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Inverse Problems: Visibility and Invisibility

Gunther Uhlmann

Abstract

This survey article expands on the lectures given at Biarritz in June, 2012, on "Inverse Problems: Visibility and Invisibility". The first inverse problem we consider is whether one can determine the electrical conductivity of a medium by making voltage and current measurements at the boundary. This is called electrical impedance tomography (EIT) and also Calderón's problem since the famous analyst proposed it in the mathematical literature [38]. The second is on travel time tomography. The question is whether one can determine the anisotropic index of refraction of a medium by measuring the travel times of waves going through the medium. This can be recast as a geometry problem, the boundary rigidity problem. Can we determine a Riemannian metric of a compact Riemannian manifold with boundary by measuring the distance function between boundary points? These two inverse problems concern *visibility*, that is whether we can determine the internal properties of a medium by making measurements at the boundary. The last topic of this paper considers the opposite issue: *invisibility*: Can one make objects invisible to different types of waves, including light?

1. Calderón's Problem

1.1. Introduction

In 1980 A. P. Calderón published a short paper entitled "On an inverse boundary value problem" [38]. This pioneer contribution motivated many developments in inverse problems, in particular in the construction of "complex geometrical optics" solutions of partial differential equations to solve several inverse problems. We survey some these developments in this paper. In his talk at the ICM in Berlin in 1998 the author proposed 7 open problems [202] on this subject. This section is to a large extent a report on the progress made in solving these problems.

The problem that Calderón considered was whether one can determine the electrical conductivity of a medium by making voltage and current measurements at the boundary of the medium. This inverse method is known as *Electrical Impedance Tomography* (EIT). Calderón was motivated by oil prospection. In the 40's he worked as an engineer

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for Yacimientos Petroliferos Fiscales (YPF), the state oil company of Argentina, and he thought about this problem then although he did not publish his results until many years later. For use of electrical methods in geophysical prospection see [213]. Parenthetically Calderón said in his speech accepting the "Doctor Honoris Causa" of the Universidad Autónoma de Madrid that his work at YPF had been very interesting but he was not well treated there; he would have stayed at YPF otherwise [39]. It goes without saying that the bad treatment of Calderón by YPF was very fortunate for Mathematics! EIT also arises in medical imaging given that human organs and tissues have quite different conductivities [104]. One exciting potential application is the early diagnosis of breast cancer [215]. The conductivity of a malignant breast tumor is typically 0.2 mho which is significantly higher than normal tissue which has been typically measured at 0.03 mho. Another application is to monitor pulmonary functions [95]. See the book [80] and the issue of Physiological Measurement [81] for other medical imaging applications of EIT. This inverse method has also been used to detect leaks from buried pipes [103]. We now describe more precisely the mathematical problem. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with smooth boundary (many of the results we will describe are valid for domains with Lipschitz boundaries). The electrical conductivity of Ω is represented by a bounded and positive function $\gamma(x)$. In the absence of sinks or sources of current the equation for the potential is given by

$$\nabla \cdot (\gamma \nabla u) = 0 \text{ in } \Omega \tag{1.1}$$

since, by Ohm's law, $\gamma \nabla u$ represents the current flux. Given a potential $f \in H^{\frac{1}{2}}(\partial\Omega)$ on the boundary the induced potential $u \in H^1(\Omega)$ solves the Dirichlet problem

$$\begin{aligned} \nabla \cdot (\gamma \nabla u) &= 0 \text{ in } \Omega, \\ u\Big|_{\partial\Omega} &= f. \end{aligned} \tag{1.2}$$

The Dirichlet to Neumann map, or voltage to current map, is given by

$$\Lambda_\gamma(f) = \left(\gamma \frac{\partial u}{\partial \nu}\right)\Big|_{\partial\Omega} \tag{1.3}$$

where ν denotes the unit outer normal to $\partial\Omega$. The inverse problem is to determine γ knowing Λ_γ. It is difficult to find a systematic way of prescribing voltage measurements at the boundary to be able to find the conductivity. Calderón took instead a different route. Using the divergence theorem we have

$$Q_\gamma(f) := \int_\Omega \gamma |\nabla u|^2 dx = \int_{\partial\Omega} \Lambda_\gamma(f) f \, dS \tag{1.4}$$

where dS denotes surface measure and u is the solution of (1.2). In other words $Q_\gamma(f)$ is the quadratic form associated to the linear map $\Lambda_\gamma(f)$, and to know $\Lambda_\gamma(f)$ or $Q_\gamma(f)$ for all $f \in H^{\frac{1}{2}}(\partial\Omega)$ is equivalent. $Q_\gamma(f)$ measures the energy needed to maintain the potential f at the boundary. Calderón's point of view is that if one looks at $Q_\gamma(f)$ the problem is changed to finding enough solutions $u \in H^1(\Omega)$ of the equation (1.1) in order to find γ in the interior. These are the complex geometrical optics (CGO) solutions considered in section 1.3. A short summary of the contents of this section is as follows. In section 1.2 we describe results about uniqueness, stability and reconstruction, for the boundary values of a conductivity and its normal derivative. In section 1.3 we describe the construction by Sylvester and Uhlmann [193], [171] of CGO solutions for the Schrödinger equation associated to a bounded potential. These solutions behave like Calderón's complex exponential solutions for large complex frequencies. In section 1.4 we use these solutions to prove, in

dimension $n \geq 3$, a global identifiability result, stability estimates and a reconstruction method for the inverse problem. We also describe an extension of the identifiability result to non-linear conductivities [185] and give other applications of CGO solutions. In section 1.5 we consider the partial data problem, that is the case when the DN map is measured on a part of the boundary. We describe the results of [108] for the non-linear problem in dimension three or larger. This uses a larger class of CGO solutions, having a non-linear phase function that are constructed using Carleman estimates. We also review the article [51] for the linearized problem with partial data. In section 1.6 we consider the two dimensional case. In particular we describe briefly the recent work of Astala and Päivärinta proving uniqueness for bounded measurable coefficients, and the work of Bukhgeim proving uniqueness for a potential from Cauchy data associated to the Schrödinger equation. Finally we consider the work of Imanuvilov, Uhlmann and Yamamoto on the partial data problem in two dimensions [88], [89]. These sections deal with the case of *isotropic* conductivities. In section 1.7 we consider the case of *anisotropic* conductivities, i.e. the conductivity depends also on direction. In two dimensions that there has been substantial progress in the understanding of anisotropic problems since one can usually reduce the problem to the isotropic case by using isothermal coordinates. In dimension three the problem as pointed out in [125] is of geometric nature. We review the results of [123], [53].

1.2. Boundary Determination

Kohn and Vogelius proved the following identifiability result at the boundary [114].

Theorem 1.1. *Let $\gamma_i \in C^\infty(\overline{\Omega})$ be strictly positive. Assume $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$. Then*

$$\partial^\alpha \gamma_1 \Big|_{\partial\Omega} = \partial^\alpha \gamma_2 \Big|_{\partial\Omega}, \qquad \forall |\alpha|.$$

This settled the identifiability question for the non-linear problem in the real-analytic category. They extended the identifiability result to piecewise real-analytic conductivities in [115].

Sketch of proof of Theorem 1.1. We outline an alternative proof to the one given by Kohn and Vogelius of 1.1. In the case $\gamma \in C^\infty(\overline{\Omega})$ we know, by another result of Calderón [40], that Λ_γ is a classical pseudodifferential operator of order 1. Let (x', x^n) be coordinates near a point $x_0 \in \partial\Omega$ so that the boundary is given by $x^n = 0$. The function $\lambda_\gamma(x', \xi')$ denotes the full symbol of Λ_γ in these coordinates. It was proved in [194] that

$$\lambda_\gamma(x', \xi') = \gamma(x', 0)|\xi'| + a_0(x', \xi') + r(x', \xi') \tag{1.5}$$

where $a_0(x', \xi')$ is homogeneous of degree 0 in ξ' and is determined by the normal derivative of γ at the boundary and tangential derivatives of γ at the boundary. The term $r(x', \xi')$ is a classical symbol of order -1. Then $\gamma\Big|_{\partial\Omega}$ is determined by the principal symbol of Λ_γ and $\frac{\partial\gamma}{\partial x^n}\Big|_{\partial\Omega}$ by the principal symbol and the term homogeneous of degree 0 in the expansion of the full symbol of Λ_γ. More generally the higher order normal derivatives of the conductivity at the boundary can be determined recursively. In [125] one can find a general approach to the calculation of the full symbol of the Dirichlet to Neumann map that applies to more general situations. We note that this gives also a reconstruction procedure. We first can reconstruct γ at the boundary since $\gamma\Big|_{\partial\Omega}|\xi'|$ is the principal symbol

of Λ_γ (see (1.5)). In other words in coordinates (x', x^n) so that $\partial\Omega$ is locally given by $x^n = 0$ we have

$$\gamma(x', 0)a(x') = \lim_{s\to\infty} e^{-is<x',\omega'>}\frac{1}{s}\Lambda_\gamma(e^{is<x',\omega'>}a(x'))$$

with $\omega' \in \mathbb{R}^{n-1}$ and $|\omega'| = 1$ and a a smooth and compactly supported function. In a similar fashion, using (1.5), one can find $\frac{\partial\gamma}{\partial\nu}\Big|_{\partial\Omega}$ by computing the principal symbol of $(\Lambda_\gamma - \gamma\big|_{\partial\Omega}\Lambda_1)$ where Λ_1 denotes the Dirichlet to Neumann map associated to the conductivity 1. The other terms can be reconstructed recursively in a similar fashion. We also observe, by taking an appropriate cut-off function a above, that this procedure is *local*, that is we only need to know the DN map in an open set of the boundary to determine the Taylor series of the conductivity in that open set. This method also leads to stability estimates at the boundary [194].

Theorem 1.2. *Suppose that γ_1 and γ_2 are C^∞ functions on $\overline{\Omega} \subseteq \mathbb{R}^n$ satisfying*

i) $0 < \frac{1}{E} \leq \gamma_i \leq E$

ii) $\|\gamma_i\|_{C^2(\overline{\Omega})} \leq E$

Given any $0 < \sigma < \frac{1}{n+1}$, there exists $C = C(\Omega, E, n, \sigma)$ such that

$$\|\gamma_1 - \gamma_2\|_{L^\infty(\partial\Omega)} \leq C\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{\frac{1}{2},\frac{-1}{2}} \tag{1.6}$$

and

$$\left\|\frac{\partial\gamma_1}{\partial\nu} - \frac{\partial\gamma_2}{\partial\nu}\right\|_{L^\infty(\partial\Omega)} \leq C\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|^\sigma_{\frac{1}{2},\frac{-1}{2}}. \tag{1.7}$$

This result implies that we don't need the conductivity to be smooth to determine the conductivity and its normal derivative at the boundary. In the case γ is continuous on $\overline{\Omega}$ we can determine γ at the boundary by using the stability estimate (1.6) and an approximation argument. In the case that $\gamma \in C^1(\overline{\Omega})$ we can determine, knowing the DN map, γ and its normal derivative at the boundary using the estimate (1.7) above and an approximation argument. For other results and approaches to boundary determination of the conductivity see [5], [32], [136], [142]. In one way or another the boundary determination involves testing the DN map against highly oscillatory functions at the boundary.

1.3. Complex geometrical optics solutions with a linear phase

Motivated by Calderón exponential solutions used in [38] in the study of the linearized problem at a constant conductivity, Sylvester and Uhlmann [171, 193] constructed in dimension $n \geq 2$ complex geometrical optics (CGO) solutions of the conductivity equation for C^2 conductivities that behave like Calderón exponential solutions for large frequencies. This can be reduced to constructing solutions in the whole space (by extending $\gamma = 1$ outside a large ball containing Ω) for the Schrödinger equation with potential. We describe this more precisely below. Let $\gamma \in C^2(\mathbb{R}^n)$, γ strictly positive in $\mathbb{R}^n$ and $\gamma = 1$ for $|x| \geq R$, $R > 0$. Let $L_\gamma u = \nabla \cdot \gamma\nabla u$. Then we have

$$\gamma^{-\frac{1}{2}}L_\gamma(\gamma^{-\frac{1}{2}}) = \Delta - q \tag{1.8}$$

where

$$q = \frac{\Delta\sqrt{\gamma}}{\sqrt{\gamma}}. \tag{1.9}$$

Therefore, to construct solutions of $L_\gamma u = 0$ in $\mathbb{R}^n$ it is enough to construct solutions of the Schrödinger equation $(\Delta - q)u = 0$ with q of the form (1.9). The next result proven in [171, 193] states the existence of complex geometrical optics solutions for the Schrödinger equation associated to any bounded and compactly supported potential.

Theorem 1.3. *Let $q \in L^\infty(\mathbb{R}^n)$, $n \geq 2$, with $q(x) = 0$ for $|x| \geq R > 0$. Let $-1 < \delta < 0$. There exists $\epsilon(\delta)$ and such that for every $\rho \in \mathbb{C}^n$ satisfying*

$$\rho \cdot \rho = 0$$

and

$$\frac{\|(1+|x|^2)^{1/2} q\|_{L^\infty(\mathbb{R}^n)} + 1}{|\rho|} \leq \epsilon$$

there exists a unique solution to

$$(\Delta - q)u = 0$$

of the form

$$u = e^{x\cdot\rho}(1 + \psi_q(x,\rho)) \tag{1.10}$$

with $\psi_q(\cdot,\rho) \in L^2_\delta(\mathbb{R}^n)$. Moreover $\psi_q(\cdot,\rho) \in H^2_\delta(\mathbb{R}^n)$ and for $0 \leq s \leq 2$ there exists $C = C(n,s,\delta) > 0$ such that

$$\|\psi_q(\cdot,\rho)\|_{H^s_\delta} \leq \frac{C}{|\rho|^{1-s}}. \tag{1.11}$$

Here

$$L^2_\delta(\mathbb{R}^n) = \{f; \int (1+|x|^2)^\delta |f(x)|^2 dx < \infty\}$$

with the norm given by $\|f\|^2_{L^2_\delta} = \int (1+|x|^2)^\delta |f(x)|^2 dx$ and $H^m_\delta(\mathbb{R}^n)$ denotes the corresponding Sobolev space. Note that for large $|\rho|$ these solutions behave like Calderón's exponential solutions $e^{x\cdot\rho}$. The equation for ψ_q is given by

$$(\Delta + 2\rho\cdot\nabla)\psi_q = q(1+\psi_q). \tag{1.12}$$

The equation (1.12) is solved by constructing an inverse for $(\Delta + 2\rho\cdot\nabla)$ and solving the integral equation

$$\psi_q = (\Delta + 2\rho\cdot\nabla)^{-1}(q(1+\psi_q)). \tag{1.13}$$

Lemma 1.4. *Let $-1 < \delta < 0$, $\quad 0 \leq s \leq 1$. Let $\rho \in \mathbb{C}^n - 0$, $\rho\cdot\rho = 0$. Let $f \in L^2_{\delta+1}(\mathbb{R}^n)$. Then there exists a unique solution $u_\rho \in L^2_\delta(\mathbb{R}^n)$ of the equation*

$$\Delta_\rho u_\rho := (\Delta + 2\rho\cdot\nabla)u_\rho = f. \tag{1.14}$$

Moreover $u_\rho \in H^2_\delta(\mathbb{R}^n)$ and

$$\|u_\rho\|_{H^s_\delta(\mathbb{R}^n)} \leq \frac{C_{s,\delta}\|f\|_{L^2_{\delta+1}}}{|\rho|^{1-s}}$$

for $0 \leq s \leq 2$ and for some constant $C_{s,\delta} > 0$.

The integral equation (1.12) can then be solved in $L^2_\delta(\mathbb{R}^n)$ for large $|\rho|$ since

$$(I - (\Delta + 2\rho\cdot\nabla)^{-1}q)\psi_q = (\Delta + 2\rho\cdot\nabla)^{-1}q$$

and $\|(\Delta + 2\rho\cdot\nabla)^{-1}q\|_{L^2_\delta \to L^2_\delta} \leq \frac{C}{|\rho|}$ for some $C > 0$ where $\|\cdot\|_{L^2_\delta \to L^2_\delta}$ denotes the operator norm between $L^2_\delta(\mathbb{R}^n)$ and $L^2_\delta(\mathbb{R}^n)$. We will not give details of the proof of Lemma 1.4 here. We refer to the papers [171, 193] . We note that there has been other approaches to construct CGO solutions for the Schrödinger equation [98], [74]. These constructions don't give uniqueness of the CGO solutions that are used in the reconstruction method

of the conductivity from the DN map (see section 1.4.5). If 0 is not a Dirichlet eigenvalue for the Schrödinger equation we can also define the DN map

$$\Lambda_q(f) = \frac{\partial u}{\partial \nu}|_{\partial\Omega}$$

where u solves

$$(\Delta - q)u = 0; \quad u|_{\partial\Omega} = f.$$

More generally we can define the set of Cauchy data for the Schrödinger equation. Let $q \in L^\infty(\Omega)$. We define the Cauchy data as the set

$$\mathcal{C}_q = \left\{ \left(u\Big|_{\partial\Omega}, \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega} \right) \right\}, \tag{1.15}$$

where $u \in H^1(\Omega)$ is a solution of

$$(\Delta - q)u = 0 \text{ in } \Omega. \tag{1.16}$$

We have $\mathcal{C}_q \subseteq H^{\frac{1}{2}}(\partial\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$. If 0 is not a Dirichlet eigenvalue of $\Delta - q$, then in fact $\mathcal{C}_q$ is a graph, namely

$$\mathcal{C}_q = \{(f, \Lambda_q(f)) \in H^{\frac{1}{2}}(\partial\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)\}.$$

Complex geometrical optics for first order equations and systems under different regularity assumptions of the coefficients have been constructed in [143], [145], [200], [164], [163]. For the case of the magnetic Schrödinger operator unique identifiability of the magnetic field and the electrical potential was shown in [105] assuming that both the electrical potential and magnetic potential are both just bounded. We refer to the article a more up to date developments on this topic and the references given there.

1.4. The Calderón problem in dimension $n \geq 3$

In this section we summarize some of the basic theoretical results for Calderón's problem in dimension three or higher.

1.4.1. Uniqueness

The identifiability question was resolved in [171] for smooth enough conductivities. The result is

Theorem 1.5. *Let $\gamma_i \in C^2(\overline{\Omega})$, γ_i strictly positive, $i = 1, 2$. If $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$ then $\gamma_1 = \gamma_2$ in $\overline{\Omega}$.*

In dimension $n \geq 3$ this result is a consequence of a more general result. Let $q \in L^\infty(\Omega)$.

Theorem 1.6. *Let $q_i \in L^\infty(\Omega)$, $i = 1, 2$. Assume $\mathcal{C}_{q_1} = \mathcal{C}_{q_2}$, then $q_1 = q_2$.*

We now show that Theorem 1.6 implies Theorem 1.5. Using (1.8) we have

$$\mathcal{C}_{q_i} = \left\{ \left(f, \left(\frac{1}{2}\gamma_i^{-1}\Big|_{\partial\Omega} \frac{\partial \gamma_i}{\partial \nu}\Big|_{\partial\Omega} \right) f + \gamma_i^{-\frac{1}{2}}\Big|_{\partial\Omega} \Lambda_{\gamma_i}\left(\gamma^{-\frac{1}{2}}\Big|_{\partial\Omega} f \right) \right), \quad f \in H^{\frac{1}{2}}(\partial\Omega) \right\}.$$

Then we conclude $\mathcal{C}_{q_1} = \mathcal{C}_{q_2}$ using the the boundary identifiability result of Kohn and Vogelius [114] and its extension [194].

Proof of Theorem 1.6. Let $u_i \in H^1(\Omega)$ be a solution of

$$(\Delta - q_i)u_i = 0 \text{ in } \Omega, \qquad i = 1, 2.$$

Then using the divergence theorem we have

$$\int_\Omega (q_1 - q_2)u_1 u_2 dx = \int_{\partial\Omega} \left(\frac{\partial u_1}{\partial \nu} u_2 - u_1 \frac{\partial u_2}{\partial \nu} \right) dS. \tag{1.17}$$

Now it is easy to prove that if $\mathcal{C}_{q_1} = \mathcal{C}_{q_2}$ then the LHS of (1.17) is zero

$$\int_\Omega (q_1 - q_2)u_1 u_2 dx = 0. \tag{1.18}$$

Now we extend $q_i = 0$ in Ω^c. We take solutions of $(\Delta - q_i)u_i = 0$ in $\mathbb{R}^n$ of the form

$$u_i = e^{x\cdot\rho_i}(1 + \psi_{q_i}(x, \rho_i)), \qquad i = 1, 2 \tag{1.19}$$

with $|\rho_i|$ large, $i = 1, 2$, with

$$\begin{aligned} \rho_1 &= \frac{\eta}{2} + i\left(\frac{k+l}{2}\right) \\ \rho_2 &= -\frac{\eta}{2} + i\left(\frac{k-l}{2}\right) \end{aligned} \tag{1.20}$$

and $\eta, k, l \in \mathbb{R}^n$ such that

$$\begin{aligned} \eta \cdot k &= k \cdot l = \eta \cdot l = 0 \\ |\eta|^2 &= |k|^2 + |l|^2. \end{aligned} \tag{1.21}$$

Condition (1.21) guarantees that $\rho_i \cdot \rho_i = 0$, $i = 1, 2$. Substituting (1.19) into (1.18) we conclude

$$(\widehat{q_1 - q_2})(-k) = -\int_\Omega e^{ix\cdot k}(q_1 - q_2)(\psi_{q_1} + \psi_{q_2} + \psi_{q_1}\psi_{q_2})dx. \tag{1.22}$$

Now $\|\psi_{q_i}\|_{L^2(\Omega)} \leq \frac{C}{|\rho_i|}$. Therefore by taking $|l| \to \infty$ we obtain

$$\chi_\Omega \widehat{(q_1 - q_2)}(k) = 0 \qquad \forall\, k \in \mathbb{R}^n$$

concluding the proof. Theorem 1.5 has been extended to conductivities having 3/2 derivatives in some sense in [153], [33]. Uniqueness for conormal conductivies in $C^{1+\epsilon}$ was shown in [64]. Recently Haberman and Tataru in a very nice article [76] have extended the uniqueness result to C^1 conductivities or small in the $W^{1,\infty}$ norm. It is an open problem whether uniqueness holds in dimension $n \geq 3$ for Lipschitz or less regular conductivities. Theorem 1.6 was extended to potentials in $L^{n/2}$ and small potentials in the Fefferman-Phong class in [42]. For conormal potentials with strong singularities so that the potential is not in $L^{n/2}$, for instance almost a delta function of an hypersurface, uniqueness was shown in [64].

Similar problems for higher order operators were considered in [98] and [119].

1.4.2. Non-linear conductivities

We now give an extension of this result to conductivities that depend on the voltage. Let $\gamma(x, t)$ be a function with domain $\overline{\Omega} \times \mathbb{R}$. Let α be such that $0 < \alpha < 1$. We assume

$$\gamma \in C^{1,\alpha}(\overline{\Omega} \times [-T, T]), \quad \forall\, T, \tag{1.23}$$

$$\gamma(x, t) > 0, \ \forall\, (x, t) \in \overline{\Omega} \times \mathbb{R}. \tag{1.24}$$

Given $f \in C^{2,\alpha}(\partial\Omega)$, there exists a unique solution of the Dirichlet problem (see [61])

$$\begin{aligned} \nabla \cdot (\gamma(x,u)\nabla u) &= 0 \text{ in } \Omega, \\ u\Big|_{\partial\Omega} &= f. \end{aligned} \tag{1.25}$$

Then the Dirichlet to Neumann map is defined by

$$\Lambda_\gamma(f) = \gamma(x,f)\Big|_{\partial\Omega} \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega} \tag{1.26}$$

where u is a solution to (1.25). Sun [185] proved the following result.

