics [SU2]. This requires that we use either a range of temporal frequencies ω or that we do the experiments directly in the time domain.

The time-domain version of (1.3) is

$$(\nabla^2 - \mu_0 \epsilon \partial_t^2 - \mu_0 \sigma \partial_t)\mathcal{E} = 0. \tag{4.6}$$

The plan is to obtain a progressing wave expansion [CH] for (4.6); an expansion in functions of $\phi(x) - t$, however, results in successive coefficients differing in magnitude by the speed of light c_0. We therefore make the change of variables $\tau = c_0 t$, which converts (4.6) into

$$(\nabla^2 - n^2 \partial_\tau^2 - m\partial_\tau)\mathcal{U} = 0. \tag{4.7}$$

We are interested in the small-time behavior of $\mathcal{U}$ in the neighborhood of an interface at $x_3 = 0$. For $x_3 > 0$, where $n = 1$, we expect that $\mathcal{U}$ is composed of an incident plane wave $\mathcal{U}^i = /delta(s^i(x) - \tau)$ plus a reflected wave, which we expand in the form

$$\mathcal{U}^r(s^r(x) - \tau) = A_0^r(x)\delta(s^r(x) - \tau) + A_1^r(x)H(s^r(x) - \tau) + \cdots. \tag{4.8}$$

Here s^i and s^r are the incident and reflected phases, δ denotes the Dirac delta function, and H denotes the Heaviside function that is one for positive arguments and zero for negative arguments. We take $\mathcal{U}^i$ to be a plane wave propagating in direction $\hat{e} = (e_1, e_2, e_3)$, which implies that $s^i = \hat{e} \cdot x$. Because we take this wave to be propagating in the downward direction, e_3 is negative. Just below the interface, for a short time we expect $\mathcal{U}$ to take the form of a transmitted wave, which we also expand as

$$\mathcal{U}^t(s^t(x) - \tau) = A_0^t(x)\delta(s^t(x) - \tau) + A_1^t(x)H(s^t(x) - \tau) + \cdots. \tag{4.9}$$

Here again s^t denotes the phase of the transmitted wave. On the interface $x_3 = 0$, $\mathcal{U}$ and its first x_3 derivative are continuous. Using these conditions at the interface and forcing $\mathcal{U}$ to satisfy (4.7) results in the eikonal equation

$$(\nabla s)^2 = n^2, \tag{4.10}$$

the interface conditions

$$s^i|_{x_3=0} = s^r|_{x_3=0} = s^t|_{x_3=0}, \tag{4.11}$$

and the transport equations

$$2\nabla s \cdot \nabla A_0 + A_0 \nabla^2 s + mA_0 = 0 \tag{4.12}$$

$$2\nabla s \cdot \nabla A_1 + A_1 \nabla^2 s + mA_1 + \nabla^2 A_0 = 0. \tag{4.13}$$

Here the absence of superscripts r or t indicates that the equation in question holds for both the reflected wave and transmitted wave. Solving these equations gives us

$$s^r(x) = e_1 x_1 + e_2 x_2 - e_3 x_3 \tag{4.14}$$

$$\nabla s^t(x) = (e_1, e_2, -\sqrt{n^2 - e_1^2 - e_2^2}) \tag{4.15}$$

$$A_0^r|_{x_3=0} = \frac{1 + \sqrt{n^2 - e_1^2 - e_2^2}/e_3}{1 - \sqrt{n^2 - e_1^2 - e_2^2}/e_3} \tag{4.16}$$

$$A_0^t|_{x_3=0} = \frac{2}{1 - \sqrt{n^2 - e_1^2 - e_2^2}/e_3} \tag{4.17}$$

$$A_1^r|_{x_3=0} = \frac{\partial_{x_3} A_0^t - \partial_{x_3} A_0^r}{e_3 - \sqrt{n^2 - e_1^2 - e_2^2}}. \tag{4.18}$$

The quantities $\partial_{x_3} A_0^t$ and $\partial_{x_3} A_0^r$ appearing in (4.18) can be computed, with the help of the transport equation (4.12), to be

$$\partial_{x_3} A_0^r = (e_1 \partial_{x_1} + e_2 \partial_{x_2}) A_0^r / e_3 \tag{4.19}$$

and

$$\partial_{x_3} A_0^t = \frac{-1}{\sqrt{n^2 - e_1^2 - e_2^2}} \left((\partial_{x_3} \sqrt{n^2 - e_1^2 - e_2^2} A_0^t - m A_0^t)/2 - (e_1 \partial_{x_1} + e_2 \partial_{x_2}) A_0^t \right). \tag{4.20}$$

To obtain the medium parameters n^2 and m at a point x^0 on the surface from scattering data, we send in an incident wave that is planar in a neighborhood of x^0. We then measure the scattered field at all points on a plane $x_3 =$ constant. From this information, the short-time scattered field can be inferred in a neighborhood of x^0. The value of A_0^r at x^0 tells us, via (4.16), what the value of n^2 is at x^0. In this manner, we obtain n^2 for every point on the surface; this allows us to compute, at every point, not only A_0^t from (4.17) but also the x_1- and x_2-derivatives appearing on the right side of (4.19) and (4.20). Once these are known, $\partial_{x_3} A_0^r$ can be computed and used in the right side of (4.18); since A_1^r is also known, from (4.18) we can obtain $\partial_{x_3} A_0^t$. All quantities in (4.20) are thus known except for m and $\partial_{x_3} n^2$; evidently both quantities cannot be found from a single angle of incidence. Use of the scattered field from two angles of incidence allows us to find both m and $\partial_{x_3} n^2$.

