# Inverse result in simultaneous approximation by Baskakov-Durrmeyer-Stancu operators 

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Abstract
In the present paper, we study an inverse result in simultaneous approximation for
Baskakov-Durrmeyer-Stancu type operators.
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## 1 Introduction

Verma et al. [1] considered Baskakov-Durrmeyer-Stancu (abbr. BDS) operators for $0 \leq$ $\alpha \leq \beta$ as

$$
\begin{equation*}
D_{n, \alpha, \beta}(f, x)=\sum_{k=1}^{\infty} p_{n, k}(x) \int_{0}^{\infty} b_{n, k}(t) f\left(\frac{n t+\alpha}{n+\beta}\right) d t+p_{n, 0}(x) f\left(\frac{\alpha}{n+\beta}\right) \tag{1.1}
\end{equation*}
$$

where $b_{n, k}(t)=\frac{1}{B(k+1, n)} \frac{t^{k-1}}{(1+t)^{n+k+1}}$ and $p_{n, k}(x)=\binom{n+k-1}{k} \frac{x^{k}}{(1+x)^{n+k}}$.
For $\alpha=\beta=0$, these operators reduce to Baskakov-Durrmeyer operators $D_{n}(f, x)=$ $D_{n, 0,0}(f, x)$. Note that this case was investigated in [2]. Several other researchers have studied in this direction and obtained different approximation properties of many operators, and we mention some of them as [3-8] etc. Verma et al. [1] also studied some approximation properties, asymptotic formula and better estimates for these operators. Recently, Gupta et al. [9] and Mishra and Khatri [10] established point-wise convergence, a Voronovskaja-type asymptotic formula and an error estimate in terms of modulus of continuity of the function and investigated moments of these operators using hypergeometric series, errors estimation in simultaneous approximation, respectively.

Let $C_{v}[0, \infty)$, where $v>0$, be the class of all continuous functions defined on $[0, \infty)$ satisfying the growth condition $|f(t)|=O(1+t)^{\nu}$. The norm $\|\cdot\|_{\nu}$ on $C_{\nu}[0, \infty)$ is defined as $\|f\|_{\nu}=\sup _{0<t<\infty}|f(t)|(1+t)^{-\nu}$.

Let

$$
N_{n, \alpha, \beta}(x, t)=\sum_{k=1}^{\infty} p_{n, k}(x) b_{n, k}(t)+p_{n, 0}(x) \delta\left(\frac{n t+\alpha}{n+\beta}\right),
$$

here $\delta\left(\frac{n t+\alpha}{n+\beta}\right)$ being a type of the Dirac delta function. Then operators (1.1) can be written in the following form:

$$
D_{n, \alpha, \beta}(f, x)=\int_{0}^{\infty} N_{n, \alpha, \beta}(x, t) f\left(\frac{n t+\alpha}{n+\beta}\right) d t .
$$

The operators $D_{n, \alpha, \beta}(f, x)$ are well defined for $f \in C_{v}[0, \infty)$. It is easily checked that the operators $D_{n, \alpha, \beta}$ defined above are linear positive operators and $D_{n, \alpha, \beta}(f, x)=1$. It turns out that the order of approximation by these operators is at best $O\left(n^{-1}\right)$ as $n \rightarrow \infty$, howsoever smooth the function $f$ cab is. Throughout this paper, we denote by $C[a, b]$ the space of all continuous functions on the interval $[a, b]$, the norm $\|\cdot\|_{C[a, b]}$ denotes the sup norm on the space $C[a, b]$. For $f \in C[a, b]$ and a positive integer $k \geq 1$, the $k$ th order modulus of continuity is defined as

$$
\omega_{k}(f, \delta ; a, b)=\sup \left\{\left|\Delta_{h}^{k} f(x)\right|:|h| \leq \delta \text { and } x, x+k h \in[a, b]\right\},
$$

where $\Delta_{h}^{k} f(x)$ is $k$ th forward difference with step length $h$.
A function $f$ is said to belong to the generalized Zygmund class $\operatorname{Liz}(\alpha, k ; a, b)$ if for $\delta>0$ there exists a constant $C$ such that $\omega_{2 k}(f, \delta ; a, b) \leq C \delta^{\alpha k}$. In particular for $k=1$, we simply write $\operatorname{Lip}(\alpha, a, b)$ instead of $\operatorname{Liz}(\alpha, 1 ; a, b)$. By $C_{0}$ we mean the class of continuous functions defined on $(0, \infty)$ having a compact support and $C_{0}^{s}$ the subclass of $C_{0}$, consisting of $s$ times continuously differentiable functions with $\operatorname{supp}\left[a^{\prime}, b^{\prime}\right] \subset(a, b)$ and $[a, b] \subset(0, \infty)$. Also let

$$
G^{(s)}=\left\{g \in C_{0}^{s+2}: \operatorname{supp} g \subset\left[a^{\prime}, b^{\prime}\right]\right\} .
$$