Theorem 1.7. *Let $n \geq 3$. Assume $\gamma_i \in C^{1,1}(\overline{\Omega} \times [-T,T])$ $\forall$ $T > 0$, $i = 1,2$, and $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$. Then $\gamma_1(x,t) = \gamma_2(x,t)$ on $\overline{\Omega} \times \mathbb{R}$.*

The main idea is to linearize the Dirichlet to Neumann map at constant boundary data equal to the parameter t (then the solution of (1.25) is equal to t). Isakov [97] was the first to use a linearization technique to study an inverse parabolic problems associated to non-linear equations. The case of the Dirichlet to Neumann map associated to the Schrödinger equation with a non-linear potential was considered in [86] under some assumptions on the potential. We note that, in contrast to the linear case, one cannot reduce the study of the inverse problem of the conductivity equation (1.25) to the Schrödinger equation with a non-linear potential. The main technical lemma in the proof of Theorem 1.7 is

Lemma 1.8. *Let $\gamma(x,t)$ be as in (1.23) and (1.24). Let $1 < p < \infty$, $0 < \alpha < 1$. Let us define*

$$\gamma^t(x) = \gamma(x,t). \tag{1.27}$$

Then for any $f \in C^{2,\alpha}(\partial\Omega)$, $t \in \mathbb{R}$

$$\lim_{s\to 0} \|\frac{1}{s}\Lambda_\gamma(t+sf) - \Lambda_{\gamma^t}(f)\|_{W^{1-\frac{1}{p},p}(\partial\Omega)} = 0. \tag{1.28}$$

The proof of Theorem 1.7 follows immediately from the lemma. Namely (1.28) and the hypotheses $\Lambda_{\gamma_1} = \Lambda_{\gamma_2} \Rightarrow \Lambda_{\gamma_1^t} = \Lambda_{\gamma_2^t}$ for all $t \in \mathbb{R}$. Then using the linear result, Theorem 1.5, we conclude that $\gamma_1^t = \gamma_2^t$ proving the theorem. We remark that the reduction from the non-linear problem to the linear is also valid in the two dimensional case [99]. Using the result of Astala and Päivärinta [11], which is reviewed in section 1.6, one can extend Theorem 1.5 to $L^\infty(\Omega)$ conductivities in the two dimensional case. There are several open questions when the conductivity also depends on ∇u, see [187] for a survey of results and open problems in this direction.

1.4.3. Other applications

We give a short list of other applications to inverse problems using the CGO solutions described above for the Schrödinger equation.

- Quantum Scattering. In dimension $n \geq 3$ and in the case of a compactly supported electric potential, uniqueness for the fixed energy scattering problem was proven in [136], [146], [160]. In the earlier paper [147] this was done for small potentials. For compactly supported potentials, knowledge of the scattering amplitude at fixed energy is equivalent to knowing the Dirichlet-to-Neumann map for the Schrödinger equation measured on the boundary of a large ball containing the support of the potential (see [203], [205] for an account). Then Theorem 1.6 implies the result.

Melrose [129] suggested a related proof that uses the density of products of scattering solutions. Applications of CGO solutions to the 3-body problem were given in [207].

- Optics. The DN map associated to the Helmholtz equation $-\Delta + k^2 n(x)$ with an isotropic index of refraction n determines uniquely a bounded index of refraction in dimension $n \geq 3$.

- Optical tomography in the diffusion approximation. In this case we have $\nabla \cdot \mathcal{D}(x) \nabla u - \sigma_a(x) u - i\omega u = 0$ in Ω where u represents the density of photons, $\mathcal{D}$ the diffusion coefficient, and σ_a the optical absorption. Using the result of [171] one can show in dimension three or higher that if $\omega \neq 0$ one can recover both $\mathcal{D}$ and σ_a from the corresponding DN map. If $\omega = 0$ then one can recover one of the two parameters.

- Electromagnetics. For Maxwell's equations the analog of the DN map is the admittance map that maps the tangential component of the electric field to the tangential component of the magnetic field [176]. The admittance map for isotropic Maxwell's equations determines uniquely the isotropic electric permittivity, magnetic permeability and conductivity [148]. This system can in fact be reduced to the Schrödinger equation $\Delta - Q$ with Q an 8×8 system and Δ the Laplacian times the identity matrix [149].

- Elasticity. For the isotropic elasticity system the problem of determining the Lamé parameters from the analog of the DN map in this case which sends the displacement at the boundary to the traction of the boundary has been solved if the Lamé parameter μ is close to a constant [56], [143], [144].

- Determination of Inclusions and Obstacles. The CGO solutions constructed in Theorem 1.3 have been applied to determine inclusions for Helmholtz equations in [85] and Maxwell's equations in [214] using the enclosure method [85].

- Coupled-Physics Inverse Problems. In these problems one tries to combine the best features of two type of waves, one with high contrast and the other with high resolution to find the electromagnetic, optical or elastic properties of a medium. This combination is done through some physical principle. Examples are Photoacoustic and Thermoacoustic Tomography, Ultrasound Modulated Optical Tomography, Ultrasound Modulated Electrical Impedance Tomography, Magnetic Resonance Elastography and Transient Elastography among others. See [13], [177] for a review of some of these inverse methods. CGO solutions have been used in these hybrid methods in [15], [16], [17], [43], [111].

1.4.4. Stability

The arguments used in the proofs of Theorems 1.5, 1.6, 1.1 can be pushed further to prove the following stability estimates proven in [4].

Theorem 1.9. *Let $n \geq 3$. Suppose that $s > \frac{n}{2}$ and that γ_1 and γ_2 are C^∞ conductivities on $\overline{\Omega} \subseteq \mathbb{R}^n$ satisfying*

i) $0 < \frac{1}{E} \leq \gamma_j \leq E, j = 1, 2.$

ii) $\|\gamma_j\|_{H^{s+2}(\Omega)} \leq E, j = 1, 2.$

Then there exist $C = C(\Omega, E, n, s)$ *and* $0 < \sigma < 1$ *(*$\sigma = \sigma(n, s)$*) such that*

$$\|\gamma_1 - \gamma_2\|_{L^\infty(\Omega)} \leq C\Big(|\log \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{\frac{1}{2},\frac{-1}{2}}|^{-\sigma} + \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{\frac{1}{2},\frac{-1}{2}}\Big) \tag{1.29}$$

where $\| \quad \|_{\frac{1}{2},\frac{-1}{2}}$ *denotes the operator norm as operators from* $H^{\frac{1}{2}}(\partial\Omega)$ *to* $H^{-\frac{1}{2}}(\partial\Omega)$*.*

Notice that this logarithmic type stability estimates indicates that the problem is severely ill-posed. Mandache [128] has shown that this estimate is optimal up to the value of the exponent. There is the question of whether under some additional a-priori condition one can improve this logarithmic type stability estimate. Alessandrini and Vessella [7] have shown that this is indeed the case and one has a Lipschitz type stability estimate if the conductivity is piecewise constant with jumps on a finite number of domains. Rondi [161] has subsequently shown that the constant in the estimate grows exponentially with the number of domains. It is conjectured, and this is supported by numerical experiments, that the stability estimate should be "better" near the boundary and gets increasingly worse as one penetrated deeper into the domain (Theorem 1.2 shows that at the boundary we have Lipschitz type stability estimate.) This type of depth dependence stability estimate has been proved in [139] for the case of some electrical inclusions. For a recent review of stability issues in EIT see [6]. Theorem 1.9 is a consequence of Theorem 1.2 and the following result.

Theorem 1.10. *Assume* 0 *is not a Dirichlet eigenvalue of* $\Delta - q_i$*,* $i = 1, 2$*. Let* $s > \frac{n}{2}$*,* $n \geq 3$ *and*

$$\|q_j\|_{H^s(\Omega)} \leq M.$$

Then there exists $C = C(\Omega, M, n, s)$ *and* $0 < \sigma < 1$ *(*$\sigma = \sigma(n, s)$*) such that*

$$\|q_1 - q_2\|_{H^{-1}(\Omega)} \leq C\Big(|\log \|\Lambda_{q_1} - \Lambda_{q_2}\|_{\frac{1}{2},\frac{-1}{2}}|^{-\sigma} + \|\Lambda_{q_1} - \Lambda_{q_2}\|_{\frac{1}{2},\frac{-1}{2}}\Big). \tag{1.30}$$

1.4.5. Reconstruction

The complex geometrical optics solutions of Theorems 1.5 and 1.6 were also used by A. Nachman [136] and R. Novikov [146] to give a reconstruction procedure of the conductivity from Λ_γ. As we have already noticed in section 1.2 we can reconstruct the conductivity at the boundary and its normal derivative from the DN map. Therefore if we know Λ_γ we can determine Λ_q. We will then show how to reconstruct q from Λ_q. Once this is done, to find $\sqrt{\gamma}$, we solve the problem

$$\begin{aligned} \Delta u - qu &= 0 \text{ in } \Omega, \\ u|_{\partial\Omega} &= \sqrt{\gamma}\Big|_{\partial\Omega}. \end{aligned} \tag{1.31}$$

Let $q_1 = q$, $q_2 = 0$ in formula (1.17). Then we have

$$\int_\Omega quvdx = \int_{\partial\Omega} (\Lambda_q - \Lambda_0)\left(v\Big|_{\partial\Omega}\right) u\Big|_{\partial\Omega} dS \tag{1.32}$$

where $u, v \in H^1(\Omega)$ solve$\Delta u - qu = 0$, $\Delta v = 0$ in Ω. Here Λ_0 denotes the Dirichlet to Neumann map associated to the potential $q = 0$. We choose $\rho_i, i = 1, 2$ as in (1.21) and

(20). Take $v = e^{x\cdot\rho_1}$, $u := u_\rho = e^{x\cdot\rho_2}(1+\psi_q(x,\rho_2))$ as in Theorem 1.3. By taking $|l|\to\infty$ in (1.32) we conclude

$$\widehat{q}(-k) = \lim_{|l|\to\infty}\int_{\partial\Omega}(\Lambda_q-\Lambda_0)(e^{x\cdot\rho_1}\Big|_{\partial\Omega})u_\rho\Big|_{\partial\Omega}dS.$$

So the problem is then to recover the boundary values of the solutions u_ρ from Λ_q. The idea is to find $u_\rho\Big|_{\partial\Omega}$ by looking at the exterior problem. Namely by extending $q=0$ outside Ω, u_ρ solves

$$\begin{aligned} \Delta u_\rho &= 0 \text{ in } \mathbb{R}^n-\Omega \\ \frac{\partial u_\rho}{\partial\nu}\Big|_{\partial\Omega} &= \Lambda_q(u_\rho\Big|_{\partial\Omega}). \end{aligned} \tag{1.33}$$

Also note that

$$e^{-x\cdot\rho_2}u_\rho - 1 \in L^2_\delta(\mathbb{R}^n). \tag{1.34}$$

Let $\rho\in\mathbb{C}^n-0$ with $\rho\cdot\rho=0$. Let $G_\rho(x,y)\in\mathcal{D}'(\mathbb{R}^n\times\mathbb{R}^n)$ denote the Schwartz kernel of the operator Δ_ρ^{-1}. Then we have that

$$g_\rho(x) = e^{x\cdot\rho}G_\rho(x) \tag{1.35}$$

is a Green's kernel for Δ, namely

$$\Delta g_\rho = \delta_0. \tag{1.36}$$

We write the solution of (1.33) and (1.34) in terms of single and double layer potentials using this Green's kernel. This is also called Faddeev Green's kernel [57] who considered it in the context of scattering theory. We define the single and double layer potentials

$$S_\rho f(x) = \int_{\partial\Omega} g_\rho(x-y)f(y)dS_y, \qquad x\in\mathbb{R}^n-\Omega, \tag{1.37}$$

$$D_\rho f(x) = \int_{\partial\Omega}\frac{\partial g_\rho}{\partial\nu}(x-y)f(y)dS_y, \qquad x\in\mathbb{R}^n-\Omega \tag{1.38}$$

$$B_\rho f(x) = p.v.\int_{\partial\Omega}\frac{\partial g_\rho}{\partial\nu}(x-y)f(y)dS_y, \qquad x\in\partial\Omega. \tag{1.39}$$

Nachman showed that $f_\rho = u_\rho\Big|_{\partial\Omega}$ is a solution of the integral equation

$$f_\rho = e^{x\cdot\rho} - (S_\rho\Lambda_q - B_\rho - \frac{1}{2}I)f_\rho. \tag{1.40}$$

Moreover (1.40) is an inhomogeneous integral equation of Fredholm type for f_ρ and it has a unique solution in $H^{\frac{3}{2}}(\partial\Omega)$. The uniqueness of the homogeneous equation follows from the uniqueness of the CGO solutions in Theorem 1.6.

1.5. The Partial Data Problem

In several applications in EIT one can only measure currents and voltages on part of the boundary. Substantial progress has been made recently on the problem of whether one can determine the conductivity in the interior by measuring the DN map on part of the boundary. We review here the articles [108] and [51]. The paper [36] used the method of Carleman estimates with a linear weight to prove that, roughly speaking, knowledge of the DN map in "half" of the boundary is enough to determine uniquely a C^2 conductivity. The regularity assumption on the conductivity was relaxed to $C^{1+\epsilon}, \epsilon>0$ in [109]. Stability estimates for the uniqueness result of [36] were given in [77]. Stability estimates for the

magnetic Schrödinger operator with partial data in the setting of [36] can be found in [201]. The result [36] was substantially improved in [108]. The latter paper contains a global identifiability result where it is assumed that the DN map is measured on any open subset of the boundary of a strictly convex domain for all functions supported, roughly, on the complement. We state the theorem more precisely below. The key new ingredient is the construction of a larger class of CGO solutions than the ones considered in section 1.4. Let $x_0 \in \mathbf{R}^n \setminus \overline{\mathrm{ch}\,(\Omega)}$, where $\mathrm{ch}\,(\Omega)$ denotes the convex hull of Ω. Define the front and the back faces of $\partial\Omega$ by

$$F(x_0) = \{x \in \partial\Omega;\ (x - x_0) \cdot \nu \leq 0\},\ B(x_0) = \{x \in \partial\Omega;\ (x - x_0) \cdot \nu > 0\}.$$

The main result of [108] is the following:

Theorem 1.11. *Let $n > 2$. With Ω, x_0, $F(x_0)$, $B(x_0)$ defined as above, let $q_1, q_2 \in L^\infty(\Omega)$ be two potentials and assume that there exist open neighborhoods $\widetilde{F}$, $\widetilde{B} \subset \partial\Omega$ of $F(x_0)$ and $B(x_0) \cup \{x \in \partial\Omega; (x - x_0) \cdot \nu = 0\}$ respectively, such that*

$$\Lambda_{q_1} u = \Lambda_{q_2} u \text{ in } \widetilde{F},\ \text{for all } u \in H^{\frac{1}{2}}(\partial\Omega) \cap \mathscr{E}'(\widetilde{B}). \tag{1.41}$$

Then $q_1 = q_2$.

Here $\mathscr{E}'(\widetilde{B})$ denotes the space of compactly supported distributions in $\widetilde{B}$. The proof of this result uses Carleman estimates for the Laplacian with limiting Carleman weights (LCW). The Carleman estimates allow one to construct, for large τ, a larger class of CGO solutions for the Schrödinger equation than previously used. These have the form

$$u = e^{\tau(\phi + i\psi)}(a + r), \tag{1.42}$$

where $\nabla\phi \cdot \nabla\psi = 0, |\nabla\phi|^2 = |\nabla\psi|^2$ and ϕ is the LCW. Moreover a is smooth and non-vanishing and $\|r\|_{L^2(\Omega)} = O(\frac{1}{\tau})$, $\|r\|_{H^1(\Omega)} = O(1)$. Examples of LCW are the linear phase $\phi(x) = x \cdot \omega, \omega \in S^{n-1}$, used previously, and the non-linear phase $\phi(x) = \ln|x - x_0|$, where $x_0 \in \mathbf{R}^n \setminus \overline{\mathrm{ch}\,(\Omega)}$ which was used in [108]. Any conformal transformation of these would also be a LCW. Below we give a characterization of all the LCW in $\mathbb{R}^n$, $n > 2$, see [53]. In two dimensions any harmonic function is a LCW [209]. The CGO solutions used in [108] are of the form

$$u(x, \tau) = e^{\log|x - x_0| + id(\frac{x - x_0}{|x - x_0|}, \omega)}(a + r) \tag{1.43}$$

where x_0 is a point outside the convex hull of Ω, ω is a unit vector and $d(\frac{x - x_0}{|x - x_0|}, \omega)$ denote distance. We take directions ω so that the distance function is smooth for $x \in \overline{\Omega}$.

1.5.1. Limiting Carleman weights

We only recall here the main ideas in the construction of the CGO solutions. We will denote $\tau = \frac{1}{h}$ in order to use the standard semiclassical notation. Let $P_0 = -h^2\Delta$, where $h > 0$ is a small semi-classical parameter. The weighted L^2-estimate

$$\|e^{\phi/h} u\| \leq C\|e^{\phi/h} P_0 u\|$$

is of course equivalent to the unweighted estimate for a conjugated operator:

$$\|v\| \leq C\|e^{\phi/h} P_0 e^{-\phi/h} v\|.$$

The semi-classical principal symbol of P_0 is $p(x, \xi) = \xi^2$, and that of the conjugated operator $e^{\phi/h} P_0 e^{-\phi/h}$ is

$$p(x, \xi + i\phi'(x)) = a(x, \xi) + ib(x, \xi),$$

where

$$a(x,\xi) = \xi^2 - \phi'(x)^2,\ b(x,\xi) = 2\xi \cdot \phi'(x).$$

Here we denote by ϕ' the gradient of ϕ. Write the conjugated operator as $A + iB$, with A and B formally selfadjoint and with a and b as their associated principal symbols. Then

$$\|(A+iB)u\|^2 = \|Au\|^2 + \|Bu\|^2 + (i[A,B]u|u).$$

The principal symbol of $i[A,B]$ is $h\{a,b\}$, where $\{\cdot,\cdot\}$ denotes the Poisson bracket. In order to get enough negativity to satisfy Hörmander's solvability condition we require that

$$a(x,\xi) = b(x,\xi) = 0 \Rightarrow \{a,b\} \leq 0.$$

It is then indeed possible to get an a-priori estimate for the conjugated operator. We are led to the limiting case since we need to have CGO solutions for both ϕ and $-\phi$.

Definition 1.12. *ϕ is a limiting Carleman weight (LCW) on some open set Ω if $\nabla\phi(x)$ is non-vanishing there and we have*

$$a(x,\xi) = b(x,\xi) = 0 \Rightarrow \{a,b\}(x) = 0, x \in \Omega.$$

We remark that if ϕ is a LCW so is $-\phi$. In [53] we have classified locally all the LCW in Euclidean space.

Theorem 1.13. *Let Ω be an open subset of $\mathbb{R}^n$, $n \geq 3$. The limiting Carleman weights in Ω are locally of the form*

$$\phi(x) = a\phi_0(x - x_0) + b$$

where $a \in \mathbb{R} \setminus \{0\}$ and ϕ_0 is one of the following functions:

$$\begin{array}{llll} & \langle x,\xi\rangle, & \arg\langle x, \omega_1 + i\omega_2\rangle, & \\ \log|x|, & \dfrac{\langle x,\xi\rangle}{|x|^2}, & \arg\left(e^{i\theta}(x+i\xi)^2\right), & \log\dfrac{|x+\xi|^2}{|x-\xi|^2} \end{array}$$

with ω_1, ω_2 orthogonal unit vectors, $\theta \in [0, 2\pi)$ and $\xi \in \mathbb{R}^n \setminus \{0\}$.

As noted earlier, in two dimensions, any harmonic function with a non-vanishing gradient is a limiting Carleman weight.

1.5.2. Construction of CGO Solutions with a non-linear phase

A key ingredient in the construction of a richer family of CGO solutions is the following Carleman estimate.

Proposition 1.14. *Let $\phi \in C^\infty(\text{neigh}\,(\overline{\Omega}))$ be an LCW, $P = -h^2\Delta + h^2 q$, $q \in L^\infty(\Omega)$. Then, for $u \in C^\infty(\overline{\Omega})$, with $u_{|\partial\Omega} = 0$, we have*

$$\begin{aligned} -\tfrac{h^3}{C}((\phi'_x \cdot \nu)e^{\phi/h}\partial_\nu u|e^{\phi/h}\partial_\nu u)_{\partial\Omega_-} + \tfrac{h^2}{C}(\|e^{\phi/h}u\|^2 + \|e^{\phi/h}h\nabla u\|^2) \\ \leq Ch^3((\phi'_x \cdot \nu)e^{\phi/h}\partial_\nu u|e^{\phi/h}\partial_\nu u)_{\partial\Omega_+} + \|e^{\phi/h}Pu\|^2, \end{aligned} \tag{1.44}$$

where norms and scalar products are in $L^2(\Omega)$ unless a subscript A (like for instance $A = \partial\Omega_-$) indicates that they should be taken in $L^2(A)$. Here

$$\partial\Omega_\pm = \{x \in \partial\Omega;\ \pm\nu(x) \cdot \phi'(x) \geq 0\}.$$

The proof of existence of solutions of the form (1.42) follows by using the Hahn-Banach theorem for the adjoint equation $e^{-\phi/h}Pe^{\phi/h}u = v$. Let ϕ be a LCW and write $p(x, \xi + \phi'(x)) = a(x, \xi) + ib(x, \xi)$. Then we know that a and b are in involution on their common zero set, and in this case it is well-known and exploited in [54] that we can find plenty of local solutions to the Hamilton-Jacobi system

$$a(x, \psi'(x)) = 0, b(x, \psi'(x)) = 0 \Leftrightarrow \psi'^2 = \phi'^2, \psi' \cdot \phi' = 0 \tag{1.45}$$

We need the following more global statement:

Proposition 1.15. *Let $\phi \in C^\infty(\mathrm{neigh}\,(\overline{\Omega}))$ be a LCW, where Ω is a domain in $\mathbb{R}^n$ and define the hypersurface $G = p^{-1}(C_0)$ for some fixed value of C_0. Assume that each integral curve of $\phi' \cdot \nabla_x$ through a point in Ω also intersects G and that the corresponding projection map $\Omega \to G$ is proper. Then we get a solution of (1.45) in $C^\infty(\Omega)$ by solving first $g'(x)^2 = \phi'(x)^2$ on G and then defining ψ by $\psi_{|G} = g$, $\phi'(x) \cdot \partial_x \psi = 0$. The vector fields $\phi' \cdot \partial_x$ and $\psi' \cdot \partial_x$ commute.*

This result will be applied with a new domain Ω that contains the original one.