Let us consider the layer-stripping algorithm in the case when a complete set of incident fields are used and measurements of the corresponding scattered fields are made on a plane. We assume measurements are made at N frequencies. For experiments with stepped-frequency radar, for example, N can range from 51 to 801 [Jz]. The algorithm proceeds as follows.

Step 1. From the measurements at frequencies $k_0, k_1, \ldots, k_N$, construct an approximation to each scattering operator $S(k_n)$, $n = 0, 1, \ldots, N$. In practice, one would represent $S(k_n)$ by its matrix with respect to some basis. Such a basis could perhaps be constructed from antenna beam patterns for a large number of incident angles. The operator $\hat{S}$, for example, is the representation of S in a Fourier basis.

Step 2. For each of at least two incident directions $\hat{e}_j$, $j = 1, 2, \ldots, J$, choose an incident field that looks like $\exp(ik_n \hat{e}_j \cdot x)$ in the neighborhood of some point x_0 on the surface. Apply $S(k_n)$ to these incident fields to obtain the scattered field $E_{sc}(k_n, x)$.

Step 3. Fourier transform into the time domain to obtain $U^r(\tau, x)$. In practice, one can do this by first synthesizing an approximate delta function in the form

$$\delta(\tau) \approx \sum_{n=1}^{N} w_n e^{ik_n \tau}, \tag{4.21}$$

where the w_n are, for example, Hamming weights [OS]. Then the field

$$U^r(\tau, x) \approx \sum_{n=1}^{N} E_{sc}(k_n, x) w_n e^{ik_n \tau} \tag{4.22}$$

is locally the response to the incident approximate delta function (4.21).

Step 4. Extract the coefficients $A_0^r(x_0, \hat{e}_j)$ and $A_1^r(x_0, \hat{e}_j)$. This can be done, for example, by the least squares minimization

$$\min_{A_0^r, A_1^r} \int_0^T |U^r(\tau, x_0) - A_0^r(x_0, \hat{e}_j)\delta(s^r(x_0) - \tau) - A_1^r(x_0, \hat{e}_j) H(s^r(x_0) - \tau)|^2 d\tau, \tag{4.23}$$

where for U^r one uses (4.22), for s^r one uses (4.14), for δ one uses (4.21), and for the Heaviside function H one uses

$$H(\tau) \approx \sum_{n=1}^{N} \frac{w_n}{ik_n} e^{ik_n \tau}. \tag{4.24}$$

Step 5. From $A_0^r(x_0, \hat{e}_j)$ and $A_1^r(x_0, \hat{e}_j)$ for $j = 1, 2, \ldots, J$, determine $n^2(x_0)$, $m(x_0)$, and $\partial_{x_3} n^2(x_0)$. If $J > 2$ so that the system is overdetermined, one can use least squares to find the best fit.

Step 6. Repeat steps 2) through 5) for a large number of points x_0 on the surface.

Step 7. For each k_n, synthesize the subsurface data either from a Riccati equation for $S(k_n)$ such as (4.3), or use (3.3) or (3.4) to convert $S(k_n)$ to $\Lambda(k_n)$, use (4.2), and convert back to $S(k_n)$ with (3.3) or (3.4). Again, in practice, the operators $S(k_n)$ and $\Lambda(k_n)$ would be represented as matrices with respect to some basis, and equations (3.3), (3.4), and (4.2) would be approximated as matrix equations.

Step 8. Repeat, starting with step 2).

Although the above algorithm may seem ready to implement, it cannot be used in its present form because it is UNSTABLE. This is partly because of the multiplication by $|\xi|^2$ on the right side of (4.3) or equivalently, because of the x_1 and x_2 derivatives appearing on the right side of (4.2). This is similar to the situation in [YL]; this type of instability can be overcome to some extent by smoothing in the x_1 and x_2 directions, as discussed in [C].

Even when the problem is independent of x_1 and x_2, however, one expects the methods to be unstable, due to the fact that only a little of the energy put into the system on the top can propagate to great depths. Thus one expects the boundary data and scattering data to contain little information about the deeper regions. There may be methods, such as those of [SCII, SWG], for overcoming this instability to some extent. Finally, there may be difficulties connected with using bandlimiting data as described in [PSS]. Investigation of methods for overcoming the instability is left for the future.

APPENDIX 1. CONSTRUCTION OF SCATTERING SOLUTIONS

In this Appendix, we will construct outgoing solutions of (1.4) and (2.3). We do this with the help of the unperturbed Green's function.

Construction of the scattering Green's function. The unperturbed scattering Green's function $G_0(x,y) = G_0(x_1 - y_1, x_2 - y_2, x_3, y_3)$ satisfies

$$(\nabla^2 + k^2 n_0^2(x_3) + ikm_0(x_3))G_0(x,y) = -\delta(x-y) \tag{A1.1}$$

and is outgoing at infinity. Here $n_0^2 = 1$ and $m_0 = 0$ for $x_3 > 0$, and $n_0^2 = n_-^2$ and $m_0 = m_-$ for $x_3 < 0$. Equation (A1.1) can be Fourier transformed in the x_1 and x_2 variables, which yields

$$(\partial_{x_3}^2 + \lambda_0^2)\hat{G}_0(\xi, x_3, y_3) = -\delta(x_3 - y_3), \tag{A1.2}$$

where we have written $\lambda_0^2 = -|\xi|^2 + k^2 n_0^2 + ikm_0$. The general solution of (A1.2) is

$$\hat{G}_0^+(\xi, x_3, y_3) = A^+(\xi)e^{i\lambda_+ x_3} + B^+(\xi)e^{-i\lambda_+ x_3} \tag{A1.3}$$

for $x_3 > 0$ and

$$\hat{G}_0^-(\xi, x_3, y_3) = A^-(\xi)e^{i\lambda_- x_3} + B^-(\xi)e^{-i\lambda_- x_3}. \tag{A1.4}$$