For $f \in C_{0}^{s}$ with $\operatorname{supp} f \subset\left[a^{\prime}, b^{\prime}\right]$, Peetre's $K$-functionals are defined as

$$
K_{s}(\xi, f ; a, b)=\inf _{g \in G^{s}}\left\{\left\|f^{(s)}-g^{(s)}\right\|_{C\left[a^{\prime}, b^{\prime}\right]}+\xi\left(\left\|g^{(s)}\right\|_{C\left[a^{\prime}, b^{\prime}\right]}+\left\|g^{(s+2)}\right\|_{C\left[a^{\prime}, b^{\prime}\right]}\right)\right\}, \quad 0<\xi \leq 1 .
$$

For $0<\alpha<2$ and $f \in C_{0}^{s}$ with $\operatorname{supp} f \subset\left[a^{\prime}, b^{\prime}\right]$, we say that $f \in C_{0}^{s}\left(\alpha, k+1 ; a^{\prime}, b^{\prime}\right)$ if

$$
\left\|f^{(s)}\right\|_{\alpha, s} \equiv \sup _{0<\xi \leq 1} \xi^{-\alpha / 2} K_{s}(\xi, f)<\infty .
$$

## 2 Auxiliary results

In the sequel we shall need several lemmas.

Lemma 1 [10] For $n>0, m>0$ and $s \geq 0$, we have

$$
\begin{equation*}
D_{n}\left(t^{s}, x\right)=\frac{\Gamma(n-s+1) \Gamma(s+1)}{\Gamma(n+1)}\left[(1+x)^{s}{ }_{2} F_{1}\left(1-n,-s ; 1 ; \frac{x}{1+x}\right)-(1+x)^{-n}\right] . \tag{2.1}
\end{equation*}
$$

Moreover,

$$
\begin{equation*}
D_{n}\left(t^{s}, x\right)=\frac{(n+s-1)!(n-s)!}{n!(n-1)!} x^{s}+\frac{s(s-1)(n+s-2)!(n-s)!}{n!(n-1)!} x^{s-1}+O\left(n^{-m}\right) \tag{2.2}
\end{equation*}
$$

Lemma 2 [10] For $0 \leq \alpha \leq \beta$ and $m>0$, we have

$$
\begin{aligned}
& D_{n, \alpha, \beta}\left(t^{s}, x\right) \\
&= x^{s} \frac{n^{s}}{(n+\beta)^{s}} \frac{(n+s-1)!(n-s)!}{n!(n-1)!} \\
&+x^{s-1}\left\{s(s-1) \frac{n^{s}}{(n+\beta)^{s}} \frac{(n+s-2)!(n-s)!}{n!(n-1)!}+s \alpha \frac{n^{s-1}}{(n+\beta)^{s}} \frac{(n+s-2)!(n-s+1)!}{n!(n-1)!}\right\} \\
&+x^{s-2}\left\{s(s-1)^{2} \alpha \frac{n^{s-1}}{(n+\beta)^{s}} \frac{(n+s-3)!(n-s+1)!}{n!(n-1)!}\right. \\
&\left.+\frac{s(s-1)}{2} \alpha^{2} \frac{n^{s-2}}{(n+\beta)^{s}} \frac{(n+s-3)!(n-s+2)!}{n!(n-1)!}\right\}+O\left(n^{-m}\right) .
\end{aligned}
$$

Lemma 3 [11] For $m \in \mathbb{N} \cup\{0\}$, if

$$
U_{n, m}(x)=\sum_{k=0}^{\infty} p_{n, k}(x)\left(\frac{k}{n}-x\right)^{m},
$$

then $U_{n, 0}(x)=1, U_{n, 1}(x)=0$, and we have the recurrence relation:

$$
n U_{n, m+1}(x)=x(1+x)\left[U_{n, m}^{\prime}(x)+m U_{n, m-1}(x)\right] .
$$

Consequently, $U_{n, m}(x)=O\left(n^{-[(m+1) / 2]}\right)$, where $[m]$ is an integral part of $m$.