Next consider the WKB-problem

$$P_0(e^{\frac{1}{h}(-\phi+i\psi)}a(x)) = e^{\frac{1}{h}(-\phi+i\psi)}\mathcal{O}(h^2). \tag{1.46}$$

The transport equation for a is of Cauchy-Riemann type along the two-dimensional integral leaves of $\{\phi' \cdot \partial_x, \psi' \cdot \partial_x\}$. We have solutions that are smooth and everywhere $\neq 0$. (See [54].) The existence result for $e^{\phi/h}Pe^{-\phi/h}$ mentioned in one of the remarks after Proposition 1.14 permits us to replace the right hand side of (1.46) by zero; more precisely, we can find $r = \mathcal{O}(h)$ in the semi-classical Sobolev space H^1 equipped with the norm $\|r\| = \|\langle hD\rangle r\|$, such that

$$P(e^{\frac{1}{h}(-\phi+i\psi)}(a + r)) = 0. \tag{1.47}$$

1.5.3. The uniqueness proof

We sketch the proof for the case that $\widetilde{B} = \partial\Omega$. All the arguments in this section are in dimension $n > 2$. The arguments are similar to those of [36] using the new CGO solutions. Let ϕ be an LCW for which the constructions of section 1.5.2 are available. Let $q_1, q_2 \in L^\infty(\Omega)$ be as in Theorem 1.11 with

$$\Lambda_{q_1}(f) = \Lambda_{q_2}(f) \text{ in } \partial\Omega_{-,\epsilon_0},\ \forall f \in H^{\frac{1}{2}}(\partial\Omega), \tag{1.48}$$

where

$$\begin{aligned} \partial\Omega_{-,\epsilon_0} &= \{x \in \partial\Omega;\ \nu(x) \cdot \phi'(x) < \epsilon_0\} \\ \partial\Omega_{+,\epsilon_0} &= \{x \in \partial\Omega;\ \nu(x) \cdot \phi'(x) \geq \epsilon_0\}. \end{aligned}$$

Let

$$u_2 = e^{\frac{1}{h}(\phi+i\psi_2)}(a_2 + r_2)$$

solve

$$(\Delta - q_2)u_2 = 0 \text{ in } \Omega,$$

with $\|r_2\|_{H^1} = \mathcal{O}(h)$. Let u_1 solve

$$(\Delta - q_1)u_1 = 0 \text{ in } \Omega,\ u_{1|\partial\Omega} = u_{2|\partial\Omega}.$$

Then according to the assumptions in the theorem, we have $\partial_\nu u_1 = \partial_\nu u_2$ in $\partial\Omega_{-,\epsilon_0}$ if $\epsilon_0 > 0$ has been fixed sufficiently small and we choose $\phi(x) = \ln|x - x_0|$.

Put $u = u_1 - u_2$, $q = q_2 - q_1$, so that

$$(\Delta - q_1)u = qu_2,\ u_{|\partial\Omega} = 0,\ \mathrm{supp}\,(\partial_\nu u_{|\partial\Omega}) \subset \partial\Omega_{+,\epsilon_0}. \tag{1.49}$$

For $v \in H^1(\Omega)$ with $\Delta v \in L^2$, we get from Green's formula

$$\begin{aligned} \int_\Omega qu_2\overline{v}dx &= \int_\Omega (\Delta - q_1)u\overline{v}dx \\ &= \int_\Omega u\,\overline{(\Delta - \overline{q}_1)v}dx + \int_{\partial\Omega_{+,\epsilon_0}} \partial_\nu u\overline{v}dS. \end{aligned} \tag{1.50}$$

Similarly, we choose

$$v = e^{-\frac{1}{h}(\phi + i\psi_1)}(a_1 + r_1),$$

with

$$(\Delta - \overline{q}_1)v = 0.$$

Then

$$\int_\Omega qe^{\frac{i}{h}(\psi_1+\psi_2)}(a_2 + r_2)(\overline{a_1 + r_1})dx = \int_{\partial\Omega_{+,\epsilon_0}} \partial_\nu u\, e^{-\frac{1}{h}(\phi - i\psi_1)}(\overline{a_1 + r_1})dS. \tag{1.51}$$

Assume that ψ_1, ψ_2 are slightly h-dependent with

$$\frac{1}{h}(\psi_1 + \psi_2) \to f,\ h \to 0.$$

The left hand side of (1.51) tends to

$$\int_\Omega qe^{if}a_2\overline{a}_1dx,$$

when $h \to 0$. The modulus of the right hand side is

$$\leq \|a_1 + r_1\|_{L^2(\partial\Omega_{+,\epsilon_0})}\Big(\int_{\partial\Omega_{+,\epsilon_0}} e^{-2\phi/h}|\partial_\nu u|^2dS\Big)^{\frac{1}{2}}.$$

Here the first factor is bounded when $h \to 0$. In the Carleman estimate (1.44) we can replace ϕ by $-\phi$ and make the corresponding permutation of $\partial\Omega_-$ and $\partial\Omega_+$. Applying this variant to the equation (1.49), we see that the second factor tends to 0, when $h \to 0$. Thus,

$$\int_\Omega e^{if(x)}a_2(x)\overline{a_1(x)}q(x)dx = 0.$$

Here we can arrange it so that f, a_2, a_1 are real-analytic and so that a_1, a_2 are non-vanishing. Moreover if f can be attained as a limit of $(\psi_1 + \psi_2)/h$ when $h \to 0$, so can λf for any $\lambda > 0$. Thus we get the conclusion

$$\int_\Omega e^{i\lambda f(x)}a_2(x)\overline{a_1(x)}q(x)dx = 0. \tag{1.52}$$

To show that $q = 0$ one uses arguments of analytic microlocal analysis [108]. In [9] it was shown that if the potential is known in a neighborhood of the boundary and the DN map is measured on any open subset with Dirichlet data supported in the same set, the potential can be reconstructed from this data. It is an open problem whether this is valid in the general case. Isakov [96] proved a uniqueness result in dimension three or higher when the DN map is given on an arbitrary part of the boundary assuming that the remaining part is an open subset of a plane or a sphere and the DN map is measured on the plane or sphere. The case of partial data on a slab was studied in [127]. The DN map with partial data for the magnetic Schrödinger operator was studied in [52], [110], [201], [118]. The case of the polyharmonic operator was considered in [119]. We also mention

that in [67] (resp. [102]) CGO approximate solutions are concentrated near planes (resp. spheres) and provided some local results related to the local DN map. It would be very interesting to extend the partial data result to systems. See [165] for Dirac systems, [41] for Maxwell and [92] for elasticity. Using methods of hyperbolic geometry similar to [102] it is shown in [83] that one can reconstruct inclusions from the *local* DN map using CGO solutions that decay exponentially inside a ball and grow exponentially outside. These are called *complex spherical waves*. A numerical implementation of this method has been done in [83]. The construction of complex spherical waves can also be done using the CGO solutions constructed in [108]. This was done in [208] in order to detect elastic inclusions, in [209] to detect inclusions in the two dimensional case for a large class of systems with inhomogeneous background, and in [166] for the case of inclusions contained in a slab. We mention that methods of hyperbolic geometry have been also studied earlier in the works [19], [59], [167].

1.5.4. The Linearized Calderón Partial Data Problem

It is an open problem in dimension $n \geq 3$ that if the Dirichlet to Neumann map for the conductivity or potential is measured on an open non-empty subset of the boundary for Dirichlet data supported in that set we can determine uniquely the potential. In this section we consider the linearized version of this problem, generalizing Calderón's approach. We add the constraint that the restriction of the harmonic functions to the boundary vanishes on any fixed closed proper subset of the boundary. We show that the product of these harmonic functions is dense. More precisely

Theorem 1.16. *Let Ω be a connected bounded open set in $\mathbb{R}^n$, $n \geq 2$, with smooth boundary. The set of products of harmonic functions on Ω which vanish on a closed proper subset $\Gamma \subsetneq \partial\Omega$ of the boundary is dense in $L^1(\Omega)$.*

Sketch of the Proof. We take $f \in L^1(\Omega)$. Assume

$$\int_\Omega fuvdx = 0, \tag{1.53}$$

for all harmonic functions u, v with $u|_\Gamma = v|_\Gamma = 0$. First one proves a local result. Fix a point x_0 on the boundary. It is shown that if $f = 0$ in a neighborhood of x_0 then $f = 0$ in the whole domain. See [51] for the proof. We now extend $f = 0$ outside Ω. We reduce the problem to the case where the point x_0 has a hyperplane tangent to the boundary at the point x_0. We use Calderón's exponential solutions for all the possible complex frequencies ρ such that $\rho \cdot \rho = 0$ (previously we used in sections 1.4 and 1.5.3 the cancellation of the real parts when taking the products). Using these solutions and (1.53) one obtains a decay of the Bargmann-Segal transform of f (see [174])

$$Tf(z) = \int_{\mathbb{R}^n} e^{-\frac{1}{2h}(z-y)^2} f(y)dy \tag{1.54}$$

for certain complex directions. Using the watermelon approach [106], [175], one then shows that there is an exponential decay of this transform for other directions implying that the point (x_0, ν), where ν is the normal to the the point x_0, is not in the analytic wave front set of f in contradiction to the microlocal version of Holmgren's uniqueness theorem [82], [174]. We explain below some of the details of the proof. One can assume that $\Omega \setminus \{x_0\}$ is on one side of the tangent hyperplane $T_{x_0}(\Omega)$ at x_0 by making a conformal transformation. Pick $a \in \mathbb{R}^n \setminus \overline{\Omega}$ which sits on the line segment in the direction of the outward normal to

$\partial\Omega$ at x_0; there is a ball $B(a,r)$ such that $\partial B(a,r)\cap\overline{\Omega}=\{x_0\}$, and there is a conformal transformation

$$\psi:\mathbb{R}^n\setminus B(a,r)\to\overline{B(a,r)} \tag{1.55}$$

which fixes x_0 and exchanges the interior and the exterior of the ball $B(a,r)$. The hyperplane $H:(x-x_0)\cdot(a-x_0)=0$ is tangent to $\psi(\Omega)$, and the image $\psi(\Omega)\setminus\{x_0\}$ by the conformal transformation lies inside the ball $B(a,r)$, therefore on one side of H. The fact that functions are supported on the boundary close to x_0 is left unchanged. Since a function is harmonic on Ω if and only if its Kelvin transform

$$u^*=r^{n-2}|x-a|^{-n+2}u\circ\psi$$

is harmonic on $\psi(\Omega)$, (1.53) becomes

$$0=\int\limits_{\Omega}fuv\,dx=\int\limits_{\psi(\Omega)}r^4|x-a|^{-4}f\circ\psi\,u^*v^*\,dx$$

for all harmonic functions u^*,v^* on $\psi(\Omega)$. If one can prove that if $|x-a|^{-4}f\circ\psi$ vanishes close to x_0 then so does f. Moreover, by scaling one can assume that Ω is contained in a ball of radius 1. Our setting will therefore be as follows: $x_0=0$, the tangent hyperplane at x_0 is given by $x_1=0$ and

$$\Omega\subset\Big\{x\in\mathbb{R}^n:|x+e_1|<1\Big\},\quad \Gamma=\Big\{x\in\partial\Omega:x_1\geq-2c\Big\}. \tag{1.56}$$

The prime will be used to denote the last $n-1$ variables so that $x=(x_1,x')$ for instance. The Laplacian on $\mathbb{R}^n$ has $p(\xi)=\xi^2$ as a principal symbol, if we still denote by $p(\zeta)=\zeta^2$ the continuation of this principal symbol on $\mathbb{C}^n$, we consider

$$p^{-1}(0)=\Big\{\zeta\in\mathbb{C}^n:\zeta^2=0\Big\}.$$

In dimension $n=2$, this set is the union of two complex lines

$$p^{-1}(0)=\mathbb{C}\gamma\cup\mathbb{C}\overline{\gamma}$$

where $\gamma=ie_1+e_2=(i,1)\in\mathbb{C}^2$. Note that $(\gamma,\overline{\gamma})$ is a basis of $\mathbb{C}^2$: the decomposition of a complex vector in this basis reads

$$\zeta=\zeta_1e_1+\zeta_2e_2=\frac{\zeta_2-i\zeta_1}{2}\,\gamma+\frac{\zeta_2+i\zeta_1}{2}\,\overline{\gamma}. \tag{1.57}$$

Similarly for $n\geq2$, the differential of the map

$$\begin{aligned} s:p^{-1}(0)\times\ \ p^{-1}(0)\ &\to\mathbb{C}^n\\ (\zeta,\eta)\ &\mapsto\zeta+\eta\end{aligned}$$

at (ζ_0,η_0) is surjective

$$\begin{aligned} Ds(\zeta_0,\eta_0):T_{\zeta_0}p^{-1}(0)\times T_{\eta_0}p^{-1}\quad(0)\ &\to\mathbb{C}^n\\ (\zeta,\eta)\ &\mapsto\zeta+\eta\end{aligned}$$

provided $\mathbb{C}^n=T_{\zeta_0}p^{-1}(0)+T_{\eta_0}p^{-1}(0)$, i.e. provided ζ_0 and η_0 are linearly independent. In particular, this is the case if $\zeta_0=\gamma$ and $\eta_0=-\overline{\gamma}$; as a consequence all $z\in\mathbb{C}^n$, $|z-2ie_1|<2\varepsilon$ may be decomposed as a sum of the form

$$z=\zeta+\eta,\qquad\text{with }\zeta,\eta\in p^{-1}(0),\ |\zeta-\gamma|<C\varepsilon,\ |\eta+\overline{\gamma}|<C\varepsilon \tag{1.58}$$

provided $\varepsilon>0$ is small enough. The exponentials with linear weights

$$e^{-\frac{i}{h}x\cdot\zeta},\quad\zeta\in p^{-1}(0)$$

are harmonic functions. We need to add a correction term in order to obtain harmonic functions u satisfying the boundary requirement $u|_\Gamma = 0$. Let $\chi \in \mathcal{D}(\mathbb{R}^n)$ be a cutoff function which equals 1 on Γ, and consider the solution w to the Dirichlet problem

$$\Delta w = 0 \quad \text{ in } \Omega, w|_{\partial\Omega} = -(e^{-\frac{i}{h}x\cdot\zeta}\chi)|_{\partial\Omega}. \tag{1.59}$$

The function

$$u(x,\zeta) = e^{-\frac{i}{h}x\cdot\zeta} + w(x,\zeta)$$

is harmonic and satisfies $u|_\Gamma = 0$. We have the following bound on w:

$$\begin{aligned} \|w\|_{H^1(\Omega)} &\leq C_1 \|e^{-\frac{i}{h}x\cdot\zeta}\chi\|_{H^{\frac{1}{2}}(\partial\Omega)} \\ &\leq C_2(1+h^{-1}|\zeta|)^{\frac{1}{2}}\, e^{\frac{1}{h}H_K(\mathrm{Im}\zeta)} \end{aligned} \tag{1.60}$$

where H_K is the supporting function of the compact subset $K = \operatorname{supp}\chi \cap \partial\Omega$ of the boundary

$$H_K(\xi) = \sup_{x\in K} x\cdot\xi, \quad \xi \in \mathbb{R}^n.$$

In particular, if we take χ to be supported in $x_1 \leq -c$ and equal to 1 on $x_1 \leq -2c$ then the bound (1.60) becomes

$$\|w\|_{H^1(\Omega)} \leq C_2(1+h^{-1}|\zeta|)^{\frac{1}{2}}\, e^{-\frac{c}{h}\mathrm{Im}\zeta_1}\, e^{\frac{1}{h}|\mathrm{Im}\zeta'|} \quad \text{ when } \mathrm{Im}\zeta_1 \geq 0. \tag{1.61}$$

The starting point is the cancellation of the integral

$$\int_\Omega f(x)u(x,\zeta)u(x,\eta)\,dx = 0, \quad \zeta,\eta \in p^{-1}(0) \tag{1.62}$$

which may be rewritten in the form

$$\begin{aligned} \int_\Omega f(x)e^{-\frac{i}{h}x\cdot(\zeta+\eta)}\,dx = &- \int_\Omega f(x)e^{-\frac{i}{h}x\cdot\zeta}w(x,\eta)\,dx \\ &- \int_\Omega f(x)e^{-\frac{i}{h}x\cdot\eta}w(x,\zeta)\,dx - \int_\Omega f(x)w(x,\zeta)w(x,\eta)\,dx. \end{aligned}$$

This allows to give a bound on the left-hand side

$$\begin{aligned} \left|\int_\Omega f(x)e^{-\frac{i}{h}x\cdot(\zeta+\eta)}\,dx\right| &\leq \|f\|_{L^\infty(\Omega)}\Big(\|e^{-\frac{i}{h}x\cdot\zeta}\|_{L^2(\Omega)}\|w(x,\eta)\|_{L^2(\Omega)} \\ &+\|e^{-\frac{i}{h}x\cdot\eta}\|_{L^2(\Omega)}\|w(x,\zeta)\|_{L^2(\Omega)} + \|w(x,\eta)\|_{L^2(\Omega)}\|w(x,\zeta)\|_{L^2(\Omega)}\Big). \end{aligned}$$

Thus using (1.61)

$$\begin{aligned} \left|\int_\Omega f(x)e^{-\frac{i}{h}x\cdot(\zeta+\eta)}\,dx\right| \leq C_3\|f\|_{L^\infty(\Omega)}(1 &+ h^{-1}|\eta|)^{\frac{1}{2}}(1+h^{-1}|\zeta|)^{\frac{1}{2}} \\ &\times\ e^{-\frac{c}{h}\min(\mathrm{Im}\zeta_1,\mathrm{Im}\eta_1)}\, e^{\frac{1}{h}(|\mathrm{Im}\zeta'|+|\mathrm{Im}\eta'|)} \end{aligned}$$

when $\mathrm{Im}\zeta_1 \geq 0, \mathrm{Im}\eta_1 \geq 0$ and $\zeta,\eta \in p^{-1}(0)$. In particular, if $|\zeta - a\gamma| < C\varepsilon a$ and $|\eta + a\overline{\gamma}| < C\varepsilon a$ with $\varepsilon \leq 1/2C$ then

$$\left|\int_\Omega f(x)e^{-\frac{i}{h}x\cdot(\zeta+\eta)}\,dx\right| \leq C_4 h^{-1}\|f\|_{L^\infty(\Omega)}e^{-\frac{ca}{2h}}\, e^{\frac{2C\varepsilon a}{h}}.$$

Take $z \in \mathbb{C}^n$ with $|z - 2ae_1| < 2\varepsilon a$ and with ε small enough. Once rescaled the decomposition (1.58) gives

$$z = \zeta + \eta, \quad \zeta, \eta \in p^{-1}(0),\ |\zeta - a\gamma| < C\varepsilon a,\ |\eta + a\overline{\gamma}| < C\varepsilon a,$$

and we therefore get the estimate

$$\left| \int_\Omega f(x) e^{-\frac{i}{h} x\cdot z}\, dx \right| \leq C_4 h^{-1} \|f\|_{L^\infty(\Omega)} e^{-\frac{ca}{2h}}\, e^{\frac{2C\varepsilon a}{h}} \tag{1.63}$$

for all $z \in \mathbb{C}^n$ such that $|z - 2aie_1| < 2\varepsilon a$. This implies that the Bargmann-Segal transform of f satisfies

$$|Tf(z)| \leq C|f\|_{L^\infty(\Omega)} e^{\frac{1}{2h}(|Imz|^2 - |Rez|^2 - \frac{ca}{2})} \tag{1.64}$$

for some $\epsilon, a, c > 0$ and for all $z \in \mathbb{C}^n$ such that $|z - 2aie_1| < 2\varepsilon a$. By the definition of the analytic wave front set, the last estimate says that the point $(0, 2ae_1)$ is not in the analytic wave front set of f. By Kashiwara's watermelon theorem [175], [106], since f is supported in the half space $x_1 \leq 0$, if 0 is in the support of f then $(0, \nu)$ with ν the unit normal to the boundary is also in the analytic wave front set but this is a contradiction since $2ae_1$ is also normal to $x_1 = 0$. Therefore 0 is not in the support of f and f vanishes in a neighborhood of 0.

1.6. The Calderón Problem in Two Dimensions

Astala and Päivärinta [11], in a seminal contribution, have recently extended significantly the uniqueness result of [135] for conductivities having two derivatives in an appropriate sense and the result of [34] for conductivities having one derivative in appropriate sense, by proving that any L^∞ conductivity in two dimensions can be determined uniquely from the DN map. We remark that the method of [135] and [34] uses, besides CGO solutions, the $\overline{\partial}$ method introduced in one dimension by Beals and Coifman [21] and generalized to several dimensions in [1], [137], [22], [199]. The $\overline{\partial}$ method has been used in numerical reconstruction procedures in two dimensions in [94], [173] among others. The proof of [11] relies also on construction of CGO solutions for the conductivity equation with L^∞ coefficients and the $\overline{\partial}$ method. This is done by transforming the conductivity equation to a quasi-regular map. Let $\mathcal{D}$ be the unit disk in the plane. Then we have

Lemma 1.17. *Assume $u \in H^1(\mathcal{D})$ is real valued and satisfies the conductivity equation on $\mathcal{D}$. Then there exists a function $v \in H^1(\mathcal{D})$, unique up to a constant, such that $f = u + iv$ satisfies the Beltrami equation*

$$\overline{\partial} f = \mu \overline{\partial f}, \tag{1.65}$$

where $\mu = (1-\gamma)/(1+\gamma)$. Conversely, if $f \in H^1(\mathcal{D})$ satisfies (1.65) with a real-valued μ, then $u = Ref$ and $v = Imf$ satisfy

$$\nabla \cdot \gamma \nabla u = 0 \quad \textit{and} \quad \nabla \cdot \frac{1}{\gamma} \nabla v = 0, \tag{1.66}$$

respectively, where $\gamma = (1-\mu)/(1+\mu)$.

Let us denote $\kappa = ||\mu||_{L^\infty} < 1$. Then (1.65) means that f is a quasi-regular map. The function v is called the γ-harmonic conjugate of u and it is unique up to a constant. Astala and Päivärinta consider the μ-Hilbert transform $\mathcal{H}_\mu : H^{1/2}(\partial\Omega) \to H^{1/2}(\partial\Omega)$ that is defined by

$$\mathcal{H}_\mu : u\Big|_{\partial\Omega} \mapsto v\Big|_{\partial\Omega}$$

and show that the DN map Λ_γ determines $\mathcal{H}_\mu$ and vice versa. Below we use the complex notation $z = x_1 + ix_2$. Moreover, for the equation (1.65), it is shown that for every $k \in \mathbb{C}$ there are complex geometrical optics solutions of the Beltrami equation that have the form

$$f_\mu(z,k) = e^{ikz} M_\mu(z,k), \tag{1.67}$$

where

$$M_\mu(z,k) = 1 + \mathcal{O}\left(\frac{1}{z}\right) \text{ as } |z| \to \infty. \tag{1.68}$$

More precisely, they prove that

Theorem 1.18. *For each $k \in \mathbb{C}$ and for each $2 < p < 1 + 1/k$ the equation (1.65) admits a unique solution $f \in W^{1,p}_{loc}(\mathbb{C})$ of the form (1.67) such that the asymptotic formula (1.68) holds true.*

In the case of non-smooth coefficients the function M_μ grows sub-exponentially in k. Astala and Päivärinta introduce the "transport matrix" to deal with this problem. Using a result of Bers connecting pseudoanalytic functions with quasi-regular maps they show that this matrix is determined by the Hilbert transform H_μ and therefore the DN map. Then they use the transport matrix to show that Λ_γ determines uniquely γ. See [11] for more details. Logarithmic type stability estimates for Hölder conductivities of positive exponent have been given in [20].