for $x_3 < 0$. The coefficients $A^\pm$ and $B^\pm$, however, depend on whether y_3 is positive or negative and are different in the regions separated by the origin and the point $x_3 = y_3$. When x_3 is bigger than both 0 and y_3, the condition that $\hat{G}_0$ be upgoing implies that B^+ is zero; when x_3 is less than both 0 and y_3, the condition that $\hat{G}_0$ be downgoing implies that A^- is zero. $\hat{G}_0$ and its x_3 derivative are continuous except at $x_3 = y_3$, where $\hat{G}_0$ is continuous but its x_3 derivative jumps by one. Solving for the As and Bs in both cases results in

$$\hat{G}_0(\xi, x_3, y_3) = \frac{i}{2\lambda_+}\begin{cases} R(\lambda_+, \lambda_-)e^{i\lambda_+(x_3+y_3)} + e^{i\lambda_+|x_3-y_3|}, & \text{for } x_3 > 0 \\ T(\lambda_+, \lambda_-)e^{i\lambda_+ y_3}e^{-i\lambda_- x_3}, & \text{for } x_3 < 0 \end{cases} \tag{A1.5}$$

for the case when $y_3 > 0$ and

$$\hat{G}_0(\xi, x_3, y_3) = \frac{i}{2\lambda_-}\begin{cases} T(\lambda_-, \lambda_+)e^{-i\lambda_- y_3}e^{i\lambda_+ x_3}, & \text{for } x_3 > 0 \\ e^{i\lambda_-|x_3-y_3|} + R(\lambda_-, \lambda_+)e^{-i\lambda_-(x_3+y_3)}, & \text{for } x_3 < 0 \end{cases} \tag{A1.6}$$

for the case when $y_3 < 0$, where

$$T(\lambda_1, \lambda_2) = \frac{2\lambda_2}{\lambda_1 + \lambda_2} \tag{A1.7}$$

and

$$R(\lambda_1, \lambda_2) = \frac{\lambda_2 - \lambda_1}{\lambda_1 + \lambda_2}. \tag{A1.8}$$

Note that since the imaginary parts of λ_+ and λ_- are nonnegative, the exponents in (A1.5) and (A1.6) are decaying. The Green's function itself is obtained from its Fourier transform by

$$G_0(x, y) = \frac{1}{(2\pi)^2} \int e^{i(x'-y')\cdot\xi} \hat{G}_0(\xi, x_3, y_3) d\xi.$$

Construction of scattering solutions. For an incident wave E^0, a scattering solution E of (1.4) can be defined as the solution to the integral equation

$$E(x) = E^0(x) + \int G_0(x, y) V(y) E(y) dy, \tag{A1.9}$$

where we have written $V(y) = k^2(n^2(y) - n_-^2) + ik(m(y) - m_-)$.

Similarly, for (2.3), the scattering solution $G(x, y)$ at x due to a point source at y should satisfy the resolvent equation

$$G(x, y) = G_0(x, y) + \int G_0(x, z) V(z) G(z, y) dz. \tag{A1.10a}$$

The Green's function G, however, has a singularity at $x = y$, which causes some technical problems. We therefore write the resolvent equation in terms of the scattered wave G_{sc}, which is defined by $G = G_0 + G_{sc}$:

$$G_{sc} = \int G_0 V G_0 + \int G_0 V G_{sc}. \tag{A1.10b}$$

In order to use (A1.9) to define E and (A1.10b) to define G, we must show that both equations have unique solutions.

In order to do this, we will need the following spaces that are weighted in the x_3 variable.

$$L^{2,s}(\mathbf{R}^3) = \{u : (1 + |x_3|^2)^{s/2} u \in L^2(\mathbf{R}^3)\}$$
$$H^{1,s} = \{u : D^\alpha u \in L^{2,s}, |\alpha| \leq 1\},$$

where we use the multi-index notation $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $|\alpha| = |\alpha_1| + |\alpha_2| + |\alpha_3|$, and $D^\alpha = (\partial/\partial x_1)^{\alpha_1} (\partial/\partial x_2)^{\alpha_2} (\partial/\partial x_3)^{\alpha_3}$. Here L^2 denotes the space of square-integrable functions.

Proposition A1.1. If V is a bounded function of compact support, the operator $G_0 V$ is compact in $H^{1,-s}(\mathbf{R}^3)$ for any $s > 1/2$.

Proof. Because we are making the (unnecessary but simplifying) assumption that V has compact support in the lower half-space, we write $G_0 V$ as $G_0 \chi V$, where χ is the function that is one on the support of V and zero everywhere else. We then follow the ideas of [Ag]: first we show that multiplication by V is a compact operator mapping $H^{1,-s}$

into $L^{2,s}$; then we show that the operator $G_0\chi$ is a bounded operator mapping $L^{2,s}$ into $H^{1,-s}$. Hence the product operator $G_0\chi V = G_0 V$ is a compact operator on $H^{1,-s}$.

Multiplication by V is a compact operator from $H^{1,-s}$ to $L^{2,s}$ under much more general conditions (see [S]). Here, however, we can simply rely on the Sobolev imbedding theorem [Ad, p. 144].