Lemma 4 [1] For $m \in \mathbb{N} \cup\{0\}$, if

$$
\begin{aligned}
\mu_{n, m}(x) & =D_{n, \alpha, \beta}\left((t-x)^{m}, x\right) \\
& =\sum_{k=1}^{\infty} p_{n, k}(x) \int_{0}^{\infty} b_{n, k}(t)\left(\frac{n t+\alpha}{n+\beta}-x\right)^{m} d t+p_{n, 0}(x)\left(\frac{\alpha}{n+\beta}-x\right)^{m},
\end{aligned}
$$

then

$$
\mu_{n, 0}(x)=1, \quad \mu_{n, 1}(x)=\frac{\alpha-\beta x}{n+\beta}
$$

and for $n>m$ we have the recurrence relation:

$$
\begin{aligned}
(n-m)\left(\frac{n+\beta}{n}\right) \mu_{n, m+1}(x)= & x(1+x)\left[\mu_{n, m}^{\prime}(x)+m \mu_{n, m-1}(x)\right] \\
& +\left[(m+n x)+\left(\frac{n+\beta}{n}\right)\left(\frac{\alpha}{n+\beta}-x\right)(n-2 m)\right] \mu_{n, m}(x) \\
& -\left(\frac{\alpha}{n+\beta}-x\right)\left[\left(\frac{\alpha}{n+\beta}-x\right)\left(\frac{n+\beta}{n}\right)-1\right] m \mu_{n, m-1}(x) .
\end{aligned}
$$

From the recurrence relation, it is easily verified that for all $x \in[0, \infty)$, we have

$$
\mu_{n, m}(x)=O\left(n^{-[(m+1) / 2]}\right) .
$$

Lemma 5 [11] There exist polynomials $q_{i, j, s}(x)$ on $[0, \infty)$, independent of $n$ and $k$, such that

$$
x^{s}(1+x)^{s} \frac{d^{s}}{d x^{s}} p_{n, k}(x)=\sum_{\substack{2 i+j \leq s \\ i, j \geq 0}} n^{i}(k-n x)^{j} q_{i, j, s}(x) p_{n, k}(x) .
$$

Lemma 6 Let $0<a<a^{\prime}<a^{\prime \prime}<b^{\prime \prime}<b^{\prime}<b<\infty$ and $f^{(s)} \in C_{0}$ with $\operatorname{supp} f \subset\left[a^{\prime \prime}, b^{\prime \prime}\right]$. If

$$
\left\|D_{n, \alpha, \beta}^{(s)}(f, \star)-f^{(s)}\right\|_{C[a, b]}=O\left(n^{-\alpha / 2}\right),
$$

then

$$
\begin{equation*}
K_{s}(\xi, f)=C_{1}\left\{n^{-\alpha / 2}+n \xi K_{s}\left(n^{-1}, f\right)\right\} . \tag{2.3}
\end{equation*}
$$

Consequently, $K_{s}(\xi, f) \leq C_{2} \xi^{\alpha / 2}$, i.e., $f \in C_{0}^{s}\left(\alpha, 1 ; a^{\prime}, b^{\prime}\right)$, where $C_{1}$ and $C_{2}$ are some positive constants.

Proof To prove (2.3), it is sufficient to show that

$$
K_{s}(\xi, f) \leq C_{1}\left\{n^{-\alpha / 2}+n \xi K_{s}\left(n^{-1}, f\right)\right\} \quad \text { for all } n \text { sufficiently large. }
$$

Since $\operatorname{supp} f \subset\left[a^{\prime \prime}, b^{\prime \prime}\right]$, therefore by Theorem 2 there exists a function $e^{(i)} \in G^{(s)}$ such that for $i=s$ and $i=s+2$,

$$
\left\|D_{n, \alpha, \beta}^{(i)}(f, \star)-e^{(i)}\right\|_{C[a, b]} \leq C_{3} n^{-1},
$$

which implies that

$$
\begin{aligned}
K_{s}(\xi, f) \leq & 3 C_{3} n^{-1}+\left\|D_{n, \alpha, \beta}^{(s)}(f, \star)-f^{(s)}\right\|_{C\left[a^{\prime}, b^{\prime}\right]} \\
& +\xi\left(\left\|D_{n, \alpha, \beta}^{(s)}(f, \star)\right\|_{C\left[a^{\prime}, b^{\prime}\right]}+\left\|D_{n, \alpha, \beta}^{(s+2)}(f, \star)\right\|_{C\left[a^{\prime}, b^{\prime}\right]}\right)
\end{aligned}
$$

Thus, it is sufficient to show that there exits a constant $C_{4}$ such that for each $g \in G^{(s)}$,

$$
\begin{equation*}
\left\|D_{n, \alpha, \beta}^{(s+2)}(f, \star)\right\|_{C\left[a^{\prime}, b^{\prime}\right]} \leq C_{4} n\left(\left\|f^{(s)}-g^{(s)}\right\|_{C\left[a^{\prime}, b^{\prime}\right]}+n^{-1}\left\|g^{(s+2)}\right\|_{C\left[a^{\prime}, b^{\prime}\right]}\right) \tag{2.4}
\end{equation*}
$$

In fact, by the linearity property, we have

$$
\begin{equation*}
\left\|D_{n, \alpha, \beta}^{(s+2)}(f, \star)\right\|_{C\left[a^{\prime}, b^{\prime}\right]} \leq\left\|D_{n, \alpha, \beta}^{(s+2)}(f-g, \star)\right\|_{C\left[a^{\prime}, b^{\prime}\right]}+\left\|D_{n, \alpha, \beta}^{(s+2)}(g, \star)\right\|_{C\left[a^{\prime}, b^{\prime}\right]} \tag{2.5}
\end{equation*}
$$