1.6.1. Bukhgeim's Result

In a recent breakthrough, Bukhgeim [35] proved that a potential in $W^{2,p}(\Omega), p > 2$ can be uniquely determined from the set of Cauchy data as defined in (1.15). An earlier result [189] gave this for a generic class of potentials. As before, if two potentials q_1, q_2 have the same set of Cauchy data, we have

$$\int_\Omega (q_1 - q_2) u_1 u_2 dx = 0 \tag{1.69}$$

where $u_i, i = 1, 2$, are solutions of the Schrödinger equation. Assume now that $0 \in \Omega$. Bukhgeim takes CGO solutions of the form

$$u_1(z,k) = e^{z^2 k}(1 + \psi_1(z,k)), \quad u_2(z,k) = e^{-\bar{z}^2 k}(1 + \psi_2(z,k)) \tag{1.70}$$

where $z, k \in \mathbb{C}$ and we have used the complex notation $z = x_1 + ix_2$. Moreover ψ_1 and ψ_2 decay uniformly in Ω, in an appropriate sense, for $|k|$ large. Note that the weight $z^2 k$ in the exponential is a limiting Carleman weight since it is a harmonic function but it is singular at 0 since its gradient vanishes there. Substituting (1.70) into (1.69) we obtain

$$\int_\Omega e^{2i\tau x_1 x_2}(q_1 - q_2)(1 + \psi_1 + \psi_2 + \psi_1\psi_2) dx = 0.$$

Now using the decay of ψ_i in τ, $i = 1, 2$, and applying stationary phase (the phase function $x_1 x_2$ that has a non-degenerate critical point at 0) we obtain $q_1(0) = q_2(0) = 0$ in Ω. Of course we can do this at any point of Ω proving the result. This result also shows that complex conductivities can be determined uniquely from the DN map. Francini has shown in [58] that this was the case for conductivities with small imaginary part. It also implies unique determination of a potential from the fixed energy scattering amplitude in two dimensions.

1.6.2. Partial Data Problem in 2D

It is shown in [88] that for a two dimensional bounded domain the Cauchy data for the Schrödinger equation measured on an arbitrary open subset of the boundary determines uniquely the potential. This implies, for the conductivity equation, that if one measures the current fluxes at the boundary on an arbitrary open subset of the boundary produced by voltage potentials supported in the same subset, one can determine uniquely the conductivity. The paper [88] uses Carleman estimates with weights which are harmonic functions with non-degenerate critical points to construct appropriate complex geometrical optics solutions to prove the result. We describe this more precisely below. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain which consists of N smooth closed curves γ_j, $\partial\Omega = \cup_{j=}^N \gamma_j$. As before we define the set of Cauchy data for a bounded potential q by:

$$\widehat{C_q} = \left\{ \left(u|_{\partial\Omega}, \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega} \right) \mid (\Delta - q)u = 0 \text{ on } \Omega, \quad u \in H^1(\Omega) \right\}. \tag{1.71}$$

Let $\Gamma \subset \partial\Omega$ be a non-empty open subset of the boundary. Denote $\Gamma_0 = \partial\Omega \setminus \overline{\Gamma}$. The main result of [88] gives global uniqueness by measuring the Cauchy data on Γ. Let $q_j \in C^{2+\alpha}(\overline{\Omega})$, $j = 1, 2$ for some $\alpha > 0$ and let q_j be complex-valued. Consider the following sets of Cauchy data on Γ:

$$\mathcal{C}_{q_j} = \left\{ \left(u|_{\Gamma}, \frac{\partial u}{\partial \nu}\Big|_{\Gamma} \right) \mid (\Delta - q_j)u = 0 \,\text{in}\, \Omega,\, u|_{\Gamma_0} = 0,\ u \in H^1(\Omega) \right\}, \quad j = 1, 2. \tag{1.72}$$

Theorem 1.19. *Assume $\mathcal{C}_{q_1} = \mathcal{C}_{q_2}$. Then $q_1 = q_2$.*

Using Theorem 1.19 one concludes immediately, as a corollary, the following global identifiability result for the conductivity equation (1.2). This result uses that knowledge of the Dirichlet-to-Neumann map on an open subset of the boundary determines γ and its first derivatives on Γ (see [113], [194]).

Corollary 1.20. *With some $\alpha > 0$, let $\gamma_j \in C^{4+\alpha}(\overline{\Omega})$, $j = 1, 2$, be non-vanishing functions. Assume that*

$$\Lambda_{\gamma_1}(f) = \Lambda_{\gamma_2}(f) \text{ on } \Gamma \text{ for all } f \in H^{\frac{1}{2}}(\Gamma),\ \text{supp}\, f \subset \Gamma.$$

Then $\gamma_1 = \gamma_2$.

It is easy to see that Theorem 1.19 implies the analogous result to [108] in the two dimensional case. Notice that Theorem 1.19 does not assume that Ω is simply connected. An interesting inverse problem is whether one can determine the potential or conductivity in a region of the plane with holes by measuring the Cauchy data only on the accessible boundary. This is also called the obstacle problem. Let Ω, D be domains in $\mathbb{R}^2$ with smooth boundaries such that $\overline{D} \subset \Omega$. Let $V \subset \partial\Omega$ be an open set. Let $q_j \in C^{2+\alpha}(\overline{\Omega \setminus D})$, for some $\alpha > 0$ and $j = 1, 2$. Let us consider the following set of partial Cauchy data

$$\widetilde{C}_{q_j} = \{(u|_V, \frac{\partial u}{\partial \nu}|_V) | (\Delta - q_j)u = 0 \text{ in } \Omega \setminus \overline{D},\, u|_{\partial D \cup \partial\Omega \setminus V} = 0, u \in H^1(\Omega \setminus \overline{D})\}.$$

Corollary 1.21. *Assume $\widetilde{C}_{q_1} = \widetilde{C}_{q_2}$. Then $q_1 = q_2$.*

A similar result holds for the conductivity equation.

Corollary 1.22. *Let $\gamma_j \in C^{4+\alpha}(\overline{\Omega \setminus D})$ $j = 1, 2$ be non vanishing functions. Assume*

$$\Lambda_{\gamma_1}(f) = \Lambda_{\gamma_2}(f) \text{ on } V \quad \forall f \in H^{\frac{1}{2}}(\partial(\overline{\Omega} \setminus D)),\ \text{supp}\, f \subset V$$

Then $\gamma_1 = \gamma_2$.

The two dimensional case has special features since one can construct a much larger set of complex geometrical optics solutions than in higher dimensions. On the other hand, the problem is formally determined in two dimensions and therefore more difficult. The proof of Theorem 1.19 is based on the construction of appropriate complex geometrical optics solutions by Carleman estimates with degenerate weight functions.

Sketch of the Proof. For the partial data problem we need a more general class of CGO solutions than the ones constructed by Bukhgeim, since we would like to have the imaginary part of the phase vanish on Γ. So we consider more general holomorphic functions with non-degenerate critical points as phases. Let the function $\Phi(z) = \varphi(x_1, x_2) + i\psi(x_1, x_2) \in C^2(\overline{\Omega})$ be holomorphic in Ω and $\operatorname{Im} \Phi|_{\partial\Omega\setminus\widetilde{\Gamma}} = 0$. Notice that this implies $\nabla\varphi \cdot \nu = 0$ on $\partial\Omega \setminus \widetilde{\Gamma}$. We denote the set of critical points of Φ by

$$\mathcal{H} = \{z \in \overline{\Omega} | \partial_z \Phi(z) = 0\}.$$

We assume that Φ has a finite number of non-degenerate critical points in $\overline{\Omega}$, that is $\partial_z^2 \Phi(z) \neq 0$, $z \in \mathcal{H}$. We denote the critical points by $\mathcal{H} = \{\widetilde{x}_1, ..., \widetilde{x}_\ell\}$ As in the partial data problem considered in section 1.5 we construct appropriate CGO solutions by proving a Carleman estimate.

Carleman estimate Let $u \in H_0^1(\Omega)$, real valued. Then for all large $\tau > 0$:

$$\begin{aligned}\tau\|ue^{\tau\varphi}\|^2_{L^2(\Omega)} + \quad & \|ue^{\tau\varphi}\|^2_{H^1(\Omega)} + \left\|\frac{\partial u}{\partial \nu} e^{\tau\varphi}\right\|^2_{L^2(\partial\Omega\setminus\widetilde{\Gamma})} + \tau^2 \left\|\left|\frac{\partial \Phi}{\partial z}\right| ue^{\tau\varphi}\right\|^2_{L^2(\Omega)} \\ & \leq C\left(\|(\Delta u)e^{\tau\varphi}\|^2_{L^2(\Omega)} + \tau \int_{\widetilde{\Gamma}} \left|\frac{\partial u}{\partial \nu}\right|^2 e^{2\tau\varphi} d\sigma\right)\end{aligned}$$

with σ the standard measure on $\partial\Omega$. The Carleman estimate implies the existence of a solution to the boundary value problem for the Schrödinger equation

$$(\Delta - q)u = f \text{ in } \Omega; \quad u|_{\partial\Omega\setminus\widetilde{\Gamma}} = g \tag{1.73}$$

and that it satisfies an estimate. More precisely we have

Proposition 1.23. *Let* $q \in L^\infty(\Omega)$. *There exists* $\tau_0 > 0$ *such that for all* $|\tau| > \tau_0$ *there exists a solution of (1.72) such that*

$$\|ue^{-\tau\varphi}\|_{L^2(\Omega)} \leq C\Big(\|fe^{-\tau\varphi}\|_{L^2(\Omega)}/\tau + \|ge^{-\tau\varphi}\|_{L^2(\partial\Omega\setminus\widetilde{\Gamma})}\Big).$$

We next find CGO solutions of

$$(\Delta - q)u = 0 \quad \text{in } \Omega; \quad u|_{\partial\Omega\setminus\widetilde{\Gamma}} = 0 \tag{1.74}$$

of the form

$$u(x) = e^{\tau\Phi(z)}(a(z) + a_0(z)/\tau) + e^{\tau\overline{\Phi(z)}}\overline{(a(z) + a_1(z)/\tau)} + e^{\tau\varphi}u_1 + e^{\tau\varphi}u_2. \tag{1.75}$$

The functions $a, a_0, a_1 \in C^2(\overline{\Omega})$ are holomorphic in Ω and $\operatorname{Re} a|_{\partial\Omega\setminus\widetilde{\Gamma}} = 0$. Moreover

$$\|u_j\|_{L^2(\Omega)} = o\left(\frac{1}{\tau}\right), \quad \tau \to \infty, j = 1, 2. \tag{1.76}$$

Now we take two potentials q_1 and q_2 satisfying the hypothesis of Theorem 1.19. We take for the potential q_1 a solution u of the corresponding Schrödinger equation of the form

(1.75) and for the Schrödinger equation associated to q_2 a solution v of the form

$$v(x) = e^{-\tau\Phi(z)}(a(z) + b_0(z)/\tau) + e^{-\tau\overline{\Phi(z)}}\overline{(a(z) + b_1(z)/\tau)} + e^{-\tau\varphi}v_1 + e^{-\tau\varphi}v_2 \tag{1.77}$$

with v_1, v_2 satisfying the same decay for large τ as u_1, u_2. Using arguments similar to those of section 1.4.1 we get

$$\int_\Omega (q_1 - q_2)uvdx = 0. \tag{1.78}$$

Substituting (1.74) and (1.77) into this identity and applying stationary phase we conclude

Proposition 1.24. *Let $\{\widetilde{x}_1, \ldots, \widetilde{x}_\ell\}$ be the set of critical points of the function $Im\,\Phi$. Then for any potentials q_1, q_2 satisfying the hypotheses of Theorem 1.19 and for any holomorphic function a, we have*

$$2\sum_{k=1}^{\ell} \frac{\pi((q_1 - q_2)|a|^2)(\widetilde{x}_k)Re\, e^{2i\tau Im\,\Phi(\widetilde{x_k})}}{|(det\, Im\,\Phi'')(\widetilde{x_k})|^{\frac{1}{2}}} = 0, \quad \tau > 0.$$

We can choose Φ such that

$$\mathrm{Im}\,\Phi(\widetilde{x_k}) \neq \mathrm{Im}\,\Phi(\widetilde{x_j}),\ j \neq k.$$

Let $a(\widetilde{x_k}) \neq 0$. Then Proposition 1.24 implies

$$q_1(\widetilde{x_k}) = q_2(\widetilde{x_k}).$$

We then show that the non-degenerate critical points of Φ can be chosen to be a dense set concluding the sketch of the proof of the theorem.

1.7. Anisotropic Conductivities

Anisotropic conductivities depend on direction. Muscle tissue in the human body is an important example of an anisotropic conductor. For instance cardiac muscle has a conductivity of 2.3 mho in the transverse direction and 6.3 in the longitudinal direction. The conductivity in this case is represented by a positive definite, smooth, symmetric matrix $\gamma = (\gamma^{ij}(x))$ on Ω. Under the assumption of no sources or sinks of current in Ω, the potential u in Ω, given a voltage potential f on $\partial\Omega$, solves the Dirichlet problem

$$\begin{cases} \sum\limits_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(\gamma^{ij}\frac{\partial u}{\partial x_j}\right) &= 0 \text{ on } \Omega \\ u|_{\partial\Omega} &= f. \end{cases} \tag{1.79}$$

The DN map is defined by

$$\Lambda_\gamma(f) = \sum_{i,j=1}^{n} \nu^i\gamma^{ij}\frac{\partial u}{\partial x_j}\Big|_{\partial\Omega} \tag{1.80}$$

where $\nu = (\nu^1, \ldots, \nu^n)$ denotes the unit outer normal to $\partial\Omega$ and u is the solution of (1.79). The inverse problem is whether one can determine γ by knowing Λ_γ. Unfortunately, Λ_γ doesn't determine γ uniquely. This observation is due to L. Tartar (see [113] for an account). Let $\psi : \overline{\Omega} \to \overline{\Omega}$ be a C^∞ diffeomorphism with $\psi|_{\partial\Omega} = Id$ where Id denotes the identity map. We have

$$\Lambda_{\widetilde{\gamma}} = \Lambda_\gamma \tag{1.81}$$

where

$$\widetilde{\gamma} = \left(\frac{(D\psi)^T \circ \gamma \circ (D\psi)}{|\det D\psi|}\right) \circ \psi^{-1}. \tag{1.82}$$

Here $D\psi$ denotes the (matrix) differential of ψ, $(D\psi)^T$ its transpose and the composition in (1.82) is to be interpreted as multiplication of matrices. We have then a large number of conductivities with the same DN map: any change of variables of Ω that leaves the boundary fixed gives rise to a new conductivity with the same electrostatic boundary measurements. The question is then whether this is the only obstruction to unique identifiability of the conductivity. In two dimensions this has been shown for $L^\infty(\Omega)$ conductivities in [12]. This is done by reducing the anisotropic problem to the isotropic one by using isothermal coordinates [2], [191] and using Astala and Päivärinta's result in the isotropic case [11]. Earlier results were for C^3 conductivities using the result of Nachman [135] and for Lipschitz conductivities in [188] using the techniques of [34]. An extension of some of these results to quasilinear anisotropic conductivities can be found in [190]. In three dimensions, as was pointed out in [125], this is a problem of geometrical nature and makes sense for general compact Riemannian manifolds with boundary. Let (M, g) be a compact Riemannian manifold with boundary. The Laplace-Beltrami operator associated to the metric g is given in local coordinates by

$$\Delta_g u = \frac{1}{\sqrt{\det g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(\sqrt{\det g} g^{ij} \frac{\partial u}{\partial x_j} \right) \tag{1.83}$$

where (g^{ij}) is the matrix inverse of the matrix (g_{ij}). Let us consider the Dirichlet problem associated to (1.83)

$$\Delta_g u = 0 \text{ on } \Omega, \quad u|_{\partial\Omega} = f. \tag{1.84}$$

We define the DN map in this case by

$$\Lambda_g(f) = \sum_{i,j=1}^{n} \nu^i g^{ij} \frac{\partial u}{\partial x_j} \sqrt{\det g}|_{\partial\Omega} \tag{1.85}$$

The inverse problem is to recover g from Λ_g. We have

$$\Lambda_{\psi^* g} = \Lambda_g \tag{1.86}$$

where ψ is any C^∞ diffeomorphism of $\overline{M}$ which is the identity on the boundary. As usual $\psi^* g$ denotes the pull back of the metric g by the diffeomorphism ψ. In the case that M is an open, bounded subset of $\mathbb{R}^n$ with smooth boundary, it is easy to see ([125]) that for $n \geq 3$

$$\Lambda_g = \Lambda_\gamma \tag{1.87}$$

where

$$(g_{ij}) = (\det \gamma^{kl})^{\frac{1}{n-2}} (\gamma^{ij})^{-1}, \quad (\gamma^{ij}) = (\det g_{kl})^{\frac{1}{2}} (g_{ij})^{-1}. \tag{1.88}$$

In the two dimensional case there is an additional obstruction since the Laplace-Beltrami operator is conformally invariant. More precisely we have

$$\Delta_{\alpha g} = \frac{1}{\alpha} \Delta_g$$

for any function α, $\alpha \neq 0$. Therefore we have, for $n = 2$,

$$\Lambda_{\alpha(\psi^* g)} = \Lambda_g \tag{1.89}$$

for any smooth function $\alpha \neq 0$ so that $\alpha|_{\partial M} = 1$. Lassas and Uhlmann ([123]) proved that (1.86) is the only obstruction to unique identifiability of the conductivity for real-analytic manifolds in dimension $n \geq 3$. In the two dimensional case they showed that (1.89) is the only obstruction to unique identifiability for smooth Riemannian surfaces. Moreover these

results assume that the DN map is measured only on an open subset of the boundary. We state the two basic results. Let Γ be an open subset ∂M. We define for f, $\operatorname{supp} f \subseteq \Gamma$

$$\Lambda_{g,\Gamma}(f) = \Lambda_g(f)|_\Gamma.$$

Theorem 1.25 ($n = 2$)**.** *Let (M, g) be a compact Riemannian surface with boundary. Let $\Gamma \subseteq \partial M$ be an open subset. Then $\Lambda_{g,\Gamma}$ determines uniquely the conformal class of (M, g).*

Theorem 1.26 ($n \geq 3$)**.** *Let (M, g) be a real-analytic compact, connected Riemannian manifold with boundary. Let $\Gamma \subseteq \partial M$ be real-analytic and assume that g is real-analytic up to Γ. Then $\Lambda_{g,\Gamma}$ determines uniquely (M, g) up to an isometry.*

Einstein manifolds are real-analytic in the interior and it was conjectured by Lassas and Uhlmann that they were uniquely determined up to isometry by the DN map. This was proven in [69]. Notice that these results don't assume any condition on the topology of the manifold except for connectedness. An earlier result of [125] assumed that (M, g) was strongly convex and simply connected and $\Gamma = \partial M$ in both results. Theorem 1.26 was extended in [124] to non-compact, connected real-analytic manifolds with boundary. These results were extended to differential forms in [117].

1.7.1. The Calderón Problem on Manifolds

The invariant form on a Riemannian manifold with boundary (M, g) for an isotropic conductivity β is given by

$$\operatorname{div}_g(\beta \nabla_g) u = 0 \tag{1.90}$$

where div_g (resp. ∇_g) denotes divergence (resp. gradient) with respect to the Riemannian metric g. This includes the case considered by Calderón with g the Euclidean metric, the anisotropic case by taking $g^{ij} = \gamma^{ij}\beta$ and $\beta = \sqrt{\det g}$. It was shown in [188] for bounded domains of Euclidean space in two dimensions that the isometric class of (β, g) is determined uniquely by the DN map associated to (1.90). In two dimensions, when the metric g is known, it is proven in [78] that one can uniquely determine the conductivity β. Guillarmou and Tzou [71] have shown that a potential is uniquely determined for the Schrödinger equation $\Delta_g - q$, with Δ_g the Laplace-Beltrami operator associated to the metric g, generalizing the result of [78]. In dimension $n \geq 3$ it is an open problem whether one can determine the isotropic conductivity β from the corresponding DN map associated to (1.90). As before one can consider the more general problem of recovering the potential q from the DN map associated to $\Delta_g - q$. We review below the progress that has been made on this problem based on [53].

1.7.2. Complex geometrical optics on manifolds

We review the recent construction of complex geometrical optics construction for a class of Riemannian manifolds based on [53]. In this paper those Riemannian manifolds which admit limiting Carleman weights, were characterized. All such weights in Euclidean space were listed in Theorem 1.13.

Theorem 1.27. *If (M, g) is an open manifold having a limiting Carleman weight, then some conformal multiple of the metric g admits a parallel unit vector field. For simply connected manifolds, the converse is also true.*

Locally, a manifold admits a parallel unit vector field if and only if it is isometric to the product of an Euclidean interval and another Riemannian manifold. This is an instance of the de Rham decomposition [159]. Thus, if (M, g) has an LCW φ, one can choose local coordinates in such a way that $\phi(x) = x_1$ and

$$g(x_1, x') = c(x) \begin{pmatrix} 1 & 0 \\ 0 & g_0(x') \end{pmatrix},$$

where c is a positive conformal factor. Conversely, any metric of this form admits $\varphi(x) = x_1$ as a limiting weight. In the case $n = 2$, limiting Carleman weights in (M, g) are exactly the harmonic functions with non-vanishing differential. Let us now introduce the class of manifolds which admit limiting Carleman weights and for which one can prove uniqueness results. For this we need the notion of simple manifolds [168].

Definition 1.28. *A manifold (M, g) with boundary is* simple *if ∂M is strictly convex, and for any point $x \in M$ the exponential map $\exp_x$ is a diffeomorphism from some closed neighborhood of 0 in T_xM onto M.*

Definition 1.29. *A compact manifold with boundary (M, g), of dimension $n \geq 3$, is* admissible *if it is conformal to a submanifold with boundary of $\mathbb{R} \times (M_0, g_0)$ where (M_0, g_0) is a compact simple $(n-1)$-dimensional manifold.*

Examples of admissible manifolds include the following:

1. Bounded domains in Euclidean space, in the sphere minus a point, or in hyperbolic space. In the last two cases, the manifold is conformal to a domain in Euclidean space via stereographic projection.

2. More generally, any domain in a locally conformally flat manifold is admissible, provided that the domain is appropriately small. Such manifolds include locally symmetric 3-dimensional spaces, which have parallel curvature tensor so their Cotton tensor vanishes [55].

3. Any bounded domain M in $\mathbb{R}^n$, endowed with a metric which in some coordinates has the form
$$g(x_1, x') = c(x) \begin{pmatrix} 1 & 0 \\ 0 & g_0(x') \end{pmatrix},$$
with $c > 0$ and g_0 simple, is admissible.

4. The class of admissible metrics is stable under C^2-small perturbations of g_0.

The first inverse problem involves the Schrödinger operator

$$\mathcal{L}_{g,q} = \Delta_g - q,$$

where q is a smooth complex valued function on (M, g). We make the standard assumption that 0 is not a Dirichlet eigenvalue of $\mathcal{L}_{g,q}$ in M. Then the Dirichlet problem

$$\begin{cases} \mathcal{L}_{g,q} u = 0 \text{ in } M, \\ \quad u = f \text{ on } \partial M \end{cases}$$

has a unique solution for any $f \in H^{1/2}(\partial M)$, and we may define the DN map

$$\Lambda_{g,q} : f \mapsto \partial_\nu u|_{\partial M}.$$

Given a fixed admissible metric, one can determine the potential q from boundary measurements.

Theorem 1.30. *Let (M, g) be admissible, and let q_1 and q_2 be two smooth functions on M. If $\Lambda_{g,q_1} = \Lambda_{g,q_2}$, then $q_1 = q_2$.*

This result was known previously in dimensions $n \geq 3$ for the Euclidean metric [171] and for the hyperbolic metric [101]. It has been generalized to Maxwell's equations in [107].