To show that $G_0\chi$ is a bounded mapping from $L^{2,s}$ into $H^{1,-s}$, we follow the outline of the argument in [RS4]. We write $\phi = G_0\chi\psi$, which, when Fourier transformed, reads

$$\hat{\phi}(\xi, x_3) = \frac{1}{2\lambda_-}\begin{cases} \int_{-h}^{0}(R(\lambda_-,\lambda_+)e^{-i\lambda_-(x_3+y_3)} + e^{i\lambda_-|x_3-y_3|})\hat{\psi}(\xi,y_3)dy_3, & \text{for } x_3 < 0 \\ \int_{-h}^{0} T(\lambda_-,\lambda_+)e^{-i\lambda_- y_3}e^{i\lambda_+ x_3}\hat{\psi}(\xi,y_3)dy_3, & \text{for } x_3 > 0 \end{cases} \tag{A1.11}$$

where h is chosen so that the support of V is in the region $y_3 > -h$.

Because the exponentials on the right side are decaying, from (A1.11) we can draw the conclusion

$$|\hat{\phi}(\xi,x_3)| \le \frac{c}{(1+|\xi|^2)^{1/2}}\|\hat{\psi}(\xi,\cdot)\|_{L^1}, \tag{A1.12}$$

where L^1 refers to $L^1(-h,0)$. Similarly, if we differentiate ϕ, we obtain

$$|\widehat{D^\alpha\phi}(\xi,x_3)| \le c\|\hat{\psi}(\xi,\cdot)\|_{L^1} \tag{A1.13}$$

for any $|\alpha| \le 1$.

Next we relate each side of (A1.13) to a weighted norm in the x_3 variable. First, the L^1 norm on the right side can be bounded above by

$$\int (1+|x_3|^2)^{-s/2}(1+|x_3|^2)^{s/2}|\hat{\psi}(\xi,x_3)|dx_3 \le c\|\hat{\psi}(\xi,\cdot)\|_{L^{2,s}}. \tag{A1.14}$$

The left side of (A1.13), on the other hand, can be related to a weighted norm by

$$\|\widehat{D^\alpha\phi}(\xi,\cdot)\|^2_{L^{2,-s}} = \int (1+|x_3|^2)^{-s}|\widehat{D^\alpha\phi}(\xi,x_3)|^2dx_3 \le \|\widehat{D^\alpha\phi}(\xi,\cdot)\|^2_{L^\infty}\int(1+|x_3|^2)^{-s}dx_3. \tag{A1.15}$$

For $s > 1/2$, the rightmost integral of (A1.15) converges to a positive real number; thus we can rewrite (A1.15) as

$$\|\widehat{D^\alpha\phi}(\xi,\cdot)\|^2_{L^{2,-s}} \le c\|\widehat{D^\alpha\phi}(\xi,\cdot)\|^2_{L^\infty}. \tag{A1.16}$$

Using (A1.14) and (A1.16) in (A1.13), we obtain

$$\|\widehat{D^\alpha\phi}(\xi,\cdot)\|^2_{L^{2,-s}} \le c\|\hat{\psi}(\xi,\cdot)\|^2_{L^{2,s}}. \tag{A1.17}$$

Next we convert (A1.17), which involves only one-dimensional weighted norms, to a similar statement about three-dimensional weighted norms. We do this by integrating both sides with respect to x_1 and x_2, and using the Plancherel theorem to obtain

$$\int |D^\alpha\phi(x)|^2(1+|x_3|^2)^{-s}dx \le c\int|\psi(x)|^2(1+|x_3|^2)^s dx. \tag{A1.18}$$

QED

Because the medium in the lower half-space is dissipative, waves there must decay exponentially as they travel in the medium. We can see this as follows.

Lemma A1.2. If V has compact support, then in any finite-thickness slice of the lower half-plane $\{x : x_3^- < x_3 < x_3^+ < 0\}$, any solution u of (1.4) that is in $H^{1,-s}$ decays exponentially at infinity.

Proof. We consider a rectangular region along the x_1 axis outside the support of V, namely $\{x : x_1^- < x_1, 0 < x_2 < x_2^+, x_3^- < x_3 < x_3^+ < 0\}$. In this region, we write u in a Fourier series

$$u(x) = \sum_{l,p} \tilde{u}_{l,p}(x_1) e^{il\pi x_2/\Delta x_2} e^{ip\pi x_3/\Delta x_3}, \qquad (A1.20)$$

where we have written $\Delta x_2 = x_2^+ - x_2^-$ and $\Delta x_3 = x_3^+ - x_3^-$. Since u is not periodic in the x_2 or x_3 variables, this expression will not coincide with u outside the box, but this does not matter for the present purpose.

Because u satisfies the unperturbed wave equation (1.4) in the box, the Fourier coefficient $\tilde{u}_{l,p}$ satisfies the ordinary differential equation

$$(\partial_{x_1}^2 - (\frac{l\pi}{\Delta x_2})^2 - (\frac{p\pi}{\Delta x_3})^2 + k^2 n_-^2 + ikm_-)\tilde{u}_{l,p} = 0, \qquad (A1.21)$$

whose solutions are linear combinations of exponentials that either grow or decay for large x_1. Because u is in $H^{1,-s}$, the coefficients of the growing exponentials must be zero. The solution u can therefore be written

$$u(x) = \sum_{l,p} \tilde{u}_{l,p}(x_1^-) \exp\left(ix_1 \sqrt{k^2 n_-^2 + ikm_- - (\frac{l\pi}{\Delta x_2})^2 - (\frac{p\pi}{\Delta x_3})^2} \right) e^{il\pi x_2/\Delta x_2} e^{ip\pi x_3/\Delta x_3}. \qquad (A1.22)$$

To show that the right side decays exponentially in the x_1 variable, we note that the imaginary part of the square root is bounded below by $m_-/2$. A factor of $e^{-m_- x_1/4}$ can thus be pulled out of each term, and the remaining series converges.

Because (1.4) is isotropic outside the support of V, any direction can be chosen as the x_1 direction.