Applying Lemma 5, we have

$$
\begin{aligned}
\int_{0}^{\infty}\left|\frac{\partial^{s+2}}{\partial x^{s+2}} N_{n, \alpha, \beta}(x, t)\right| d t \leq & \sum_{\substack{2 i+j \leq s+2 \\
i, j \geq 0}} \sum_{\substack{ \\
i=1}}^{\infty} \frac{n^{i}\left|q_{i, j, s+2}\right|(x)}{[x(1+x)]^{s+2}} p_{n, k}(x)|k-n x|^{j} \int_{0}^{\infty} b_{n, k}(t) d t \\
& +\frac{d^{s+2}}{d x^{s+2}}\left[(1+x)^{-n}\right] .
\end{aligned}
$$

Therefore, by the Cauchy-Schwarz inequality and Lemma 3, we get

$$
\begin{equation*}
\left\|D_{n, \alpha, \beta}^{(s+2)}(f-g, \star)\right\|_{C\left[a^{\prime}, b^{\prime}\right]} \leq C_{5} n\left\|f^{(s)}-g^{(s)}\right\|_{C\left[a^{\prime}, b^{\prime}\right]}, \tag{2.6}
\end{equation*}
$$

where the constant $N_{4}$ is independent of $f$ and $g$. Next, by Taylor's expansion, we have

$$
g(t)=\sum_{i=0}^{s+1} \frac{g^{(i)}(x)}{i!}(t-x)^{i}+\frac{g^{(s+2)}(\xi)}{(s+2)!}(t-x)^{s+2}
$$

where $\xi$ lies between $t$ and $x$. Using the above expansion and the fact that

$$
\begin{equation*}
\int_{0}^{\infty} \frac{\partial^{s}}{\partial x^{s}} N_{n, \alpha, \beta}(x, t)(t-x)^{i} d t=0 \quad \text { for } s>i, \tag{2.7}
\end{equation*}
$$

we get

$$
\begin{equation*}
\left\|D_{n, \alpha, \beta}^{(s+2)}(g, \star)\right\|_{C\left[a^{\prime}, b^{\prime}\right]} \leq C_{6}\left\|g^{(s+2)}\right\|_{C\left[a^{\prime}, b^{\prime}\right]}\left\|\int_{0}^{\infty} \frac{\partial^{s+2}}{\partial x^{s+2}} N_{n, \alpha, \beta}(x, t)(t-x)^{s+2} d t\right\|_{C\left[a^{\prime}, b^{\prime}\right]} \tag{2.8}
\end{equation*}
$$

Also, by Lemmas 3, 4 and 5 and the Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
J \equiv & \int_{0}^{\infty}\left|\frac{\partial^{s+2}}{\partial x^{s+2}} N_{n, \alpha, \beta}(x, t)\right|(t-x)^{s+2} d t \\
\leq & \sum_{\substack{2 i+j \leq s+2 \\
i, j \geq 0}} \sum_{k=1}^{\infty} \frac{n^{i}\left|q_{i, j, s+2}\right|(x)}{[x(1+x)]^{s+2}} p_{n, k}(x)|k-n x|^{j} \int_{0}^{\infty} b_{n, k}(t)\left(\frac{n t+\alpha}{n+\beta}-x\right)^{s+2} d t \\
& +\frac{d^{s+2}}{d x^{s+2}}\left[\left(\frac{\alpha}{n+\beta}-x\right)^{s+2}(1+x)^{-n}\right] \\
\leq & \sum_{2 i+j \leq s+2} \frac{n^{i}\left|q_{i, j, s+2}\right|(x)}{[x(1+x)]^{s+2}}\left(\sum_{k=1}^{\infty} p_{n, k}(x)(k-n x)^{2 j}\right)^{\frac{1}{2}} \\
& \times\left(\sum_{k=1}^{\infty} p_{n, k}(x) \int_{0}^{\infty} b_{n, k}(t)\left(\frac{n t+\alpha}{n+\beta}-x\right)^{2 s+4} d t\right)^{\frac{1}{2}}\left(\int_{0}^{\infty} b_{n, k}(t) d t\right)^{\frac{1}{2}} \\
& +\frac{d^{s+2}}{d x^{s+2}}\left[\left(\frac{\alpha}{n+\beta}-x\right)^{s+2}(1+x)^{-n}\right] \\
= & C_{7} \sum_{2 i+j \leq s+2} n^{i} O\left(n^{j / 2}\right) O\left(n^{-(s+2) / 2}\right)=C_{8} O(1) .
\end{aligned}
$$

Hence

$$
\begin{equation*}
\left\|D_{n, \alpha, \beta}^{(s+2)}(g, \star)\right\|_{C\left[a^{\prime}, b^{\prime}\right]} \leq C_{9}\left\|g^{(s+2)}\right\|_{C\left[a^{\prime}, b^{\prime}\right]} . \tag{2.9}
\end{equation*}
$$

Combining the estimates (2.5)-(2.9), we get (2.4). The other consequence follows from [12]. This completes the proof of the lemma.