2. Travel Time Tomography and Boundary Rigidity

2.1. Introduction

The question of determining the sound speed or index of refraction of a medium by measuring the first arrival times of waves arose in geophysics in an attempt to determine the substructure of the Earth by measuring at the surface of the Earth the travel times of seismic waves. An early success of this inverse method was the estimate by Herglotz [79] and Wiechert and Zoeppritz [212] of the diameter of the Earth and the location of the mantle, crust and core. The assumption used in those papers is that the index of refraction (inverse proportional to the speed) depends only on the radius. A more realistic model is to assume that it depends on position. The inverse kinematic problem can be formulated mathematically as determining a Riemannian metric on a bounded domain (the Earth) given by $ds^2 = \frac{1}{c^2(x)}dx^2$, where c is a positive function, from the length of geodesics (travel times) joining points in the boundary. More recently it has been realized, by measuring the travel times of seismic waves, that the inner core of the Earth might exhibit anisotropic behavior, that is the speed of waves depends also on direction there with the fast direction parallel to the Earth's spin axis [45]. Given the complications presented by modeling the Earth as an anisotropic elastic medium we consider a simpler model of anisotropy, namely that the wave speed is given by a symmetric, positive definite matrix $g = (g_{ij})(x)$, that is, a Riemannian metric in mathematical terms. The problem is to determine the metric from the lengths of geodesics joining points in the boundary (the surface of the Earth in the motivating example). It is useful to consider a more general and geometric formulation of the problem. Let (M, g) be a compact Riemannian manifold with boundary ∂M. Let $d_g(x, y)$ denote the geodesic distance between x and y, two points in the boundary. This is defined as the infimum of the length of all sufficiently smooth curves joining the two points. The function d_g measures the first arrival time of waves joining points of the boundary. One of the inverse problems we discuss in this section is whether we can determine the Riemannian metric g knowing $d_g(x, y)$ for any $x \in \partial M$, $y \in \partial M$. This problem also arose in rigidity questions in Riemannian geometry [130], [46], [68]. The metric g cannot be determined from this information alone. We have $d_{\psi^* g} = d_g$ for any diffeomorphism $\psi : M \to M$ that leaves the boundary pointwise fixed, i.e., $\psi|_{\partial M} = Id$, where Id denotes the identity map and $\psi^* g$ is the pull-back of the metric g. The natural question is whether this is the only obstruction to unique identifiability of the metric. It is easy to see that this is not the case. Namely one can construct a metric g and find a point x_0 in M so that $d_g(x_0, \partial M) > \sup_{x,y \in \partial M} d_g(x, y)$. For such a metric, d_g is independent of a change of g in a neighborhood of x_0. The hemisphere of the round sphere is another example. Therefore it is necessary to impose some a-priori restrictions on the metric. One such restriction is to assume that the Riemannian manifold (M, g)

is **simple**, i.e., M is simply-connected, any geodesic has no conjugate points and ∂M is strictly convex. ∂M is strictly convex if the second fundamental form of the boundary is positive definite in every boundary point. R. Michel conjectured in [130] that simple manifolds are boundary distance rigid, that is d_g determines g uniquely up to an isometry which is the identity on the boundary. This is known for simple subspaces of Euclidean space (see [68]), simple subspaces of an open hemisphere in two dimensions (see [131]), simple subspaces of symmetric spaces of constant negative curvature [27], simple two dimensional spaces of negative curvature (see [47] or [150]). If one metric is close to the Euclidean metric boundary rigidity was proven in [122] that was improved in [37]. We remark that simplicity of a compact manifold with boundary can be determined from the boundary distance function. Michel's conjecture was proven in generality in [156] in two dimensions and we describe the details of the proof in section 2.2. In the case that both g_1 and g_2 are conformal to the Euclidean metric e (i.e., $(g_k)_{ij} = \alpha_k \delta_{ij}$, $k = 1, 2$ with δ_{ij} the Kronecker symbol), as mentioned earlier, the problem we are considering here is known in seismology as the inverse kinematic problem. In this case, it has been proven by Mukhometov in two dimensions [132] that if $(M, g_i), i = 1, 2$ is simple and $d_{g_1} = d_{g_2}$, then $g_1 = g_2$. More generally the same method of proof shows that if $(M, g_i), i = 1, 2$, are simple compact Riemannian manifolds with boundary and they are in the same conformal class then the metrics are determined by the boundary distance function. More precisely we have:

Theorem 2.1. *Let $(M, g_i), i = 1, 2$ be simple compact Riemannian manifolds with boundary of dimension $n \geq 2$. Assume $g_1 = \rho g_2$ for a positive, smooth function $\rho, \rho|_{\partial M} = 1$ and $d_{g_1} = d_{g_2}$ then $g_1 = g_2$.*

This result and a stability estimate were proven in [132]. We remark that in this case the diffeomorphism ψ that is present in the general case must be the identity if the metrics are conformal to each other. For related results and generalizations see [28], [26], [46], [60], [134].

In section 2.2 we consider the boundary rigidity in the two dimensional case and in section 2.3 in the higher dimensional case. In section 2.4 we discuss some results on the non-simple case where the measurements are given by the *scattering relation*. Roughly speaking one measures the point of exit and direction of exit of a geodesic for which we know the point of entrance and direction of entrance besides this we also know the travel time, that is the length of that geodesic.

2.2. Boundary Rigidity in Two Dimensions

We will sketch in this section the proof of the following result.

Theorem 2.2. *Let $(M, g_i), i = 1, 2$ be two dimensional simple compact Riemannian manifolds with boundary. Assume*

$$d_{g_1}(x, y) = d_{g_2}(x, y) \quad \forall (x, y) \in \partial M \times \partial M.$$

Then there exists a diffeomorphism $\psi : M \to M$, $\psi|_{\partial M} = Id$, so that

$$g_2 = \psi^* g_1.$$

The proof of Theorem 2.2 involves a connection between the scattering relation and the Dirichlet-to-Neumann map (DN) associated to the Laplace-Beltrami operator discussed

in section 1.7. In section 2.2.1 we define the scattering relation which quantizes the scattering operator. In Section 2.2.2 we discuss the geodesic X-ray transform and compute the commutator of the fiberwise Hilbert transform and geodesic flow (see Theorem 2.7 and Theorem 2.8). In Section 2.2.3 we discuss the main step of the proof of Theorem 2.2 which consists in showing that, under the assumptions of the theorem, we can determine the Dirichlet-to-Neumann map if we know the scattering relation. We also sketch the proof of Theorem 2.9. In section 2.2.4 we use the connection indicated in section 2.2.1, to give a characterization of the range of the geodesic X-ray transform in terms of the scattering relation and we give Fredholm type inversion formulas for the geodesic X-ray transform acting on scalar functions and vector fields.

2.2.1. The scattering relation

Suppose we have a Riemannian metric in Euclidean space which is the Euclidean metric outside a compact set. The inverse scattering problem for metrics is to determine the Riemannian metric by measuring the scattering operator (see [73]). A similar obstruction to the boundary rigidity problem occurs in this case with the diffeomorphism ψ equal to the identity outside a compact set. It was proven in [73] that from the wave front set of the scattering operator, one can determine, under some non-trapping assumptions on the metric, the **scattering relation** on the boundary of a large ball. This uses high frequency information of the scattering operator. In the semiclassical setting Alexandrova has shown for a large class of operators that the scattering operator associated to potential and metric perturbations of the Euclidean Laplacian is a semiclassical Fourier integral operator quantized by the scattering relation [8]. The scattering relation maps the point and direction of a geodesic entering the manifold to the point and direction of exit of the geodesic. We proceed to define in more detail the scattering relation and its relation with the boundary distance function. Let ν denote the unit-inner normal to ∂M. We denote by $S\left(M\right) \to M$ the unit-sphere bundle over M:

$$S(M) = \bigcup_{x \in M} S_x, \quad S_x = \{\xi \in T_x(M) : |\xi|_g = 1\}.$$

$S(M)$ is a (2 dim $M - 1$)-dimensional compact manifold with boundary, which can be written as the union $\partial\Omega\left(M\right) = \partial_+ S\left(M\right) \cup \partial_- S\left(M\right)$

$$\partial_\pm S\left(M\right) = \{(x,\xi) \in \partial S\left(M\right), \ \pm\left(\nu\left(x\right), \xi\right) \geq 0 \ \}.$$

The manifold of inner vectors $\partial_+ S\left(M\right)$ and outer vectors $\partial_- S\left(M\right)$ intersect at the set of tangent vectors

$$\partial_0 S\left(M\right) = \{(x,\xi) \in \partial S\left(M\right), \quad \left(\nu\left(x\right), \xi\right) = 0 \ \}.$$

Let (M, g) be an n-dimensional compact manifold with boundary. We say that (M, g) is **non-trapping** if each maximal geodesic is finite. Let (M, g) be non-trapping and the boundary ∂M is strictly convex. Denote by $\tau(x, \xi)$ the length of the geodesic $\gamma(x, \xi, t), t \geq 0$, starting at the point x in the direction $\xi \in S_x$. This function is smooth on $S(M) \setminus \partial_0 S(M)$. The function $\tau^0 = \tau|_{\partial S(M)}$ is equal to zero on $\partial_- S(M)$ and is smooth on $\partial_+ S(M)$. Its odd part with respect to ξ

$$\tau^0_-(x,\xi) = \frac{1}{2}\left(\tau^0(x,\xi) - \tau^0\left(x, -\xi\right)\right)$$

is a smooth function.

Definition 2.3. *Let (M,g) be non-trapping with strictly convex boundary. The scattering relation $\alpha : \partial S(M) \to \partial S(M)$ is defined by*

$$\alpha(x,\xi) = (\gamma(x,\xi,2\tau_-^0(x,\xi)), \dot\gamma(x,\xi,2\tau_-^0(x,\xi))).$$

The scattering relation is a diffeomorphism $\partial S(M) \to \partial S(M)$. Notice that $\alpha|_{\partial_+ S(M)} : \partial_+ S(M) \to \partial_- S(M)$, $\alpha|_{\partial_- S(M)} : \partial_- S(M) \to \partial_+ S(M)$ are diffeomorphisms as well. Obviously, α is an involution, $\alpha^2 = id$ and $\partial_0 S(M)$ is the hypersurface of its fixed points, $\alpha(x,\xi) = (x,\xi)$, $(x,\xi) \in \partial_0 S(M)$. A natural inverse problem is whether the scattering relation determines the metric g up to an isometry which is the identity on the boundary. This information takes into account all the travel times not just the first arrivals. In the case that (M,g) is a simple manifold, and we know the metric at the boundary (and this is determined if d_g is known, see [130], knowing the scattering relation is equivalent to knowing the boundary distance function ([130]) so that we concentrate on studying the scattering relation. We introduce the operators of even and odd continuation with respect to α:

$$A_\pm w(x,\xi) = w(x,\xi), \qquad (x,\xi) \in \partial_+ S(M),$$

$$A_\pm w(x,\xi) = \pm(\alpha^* w)(x,\xi), \quad (x,\xi) \in \partial_- S(M).$$

The scattering relation preserves the measure $|(\xi,\nu)|d\Sigma$, ($d\Sigma$ is the measure of the boundary $\partial S(M)$ induced by the metric g) and therefore the operators $A_\pm : L^2_\mu(\partial_+ S(M)) \to L^2_{|\mu|}(\partial S(M))$ are bounded, where $L^2_{|\mu|}(\partial S(M))$ is the real Hilbert space with scalar product

$$(u,v)_{L^2_{|\mu|}(\partial S(M))} = \int\limits_{\partial S(M)} |\mu|\, uv d\Sigma, \quad \mu = (\xi,\nu).$$

and $L^2_\mu(\partial_+ S(M))$ is the real Hilbert space with scalar product

$$(u,v)_{L^2_\mu(\partial_+ S(M))} = \int\limits_{\partial_+ S(M)} \mu uv d\Sigma.$$

The adjoint of $A_\pm$ is a bounded operator $A_\pm^* : L^2_{|\mu|}(\partial S(M)) \to L^2_\mu(\partial_+ S(M))$ given by

$$A_\pm^* u = (u \pm u \circ \alpha)|_{\partial_+ S(M)}.$$

2.2.2. The geodesic X-ray transform

The X-ray transform integrates a function along lines. Radon found in 1917 an inversion formula in two dimensions to determine a function knowing the X-ray transform. This formula is non-local in the sense that in order to find the function at a point x one needs to know the integral of the function along lines far from the point. Radon's inversion formula has been implemented numerically using the filtered backprojection algorithm which is used today in CT scans. Another important transform in medical imaging and other applications is the Doppler transform which integrates a vector field along lines. The motivation is ultrasound Doppler tomography. It is known that blood flow is irregular and faster around tumor tissue than in normal tissue and Doppler tomography attempts to reconstruct the blood flow pattern. Mathematically the problem is to what extend a vector field is determined from its integral along lines. In this paper we consider the case of integrating functions and vector fields along geodesics of a Riemannian metric. This arises in geophysics since the ray paths are no longer straight lines. We obtain inversion formulas

for the constant curvature case and Fredholm type formulas in general which are *non-local*. We define next the geodesic X-ray transform for any compact Riemannian manifold (M, g) with boundary of any dimension. We embed (M, g) into a compact Riemannian manifold $(\widetilde{M}, g)$ with no boundary. Let φ_t be the geodesic flow on $S(\widetilde{M})$ and $\mathcal{H} = \frac{d}{dt}\varphi_t|_{t=0}$ be the geodesic vector field. Let u^f be the solution of the boundary value problem

$$\mathcal{H}u = -f, \quad u|_{\partial_- S(M)} = 0,$$

which can be written as

$$u^f(x, \xi) = \int\limits_0^{\tau(x,\xi)} f(\varphi_t(x, \xi))dt, \quad (x, \xi) \in S(M).$$

In particular

$$\mathcal{H}\tau = -1.$$

The trace

$$If = u^f|_{\partial_+ S(M)}$$

is called **the geodesic X-ray transform** of the function f. If the manifold (M, g) is non-trapping and has a strictly convex boundary the operator $I : C^\infty(S(M)) \to C^\infty(\partial_+ S(M))$. Clearly a function f is not determined by its geodesic X-ray transform alone, since it depends on more variables than If. We consider the geodesic X-ray transform acting on symmetric tensor fields. We denote by $f_m(x, \xi)$ an homogeneous polynomial of degree m with respect to ξ, induced by the symmetric tensor field f on (M, g) of m degree :

$$f_m(x, \xi) = f_{i_1 \dots i_m}(x)\, \xi^{i_1} \dots \xi^{i_m}.$$

The operator I_m, defined by

$$I_m f = I f_m$$

is called **the geodesic X-ray transform** of the symmetric tensor field. If the manifold (M, g) is non-trapping and the boundary ∂M is strictly convex $I_m : C^\infty(M, S_m(M)) \to C^\infty(\partial_+ S(M))$, where $S_m(M)$ denotes the bundle of symmetric tensors over (M, g). It is known that any symmetric smooth enough tensor field f may be decomposed in a potential and solenoidal part [168]:

$$f = dp + h, \;\; p|_{\partial M} = 0, \;\; \delta h = 0, \tag{2.1}$$

where δ denotes the divergence and $d = \sigma\nabla$ is the symmetric part of covariant derivative. It is easy to see that the geodesic X-ray transform of the potential part dp is zero. We denote by $C^\infty_{sol}(M, S_m(M))$ the space of smooth solenoidal symmetric tensor fields. The case of the geodesic X-ray transform acting on functions independent of ξ and the geodesic X-ray transform acting on vector fields which, following the notation above, are denoted by I_0 and I_1 respectively. It is known that I_0 is injective on simple manifolds [132] and that I_1 is injective acting on solenoidal vector fields on simple manifolds [10] and for tensors of order two in [169]. Recently in [151] it has been proven injectivity of I_m for all m for simple two-dimensional manifolds. We mention also that the transform I_2 arises in the linearization of the boundary rigidity problem (see [168]). We define $\psi : S(M) \to \partial_- S(M)$ by

$$\psi(x, \xi) = \varphi_{-\tau(x,-\xi)}(x, \xi), (x, \xi) \in S(M).$$

So, φ is a retract which maps vector (x, ξ) along geodesic $\gamma(x, \xi, t)$ in back direction into incoming vector. The solution of the boundary value problem for the transport equation

$$\mathcal{H}u = 0, \quad u|_{\partial_+ S(M)} = w$$

can be written in the form

$$u = w_\psi = w \circ \psi.$$

The adjoint of the operator I_m is the bounded operator $I_m^* : L^2_\mu(\partial_+ S(M)) \to L^2(M, S_m(M))$ which is given by

$$(I_m^* w)^{i_1 \dots i_m}(x) = \int\limits_{S_x} w_\psi(x,\xi)\xi^{i_1}...\xi^{i_m} dS_x.$$

The Hilbert space $L^2(M, S_m(M))$ may be considered as subspace of $L^2(S(M))$ of homogeneous polynomials with respect to ξ of degree m. The field $I_m^* w$ is solenoidal in the sense of the theory of distributions. Notice, that the adjoint of the bounded operator $I : L^2(S(M)) \to L^2_\mu(\partial_+ S(M))$ is given by

$$I^* w = w_\psi.$$

We also remark that by the fundamental theorem of calculus we have

$$I\mathcal{H}f = (f \circ \alpha - f)|_{\partial_+ S(M)} = -A_-^* f^0, f^0 = f|_{\partial S(M)}. \tag{2.2}$$

The space $C^\infty_\alpha(\partial_+ S(M))$ is defined by

$$C^\infty_\alpha(\partial_+ S(M)) = \{w \in C^\infty(\partial_+ S(M)) : w_\psi \in C^\infty(S(M))\}.$$

In [156] the following characterization of the space of smooth solutions of the transport equation was given

Lemma 2.4.

$$C^\infty_\alpha(\partial_+ S(M)) = \{w \in C^\infty(\partial_+ S(M)) : A_+ w \in C^\infty(\partial S(M))\}.$$

In the scalar case the following result holds on the solvability of I_0^* [156].

Theorem 2.5. *Let (M, g) be a simple, compact Riemannian manifold with boundary. Then the operator $I_0^* : C^\infty_\alpha(\partial_+ S(M)) \to C^\infty(M)$ is onto.*

The analog result for vector fields was proven in [50].

Theorem 2.6. *Let (M, g) be a simple, compact Riemannian manifold with boundary. Then for any field $v \in C^\infty_{sol}(M, T(M))$ there exists a function $w \in C^\infty_\alpha(\partial_+ S(M))$ such that*

$$v = I_1^* w.$$

Now we define the Hilbert transform in the ξ variable:

$$Hu(x,\xi) = \frac{1}{2\pi}\int_{S_x} \frac{1 + (\xi,\eta)}{(\xi_\perp,\eta)} u(x,\eta) dS_x(\eta), \quad \xi \in S_x,$$

where the integral is understood as a principle value integral. Here $\perp$ means a 90° degree rotation. In coordinates $(\xi_\perp)_i = \varepsilon_{ij}\xi^j$, where

$$\varepsilon = \sqrt{\det g}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The Hilbert transform H transforms even (respectively odd) functions with respect to ξ to even (respectively odd) ones. If H_+ (respectively H_-) is the even (respectively odd) part of the operator H:

$$H_+ u(x,\xi) = \frac{1}{2\pi}\int_{S_x} \frac{(\xi,\eta)}{(\xi_\perp,\eta)} u(x,\eta) dS_x(\eta),$$

$$Hu_-(x,\xi) = \frac{1}{2\pi}\int_{S_x} \frac{1}{(\xi_\perp,\eta)} u(x,\eta) dS_x(\eta)$$

and u_+, u_- are the even and odd parts of the function u, then $H_+u = Hu_+, H_-u = Hu_-$. We introduce the notation $\mathcal{H}_\perp = (\xi_\perp, \nabla) = -(\xi, \nabla_\perp)$, where $\nabla_\perp = \varepsilon\nabla$ and ∇ is the covariant derivative with respect to the metric g. The following commutator formula for the geodesic vector field and the Hilbert transform, is a crucial ingredient in the proofs of the main theorems surveyed in this paper (see [156]).

Theorem 2.7. *Let (M, g) be a two dimensional Riemannian manifold. For any smooth function u on $S(M)$ we have the identity*

$$[H, \mathcal{H}]u = \mathcal{H}_\perp u_0 + (\mathcal{H}_\perp u)_0$$

where

$$u_0(x) = \frac{1}{2\pi}\int_{S_x} u(x,\xi) dS_x$$

is the average value.

We define

$$P_- = A_-^* H_- A_+, \quad P_+ = A_-^* H_+ A_+.$$

If the manifold (M, g) is simple, the following factorizations hold:

Theorem 2.8.

$$P_- = -\frac{1}{2\pi} I\delta_\perp I_1^*, \quad P_+ = -\frac{1}{2\pi} I_1 \nabla_\perp I_0^*. \tag{2.3}$$

2.2.3. The scattering relation and the Dirichlet-to-Neumann Map

The DN map for the Laplace-Beltrami operator was defined in section 1.7. The connection in two dimensions between the DN map and the scattering relation is given by

Theorem 2.9. *Let $(M, g_i), i = 1, 2$, be compact, simple two dimensional Riemannian manifolds with boundary. Assume that $\alpha_{g_1} = \alpha_{g_2}$. Then $\Lambda_{g_1} = \Lambda_{g_2}$.*

The proof of Theorem 2.2 is reduced then to the proof of Theorem 2.9. In fact from Theorem 2.9 and Theorem 1.25 we obtain that we can determine the conformal class of the metric up to an isometry which is the identity on the boundary. Now by Theorem 2.1 we have that the conformal factor must be one proving that the metrics are isometric via a diffeomorphism which is the identity at the boundary. In other words $d_{g_1} = d_{g_2}$ implies that $\alpha_{g_1} = \alpha_{g_2}$. By Theorem 2.9 $\Lambda_{g_1} = \Lambda_{g_2}$. By Theorem 1.25, there exists a diffeomorphism $\psi : M \longrightarrow M$, $\psi|_{\partial M} =$ Identity and a function $\beta \neq 0, \beta|_{\partial M} =$ identity such that $g_1 = \beta\psi^* g_2$. By Mukhometov's theorem $\beta = 1$ showing that $g_1 = \psi^* g_2$ proving Theorem 2.2. Before starting the proof of Theorem 2.9 we recall that Michel [131] has proven that for two dimensional Riemannian manifolds with strictly convex boundary one can determine from the boundary distance function, up to the natural obstruction, all the derivatives of the metric at the boundary. This result was generalized to any dimensions in [122]. The proof of Theorem 2.9 consists in showing that from the scattering relation we can determine the traces at the boundary of conjugate harmonic functions, which is equivalent information to knowing the DN map associated to the Laplace-Beltrami operator.

Sketch of the proof of Theorem 2.9 Let (h, h_*) be a pair of conjugate harmonic functions

on M,

$$\nabla h = \nabla_\perp h_*, \quad \nabla h_* = -\nabla_\perp h.$$

Notice, that $\delta\nabla = \triangle$ is the Laplace-Beltrami operator and $\delta\nabla_\perp = 0$. Let $I_0^* w = h$. Since $I_1\mathcal{H}_\perp h = I_1\mathcal{H}h_* = -A_-^* h_*^0$, where $h_*^0 = h_*|_{\partial M}$, we obtain from the second identity (2.3)

$$2\pi A_-^* H_+ A_+ w = -A_-^* h_*^0. \tag{2.4}$$

The following theorem gives the key to obtain the DN map from the scattering relation.