QED

Proposition A1.3. Suppose V is a bounded function of compact support, and assume m is strictly positive in the lower half-space. Then if E^0 is in $H^{1,-s}$, (A1.9) has a unique solution in $H^{1,-s}$ for $s > 1/2$. Similarly, (A1.10b) has a unique solution in the same space.

Proof. For (A1.10b), we check that the inhomogeneous term $G_0 V G_0$ is in $H^{1,-s}$ for $s > 1/2$. The Green's function G_0, being a fundamental solution of the Helmholtz equation, has no singularities worse than the $1/|x-y|$ singularity for x near y. This singularity, however, is square-integrable in three dimensions. The product VG_0 is therefore in $L^{2,s}$, so by Proposition A1.1, $G_0 V G_0$ is in $H^{1,-s}$.

By the Fredholm theorem, to show that (A1.9) and (A1.10b) each have unique solutions, we need to show that the homogeneous equation

$$\psi(x) = \int G_0(x,y)V(y)\psi(y)dy \tag{A1.23}$$

has only the trivial solution. A solution to (A1.23), however, corresponds to a solution of (1.4) with no sources and no incoming wave. To show that such a solution must be identically zero, we use an energy identity, which we obtain by multiplying (1.4) by the complex conjugate $\bar{E}$ and integrating over a cylindrical region Ω_ρ of radius ρ, whose top is a disk in the $x_3 = x_3^+$ plane and whose bottom is a disk in the plane $x_3 = -h$. Here h is chosen so that the support of the perturbation V is contained between the planes $x_3 = -h$ and $x_3 = 0$. After an application of the divergence theorem, we have

$$\int_{\Omega_\rho} (|\nabla E|^2 - k^2n^2|E|^2 - ikm|E|^2) = \int_{\partial\Omega_\rho} \bar{E}\frac{\partial E}{\partial \nu}. \tag{A1.24}$$

For E in (A1.24) we substitute a solution ψ of (A1.23), written in the form

$$\psi(x) = \frac{1}{(2\pi)^2}\int \hat{\psi}(\xi, x_3)e^{ix'\cdot\xi}d\xi. \tag{A1.25}$$

From (A1.6) we see that for $x_3 > 0$,

$$\hat{\psi}(\xi, x_3) = A(\xi)e^{i\lambda_+ x_3}$$

and for $x_3 < -h$,

$$\hat{\psi}(\xi, x_3) = B(\xi)e^{-i\lambda_- x_3}$$

for some coefficients A and B. We use these expansions in (A1.25), which is then substituted into the integrals over the top and bottom of the cylinder on the right side of (A1.24). We then let ρ go to infinity; the integrals over the vertical sides of the cylinder go to zero as ρ goes to infinity by Lemma A1.2. Finally, in the integrals over the top and bottom, we perform the x' integration. The result is

$$\int_{\Omega_\infty} (|\nabla\psi|^2 - k^2n^2|\psi|^2 - ikm|\psi|^2) = i\int (|A(\xi)|^2\lambda_+e^{-2\mathrm{Im}\lambda_+x_3^+} + |B(\xi)|^2\lambda_-e^{-2h\lambda_-})d\xi \tag{A1.26}$$

If ψ is nonzero and k is positive, the left side of (A1.26) has negative imaginary part, whereas the imaginary part of the right side is nonnegative. This shows that ψ is identically zero for k positive.

QED

If m were identically zero in some region in the lower half-space, the above argument does not rule out the possibility that ψ might be nonzero there. This could happen, for example, if k^2n^2 were equal to a constant that happened to be a Dirichlet eigenvalue for the region in which m is identically zero. In this case, there could exist a nonzero solution in that region with zero boundary values. This possibility can be ruled out by assuming smoothness of n^2 and m, so that the unique continuation principle of [RS4] holds.

Next we investigate the invertibility of the integral operators defined in (3.6). For this, we need to define the following subspaces of $L^{2,s}$ and H^s:

$$\tilde{L}_k^{2,s} = \{\sqrt{k^2 - |\cdot|^2} f : (1 + |\cdot|^2)^{(s+1)/2} f(\cdot) \in L^2\}$$

and

$$\tilde{H}_k^s = \{f : \hat{f} \in \tilde{L}_k^{2,s}\}.$$

Proposition A1.4. If m is strictly positive in the lower half-space, then the integral operators Γ and Γ_0 are both invertible operators from $H^{-1/2}(\mathbf{R}^2)$ to $\tilde{H}_k^{1/2}(\mathbf{R}^2)$.

Proof. The result for Γ_0 is clear from expression (A1.6) with $x_3 = y_3 = 0$.

To show that Γ is invertible, we write (following [N])

$$\Gamma = \Gamma_0 \left(I + \Gamma_0^{-1}(\Gamma - \Gamma_0)\right),$$

so that invertibility of Γ follows from invertibility of $I + \Gamma_0^{-1}(\Gamma - \Gamma_0)$. The difference $\Gamma - \Gamma_0$ is a compact operator on $H^{1/2}$ because it can be written in terms of the composition of the compact operator $G_0 V$ with G (see Proposition A1.1). Since $\Gamma_0^{-1}(\Gamma - \Gamma_0)$ is compact, invertibility of $I + \Gamma_0^{-1}(\Gamma - \Gamma_0)$ follows from its injectivity, which in turn follows from the injectivity of Γ.

To see that Γ is injective, we write $u(x) = \int_{y_3=0} G(x, y')f(y')dy'$, where we assume $u(x) = 0$ for $x_3 = 0$. The Lemma of section 3 shows that u satisfies (3.8) with zero boundary values; the argument of Proposition A2.2 shows that u must be identically zero in the lower half-space. We next multiply (A1.10a) by f, integrate with respect to y', and let y_3 and x_3 approach zero through negative values. We obtain

$$u(x') - \Gamma_0 f(x') = \int G_0((x', 0), z)V(z)u(z)dz,$$

which, since u is identically zero, reduces to $\Gamma_0 f = 0$. The injectivity of Γ_0, however, is clear from (A1.6).