Lemma 7 [5] Let $0<a<a^{\prime}<a^{\prime \prime}<b^{\prime \prime}<b^{\prime}<b<\infty$ and $f^{(s)} \in C_{0}$ with $\operatorname{supp} f \subset\left[a^{\prime \prime}, b^{\prime \prime}\right]$. Iff $\in C_{0}^{s}\left(\alpha, 1 ; a^{\prime}, b^{\prime}\right)$, then $f^{(s)} \in \operatorname{Lip}^{\star}\left(\alpha, a^{\prime}, b^{\prime}\right)$.

## 3 Known and inverse results

In this section, first we give some known results and then we estimate an inverse theorem in simultaneous approximation for Baskakov-Durrmeyer-Stancu operators. Now, this section is devoted to the following inverse theorem in simultaneous approximation.

Theorem 1 [9] If $s \in \mathbb{N}, f \in C_{v}[0, \infty)$ for some $v>0$, and $f^{(s)}$ exists at a point $x \in(0, \infty)$, then

$$
\begin{equation*}
\lim _{n \rightarrow \infty} D_{n, \alpha, \beta}^{(s)}(f, x)=f^{(s)}(x) . \tag{3.1}
\end{equation*}
$$

Theorem 2 [9] Let $f \in C_{v}[0, \infty)$ for some $v>0$, and $f^{(s+2)}$ exists at a point $x \in(0, \infty)$. Then

$$
\begin{align*}
& \lim _{n \rightarrow \infty} n\left(D_{n, \alpha, \beta}^{(s)}(f, x)-f^{(s)}(x)\right) \\
& \quad=s(s-1-\beta) f^{(s)}(x)+[(2 s-\beta) x+(s+\alpha)] f^{(s+1)}(x)+x(1+x) f^{(s+2)}(x) . \tag{3.2}
\end{align*}
$$

Theorem 3 [9] Let $f \in C_{v}[0, \infty)$ for some $v>0$, and $0<a<a_{1}<b_{1}<b<\infty$. Then, for sufficiently large $n$, we have

$$
\begin{equation*}
\left\|D_{n, \alpha, \beta}^{(s)}(f, \star)-f^{(s)}\right\|_{C\left[a_{1}, b_{1}\right]} \leq C_{1} \omega\left(f^{(s)}, n^{-1 / 2}, a, b\right)+C_{2} n^{-k}\|f\|_{\nu}, \tag{3.3}
\end{equation*}
$$

where $C_{1}=C_{1}(s), C_{2}=C_{2}(s, f)$.

Theorem 4 Let $0<\alpha<2,0<a_{1}<a_{2}<b_{2}<b_{1}<\infty$, and suppose $f \in C_{v}[0, \infty)$. Then in the following statements $(\mathrm{i}) \Longrightarrow$ (ii):
(i) $\left\|D_{n, \alpha, \beta}^{(s)}(f, \star)\right\|_{C\left[a_{1}, b_{1}\right]}=O\left(n^{-\alpha / 2}\right)$,
(ii) $f^{(s)} \in \operatorname{Lip}^{\star}\left(\alpha, a_{2}, b_{2}\right)$,
where $\operatorname{Lip}^{\star}\left(\alpha, a_{2}, b_{2}\right)$ denotes the Zygmund class satisfying $\omega_{2}\left(f, \delta, a_{2}, b_{2}\right) \leq C \delta^{\alpha}$.

Proof Let us choose $a^{\prime}, a^{\prime \prime}, b^{\prime}, b^{\prime \prime}$ in such a way that $a_{1}<a^{\prime}<a^{\prime \prime}<a_{2}<b_{2}<b<b^{\prime \prime}<b_{1}$. Also suppose $g \in C_{0}^{\infty}$ with supp $g \in\left[a^{\prime \prime}, b^{\prime \prime}\right]$ and $g(x)=1$ on the interval $\left[a_{2}, b_{2}\right]$. For $x \in\left[a^{\prime}, b^{\prime}\right]$ with $D \equiv \frac{d}{d x}$, we have

$$
\begin{aligned}
& D_{n, \alpha, \beta}^{(s)}(f g, x)-(f g)^{(s)}(x) \\
& \quad=D^{s}\left(D_{n, \alpha, \beta}((f g)(t)-(f g)(x)), x\right) \\
& \quad=D^{s}\left(D_{n, \alpha, \beta}(f(t)[g(t)-g(x)], x)\right)+D^{s}\left(D_{n, \alpha, \beta}(g(x)[f(t)-f(x)], x)\right) \\
& \quad=: E_{1}+E_{2} .
\end{aligned}
$$