Theorem 2.10. *Let M be a 2-dimensional simple manifold. Let $w \in C_\alpha^\infty(\partial_+ S(M))$ and h_* the harmonic continuation of function h_*^0. Then the equation (2.4) holds iff the functions $h = I_0^* w$ and h_* are conjugate harmonic functions.*

In summary we have the following procedure to obtain the DN map from the scattering relation. For a given smooth function h_*^0 on ∂M we find a solution $w \in C_\alpha^\infty(\partial_+ S(M))$ of the equation (2.4). Then the functions $h^0 = 2\pi(A_+ w)_0$ (notice, that $2\pi(A_+ w)_0 = I_0^* w|_{\partial M}$) and h_*^0 are the traces of conjugate harmonic functions. It is easy to see that this gives the DN map.

2.2.4. Range and inversion of the geodesic X-ray transform

Let $T(M)$ be the tangent bundle of M. We denote by δ the divergence operator $\delta : C^\infty(M, TM)) \to C^\infty(M)$. In local coordinates this is given by $\delta u = g^{kj}\nabla_k u_j$ using Einstein's summation convention. We define the operator $\delta_\perp : C^\infty(M, T(M)) \to C^\infty(M)$ by

$$\delta_\perp u = -\delta u_\perp.$$

Then

$$\delta_\perp \nabla_\perp f = \delta\nabla f = \Delta f, \;\; \delta_\perp \nabla f = -\delta\nabla_\perp f = 0.$$

We now give the characterization of the range of I_0 and I_1 in terms of the scattering relation only. We have that these are the projections of the operators P_-, P_+ respectively. For the details see [158].

Theorem 2.11. *Let (M, g) be simple two dimensional compact Riemannian manifold with boundary. Then i) The maps*

$$\begin{aligned} \delta_\perp I_1^* &: \; C_\alpha^\infty(\partial_+ S(M)) \to C^\infty(M), \\ \nabla_\perp I_0^* &: \; C_\alpha^\infty(\partial_+ S(M)) \to C_{sol}^\infty(M, T(M)) \end{aligned}$$

are onto. ii). A function $u \in C^\infty(\partial_+ S(M))$ belong to Range I_0 iff $u = P_- w$, $w \in C_\alpha^\infty(\partial_+ S(M))$. iii). A function $u \in C^\infty(\partial_+ S(M))$ belong to Range I_1 iff $u = P_+ w$, $w \in C_\alpha^\infty(\partial_+ S(M))$.

Proposition 2.12. *The operator $W : C_0^\infty(M) \to C^\infty(M)$, defined by*

$$Wf = (\mathcal{H}_\perp u^f)_0$$

can be extended to a smoothing operator $W : L^2(M) \to C^\infty(M)$.

We remark that in the case of constant Gaussian curvature $W = 0$ and this does not depend on whether the metric has conjugate points so that the inversion formulas of Theorem 2.13 hold for all two dimensional manifolds with boundary with constant curvature. The inversion formulas are (see [158])

Theorem 2.13. *Let (M, g) be a two-dimensional simple manifold. Then we have*

$$\begin{aligned} f + W^2 f &= \frac{1}{2\pi}\delta_\perp I_1^* w, \quad w = \frac{1}{2}\alpha^* H(I_0 f)^-|_{\partial_+ S(M)}, \quad f \in L^2(M), \\ h + (W^*)^2 h &= \frac{1}{2\pi} I_0^* w, \quad w = \frac{1}{2}\alpha^* H(I_1 \mathcal{H}_\perp h)^+|_{\partial_+ S(M)}, \; h \in H_0^1(M), \end{aligned}$$

where $W, W^* : L^2(M) \to C^\infty(M)$. *In the case of a manifold of constant curvature* $W = 0, \quad W^* = 0$.

2.2.5. Remarks

The Hilbert transform for 2-dimensional Riemannian manifolds is the map that relates the restrictions on the boundary of conjugate harmonic functions. In this sense the Hilbert transform, up to a constant, is just the Dirichlet-to-Neumann (DN) map. In section 2.2 we fixed a point x and started with the microlocal Hilbert transform on the circle S_x in the tangent space with Euclidean metric and we ended with the global Hilbert transform (the DN map). The scattering relation and the boundary distance function are determined by the singularities of the DN map associated to the wave equation for the Laplace-Beltrami operator, the so-called hyperbolic (or dynamic) Dirichlet-to-Neumann map [204]. We have found, in two dimensions, a connection between the scattering relation and the *elliptic* Dirichlet-to-Neumann map which led to a solution of the boundary rigidity problem in two dimensions. Is there a similar connection in higher dimensions?

2.3. Boundary Rigidity and Tensor Tomography in Dimensions $n \geq 3$

In [179], it was proven a local result for metrics in a small neighborhood of the Euclidean one. This result was used in [122] to prove a semiglobal solvability result assuming that one metric is close to the Euclidean and the other has bounded curvature. As it was mentioned earlier it is known [168], that a linearization of the boundary rigidity problem near a simple metric g is given by the following integral geometry problem: recover a symmetric tensor of order 2, which in any coordinates is given by $f = (f_{ij})$, by the geodesic X-ray transform

$$I_g f(\gamma) = \int f_{ij}(\gamma(t))\dot{\gamma}^i(t)\dot{\gamma}^j(t)\,\mathrm{d}t$$

known for all geodesics γ in M. In this section we denote by I_g the geodesic X-ray transform of tensors of order two. It can be easily seen that $I_g dv = 0$ for any vector field v with $v|_{\partial M} = 0$, where dv denotes the symmetric differential

$$[dv]_{ij} = \frac{1}{2}\left(\nabla_i v_j + \nabla_j v_i\right), \tag{2.5}$$

and $\nabla_k v$ denote the covariant derivatives of the vector field v. This is the linear version of the fact that d_g does not change on $(\partial M)^2 := \partial M \times \partial M$ under an action of a diffeomorphism as above. The natural formulation of the linearized problem is therefore that $I_g f = 0$ implies $f = dv$ with v vanishing on the boundary. We will refer to this property as *s-injectivity* of I_g. More precisely, we have.

Definition 2.14. *We say that I_g is* s-injective *in M, if $I_g f = 0$ and $f \in L^2(M)$ imply $f = dv$ with some vector field $v \in H_0^1(M)$.*

Any symmetric tensor $f \in L^2(M)$ admits an orthogonal decomposition $f = f^s + dv$ into a *solenoidal* and *potential* parts with $v \in H_0^1(M)$, and f^s divergence free, i.e., $\delta f^s = 0$, where δ is the adjoint operator to $-d$ given by $[\delta f]_i = g^{jk}\nabla_k f_{ij}$ [168]. Therefore, I_g is s-injective, if it is injective on the space of solenoidal tensors. The inversion of I_g is a problem of independent interest in integral geometry, also called *tensor tomography*. We first survey the recent results on this problem. S-injectivity, respectively injectivity for 1-tensors (1-forms) and functions is known, see [168] for references. S-injectivity of I_g was proved in [155] for metrics with negative curvature, in [168] for metrics with small curvature and in [171] for Riemannian surfaces with no focal points. A conditional and non-sharp stability estimate for metrics with small curvature is also established in [168]. In [180], stability estimates for s-injective metrics (see (2.9) below) were shown and sharp estimates about the recovery of a 1-form $f = f_j dx^j$ and a function f from the associated $I_g f$ which is defined by

$$I_g f(\gamma) = \int f_i(\gamma(t))\dot{\gamma}^i(t))\ dt.$$

The stability estimates proven in [180] were used to prove local uniqueness for the boundary rigidity problem near any simple metric g with s-injective I_g. Similarly to [198], we say that f is analytic in the set K (not necessarily open), if it is real analytic in some neighborhood of K. The results that follow in this section are based on [182]. The first main result we discuss is about s-injectivity for simple analytic metrics.

Theorem 2.15. *Let g be a simple, real analytic metric in M. Then I_g is s-injective.*

Sketch of the proof. Note that a simple metric g in M can be extended to a simple metric in some M_1 with $M \Subset M_1$. A simple manifold is diffeomorphic to a (strictly convex) domain $\Omega \subset \mathbf{R}^n$ with the Euclidean coordinates x in a neighborhood of Ω and a metric $g(x)$ there. For this reason, it is enough to prove the results of this section for domains Ω in $\mathbf{R}^n$ provided with a Riemannian metric g. The proof of Theorem 2.15 is based on the following. For smooth metrics, the normal operator $N_g = I_g^* I_g$ is a pseudodifferential operator with a non-trivial null space which is given by

$$(N_g f)_{kl}(x) = \frac{2}{\sqrt{\det g}} \int \frac{f^{ij}(y)}{d_g(x,y)^{n-1}} \frac{\partial d_g}{\partial y^i} \frac{d_g}{\partial y^j} \frac{\partial d_g}{\partial x^k} \frac{\partial d_g}{\partial x^l} \det \frac{\partial^2 (d_g^2/2)}{\partial x \partial y}\, \mathrm{d}y, \quad x \in \Omega. \tag{2.6}$$

In the case that the metric g is real-analytic, N_g is an analytic pseudodifferential operator with a non-trivial kernel. We construct an analytic parametrix, using the analytic pseudodifferential calculus in [198], that allows us to reconstruct the solenoidal part of a tensor field from its geodesic X-ray transform, up to a term that is analytic near Ω. If $I_g f = 0$, we show that for some v vanishing on $\partial\Omega$, $\widetilde{f} := f - dv$ must be flat at $\partial\Omega$ and analytic in $\bar{\Omega}$, hence $\widetilde{f} = 0$. This is similar to the known argument that an analytic elliptic pseudodifferential operator resolves the analytic singularities, hence cannot have compactly supported functions in its kernel. In our case we have a non-trivial kernel, and complications due to the presence of a boundary, in particular a lost of one derivative. For more details see [182]. □

As shown in [180], the s-injectivity of I_g for analytic simple g implies a stability estimate for I_g. In next theorem we show something more, namely that we have a stability estimate for g in a neighborhood of each analytic metric, which leads to stability estimates for generic metrics. As above, let $M_1 \Supset M$ be a compact manifold which is a neighborhood of M and g extends as a simple metric there. We always assume that our tensors are

extended as zero outside M, which may create jumps at ∂M. Define the *normal operator* $N_g = I_g^* I_g$, where I_g^* denotes the operator adjoint to I_g with respect to an appropriate measure. We showed in [180] that N_g is a pseudo-differential operator in M_1 of order -1. We introduce the norm $\|\cdot\|_{\widetilde{H}^2(M_1)}$ of $N_g f$ in $M_1 \supset M$ in the following way. Choose $\chi \in C_0^\infty$ equal to 1 near ∂M and supported in a small neighborhood of ∂M and let $\chi = \sum_{j=1}^J \chi_j$ be a partition of χ such that for each j, on $\operatorname{supp}\chi_j$ we have coordinates (x_j', x_j^n), with x_j^n a normal coordinate. Set

$$\|f\|_{\widetilde{H}^1}^2 = \int \sum_{j=1}^J \chi_j \left(\sum_{i=1}^{n-1} |\partial_{x_j^i} f|^2 + |x_j^n \partial_{x_j^n} f|^2 + |f|^2 \right) \mathrm{d}x, \tag{2.7}$$

$$\|N_g f\|_{\widetilde{H}^2(M_1)} = \sum_{i=1}^n \|\partial_{x^i} N_g f\|_{\widetilde{H}^1} + \|N_g f\|_{H^1(M_1)}. \tag{2.8}$$

In other words, in addition to derivatives up to order 1, $\|N_g f\|_{\widetilde{H}^2(M_1)}$ includes also second derivatives near ∂M but they are realized as first derivatives of $\nabla N_g f$ tangent to ∂M. The reason to use the $\widetilde{H}^2(M_1)$ norm, instead of the stronger $H^2(M_1)$ norm, is that this allows us to work with $f \in H^1(M)$, not only with $f \in H_0^1(M)$, since for such f, extended as 0 outside M, we still have that $N_g f \in \widetilde{H}^2(M_1)$, see [180]. On the other hand, $f \in H^1(M)$ implies $N_g f \in \widetilde{H}^2(M_1)$ despite the possible jump of f at ∂M. Our stability estimate for the linearized inverse problem is as follows:

Theorem 2.16. *There exists k_0 such that for each $k \geq k_0$, the set $\mathcal{G}^k(M)$ of simple $C^k(M)$ metrics in M for which I_g is s-injective is open and dense in the $C^k(M)$ topology. Moreover, for any $g \in \mathcal{G}^k(M)$,*

$$\|f^s\|_{L^2(M)} \leq C \|N_g f\|_{\widetilde{H}^2(M_1)}, \quad \forall f \in H^1(M), \tag{2.9}$$

with a constant $C > 0$ that can be chosen locally uniform in $\mathcal{G}^k(M)$ in the $C^k(M)$ topology.

Of course, $\mathcal{G}^k(M)$ includes all real analytic simple metrics in M, according to Theorem 2.15.

Sketch of the proof. The proof of the basic estimate (2.9) is based on the following ideas. For g of finite smoothness, one can still construct a parametrix Q_g of N_g as above that allows us to reconstruct f^s from $N_g f$ up to smoothing operator terms. This is done in a way similar to that in [180] in two steps: first we invert N_g modulo smoothing operators in a neighborhood M_1 of M, and that gives us $f_{M_1}^s$, i.e., the solenoidal projection of f but associated to the manifold M_1. Next, we compare $f_{M_1}^s$ and f^s and show that one can get the latter from the former by an operator that loses one derivative. This is the same construction as in the proof of Theorem 2.15 above but the metric is only C^k, $k \gg 1$. After applying the parametrix Q_g, the equation for recovering f^s from $N_g f$ is reduced to solving the Fredholm equation

$$(\mathcal{S}_g + K_g) f = Q_g N_g f, \quad f \in \mathcal{S}_g L^2(M) \tag{2.10}$$

where $\mathcal{S}_g$ is the projection to solenoidal tensors, similarly we denote by $\mathcal{P}_g$ the projection onto potential tensors. Here, K_g is a compact operator on $\mathcal{S}_g L^2(M)$. We can write this as an equation in the whole $L^2(M)$ by adding $\mathcal{P}_g f$ to both sides above to get

$$(I + K_g) f = (Q_g N_g + \mathcal{P}_g) f. \tag{2.11}$$

Then the solenoidal projection of the solution of (2.11) solves (2.10). A finite rank modification of K_g above can guarantee that $I + K_g$ has a trivial kernel, and therefore is

invertible, if and only if N_g is s-injective. The problem then reduces to that of invertibility of $I + K_g$. The operators above depend continuously on $g \in C^k$, $k \gg 1$. Since for g analytic, $I + K_g$ is invertible by Theorem 2.15, it would still be invertible in a neighborhood of any analytic g, and estimate (2.9) is true with a locally uniform constant. Analytic (simple) metrics are dense in the set of all simple metrics, and this completes the sketch of the proof of Theorem 2.16. For more details see [182]. □

The analysis of I_g can also be carried out for symmetric tensors of any order, see e.g., [168] and [170]. Since we are motivated by the boundary rigidity problem, and to simplify the exposition, we study only tensors of order 2. Theorem 2.16 and especially estimate (2.9) allow us to prove the following local generic uniqueness result for the non-linear boundary rigidity problem.

Theorem 2.17. *Let k_0 and $\mathcal{G}^k(M)$ be as in Theorem 2.16. There exists $k \geq k_0$, such that for any $g_0 \in \mathcal{G}^k$, there is $\varepsilon > 0$, such that for any two metrics g_1, g_2 with $\|g_m - g_0\|_{C^k(M)} \leq \varepsilon$, $m = 1, 2$, we have the following:*

$$d_{g_1} = d_{g_2} \text{ on } (\partial M)^2 \quad \text{implies } g_2 = \psi_* g_1 \tag{2.12}$$

with some $C^{k+1}(M)$-diffeomorphism $\psi : M \to M$ fixing the boundary.

Sketch of the proof. We prove Theorem 2.17 by linearizing and using Theorem 2.16, and especially (2.9), see also [180]. This requires first to pass to special semigeodesic coordinates related to each metric in which $g_{in} = \delta_{in}$, $\forall i$. We denote the corresponding pull-backs by g_1, g_2 again. Then we show that if g_1 and g_2 have the same distance on the boundary, then $g_1 = g_2$ on the boundary with all derivatives. As a result, for $f := g_1 - g_2$ we get that $f \in C_0^l(\bar{\Omega})$ with $l \gg 1$, if $k \gg 1$; and $f_{in} = 0$, $\forall i$. Then we linearize to get

$$\|N_{g_1} f\|_{L^\infty(\Omega_1)} \leq C\|f\|_{C^1}^2,$$

where $\Omega_1 \supset \bar{\Omega}$ is as above. Combine this with (2.9) and interpolation estimates, to get $\forall \mu < 1$,

$$\|f^s\|_{L^2} \leq C\|f\|_{L^2}^{1+\mu}.$$

One can show that tensors satisfying $f_{in} = 0$ also satisfy $\|f\|_{L^2} \leq C\|f^s\|_{H^2}$, and using this, and interpolation again, we get

$$\|f\|_{L^2} \leq C\|f\|_{L^2}^{1+\mu'}, \quad \mu' > 0.$$

This implies $f = 0$ for $\|f\| \ll 1$. Note that the condition $f \in C_0^l(\bar{\Omega})$ is used to make sure that f, extended as zero in $\Omega_1 \setminus \Omega$, is in $H_0^l(\Omega)$, and then use this fact in the interpolation estimates. Again, for more details see [182]. □

Finally, in [182] it is proven a conditional stability estimate of Hölder type. A similar estimate near the Euclidean metric was proven in [211] based on the approach in [179].

Theorem 2.18. *Let k_0 and $\mathcal{G}^k(M)$ be as in Theorem 2.16. Then for any $\mu < 1$, there exits $k \geq k_0$ such that for any $g_0 \in \mathcal{G}^k$, there is an $\varepsilon_0 > 0$ and $C > 0$ with the property that that for any two metrics g_1, g_2 with $\|g_m - g_0\|_{C(M)} \leq \varepsilon_0$, and $\|g_m\|_{C^k(M)} \leq A$, $m = 1, 2$, with some $A > 0$, we have the following stability estimate*

$$\|g_2 - \psi_* g_1\|_{C^2(M)} \leq C(A)\|d_{g_1} - d_{g_2}\|_{C(\partial M \times \partial M)}^{\mu}$$

with some diffeomorphism $\psi : M \to M$ fixing the boundary.

Sketch of the proof. To prove Theorem 2.18, we basically follow the uniqueness proof sketched above by showing that each step is stable. The analysis is more delicate near pairs of points too close to each other. An important ingredient of the proof is stability at the boundary, that is also of independent interest:

Theorem 2.19. *Let g_0 and g_1 be two simple metrics in Ω, and $\Gamma \subset\subset \Gamma' \subset \partial\Omega$ be two sufficiently small open subsets of the boundary. Then for some diffeomorphism ψ fixing the boundary,*

$$\left\|\partial_{x^n}^k(\psi_* g_1 - g_0)\right\|_{C^m(\bar{\Gamma})} \leq C_{k,m}\left\|d_{g_1}^2 - d_{g_0}^2\right\|_{C^{m+2k+2}\left(\overline{\Gamma'\times\Gamma'}\right)},$$

where $C_{k,m}$ depends only on Ω and on a upper bound of g_0, g_1 in $C^{m+2k+5}(\bar{\Omega})$.

Theorem 2.18 can be used to obtain stability near generic simple metrics for the inverse problem of recovering g from the hyperbolic *Dirichlet-to-Neumann map* Λ_g^h. It is known that g can be recovered uniquely from Λ_g, up to a diffeomorphism as above, see e.g. [24]. This result however relies on a unique continuation theorem by Tataru [197] and it is unlikely to provide Hölder type of stability estimate as above. By using the fact that d_g is related to the leading singularities in the kernel of Λ_g^h, it was proven a Hölder stability estimate under the assumptions above, relating g and Λ_g. We refer to [181] for details. □

2.4. Lens Rigidity

For non-simple manifolds in particular, if we have conjugate points or the boundary is not strictly convex, we need to look at the behavior of all the geodesics and the scattering relation encodes this information. We proceed to define in more detail the scattering relation for non-convex manifolds and the lens rigidity problem and state our results. We note that we will also consider the case of incomplete data, that is when we don't have information about all the geodesics entering the manifold. More details can be found in [183], [184].

The scattering relation

$$\Sigma : \partial_- SM \to \overline{\partial_+ SM} \tag{2.13}$$

is defined by $\Sigma(x,\xi) = (y,\eta) = \Phi^\ell(x,\xi)$, where Φ^t is the geodesic flow, and $\ell > 0$ is the first moment, at which the unit speed geodesic through (x,ξ) hits ∂M again. If such an ℓ does not exists, we formally set $\ell = \infty$ and we call the corresponding initial condition and the corresponding geodesic *trapping*. This defines also $\ell(x,\xi)$ as a function $\ell : \partial_- SM \to [0,\infty]$. Note that Σ and ℓ are not necessarily continuous. This coincides with the scattering relation α defined in section 2.2 for strictly convex manifolds.

It is convenient to think of Σ and ℓ as defined on the whole ∂SM with $\Sigma = Id$ and $\ell = 0$ on $\overline{\partial_+ SM}$. We parametrize the scattering relation in a way that makes it independent of pulling it back by diffeomorphisms fixing ∂M pointwise. Let $\kappa_\pm : \partial_\pm SM \to B(\partial M)$ be the orthogonal projection onto the (open) unit ball tangent bundle that extends continuously to the closure of $\partial_\pm SM$. Then $\kappa_\pm$ are homeomorphisms, and we set

$$\sigma = \kappa_+ \circ \Sigma \circ \kappa_-^{-1} : \overline{B(\partial M)} \longrightarrow \overline{B(\partial M)}. \tag{2.14}$$

According to our convention, $\sigma = Id$ on $\partial(\overline{B(\partial M)}) = S(\partial M)$. We equip $\overline{B(\partial M)}$ with the relative topology induced by $T(\partial M)$, where neighborhoods of boundary points (those in $S(\partial M)$) are given by half-neighborhoods, i.e., by neighborhoods in $T(\partial M)$ intersected with $\overline{B(\partial M)}$. It is possible to define σ in a way that does not require knowledge of

$g|_{T(\partial M)}$ by thinking of any boundary vector ξ as characterized by its angle with ∂M and the direction of its tangential projection. Let $\mathcal{D}$ be an open subset of $\overline{B(\partial M)}$. A priori, the latter depends on $g|_{T(\partial M)}$. By the remark above, we can think of it as independent of $g|_{T(\partial M)}$ however. The *lens rigidity* problem we study is the following: *Given M and $\mathcal{D}$, do σ and ℓ, restricted to $\mathcal{D}$, determine g uniquely, up to a pull back of a diffeomorphism that is identity on ∂M?* The answer to this question, even when $\mathcal{D} = B(\partial M)$, is negative, see [49]. The known counter-examples are trapping manifolds. The boundary rigidity problem and the lens rigidity one are equivalent for simple metrics.

2.5. Main assumptions

Definition 2.20. *We say that $\mathcal{D}$ is* ***complete*** *for the metric g, if for any $(z, \zeta) \in T^*M$ there exists a maximal in M, finite length unit speed geodesic $\gamma : [0, l] \to M$ through z, normal to ζ, such that*

$$\{(\gamma(t), \dot\gamma(t));\ 0 \le t \le l\} \cap S(\partial M) \subset \mathcal{D}, \tag{2.15}$$

$$\text{there are no conjugate points on } \gamma. \tag{2.16}$$

We call the C^k metric g ***regular****, if a complete set $\mathcal{D}$ exists, i.e., if $\overline{B(\partial M)}$ is complete.*

If $z \in \partial M$ and ζ is conormal to ∂M, then γ may reduce to one point.