QED

APPENDIX 2. CONSTRUCTION OF THE OUTGOING SOLUTION TO THE BOUNDARY VALUE PROBLEM

In this appendix, we will construct an outgoing solution to the boundary value problem with the help of the outgoing Green's function that is zero on the boundary $x_3 = 0$. This Green's function is then used to convert the boundary value problem to an integral equation, which will be shown to have a unique solution.

The outgoing Dirichlet Green's function. In the lower half-space $\mathbf{R}^3_-$, the Green's function $g(x, y) = g(x_1 - y_1, x_2 - y_2, x_3, y_3)$ satisfies an outgoing radiation condition and the boundary value problem

$$(\nabla^2 + k^2 n_-^2 + ikm_-)g(x, y) = -\delta(x - y), \qquad (A2.1)$$

$$g(x,y)|_{x_3=0} = 0. \tag{A2.2}$$

This Green's function can be constructed by two methods. The first method is that used in Appendix 1; in particular, equation (A2.1) can be Fourier transformed in the x_1 and x_2 variables, which yields

$$(\partial_{x_3}^2 + \lambda_-^2)\hat{g}(\xi, x_3, y_3) = -\delta(x_3 - y_3), \tag{A2.3}$$

where we have written $\lambda_-^2 = -|\xi|^2 + k^2 n_-^2 + ikm_-$. For $y_3 < x_3 < 0$, the general solution of (A2.3) is

$$\hat{g}^+(\xi, x_3, y_3) = A^+(\xi)e^{i\lambda_- x_3} + B^+(\xi)e^{-i\lambda_- x_3}; \tag{A2.4}$$

for $x_3 < y_3 < 0$, it is

$$\hat{g}^-(\xi, x_3, y_3) = A^-(\xi)e^{i\lambda_- x_3} + B^-(\xi)e^{-i\lambda_- x_3}. \tag{A2.5}$$

The condition that $\hat{g}$ be downgoing as $x_3 \to -\infty$ implies that $A^- = 0$; the boundary condition (A2.2) implies that $A^+ + B^+ = 0$; $\hat{g}$ is continuous at $x_3 = y_3$ but the derivative $\partial\hat{g}/\partial x_3$ jumps by 1. Solving for the As and Bs, we obtain

$$\hat{g}(\xi, x_3, y_3) = \frac{i}{2\lambda_-}\left(e^{i\lambda_-|x_3-y_3|} - e^{-i\lambda_-(x_3+y_3)}\right). \tag{A2.6}$$

Written in this form, it is clear that when λ_- has positive imaginary part, $\hat{g}(\xi, x_3, y_3)$ decays exponentially as $x_3 \to -\infty$. Moreover, for fixed $x_3 \neq y_3$, $\hat{g}$ decays exponentially as $|\xi| \to \infty$. This Fourier transformed Green's function can also be written as

$$\hat{g}(\xi, x_3, y_3) = \frac{-1}{\lambda_-}\begin{cases} e^{-i\lambda_- y_3}\sin\lambda_- x_3, & \text{for } y_3 \leq x_3 \leq 0; \\ e^{-i\lambda_- x_3}\sin\lambda_- y_3, & \text{for } x_3 \leq y_3 \leq 0. \end{cases} \tag{A2.7}$$

The Green's function itself is then

$$g(x,y) = \frac{1}{(2\pi)^2}\int e^{i(x'-y')\cdot\xi}\hat{g}(\xi, x_3, y_3)d\xi. \tag{A2.8}$$

This same Green's function can also be constructed by the method of images. For a point $y = (y_1, y_2, y_3)$ in the lower half-space, the corresponding image point is $\tilde{y} = (y_1, y_2, -y_3)$. Then we can write the Green's function as

$$g(x,y) = \frac{1}{4\pi}\left(\frac{e^{i(k^2n_-^2+ikm_-)^{1/2}|x-y|}}{|x-y|} - \frac{e^{i(k^2n_-^2+ikm_-)_-^{1/2}|x-\tilde{y}|}}{|x-\tilde{y}|}\right). \tag{A2.9}$$

It is clear from this expression that g decays exponentially at infinity.

To see that these two representations are the same, we recall that the free space Green's function can be Fourier transformed as

$$\frac{e^{i\gamma|x|}}{4\pi|x|} = \frac{1}{(2\pi)^3}\int \frac{e^{ix\cdot\zeta}}{|\zeta|^2 - |\gamma|^2}d\zeta. \tag{A2.10}$$

In this Fourier transform integral, we can do the ζ_3 integral first; it is

$$\frac{1}{2\pi}\int \frac{e^{ix_3\zeta_3}}{\zeta_3^2-(\gamma^2-|\xi|^2)}d\zeta_3, \tag{A2.11}$$

where we have written $\xi=(\zeta_1,\zeta_2)$. This one-dimensional integral can be done by contour integration; it is equal to

$$\frac{i\exp(i|x_3|\sqrt{\gamma^2-|\xi|^2})}{2\sqrt{\gamma^2-|\xi|^2}}. \tag{A2.12}$$

To relate (A2.12) to (A2.6), we let $\gamma=(k^2n_-^2+ikm_-)^{1/2}$; we then substitute $x_3-\tilde{y}_3$ for x_3 in (A2.12) and subtract the resulting expression from the one obtained by substituting x_3-y_3 for x_3.