By the Leibniz formula, we have

$$
\begin{aligned}
E_{1} & =\frac{\partial^{s}}{\partial x^{s}} \int_{0}^{\infty} N_{n, \alpha, \beta}(x, t) f(t)[g(t)-g(x)] d t \\
& =\sum_{i=0}^{s}\binom{s}{i} \int_{0}^{\infty} N_{n, \alpha, \beta}^{(i)}(x, t) \frac{\partial^{s-i}}{\partial x^{s-i}}[f(t)(g(t)-g(x))] d t
\end{aligned}
$$

$$
\begin{aligned}
& =-\sum_{i=0}^{s-1}\binom{s}{i} g^{(s-i)}(x) D_{n, \alpha, \beta}^{(i)}(f, x)+\int_{0}^{\infty} N_{n, \alpha, \beta}^{(s)}(x, t) f(t)(g(t)-g(x)) d t \\
& =: E_{3}+E_{4} .
\end{aligned}
$$

Applying Theorem 3, we have

$$
E_{3}=-\sum_{i=0}^{s-1}\binom{s}{i} g^{(s-i)}(x) f^{(i)}(x)+O\left(n^{-\alpha / 2}\right)
$$

uniformly in $x \in\left[a^{\prime}, b^{\prime}\right]$. By Taylor's expansion of $f(t)$ and $g(t)$, we have

$$
f(t)=\sum_{i=0}^{s} \frac{f^{(i)}(x)}{i!}(t-x)^{i}+O(t-x)^{s}
$$

and

$$
g(t)=\sum_{i=0}^{s+1} \frac{g^{(i)}(x)}{i!}(t-x)^{i}+O(t-x)^{s+1}
$$

Substituting the above expansions in $E_{4}$ and using Theorem 2, the Schwarz inequality and Lemma 4, we obtain

$$
\begin{aligned}
E_{4} & =\sum_{i=0}^{s} \frac{g^{(i)}(x) f^{(s-i)}(x)}{i!(s-i)!} s!+O\left(n^{-1 / 2}\right) \\
& =\sum_{i=0}^{s}\binom{s}{i} g^{(i)}(x) f^{(s-i)}(x)+O\left(n^{-\alpha / 2}\right),
\end{aligned}
$$

uniformly in $x \in\left[a^{\prime}, b^{\prime}\right]$. Again using the Leibniz formula, we have

$$
\begin{aligned}
E_{2} & =\sum_{i=0}^{s}\binom{s}{i} \int_{0}^{\infty} N_{n, \alpha, \beta}^{(i)}(x, t) \frac{\partial^{s-i}}{\partial x^{s-i}}[g(t)(f(t)-f(x))] d t \\
& =\sum_{i=0}^{s}\binom{s}{i} g^{(s-i)}(x) D_{n, \alpha, \beta}^{(i)}(f, x)-(f g)^{s}(x) \\
& =\sum_{i=0}^{s}\binom{s}{i} g^{(s-i)}(x) f^{(i)}(x)-(f g)^{s}(x)+O\left(n^{-\alpha / 2}\right) \\
& =O\left(n^{-\alpha / 2}\right)
\end{aligned}
$$

uniformly in $x \in\left[a^{\prime}, b^{\prime}\right]$. Combining the above estimates, we get

$$
\left\|D_{n, \alpha, \beta}^{(s)}(f g, \star)-(f g)^{s}\right\|_{C\left[a^{\prime}, b^{\prime}\right]}=O\left(n^{-\alpha / 2}\right)
$$

Thus by Lemmas 5 and 7, we have $(f g)^{(s)} \in \operatorname{Lip}^{\star}\left(\alpha, a^{\prime}, b^{\prime}\right)$ also $g(x)=1$ on the interval $\left[a_{2}, b_{2}\right]$, and it proves that $f^{(s)} \in \operatorname{Lip}^{\star}\left(\alpha, a_{2}, b_{2}\right)$. This completes the validity of the implication (i) $\Longrightarrow$ (ii) for the case $0<\alpha \leq 1$.