Topological Condition (T): Any path in M connecting two boundary points is homotopic to a polygon $c_1 \cup \gamma_1 \cup c_2 \cup \gamma_2 \cup \cdots \cup \gamma_k \cup c_{k+1}$ with the properties that for any j, (i) c_j is a path on ∂M; (ii) $\gamma_j : [0, l_j] \to M$ is a geodesic lying in M^{int} with the exception of its endpoints and is transversal to ∂M at both ends; moreover, $\kappa_-(\gamma_j(0), \dot\gamma_j(0)) \in \mathcal{D}$; Notice that (T) is an open condition w.r.t. g, i.e., it is preserved under small C^2 perturbations of g. To define the $C^K(M)$ norm below in a unique way, we choose and fix a finite atlas on M.

2.5.1. Results about the linear problem

We refer to [183] for more details about the results in this section. It turns out that a linearization of the lens rigidity problem is again the problem of s-injectivity of the ray transform I. Here and below we sometimes drop the subscript g. Given $\mathcal{D}$ as above, we denote by $I_{\mathcal{D}}$ (or $I_{g,\mathcal{D}}$) the ray transform I restricted to the maximal geodesics issued from $(x, \xi) \in \kappa_-^{-1}(\mathcal{D})$. The first result of this section generalizes Theorem 2.15.

Theorem 2.21. *Let g be an analytic, regular metric on M. Let $\mathcal{D}$ be complete and open. Then $I_{\mathcal{D}}$ is s-injective.*

Sketch of the proof. Since we know integrals over a subset of geodesics only, this creates difficulties with cut-offs in the phase variable that cannot be analytic. For this reason, the proof of Theorem 2.21 is different from that of Theorem 2.15. Let g be an analytic regular metrics in M, and let $M_1 \Supset M$ be the manifold where g is extended analytically. There is an analytic atlas in M, and ∂M can be assumed to be analytic, too. In other words, now $(M, \partial M, g)$ is a real analytic manifold with boundary. We denote by $\mathcal{A}(M)$ (respectively $\mathcal{A}(M_1)$) the set of analytic functions on M (respectively M_1). Next, $f^s_{M_1}$ denotes the solenoidal part of the tensor f, extended as zero to M_1, in the manifold M_1. The main step is to show that $I_{\mathcal{D}} f = 0$ implies $f^s \in \mathcal{A}(M)$. In order to do that one shows

that $f^s_{M_1} \in \mathcal{A}(M_1)$. Let us first notice, that in $M_1 \setminus M$, $f^s_{M_1} = -dv_{M_1}$, where v_{M_1} satisfies $\delta dv_{M_1} = 0$ in $M_1 \setminus M$, $v|_{\partial M_1} = 0$ since $f = 0$ in $M_1 \setminus M$. Therefore, v_{M_1} is analytic up to ∂M_1. Therefore, we only need to show that $f^s_{M_1}$ is analytic in the interior of M_1. Below, $\mathrm{WF_A}(f)$ stands for the analytic wave front set of f, see [174, 198]. The crucial point is the following microlocal analytic regularity result.

Proposition 2.22. *Let γ_0 be a fixed maximal geodesic in M with endpoints on ∂M, without conjugate points, and let $I_g f(\gamma) = 0$ for $\gamma \in \mathrm{neigh}(\gamma_0)$. Let g be analytic in* $\mathrm{neigh}(\gamma_0)$. *Then*

$$N^*\gamma_0 \cap \mathrm{WF_A}(f^s_{M_1}) = \emptyset. \tag{2.17}$$

Sketch of the proof. Set $f = f^s_{M_1}$. Let U_ε be a tubular neighborhood of γ_0, and $x = (x', x^n)$ be semigeodesic coordinates in it such that $x' = 0$ on γ_0. Fix $x_0 \in \gamma_0 \cap M$. We can assume that $x_0 = 0$ and $g_{ij}(0) = \delta_{ij}$. Then we can assume that $U_\varepsilon = \{-l_1 - \varepsilon < x^n < l_2 + \varepsilon,\ |x'| < \varepsilon\}$ with the part of γ_0 corresponding to $x^n \notin [-l_1, l_2]$ outside M. Fix $\xi^0 = ((\xi^0)', 0)$ with $\xi^0_n = 0$. We will show that

$$(0, \xi^0) \notin \mathrm{WF_A}(f). \tag{2.18}$$

We choose a local chart for the geodesics close to γ_0. Set first $Z = \{x^n = 0;\ |x'| < 7\varepsilon/8\}$, and denote the x' variable on Z by z'. Then z', θ' (with $|\theta'| \ll 1$) are local coordinates in $\mathrm{neigh}(\gamma_0)$ determined by $(z', \theta') \to \gamma_{(z',0),(\theta',1)}$ where the latter denotes the geodesic through the point $(z', 0)$ in the direction $(\theta', 1)$. Let $\chi_N(z')$ be a smooth cut-off function equal to 1 for $|z'| \le 3\varepsilon/4$ and supported in Z, also satisfying $|\partial^\alpha \chi_N| \le (CN)^{|\alpha|}$, $|\alpha| \le N$. Set $\theta = (\theta', 1)$, $|\theta'| \ll 1$, and multiply

$$If\left(\gamma_{(z',0),\theta}\right) = 0$$

by $\chi_N(z') e^{\mathrm{i}\lambda z'\cdot\xi'}$, where $\lambda > 0$, ξ' is in a complex neighborhood of $(\xi^0)'$, and integrate w.r.t. z' to get

$$\iint e^{\mathrm{i}\lambda z'\cdot\xi'} \chi_N(z') f_{ij}\left(\gamma_{(z',0),\theta}(t)\right) \dot\gamma^i_{(z',0),\theta}(t) \dot\gamma^j_{(z',0),\theta}(t)\, \mathrm{d}t\, \mathrm{d}z' = 0. \tag{2.19}$$

Set $x = \gamma_{(z',0),\theta}(t)$. If $\theta' = 0$, we have $x = (z', t)$. By a perturbation argument, for θ' fixed and small enough, (t, z') are analytic local coordinates, depending analytically on θ'. In particular, $x = (z' + t\theta', t) + O(|\theta'|)$ but this expansion is not enough for the analysis below. Performing a change of variables in (2.19), we get

$$\int e^{\mathrm{i}\lambda z'(x,\theta')\cdot\xi'} a_N(x, \theta') f_{ij}(x) b^i(x, \theta') b^j(x, \theta')\, \mathrm{d}x = 0 \tag{2.20}$$

for $|\theta'| \ll 1$, $\forall\lambda$, $\forall\xi'$, where, for $|\theta'| \ll 1$, the function $(x, \theta') \mapsto a_N$ is positive for x in a neighborhood of γ_0, vanishing for $x \notin U_\varepsilon$, and satisfies the same estimate as χ_N. The vector field b is analytic, and $b(0, \theta') = \theta$, $a_N(0, \theta') = 1$. To clarify the approach, note that if g is Euclidean in $\mathrm{neigh}(\gamma_0)$, then (2.20) reduces to

$$\int e^{\mathrm{i}\lambda(\xi', -\theta'\cdot\xi')\cdot x} \chi f_{ij}(x) \theta^i \theta^j\, \mathrm{d}x = 0,$$

where $\chi = \chi(x' - x^n\theta')$. Then $\xi = (\xi', -\theta'\cdot\xi')$ is perpendicular to $\theta = (\theta', 1)$. This implies that

$$\int e^{\mathrm{i}\lambda\xi\cdot x} \chi f_{ij}(x) \theta^i(\xi) \theta^j(\xi)\, \mathrm{d}x = 0 \tag{2.21}$$

for any function $\theta(\xi)$ defined near ξ^0, such that $\theta(\xi)\cdot\xi = 0$. This has been noticed and used before if g is close to the Euclidean metric (with $\chi = 1$), see e.g., [179]. We will assume that $\theta(\xi)$ is analytic. A simple argument (see e.g. [168, 179]) shows that a

constant symmetric tensor f_{ij} is uniquely determined by the numbers $f_{ij}\theta^i\theta^j$ for finitely many θ's (actually, for $N' = (n+1)n/2$ θ's); and in any open set on the unit sphere, there are such θ's. On the other hand, f is solenoidal. To simplify the argument, assume for a moment that f vanishes on ∂M. Then $\xi^i \widehat{f}_{ij}(\xi) = 0$. Therefore, combining this with (2.21), we need to choose $N = n(n-1)/2$ vectors $\theta(\xi)$, perpendicular to ξ, that would uniquely determine the tensor $\widehat{f}$ on the plane perpendicular to ξ. To this end, it is enough to know that this choice can be made for $\xi = \xi^0$, then it would be true for $\xi \in \mathrm{neigh}(\xi^0)$. This way, $\xi^i \widehat{f}_{ij}(\xi) = 0$ and the N equations (2.21) with the so chosen $\theta_p(\xi)$, $p = 1, \dots, N$, form a system with a tensor-valued symbol elliptic near $\xi = \xi^0$. The C^∞ ΨDO calculus easily implies the statement of the lemma in the C^∞ category, and the complex stationary phase method below, or the analytic ΨDO calculus in [198] with appropriate cut-offs in ξ, implies the lemma in this special case (g locally Euclidean). The general case is considered in [183], and is based on an application of a complex stationary phase argument [174] to (2.20) as in [108]. □

Proposition 2.22 makes it possible to prove that $f^s \in \mathcal{A}(M)$. We combine this with a boundary determination theorem for tensors, a linear version of Theorem 2.28 below, to conclude that then $f = 0$. □

Next, we formulate a stability estimate in the spirit of Theorem 2.16. We need first to parametrize (a complete subset of) the geodesics issued from $\mathcal{D}$ in a different way that would make them a manifold. The parametrization provided by $\mathcal{D}$ is inconvenient near the directions tangent to ∂M. Let H_m be a finite collection of smooth hypersurfaces in M_1^{int}. Let $\mathcal{H}_m$ be an open subset of $\{(z,\theta) \in SM_1;\ z \in H_m, \theta \notin T_z H_m\}$, and let $\pm l_m^\pm(z,\theta) \ge 0$ be two continuous functions. Let $\Gamma(\mathcal{H}_m)$ be the set of geodesics

$$\Gamma(\mathcal{H}_m) = \Big\{\gamma_{z,\theta}(t);\ l_m^-(z,\theta) \le t \le l_m^+(z,\theta),\ (z,\theta) \in \mathcal{H}_m\Big\}, \tag{2.22}$$

that, depending on the context, is considered either as a family of curves, or as a point set. We also assume that each $\gamma \in \Gamma(\mathcal{H}_m)$ is a simple geodesic (no conjugate points). If g is simple, then one can take a single $H = \partial M_1$ with $l^- = 0$ and an appropriate $l^+(z,\theta)$. If g is regular only, and Γ is any complete set of geodesics, then any small enough neighborhood of a simple geodesic in Γ has the properties listed in the paragraph above and by a compactness argument one can choose a finite complete set of such $\Gamma(\mathcal{H}_m)$'s, that is included in the original Γ. Given $\mathcal{H} = \{\mathcal{H}_m\}$ as above, we consider an open set $\mathcal{H}' = \{\mathcal{H}'_m\}$, such that $\mathcal{H}'_m \Subset \mathcal{H}_m$, and let $\Gamma(\mathcal{H}'_m)$ be the associated set of geodesics defined as in (2.22), with the same $l_m^\pm$. Set $\Gamma(\mathcal{H}) = \cup\Gamma(\mathcal{H}_m)$, $\Gamma(\mathcal{H}') = \cup\Gamma(\mathcal{H}'_m)$. The restriction $\gamma \in \Gamma(\mathcal{H}'_m) \subset \Gamma(\mathcal{H}_m)$ can be modeled by introducing a weight function α_m in $\mathcal{H}_m$, such that $\alpha_m = 1$ on $\mathcal{H}'_m$, and $\alpha_m = 0$ otherwise. More generally, we allow α_m to be smooth but still supported in $\mathcal{H}_m$. We then write $\alpha = \{\alpha_m\}$, and we say that $\alpha \in C^k(\mathcal{H})$, if $\alpha_m \in C^k(\mathcal{H}_m)$, $\forall m$. We consider $I_{\alpha_m} = \alpha_m I$, or more precisely, in the coordinates $(z,\theta) \in \mathcal{H}_m$,

$$I_{\alpha_m} f = \alpha_m(z,\theta) \int_0^{l_m(z,\theta)} \Big\langle f(\gamma_{z,\theta}), \dot\gamma_{z,\theta}^2 \Big\rangle \,\mathrm{d}t, \quad (z,\theta) \in \mathcal{H}_m. \tag{2.23}$$

Next, we set

$$I_\alpha = \{I_{\alpha_m}\}, \quad N_{\alpha_m} = I_{\alpha_m}^* I_{\alpha_m} = I^*|\alpha_m|^2 I, \quad N_\alpha = \sum N_{\alpha_m}, \tag{2.24}$$

where the adjoint is taken w.r.t. the measure $\mathrm{d}\mu := |\langle \nu(z), \theta\rangle| \,\mathrm{d}S_z \,\mathrm{d}\theta$ on $\mathcal{H}_m$, $\mathrm{d}S_z \,\mathrm{d}\theta$ being the induced measure on SM, and $\nu(z)$ being a unit normal to H_m. S-injectivity of N_α is

equivalent to s-injectivity for I_α, which in turn is equivalent to s-injectivity of I restricted to $\operatorname{supp}\alpha$.

Theorem 2.23. *(a) Let $g = g_0 \in C^k$, $k \gg 1$ be regular, and let $\mathcal{H}' \Subset \mathcal{H}$ be as above with $\Gamma(\mathcal{H}')$ complete. Fix $\alpha = \{\alpha_m\} \in C^\infty$ with $\mathcal{H}'_m \subset \operatorname{supp}\alpha_m \subset \mathcal{H}_m$. Then if I_α is s-injective, we have*

$$\|f^s\|_{L^2(M)} \le C\|N_\alpha f\|_{\widetilde{H}^2(M_1)}. \tag{2.25}$$

(b) Assume that $\alpha = \alpha_g$ in (a) depends on $g \in C^k$, so that $C^k(M_1) \ni g \to C^l(\mathcal{H}) \ni \alpha_g$ is continuous with $l \gg 1$, $k \gg 1$. Assume that $I_{g_0,\alpha_{g_0}}$ is s-injective. Then estimate (2.25) remains true for g in a small enough neighborhood of g_0 in $C^k(M_1)$ with a uniform constant $C > 0$.

The theorem above allows us to formulate a generic result:

Theorem 2.24. *Let $\mathcal{G} \subset C^k(M)$ be an open set of regular Riemannian metrics on M such that (T) is satisfied for each one of them. Let the set $\mathcal{D}' \subset \partial SM$ be open and complete for each $g \in \mathcal{G}$. Then there exists an open and dense subset $\mathcal{G}_s$ of $\mathcal{G}$ such that $I_{g,\mathcal{D}'}$ is s-injective for any $g \in \mathcal{G}_s$.*

Of course, the set $\mathcal{G}_s$ includes all real analytic metrics in $\mathcal{G}$.

Corollary 2.25. *Let $\mathcal{R}(M)$ be the set of all regular C^k metrics on M satisfying (T) equipped with the $C^k(M_1)$ topology. Then for $k \gg 1$, the subset of metrics for which the X-ray transform I over all simple geodesics through all points in M is s-injective, is open and dense in $\mathcal{R}(M)$.*

2.6. Results about the non-linear lens rigidity problem

Using the results above, we prove the following about the lens rigidity problem on manifolds satisfying the assumptions in Section 2.5. More details can be found in [184]. Theorem 2.26 below says, loosely speaking, that for the classes of manifolds and metrics we study, the uniqueness question for the non-linear lens rigidity problem can be answered locally by linearization. This is a non-trivial implicit function type of theorem however because our success heavily depends on the a priori stability estimate that the s-injectivity of $I_\mathcal{D}$ implies; see Theorem 2.23; and the latter is based on the hypoelliptic properties of $I_\mathcal{D}$. We work with two metrics g and $\widehat{g}$; and will denote objects related to $\widehat{g}$ by $\widehat{\sigma}$, $\widehat{\ell}$, etc.

Theorem 2.26. *Let (M, g_0) satisfy the topological assumption (T), with $g_0 \in C^k(M)$ a regular Riemannian metric with $k \gg 1$. Let $\mathcal{D}$ be open and complete for g_0, and assume that there exists $\mathcal{D}' \Subset \mathcal{D}$ so that $I_{g_0,\mathcal{D}'}$ is s-injective. Then there exists $\varepsilon > 0$, such that for any two metrics g, $\widehat{g}$ satisfying*

$$\|g - g_0\|_{C^k(M)} + \|\widehat{g} - g_0\|_{C^k(M)} \le \varepsilon, \tag{2.26}$$

the relations

$$\sigma = \widehat{\sigma}, \quad \ell = \widehat{\ell} \quad \text{on } \mathcal{D}$$

imply that there is a C^{k+1} diffeomorphism $\psi : M \to M$ fixing the boundary such that

$$\widehat{g} = \psi^* g.$$

By Theorem 2.24, the requirement that $I_{g_0,\mathcal{D}'}$ is s-injective is a generic one for g_0. Therefore, Theorems 2.26 and 2.24 combined imply that there is local uniqueness, up to isometry, near a generic set of regular metrics.

Corollary 2.27. *Let $\mathcal{D}' \Subset \mathcal{D}$, $\mathcal{G}$, $\mathcal{G}_s$ be as in Theorem 2.24. Then the conclusion of Theorem 2.26 holds for any $g_0 \in \mathcal{G}_s$.*

2.6.1. Boundary determination of the jet of g

The first step of the proof of Theorem 2.26 is to determine all derivatives of g on ∂M. The following theorem is interesting by itself. Notice that g below does not need to be analytic or generic.

Theorem 2.28. *Let (M, g) be a compact Riemannian manifold with boundary. Let $(x_0, \xi_0) \in S(\partial M)$ be such that the maximal geodesic γ_{x_0,ξ_0} through it is of finite length, and assume that x_0 is not conjugate to any point in $\gamma_{x_0,\xi_0} \cap \partial M$. If σ and ℓ are known on some neighborhood of (x_0, ξ_0), then the jet of g at x_0 in boundary normal coordinates is determined uniquely.*

Sketch of the proof of Theorem 2.28. To make the arguments below more transparent, assume that the geodesic γ_0 issued from (x_0, ξ_0) hits ∂M for the first time transversally at $\gamma_0(l_0) = y_0$, $l_0 > 0$. Then y_0 is the only point on ∂M reachable from (x_0, ξ_0), and x_0, y_0 are not conjugate points on γ_0 by assumption. Assume also that γ_0 is tangent of finite order at x_0. Then there is a half neighborhood V of x_0 on ∂M visible from y_0. The latter is not always true if γ_0 is tangent to ∂M of infinite order at x_0. Choose local boundary normal coordinates near x_0 and y_0, and let g_0 be the Euclidean metric in each of them w.r.t. to the so chosen coordinates. We can then consider a representation of Σ, denoted by $\Sigma^\sharp$ below, defined locally on $\mathbb{R}^{n-1} \times S^{n-1}$, with values on another copy of the same space. If $(x, \theta) \in \mathbb{R}^{n-1} \times S^{n-1}$, then the associated vector at $x \in \partial M$ is $\xi = \theta/|\theta|_g$; and $\Sigma^\sharp(x, \theta) = \Sigma(x, \xi)$. The same applies to the second component of $\Sigma^\sharp(x, \theta)$. Namely, if $(y, \eta) = \Sigma(x, \xi)$, then we set $\omega = \eta/|\eta|_{g_0}$, then $\Sigma^\sharp : (x, \theta) \mapsto (y, \omega)$. Similarly, we set $\ell^\sharp(x, \theta) = \ell(x, \xi)$. Let also θ_0 and ω_0 correspond to ξ_0 and η_0, respectively, where $\Sigma(x_0, \xi_0) = (y_0, \eta_0)$. Set $\tau(x) := \tau(x, y_0)$, where τ is the smooth travel time function localized near $x = x_0$ such that $\tau(x_0, y_0) = l_0$. Then τ is well defined in a small neighborhood of x_0 by the implicit function theorem and the assumption that x_0 and y_0 are not conjugate on γ_0. In the normal boundary coordinates $x = (x', x^n)$ near x_0, $g_{in} = \delta_{in}$, $\forall i$. Since x_0 and y_0 are not conjugate, for $\eta \in S_{y_0}M$ close enough to η_0, the map $\eta \mapsto x \in \partial M$ is a local diffeomorphism as long as the geodesic connecting x and y_0 is not tangent to ∂M at x. Moreover, that map is known, being the inverse of Σ. Similarly, the map $S^{n-1} \ni \omega \mapsto x$ is a local diffeomorphism and is also known. Then we know $(x, -\theta) = \Sigma^\sharp(y_0, -\omega)$, and we know $\ell^\sharp(y_0, -\omega) = \ell^\sharp(x, \theta) = \tau(x)$. Then we can recover $\operatorname{grad}' \tau = -\theta'/|\theta|_g$, where the prime stands for tangential projection as usual. Taking the limit $\omega \to \omega_0$, we recover $|\theta_0|_g^2 = g_{\alpha\beta}\theta_0^\alpha\theta_0^\beta$. We use again the fact that a symmetric $n \times n$ tensor f_{ij} can be recovered by knowledge of $f_{ij}p_k^i p_k^j$ for $N = n(n+1)/2$ "generic" vectors p_k, $k = 1, \dots, N$; and such N vectors exist in any open set on S^{n-1}, see e.g. [184]. Thus choosing appropriate $n(n-1)/2$ perturbations of θ_0's, we recover $g(x_0)$. Thus, we recover g in a neighborhood of x_0 as well; we can assume that V covers that neighborhood. Note that we know all tangential derivatives of g in $V \ni x_0$. Then τ solves the eikonal equation

$$g^{\alpha\beta}\tau_{x^\alpha}\tau_{x^\beta} + \tau_{x^n}^2 = 1. \tag{2.27}$$

Next, in V, we know τ_{x^α}, $\alpha \le n-1$, we know g, therefore by (2.27), we get $\tau_{x^n}^2$. It is easy to see that $\tau_{x^n} \le 0$ on the visible part, so we recover τ_{x^n} there. We therefore know the

tangential derivatives of τ_{x^n} on ∂M near x_0. Differentiate (2.27) w.r.t. x^n at $x = x_0$ to get

$$\left[\frac{dg^{\alpha\beta}}{dx^n}\tau_{x^\alpha}\tau_{x^\beta} + 2g^{\alpha\beta}\tau_{x^\alpha x^n}\tau_{x^\beta} + 2\tau_{x^n x^n}\tau_{x^n}\right]\Bigg|_{x=x_0} = 0. \tag{2.28}$$

Since γ_0 is tangent to ∂M at x_0, we have $\tau_{x^n}(x_0) = 0$ by (2.27). The third term in the l.h.s. of (2.28) therefore vanishes. Therefore the only unknown term in (2.28) is $\gamma^{\alpha\beta} := dg^{\alpha\beta}/dx^n$ at $x = x_0$. Since $\tau_{x^\alpha}(x_0) = -\xi_0$, using the fact that $\operatorname{grad}\tau(x_0) = -\xi_0$ again, we get that we have to determine $\gamma^{\alpha\beta}$ from $\gamma_{\alpha\beta}\xi_0^\alpha\xi_0^\beta$. This is possible if as above, we repeat the construction and replace ξ_0 by a finite number of vectors, close enough to ξ_0. So we get an explicit formula for $\partial g/\partial x^i|_{\partial M}$ in fact. Next, for $x \in V$ but not on ∂V, we can recover $\tau_{x^n x^n}(x)$ by (2.28) because $\tau_{x^n}(x) < 0$. By continuity, we recover $\tau_{x^n x^n}(x_0)$, therefore we know $\tau_{x^n x^n}$ near x_0, and all tangential derivatives of the latter. We differentiate (2.28) w.r.t. x^n again, and as above, recover $d^2g/d(x^n)^2|_{\partial M}$ near x_0. Then we recover $d^3\tau/d(x^n)^3$, etc. In the general case, we repeat those arguments with ξ_0 replaced by $\xi_0 + \varepsilon\nu$, where ν is the interior unit normal, and take the limit $\varepsilon \to 0$. □