Construction of the outgoing solution to the boundary value problem. We construct the outgoing solution to the boundary value problem as the solution to an integral equation. This integral equation is obtained by multiplying (2.4) by g and (A2.1) by u, subtracting the resulting equations, and applying Green's theorem. After using the boundary conditions at infinity and (2.5), we obtain

$$u(x)=-\int_{y_3<0} g(x,y)V(y)u(y)dy-\int_{y_3=0} f(y)\frac{\partial g(x,y)}{\partial y_3}dy, \tag{A2.13}$$

where we have written $V(y)=k^2(n^2(y)-n_-^2)+ik(m(y)-m_-)$. We can write this equation in more compact notation by writing the first term on the right-hand side in operator notation as gVu. This equation can be used to define u.

First, we show that a solution u of (A2.13) has the desired properties. It is clearly outgoing. To see that u satisfies the correct boundary condition, we evaluate (A2.13) at $x_3=0$. From (A2.2) or (A2.7) we see that the Green's function is zero when $x_3=0$. Thus the entire contribution to u comes from the second term on the right side of (A2.13). Again from (A2.7) and (A2.8) we see that the normal derivative of g on the surface $y_3=0$ is

$$\left.\frac{\partial g(x,y)}{\partial y_3}\right|_{x_3=0}=\frac{-1}{(2\pi)^2}\int e^{i(x'-y')\cdot\xi}e^{-i\lambda_- x_3}d\xi, \tag{A2.14}$$

which, as $x_3\to 0$, becomes a negative delta function supported at $x'=y'$.

QED

Next we show that equation (A2.13) has a unique solution in H^1.

Proposition A2.1. If V is a bounded function of compact support, the operator gV is compact in $H^1(\mathbf{R}^3_-)$.

Proof. We follow the ideas of [Ag]: first we show that multiplication by V is a compact operator mapping H^1 into L^2; then we show that the operator g is a bounded operator mapping L^2 into H^1. Hence the product operator gV is a compact operator on H^1.

To see that multiplication by V is a compact mapping from H^1 into L^2, we simply invoke the Sobolev imbedding theorem [Ad].

To show that the operator g maps L^2 into H^1, we begin by writing $\phi = g\psi$. The two-dimensional Fourier transform of this is $\hat{\phi} = \hat{g}\hat{\psi}$. From (A2.6) we see that $\hat{g}$ is bounded and decays for large $|\xi|$ like $1/|\xi|$; this shows immediately (with the help of the Plancherel theorem) that $\|\phi\|_{L^2(\mathbf{R}^3_-)} \le c\|\psi\|_{L^2(\mathbf{R}^3_-)}$ and $\|\partial\phi/\partial x_i\|_{L^2(\mathbf{R}^3_-)} \le c\|\psi\|_{L^2(\mathbf{R}^3_-)}$ for $i = 1, 2$.

To show the same thing for the x_3 derivative, we write $\hat{g} = i(2\lambda)^{-1}(h_1 - h_2)$, where $h_1(\xi, x_3, y_3) = \exp(i\lambda_-|x_3 - y_3|)$ and $h_2(\xi, x_3, y_3) = \exp(-i\lambda_-(x_3 + y_3))$. With this notation, we have $\hat{\phi} = \hat{\phi}_1 + \hat{\phi}_2$, where

$$\hat{\phi}_1(\xi, x_3) = \frac{i}{2\lambda_-}\int_{-\infty}^{0} h_1(\xi, x_3, y_3)\hat{\psi}(\xi, y_3)dy_3 \tag{A2.15}$$

and

$$\hat{\phi}_2(\xi, x_3) = \frac{i}{2\lambda_-}\int_{-\infty}^{0} h_2(\xi, x_3, y_3)\hat{\psi}(\xi, y_3)dy_3. \tag{A2.16}$$

Differentiation of ϕ_2 with respect to x_3 gives

$$\frac{\partial\hat{\phi}_2(\xi, x_3)}{\partial x_3} = \frac{i}{2}e^{-i\lambda_- x_3}\int_{-\infty}^{0} e^{-i\lambda_- y_3}\hat{\psi}(\xi, y_3)dy_3; \tag{A2.17}$$

since both exponentials in this expression are decaying, we have

$$\|\frac{\partial\hat{\phi}_2(\xi,\cdot)}{\partial x_3}\|^2_{L^2(\mathbf{R}_-)} \le c\|\hat{\psi}(\xi,\cdot)\|^2_{L^2(\mathbf{R}_-)}.$$

Using the Plancherel theorem and integrating over x_1 and x_2 then shows that

$$\|\partial\phi_2/\partial x_3\|_{L^2(\mathbf{R}^3_-)} \le c\|\psi\|_{L^2(\mathbf{R}^3_-)}.$$

Differentiation of ϕ_1 with respect to x_3 gives

$$\frac{\partial\hat{\phi}_1(\xi, x_3)}{\partial x_3} = \frac{i}{2}\int_{-\infty}^{0} \mathrm{sgn}(x_3 - y_3)e^{i\lambda_-|x_3-y_3|}\hat{\psi}(\xi, y_3)dy_3 \tag{A2.18}$$

We extend $\hat{\psi}$ to the whole real line by defining it to be zero for $y_3 > 0$. This allows us to extend the region of integration on the right side of (A2.18) to the whole real line, so that the right side becomes a convolution. We then Fourier transform in the x_3 variable, obtaining

$$\hat{\Phi}(\xi, \eta) = \frac{2i(\alpha - \eta)}{\beta^2 + (\eta - \alpha)^2}\hat{\Psi}(\xi, \eta), \tag{A2.19}$$

where $\hat{\Phi}$ and $\hat{\Psi}$ are the one-dimensional Fourier transforms of $\partial\hat{\phi}_1/\partial x_3$ and $\hat{\psi}$, respectively, and where we have have used $\alpha = \alpha(\xi)$ and $\beta = \beta(\xi)$ for the respective real and imaginary parts of λ_-. Taking L^2 norms of both sides of (A2.19), in the η variable, we see that