To prove the result for $1<\alpha<2$ for any interval $\left[a^{*}, b^{*}\right] \subset\left(a_{1}, b_{1}\right)$, let $a_{2}^{*}, b_{2}^{*}$ be such that $\left(a_{2}, b_{2}\right) \subset\left(a_{2}^{*}, b_{2}^{*}\right)$ and $\left(a_{2}^{*}, b_{2}^{*}\right) \subset\left(a_{1}^{*}, b_{1}^{*}\right)$. Letting $\delta>0$ we shall prove the assertion
$\alpha<2$. From the previous case it implies that $f^{(s)}$ exists and belongs to $\operatorname{Lip}\left(1-\delta, a_{1}^{*}, b_{1}^{*}\right)$. Let $g \in C_{0}^{\infty}$ be such that $g(x)=1$ on the interval $\left[a_{2}, b_{2}\right]$ and supp $g \subset\left(a_{2}^{*}, b_{2}^{*}\right)$. If $\chi(t)$ denotes the characteristic function of the interval $\left[a_{1}^{*}, b_{1}^{*}\right]$, we have

$$
\begin{aligned}
\left\|D_{n, \alpha, \beta}^{(s)}(f g, x)-(f g)^{(s)}(x)\right\|_{C\left[a_{2}^{*}, b_{2}^{*}\right]} \leq & \left\|D^{s}\left(D_{n, \alpha, \beta}(f(t)[g(t)-g(x)], x)\right)\right\|_{C\left[a_{2}^{*}, b, b_{2}^{*}\right]} \\
& +\left\|D^{s}\left(D_{n, \alpha, \beta}(g(x)[f(t)-f(x)], x)\right)\right\|_{C\left[a_{2}^{*}, b_{2}^{*}\right]} \\
= & F_{1}+F_{2} .
\end{aligned}
$$

Using the linearity property, the Leibniz formula and Theorem 3, we have

$$
\begin{aligned}
F_{1} & =\left\|D^{s}\left(g(x) D_{n, \alpha, \beta}(f, x)-(f g)(x) D_{n, \alpha, \beta}(1, x)\right)\right\|_{C\left[a_{2}^{*}, b_{2}^{*}\right]} \\
& =\left\|\sum_{i=0}^{s}\binom{s}{i} g^{(s-i)}(x) D_{n, \alpha, \beta}^{(i)}(f, x)-(f g)^{(s)}\right\|_{C\left[a_{2}^{*}, b_{2}^{*}\right]} \\
& =\left\|\sum_{i=0}^{s}\binom{s}{i} g^{(s-i)}(x) f^{(i)}(x)-(f g)^{(s)}\right\| \|_{C\left[a_{2}^{*}, b_{2}^{*}\right]}+O\left(n^{-\alpha / 2}\right)=O\left(n^{-\alpha / 2}\right) .
\end{aligned}
$$

Applying the Leibniz formula and Theorem 2, we get

$$
\begin{aligned}
F_{2} & =\left\|-\sum_{i=0}^{s-1}\binom{s}{i} g^{(s-i)}(x) D_{n, \alpha, \beta}^{(i)}(f, x)+D_{n, \alpha, \beta}^{(s)}(f(t)[g(t)-g(x)] \chi(t), x)\right\|_{C\left[a_{2}^{*}, b_{2}^{*}\right]}+O\left(n^{-1}\right) \\
& =:\left\|F_{3}+F_{4}\right\|_{C\left[a_{2}^{*}, b_{2}^{*}\right]}+O\left(n^{-1}\right) .
\end{aligned}
$$

Then by Theorem 3, we have

$$
F_{3}=-\sum_{i=0}^{s-1}\binom{s}{i} g^{(s-i)}(x) f^{(i)}(x)+O\left(n^{-\alpha / 2}\right)
$$

uniformly in $x \in\left[a_{2}^{*}, b_{2}^{*}\right]$. Applying Taylor's expansion of $f(t)$, we have

$$
\begin{aligned}
F_{4}= & \int_{0}^{\infty} N_{n, \alpha, \beta}^{(s)}(x, t)[f(t)(g(t)-g(x)) \chi(t)] d t \\
= & \sum_{i=0}^{s} \frac{f^{(i)}(x)}{i!} \int_{0}^{\infty} N_{n, \alpha, \beta}^{(s)}(x, t)(t-x)^{i}(g(t)-g(x)) \chi(t) d t \\
& +\int_{0}^{\infty} N_{n, \alpha, \beta}^{(s)}(x, t)\left[\frac{f^{(s)}(\xi)-f^{(s)}(x)}{s!}\right](t-x)^{s}(g(t)-g(x)) \chi(t) d t \\
= & F_{5}+F_{6},
\end{aligned}
$$

where $\xi$ lies between $t$ and $x$. Using Theorem 2, we get

$$
\begin{aligned}
F_{5}= & \sum_{i=0}^{s} \frac{f^{(i)}(x)}{i!} \int_{0}^{\infty} N_{n, \alpha, \beta}^{(s)}(x, t)(t-x)^{i}(g(t)-g(x)) d t+O\left(n^{-1}\right), \\
& \text { uniformly in } x \in\left[a_{2}^{*}, b_{2}^{*}\right] \\
= & F_{7}+O\left(n^{-1}\right) .
\end{aligned}
$$