Sketch of the proof of Theorem 2.26. We first find suitable metric $\widehat{g}_1$ isometric to $\widehat{g}$, and then we show that $\widehat{g}_1 = g$. First, we can always assume that g and $\widehat{g}$ have the same boundary normal coordinates near ∂M. By [48], there is a metric h isometric to $\widehat{g}$ so that h is solenoidal w.r.t. g. Moreover, $h = \widehat{g} + O(\varepsilon)$. By a standard argument, by a diffeomorphism that identifies normal coordinates near ∂M for h and g, and is identity away from some neighborhood of the boundary, we find a third $\widehat{g}_1$ isometric to h (and therefore to $\widehat{g}$), so that $\widehat{g}_1 = \widehat{g}$ near ∂M, and $\widehat{g}_1 = h$ away from some neighborhood of ∂M (and there is a region that $\widehat{g}_1$ is neither). Then $\widehat{g}_1 - h$ is as small as $g - h$, more precisely,

$$\|\widehat{g}_1 - h\|_{C^{k-3}} \le C\|g - h\|_{C^{k-1}}, \quad k \gg 1. \tag{2.29}$$

Set

$$f = h - g, \quad \widetilde{f} = \widehat{g}_1 - g. \tag{2.30}$$

Estimate (2.29) implies

$$\|\widetilde{f} - f\|_{C^{l-3}} \le C\|f\|_{C^{l-1}}, \quad \forall l \le k. \tag{2.31}$$

By (2.26), (2.31),

$$\|f\|_{C^{k-1}} \le C\varepsilon, \quad \|\widetilde{f}\|_{C^{k-3}} \le C\varepsilon. \tag{2.32}$$

By Theorem 2.28,

$$\partial^\alpha \widetilde{f} = 0 \quad \text{on } \partial M \text{ for } |\alpha| \le k - 5. \tag{2.33}$$

It is known [168] that $2dv$ is the linearization of $\psi_\tau^* g$ at $\tau = 0$, where ψ_τ is a smooth family of diffeomorphisms, and $v = d\psi_\tau/d\tau$ at $\tau = 0$. Next proposition is therefore a version of Taylor's expansion:

Proposition 2.29. *Let $\widehat{g}$ and g be in C^k, $k \ge 2$ and isometric, i.e.,*

$$\widehat{g} = \psi^* g$$

for some diffeomorphism ψ fixing ∂M. Set $f = \widehat{g} - g$. Then there exists v vanishing on ∂M, so that

$$f = 2dv + f_2,$$

and for g belonging to any bounded set U in C^k, there exists $C(U) > 0$, such that

$$\|f_2\|_{C^{k-2}} \le C(U)\|\psi - \mathrm{Id}\|^2_{C^{k-1}}, \quad \|v\|_{C^{k-1}} \le C(U)\|\psi - \mathrm{Id}\|_{C^{k-1}}.$$

We will sketch now the rest of the proof of Theorem 2.26. We apply Proposition 2.29 to h and $\widehat{g}_1$ to get

$$\widetilde{f} = f + 2dv + f_2, \quad \|f_2\|_{C^{l-3}} \leq C\|f\|^2_{C^{l-1}}, \quad \forall l \leq k. \tag{2.34}$$

In other words, $\widetilde{f}^s = f$ up to $O(\|f\|^2)$. We can assume that g is extended smoothly on $M_1 \Supset M$. Next, with g extended as above, we extend $\widehat{g}_1$ so that $\widehat{g}_1 = g$ outside M. This can be done in a smooth way by Theorem 2.28. The next step is to reparametrize the scattering relation. We show that one can extend the maximal geodesics of g, respectively $\widehat{g}_1$, outside M (where $g = \widehat{g}_1$), and since the two metrics have the same scattering relation and travel times, they will still have the same scattering relation and travel times if we locally push ∂M a bit outside M. Then we can arrange that the new pieces of ∂M are transversal to the geodesics close to a fixed one, which provides a smooth parametrization. By a compactness argument, one can do this near finitely many geodesics issued from point on $\mathcal{D}$, and still have a complete set. This puts as in the situation of Theorem 2.23, where the set of geodesics is parametrized by $\alpha = \{\alpha_j\}$. Next, we linearize the energy functional near each geodesic (in our set of data) related to g. Using the assumption that g and $\widehat{g}_1$ have the same scattering relation and travel times, we deduct

$$\|N_{\alpha_j}\widetilde{f}\|_{L^\infty} \leq C\|\widetilde{f}\|^2_{C^1}, \quad \forall j. \tag{2.35}$$

Using interpolation inequalities, and the fact that the extension of $\widetilde{f}$ outside M is smooth enough across ∂M as a consequence of the boundary recovery, we get by (2.35), and (2.31),

$$\|N_\alpha \widetilde{f}\|_{\widetilde{H}^2(M_1)} \leq C\|\widetilde{f}\|^{3/2}_{C^3} \leq C'\|f\|^{3/2}_{C^5}. \tag{2.36}$$

Since $I_{g_0,\mathcal{D}'}$ is s-injective, so is N_α, related to g_0, by the support properties of α. Now, since g is close enough to g_0 with s-injective N_α by (2.26), N_α (the one related to g) is s-injective as well by Theorem 2.23. Therefore, by (2.36) and (2.25),

$$\|f^s\|_{L^2(M)} \leq C\|N_\alpha \widetilde{f}\|_{\widetilde{H}^2} \leq C'\|f\|^{3/2}_{C^5}. \tag{2.37}$$

A decisive moment of the proof is that by Proposition 2.29, see (2.34), $\widetilde{f}^s = f + f_2^s$, the latter being the solenoidal projection of f_2. Therefore,

$$\|\widetilde{f}^s\|_{L^2(M)} \geq \|f\|_{L^2(M)} - C\|f\|^2_{C^2}.$$

Together with (2.37), this yields

$$\|f\|_{L^2(M)} \leq C\left(\|f\|^2_{C^2} + \|f\|^{3/2}_{C^5}\right) \leq C'\|f\|^{3/2}_{C^5}$$

because the C^5 norm of f is uniformly bounded when $\varepsilon \leq 1$. Using interpolation again, we easily deduct $\|f\|_{L^2(M)} \geq 1/C$ if $f \neq 0$. This contradicts (2.32) if $\varepsilon \ll 1$. Now, $f = 0$ implies $h = g$, therefore, g and $\widehat{g}$ are isometric. This concludes the sketch proof of Theorem 2.26. □

3. Invisibility for Electrostatics

We discuss here only invisibility results for electrostatics. For similar results for electromagnetic waves, acoustic waves, quantum waves, etc., see the review papers [62], [63] and the references given there. The fact that the boundary measurements do not change, when a conductivity is pushed forward by a smooth diffeomorphism leaving the boundary fixed (see section 1.7), can already be considered as a weak form of invisibility. Different

media appear to be the same, and the apparent location of objects can change. However, this does not yet constitute real invisibility, as nothing has been hidden from view. In invisibility cloaking the aim is to hide an object inside a domain by surrounding it with a material so that even the presence of this object can not be detected by measurements on the domain's boundary. This means that all boundary measurements for the domain with this cloaked object included would be the same as if the domain were filled with a homogeneous, isotropic material. Theoretical models for this have been found by applying diffeomorphisms having singularities. These were first introduced in the framework of electrostatics, yielding counterexamples to the anisotropic Calderón problem in the form of singular, anisotropic conductivities in $\mathbb{R}^n, n \geq 3$, indistinguishable from a constant isotropic conductivity in that they have the same Dirichlet-to-Neumann map [65, 66]. The same construction was rediscovered for electromagnetism in [154], with the intention of actually building such a device with appropriately designed metamaterials; a modified version of this was then experimentally demonstrated in [172]. (See also [126] for a somewhat different approach to cloaking in the high frequency limit.) The first constructions in this direction were based on blowing up the metric around a point [124]. In this construction, let (M, g) be a compact 2-dimensional manifold with non-empty boundary, let $x_0 \in M$ and consider the manifold

$$\widetilde{M} = M \setminus \{x_0\}$$

with the metric

$$\widetilde{g}_{ij}(x) = \frac{1}{d_M(x, x_0)^2} g_{ij}(x),$$

where $d_M(x, x_0)$ is the distance between x and x_0 on (M, g). Then $(\widetilde{M}, \widetilde{g})$ is a complete, non-compact 2-dimensional Riemannian manifold with the boundary $\partial\widetilde{M} = \partial M$. Essentially, the point x_0 has been "pulled to infinity". On the manifolds M and $\widetilde{M}$ we consider the boundary value problems

$$\begin{cases} \Delta_g u = 0 & \text{in } M, \\ u = f & \text{on } \partial M, \end{cases} \qquad \text{and} \qquad \begin{cases} \Delta_{\widetilde{g}} \widetilde{u} = 0 & \text{in } \widetilde{M}, \\ \widetilde{u} = f & \text{on } \partial\widetilde{M}, \\ \widetilde{u} \in L^\infty(\widetilde{M}). \end{cases}$$

These boundary value problems are uniquely solvable and define the DN maps

$$\Lambda_{M,g} f = \partial_\nu u|_{\partial M}, \quad \Lambda_{\widetilde{M},\widetilde{g}} f = \partial_\nu \widetilde{u}|_{\partial\widetilde{M}}$$

where ∂_ν denotes the corresponding conormal derivatives. Since, in the two dimensional case, functions which are harmonic with respect to the metric g stay harmonic with respect to any metric which is conformal to g, one can see that $\Lambda_{M,g} = \Lambda_{\widetilde{M},\widetilde{g}}$. This can be seen using e.g. Brownian motion or capacity arguments. Thus, the boundary measurements for (M, g) and $(\widetilde{M}, \widetilde{g})$ coincide. This gives a counter example for the inverse electrostatic problem on Riemannian surfaces – even the topology of possibly non-compact Riemannian surfaces can not be determined using boundary measurements (see Fig. 1). The above example can be thought as a "hole" in a Riemann surface that does not change the boundary measurements. Roughly speaking, mapping the manifold $\widetilde{M}$ smoothly to the set $M \setminus \overline{B}_M(x_0, \rho)$, where $B_M(x_0, \rho)$ is a metric ball of M, and by putting an object in the obtained hole $\overline{B}_M(x_0, \rho)$, one could hide it from detection at the boundary. This observation was used in [65, 66], where "undetectability" results were introduced in three dimensions, using degenerations of Riemannian metrics, whose singular limits can be

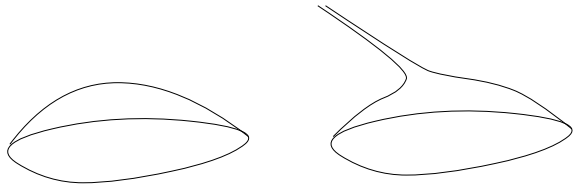

Figure 3.1: Blowing up a metric at a point, after [124]. The electrostatic boundary measurements on the boundary of the surfaces, one compact and the other noncompact but complete, coincide.

considered as coming directly from singular changes of variables. The degeneration of

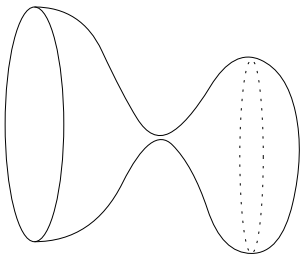

Figure 3.2: A typical member of a family of manifolds developing a singularity as the width of the neck connecting the two parts goes to zero.

the metric (see Fig. 2) can be obtained by considering surfaces (or manifolds in the higher dimensional cases) with a thin "neck" that is pinched. At the limit the manifold contains a pocket about which the boundary measurements do not give any information. If the collapsing of the manifold is done in an appropriate way, we have, in the limit, a singular Riemannian manifold which is indistinguishable in boundary measurements from a flat surface. Then the conductivity which corresponds to this metric is also singular at the pinched points, cf. the first formula in (3.3). The electrostatic measurements on the boundary for this singular conductivity will be the same as for the original regular conductivity corresponding to the metric g. To give a precise, and concrete, realization of this idea, let $B(0,R) \subset \mathbb{R}^3$ denote the open ball with center 0 and radius R. We use in the sequel the set $N = B(0,2)$, the region at the boundary of which the electrostatic measurements will be made, decomposed into two parts, $N_1 = B(0,2) \setminus \overline{B}(0,1)$ and $N_2 = B(0,1)$. We call the interface $\Sigma = \partial N_2$ between N_1 and N_2 the *cloaking surface*. We also use a "copy" of the ball $B(0,2)$, with the notation $M_1 = B(0,2)$, another ball $M_2 = B(0,1)$, and the disjoint union M of M_1 and M_2. (We will see the reason for distinguishing between N and M.) Let $g_{jk} = \delta_{jk}$ be the Euclidian metrics in M_1 and M_2 and let $\gamma = 1$ be the corresponding isotropic homogeneous conductivity. We define a singular transformation

$$F_1 : M_1 \setminus \{0\} \to N_1, \qquad F_1(x) = (\frac{|x|}{2} + 1)\frac{x}{|x|}, \quad 0 < |x| \leq 2. \tag{3.1}$$

We also consider a regular transformation (diffeomorphism) $F_2 : M_2 \mapsto N_2$, which for

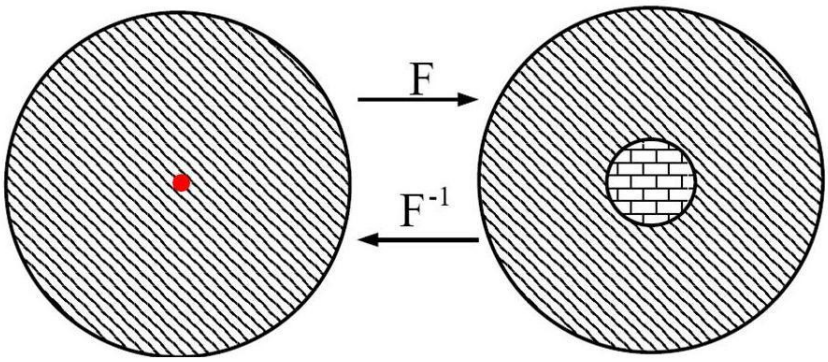

Figure 3.3: Map $F_1 : B(0,2) \setminus \{0\} \to B(0,2) \setminus \overline{B}(0,1)$

simplicity we take to be the identity map $F_2 = Id$. Considering the maps F_1 and F_2 together, $F = (F_1, F_2)$, we define a map $F : M \setminus \{0\} = (M_1 \setminus \{0\}) \cup M_2 \to N \setminus \Sigma$. The push-forward $\widetilde{g} = F_* g$ of the metric g in M by F is the metric in N given by

$$(F_* g)_{jk}(y) = \sum_{p,q=1}^{n} \frac{\partial F^p}{\partial x^j}(x) \frac{\partial F^q}{\partial x^k}(x) g_{pq}(x) \Bigg|_{x=F^{-1}(y)} . \tag{3.2}$$

This metric gives rise to a conductivity $\widetilde{\sigma}$ in N which is singular in N_1,

$$\widetilde{\sigma} = \begin{cases} |\widetilde{g}|^{1/2} \widetilde{g}^{jk} & \text{for } x \in N_1, \\ \delta^{jk} & \text{for } x \in N_2. \end{cases} \tag{3.3}$$

Thus, F forms an invisibility construction that we call "blowing up a point". Denoting by $(r, \phi, \theta) \mapsto (r \sin\theta \cos\phi, r \sin\theta \sin\phi, r \cos\theta)$ the spherical coordinates, we have

$$\widetilde{\sigma} = \begin{pmatrix} 2(r-1)^2 \sin\theta & 0 & 0 \\ 0 & 2\sin\theta & 0 \\ 0 & 0 & 2(\sin\theta)^{-1} \end{pmatrix}, \quad 1 < |x| \leq 2. \tag{3.4}$$

Note that the anisotropic conductivity $\widetilde{\sigma}$ is singular degenerate on Σ in the sense that it is not bounded from below by any positive multiple of I. (See [112] for a similar calculation.) The Euclidian conductivity δ^{jk} in N_2 (3.3) could be replaced by any smooth conductivity bounded from below and above by positive constants. This would correspond to cloaking of a general object with non-homogeneous, anisotropic conductivity. Here, we use the Euclidian metric just for simplicity. Consider now the *Cauchy data* of all solutions in the Sobolev space $H^1(N)$ of the conductivity equation corresponding to $\widetilde{\sigma}$, that is,

$$C_1(\widetilde{\sigma}) = \{(u|_{\partial N}, \nu \cdot \widetilde{\sigma} \nabla u|_{\partial N}) \ : \ u \in H^1(N), \ \nabla \cdot \widetilde{\sigma} \nabla u = 0\},$$

where ν is the Euclidian unit normal vector of ∂N.

Theorem 3.1. *([66]) The Cauchy data of all H^1-solutions for the conductivities $\widetilde{\sigma}$ and γ on N coincide, that is, $C_1(\widetilde{\sigma}) = C_1(\gamma)$.*

This means that all boundary measurements for the homogeneous conductivity $\gamma = 1$ and the degenerated conductivity $\widetilde{\sigma}$ are the same. The result above was proven in [65, 66] for the case of dimension $n \geq 3$. The same basic construction works in the two dimensional case [112]. Fig. 3.4 portrays an analytically obtained solution on a disc with

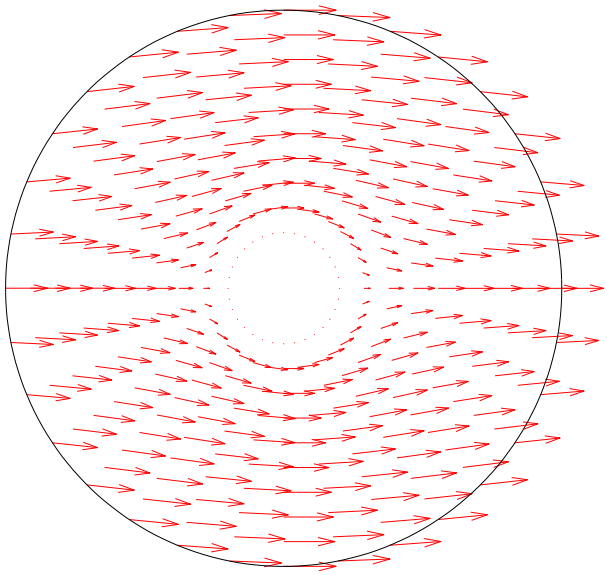

Figure 3.4: Analytic solutions for the currents

conductivity $\widetilde{\sigma}$. As seen in the figure, no currents appear near the center of the disc, so that if the conductivity is changed near the center, the measurements on the boundary ∂N do not change. The above invisibility result is valid for a more general class of singular cloaking transformations. A general class, sufficing at least for electrostatics, is given by the following result from [66]:

Theorem 3.2. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, and $g = (g_{ij})$ a smooth metric on Ω bounded from above and below by positive constants. Let $D \subset\subset \Omega$ be such that there is a C^∞-diffeomorphism $F : \Omega \setminus \{y\} \to \Omega \setminus \overline{D}$ satisfying $F|_{\partial\Omega} = Id$ and such that*

$$dF(x) \geq c_0 I, \quad det\,(dF(x)) \geq c_1\, dist_{\mathbb{R}^n}\,(x,y)^{-1} \tag{3.5}$$

where dF is the Jacobian matrix in Euclidian coordinates on $\mathbb{R}^n$ and $c_0, c_1 > 0$. Let $\widehat{g}$ be a metric in Ω which coincides with $\widetilde{g} = F_* g$ in $\Omega \setminus \overline{D}$ and is an arbitrary regular positive definite metric in D^{int}. Finally, let σ and $\widehat{\sigma}$ be the conductivities corresponding to g and $\widehat{g}$, cf. (1.88). Then,

$$C_1(\widehat{\sigma}) = C_1(\sigma).$$

The key to the proof of Theorem 3.2 is a removable singularities theorem that implies that solutions of the conductivity equation in $\Omega \setminus \overline{D}$ pull back by this singular transformation to solutions of the conductivity equation in the whole Ω. Returning to the case $\Omega = N$ and the conductivity given by (3.3), similar types of results are valid also for a more general class of solutions. Consider an unbounded quadratic form, A in $L^2(N, |\widetilde{g}|^{1/2}dx)$,

$$A_{\widetilde{\sigma}}[u,v] = \int_N \widetilde{\sigma}\nabla u \cdot \nabla v\, dx$$

defined for $u, v \in \mathcal{D}(A_{\widetilde{\sigma}}) = C_0^\infty(N)$. Let $\overline{A}_{\widetilde{\sigma}}$ be the closure of this quadratic form and say that

$$\nabla \cdot \widetilde{\sigma}\nabla u = 0 \quad \text{in } N$$

is satisfied in the finite energy sense if there is $u_0 \in H^1(N)$ supported in N_1 such that $u - u_0 \in \mathcal{D}(\overline{A}_{\widetilde{\sigma}})$ and

$$\overline{A}_{\widetilde{\sigma}}[u - u_0, v] = -\int_N \widetilde{\sigma}\nabla u_0 \cdot \nabla v \, dx, \quad \text{for all } v \in \mathcal{D}(\overline{A}_{\widetilde{\sigma}}).$$

Then the Cauchy data set of the finite energy solutions, denoted by

$$C_{f.e.}(\widetilde{\sigma}) = \Big\{ (u|_{\partial N}, \nu \cdot \widetilde{\sigma}\nabla u|_{\partial N}) \,|u \text{ is a finite energy solution of } \nabla \cdot \widetilde{\sigma}\nabla u = 0 \Big\},$$

coincides with the Cauchy data $C_{f.e.}(\gamma)$ corresponding to the homogeneous conductivity $\gamma = 1$, that is,

$$C_{f.e.}(\widetilde{\sigma}) = C_{f.e.}(\gamma). \tag{3.6}$$

Kohn, Shen, Vogelius and Weinstein [112] in an interesting article have considered the case when instead of blowing up a point one stretches a small ball into the cloaked region. In this case the conductivity is non-singular and one gets "almost" invisibility with a precise estimate in terms of the radius of the small ball.

3.1. Quantum Shielding

In [64], using CGO solutions, uniqueness was proven for the Calderón problem for Schrödinger operators having a more singular class of potentials, namely potentials conormal to submanifolds of $\mathbb{R}^n, n \geq 3$. However, for more singular potentials, there are counterexamples to uniqueness. It was constructed in [64] a class of potentials that shield any information about the values of a potential on a region D contained in a domain Ω from measurements of solutions at $\partial\Omega$. In other words, the boundary information obtained outside the shielded region is independent of $q|_D$. On $\Omega \backslash D$, these potentials behave like $q(x) \sim -Cd(x, \partial D)^{-2-\epsilon}$ where d denotes the distance to ∂D and C is a positive constant. In D, Schrödinger's cat could live forever. From the point of view of quantum mechanics, q represents a potential barrier so steep that no tunneling can occur. From the point of view of optics and acoustics, no sound waves or electromagnetic waves will penetrate, or emanate from, D. However, this construction should be thought of as shielding, not cloaking, since the potential barrier that shields $q|_D$ from boundary observation is itself detectable.

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