$\|\hat{\Phi}(\xi,\cdot)\|_{L^2(\mathbf{R}_-)} \le c\|\hat{\Psi}(\xi,\cdot)\|_{L^2(\mathbf{R}_-)}$. Using the Plancherel theorem and integrating over x_1 and x_2 then shows that

$$\|\frac{\partial \phi_1}{\partial x_3}\|_{L^2(\mathbf{R}^3_-)} \le c\|\psi\|_{L^2(\mathbf{R}^3_-)}. \tag{A2.20}$$

QED

Proposition A2.2. If m is strictly positive, equation (A2.13) has a unique solution in $H^1(\mathbf{R}^3_-)$.

Proof. The Fredholm alternative guarantees that (A2.13) has a unique solution provided that the corresponding homogeneous equation has only the zero solution. A solution of the homogeneous equation is also a solution u of (2.4) and (2.5) with $f = 0$. To show that such a u must be identically zero, we use an energy argument. The procedure for obtaining this energy identity is to multiply (2.4) by the complex conjugate $\bar{u}$ and integrate over C_h, a cylindrical region with radius $|x'| = h^2$ and extending from $x_3 = 0$ to $x_3 = -h$. After using Green's theorem, we obtain

$$\int_{C_h} (|\nabla u|^2 - q|u|^2) = \int_{\partial C_h} \bar{u}\frac{\partial u}{\partial \nu}, \tag{A2.21}$$

where ν is the outward unit normal to C_h.

The right side of (A2.21) has three parts, corresponding to the parts of the boundary of C_h. The integral over the circle on the plane $x_3 = 0$ contributes nothing because u is zero there. Similarly, the integral over the circle on the plane $x_3 = -h$ goes to zero in the limit $h \to \infty$ because of the radiation condition in the lower half-plane. The integral over the side of the cylinder also vanishes because of the large-ρ asymptotics of u. Thus the right side of (A2.21) vanishes as $h \to \infty$. Thus, in the limit, we have

$$\int_{x_3<0} (|\nabla u|^2 - q|u|^2)dx = 0. \tag{A2.22}$$

Both the real and imaginary parts of the left side of (A2.22) must be zero; since for k positive, q has positive imaginary part km, $|u|$ must be zero.

QED

As discussed in Appendix 1, the hypothesis that m be strictly positive can be replaced by smoothness assumptions on n^2 and m.

APPENDIX 3. PROPERTIES OF THE SCATTERING OPERATOR

In Appendix 1 we saw that an incident field in $H^{1,-s}(\mathbf{R}^3)$, $s > 1/2$, gives rise to a scattered field in the same space. Because these spaces are weighted only in the x_3 variable, the restriction of a function in such a space to any horizontal (i.e., fixed x_3) plane is in $H^{1/2}(\mathbf{R}^2)$ [Ad]. The Fourier transform of this space is

$$L^2_{1/2}(\mathbf{R}^2) = \{u(\xi) : (1 + |\xi|^2)^{1/4} u \in L^2(\mathbf{R}^2)\}.$$

Thus the operator $\hat{S}$ maps $L^2_{1/2}(\mathbf{R}^2)$ to itself.

With this information, equation (3.3) can be interpreted as follows. Since both $\hat{\Lambda}$ and multiplication by $i\lambda_+$ are maps from $L^2_{1/2}$ to $L^2_{-1/2}$, each side of equation (3.3) is a map on $L^2_{1/2}$ followed by a map from $L^2_{1/2}$ to $L^2_{-1/2}$.

Theorem. The projection $P\hat{S}P$ of the scattering operator onto $\{f : (k^2 - |\xi|^2)^{1/4} f \in L^2(|\xi| < k)$ has norm less than one.

Proof. We use the energy identity (A1.24) applied to a region that in the limit becomes the entire lower half-space. In the right side we use expression (A1.25), where, on the surface $x_3 = 0$,

$$\hat{\psi}(\xi, 0) = (\hat{S}f)(\xi) + f(\xi). \tag{A3.1}$$

The energy identity thus becomes

$$\begin{aligned}(2\pi)^2 \int_{x_3<0} (|\nabla\psi|^2 - k^2 n^2 |\psi|^2 - ikm|\psi|^2) = \\ i \int_{|\xi|<k} \left[(|\hat{S}f(\xi)|^2 - |f(\xi)|^2) + 2iIm(\bar{f}\hat{S}f) \right] \sqrt{k^2 - |\xi|^2} d\xi - \\ \int_{|\xi|>k} \left[(|\hat{S}f(\xi)|^2 - |f(\xi)|^2) + 2iIm(\bar{f}\hat{S}f) \right] \sqrt{|\xi|^2 - k^2} d\xi.\end{aligned} \tag{A3.2}$$

Here we assume that f is zero for $|\xi| > k$. Thus the right side of (A3.2) reduces to

$$i \int_{|\xi|<k} \left[(|\hat{S}f(\xi)|^2 - |f(\xi)|^2) + 2iIm(\bar{f}\hat{S}f) \right] \sqrt{k^2 - |\xi|^2} d\xi - \int_{|\xi|>k} |\hat{S}f(\xi)|^2 \sqrt{|\xi|^2 - k^2} d\xi. \tag{A3.3}$$

The imaginary part of the expression is $\int_{|\xi|<k} (|\hat{S}f(\xi)|^2 - |f(\xi)|^2) d\xi$, which must be negative since the left side of (A3.2) is negative.

QED

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