Again using Taylor's expansion of $g(t) \in C_{0}^{\infty}$ and using the fact that $\varepsilon(t, x) \rightarrow 0$ as $t \rightarrow x$, we have

$$
\begin{aligned}
F_{7}= & \sum_{i=0}^{s} \frac{f^{(i)}(x)}{i!} \int_{0}^{\infty} N_{n, \alpha, \beta}^{(s)}(x, t)(t-x)^{i} \\
& \times\left[g(x)+\sum_{j=1}^{s+2} \frac{g^{(j)}(x)}{j!}(t-x)^{j}+\varepsilon(t, x)(t-x)^{s+2}-g(x)\right] d t \\
= & \sum_{i=0}^{s} \frac{f^{(i)}(x)}{i!} \sum_{j=1}^{s+2} \frac{g^{(j)}(x)}{j!} \int_{0}^{\infty} N_{n, \alpha, \beta}^{(s)}(x, t)(t-x)^{i+j} d t \\
& +\sum_{i=0}^{s} \frac{f^{(i)}(x)}{i!} \int_{0}^{\infty} N_{n, \alpha, \beta}^{(s)}(x, t) \varepsilon(t, x)(t-x)^{i+s+2} d t \\
= & F_{8}+F_{9} .
\end{aligned}
$$

Since $\int_{0}^{\infty} \frac{\partial^{s}}{\partial x^{s}} N_{n, \alpha, \beta}(x, t)(t-x)^{k} d t=0$ for every $s>k$, therefore by Theorem 2 and Lemma 2, we have

$$
\begin{aligned}
F_{8} & =\sum_{j=1}^{s} \frac{g^{(j)}(x) f^{(s-j)}(x)}{j!(s-j)!} s!+O\left(n^{-1}\right) \\
& =\sum_{j=1}^{s}\binom{s}{i} g^{(j)}(x) f^{(s-j)}(x)+O\left(n^{-1}\right),
\end{aligned}
$$

uniformly in $x \in\left[a_{2}^{*}, b_{2}^{*}\right]$. Also as in the proof of Theorem 1 , it can be easily shown that

$$
F_{9}=O\left(n^{-\alpha / 2}\right)
$$

uniformly in $x \in\left[a_{2}^{*}, b_{2}^{*}\right]$. Next, using Lemma 5 , the mean value theorem, the Schwarz inequality and Lemma 4, we have

$$
\begin{aligned}
& \left\|\int_{0}^{\infty} N_{n, \alpha, \beta}^{(s)}(x, t)\left[\frac{f^{(s)}(\xi)-f^{(s)}(x)}{s!}\right](t-x)^{s}(g(t)-g(x)) \chi(t) d t\right\|_{C\left[a_{2}^{*}, b_{2}^{*}\right]} \\
& \quad \leq \sum_{\substack{2 i+j \leq s \\
i, j \geq 0}} n^{i+j} \frac{\left|q_{i, j, s}(x)\right|}{[x(1+x)]^{s}} \\
& \quad \times\left\|\int_{0}^{\infty} N_{n, \alpha, \beta}(x, t)|t-x|^{\delta+s+1} \frac{\left|f^{(s)}(\xi)-f^{(s)}(x)\right|}{s!}\left|g^{\prime}(\eta)\right| \chi(t) d t\right\|_{C\left[a_{2}^{*}, b_{2}^{*}\right]}=O\left(n^{-\delta / 2}\right),
\end{aligned}
$$

where $\eta$ lies between $t$ and $x$, and choose $\delta$ such that $0 \leq \delta \leq 2-\alpha$. Combining the above estimates, we get

$$
\left\|D_{n, \alpha, \beta}^{(s)}(f g, \star)-(f g)^{s}\right\|_{C\left[a_{2}^{*}, b_{2}^{*}\right]}=O\left(n^{-\alpha / 2}\right) .
$$

Since supp $f g \subset\left(a_{2}^{*}, b_{2}^{*}\right)$, therefore by Lemmas 6 and 7 , we have $(f g)^{(s)} \in \operatorname{Lip}^{\star}\left(\alpha, 1, a_{2}^{*}, b_{2}^{*}\right)$ also $g(x)=1$ on the interval $\left[a_{2}, b_{2}\right]$, which proves that $f^{(s)} \in \operatorname{Lip}^{\star}\left(\alpha, a_{2}, b_{2}\right)$. This completes the
validity of the implication (i) $\Longrightarrow$ (ii) for the case $1<\alpha<2$. This completes the proof of the theorem.

## Competing interests

The authors declare that they have no competing interests.

## Authors' contributions

VNM, KK, LNM and Deepmala computed the auxiliary results and inverse theorem in simultaneous approximation for Baskakov-Durrmeyer-Stancu Operators. VNM and Deepmala conceived of the study and participated in its design and coordination. VNM, KK, LNM and Deepmala contributed equally and significantly in writing this manuscript. All the authors drafted the manuscript, read and approved the final version of manuscript in JIA.

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