

Inverse Scattering Problem for the Schrödinger Equation with Magnetic Potential at a Fixed Energy

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Abstract: In this article we consider the Schrödinger operator in $R^n, n \geq 3$, with electric and magnetic potentials which decay exponentially as $|x| \rightarrow \infty$. We show that the scattering amplitude at fixed positive energy determines the electric potential and the magnetic field.

1. Introduction

Consider the Schrödinger equation in $R^n, n \geq 3$, with magnetic potential $A(x) = (A_1(x), \dots, A_n(x))$ and electric potential $V(x)$:

$$-\sum_{j=1}^n \left(\frac{\partial}{\partial x_j} + iA_j(x) \right)^2 u + V(x)u = k^2 u, \tag{1}$$

$k > 0$, or equivalently

$$-\Delta u - 2i \sum_{j=1}^n A_j(x) \frac{\partial u}{\partial x_j} + q(x)u = k^2 u, \tag{1'}$$

where

$$q(x) = \sum_{j=1}^n \left(A_j^2(x) - i \frac{\partial A_j}{\partial x_j} \right) + V(x). \tag{2}$$

We will assume that the potentials A and V are real-valued and exponentially decreasing, i.e.

$$\left| \frac{\partial^\alpha V(x)}{\partial x^\alpha} \right| \leq C_\alpha e^{-\delta|x|}, \quad \left| \frac{\partial^\beta A_j}{\partial x^\beta} \right| \leq C_\beta e^{-\delta|x|}, \quad j = 1, \dots, n, \tag{3}$$

for $0 \leq |\alpha| \leq P, 0 \leq |\beta| \leq P + 1$, where $P = n + 4$. We consider the solutions of (1) of the form

$$u = e^{ik\omega \cdot x} + v(x, \omega, k), \tag{4}$$

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where v is the outgoing solution of

$$-\Delta v - 2i \sum_{j=1}^n A_j(x) \frac{\partial v}{\partial x_j} + (q(x) - k^2)v = e^{ik\omega \cdot x} \left(-2k \sum_{j=1}^n \omega_j A_j(x) - q(x) \right) \quad (5)$$

obtained by the limiting absorption method. By this argument v exists and is unique whenever k^2 is not an embedded eigenvalue, and, combining Sect. 5 of Hörmander [4] with the proof of Theorem 3.3 of Agmon [1], one sees that (3) implies there are no embedded eigenvalues. Representing v in terms of the outgoing fundamental solution of $\Delta + k^2$, it follows that as $|x| \rightarrow \infty$,

$$v(x, \omega, k) = \frac{e^{ik|x|}}{|x|^{\frac{n-1}{2}}} \left(a\left(\frac{x}{|x|}, \omega, k\right) + O\left(\frac{1}{|x|}\right) \right), \quad (6)$$

where $a(\theta, \omega, k)$ is defined to be the scattering amplitude. Our objective is to prove

Theorem 1. Fix $k > 0$. Then one can recover $V(x)$ and the magnetic field $B = \text{curl } A$ from the scattering amplitude $a(\theta, \omega, k)$, $(\theta, \omega) \in S^{n-1} \times S^{n-1}$.

Note that, if A and A' satisfy (3) and $\text{curl } A = \text{curl } A'$, then $A' - A$ is the gradient of function φ satisfying

$$\left| \frac{\partial^p \varphi}{\partial x^p} \right| \leq C_p e^{-\delta|x|}, \quad 0 \leq |p| \leq P. \quad (7)$$

To see that changing A to $A' = A + \frac{\partial \varphi}{\partial x}$ does not change the scattering amplitude note that, if one replaces $u(x)$ by $w(x) = u(x)e^{-i\varphi(x)}$, then $w(x)$ will satisfy

$$-\left(\frac{\partial}{\partial x} + iA(x) + i\frac{\partial \varphi}{\partial x} \right)^2 w + V(x)w = k^2 w.$$

However, this does not change the scattering amplitude, since

$$\begin{aligned} w &= u(x)e^{-i\varphi(x)} = e^{-i\varphi(x)} \left(e^{ik\omega \cdot x} + a\left(\frac{x}{|x|}, \omega, k\right) \frac{e^{ik|x|}}{|x|^{\frac{n-1}{2}}} + O\left(\frac{1}{|x|^{\frac{n+1}{2}}}\right) \right) \\ &= e^{ik\omega \cdot x} + a\left(\frac{x}{|x|}, \omega, k\right) \frac{e^{ik|x|}}{|x|^{\frac{n-1}{2}}} + O\left(\frac{1}{|x|^{\frac{n+1}{2}}}\right). \end{aligned}$$

In this article as in [2] we will use $h(\xi, k\omega, k)$, the Fourier transform of $-(\Delta + k^2)v$, to study the scattering amplitude. Since v is obtained by limiting absorption,

$$v(x, \omega, k) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{h(\xi, k\omega, k) e^{ix \cdot \xi}}{|\xi|^2 - k^2 - i0} d\xi, \quad (8)$$

and, taking the asymptotics of (8) when $\theta = x/|x|$ is fixed and $|x| \rightarrow \infty$, one obtains

$$a(\theta, \omega, k) = C_{n,k} h(k\theta, k\omega, k), \quad C_{n,k} = \frac{1}{4\pi} \left(\left(\frac{k}{2\pi} \right)^{\frac{1}{2}} e^{-\frac{i\pi}{4}} \right)^{n-3}. \quad (9)$$

From (5) one sees that h satisfies

$$h(\xi, \zeta, k) + \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{q_0(\xi - \eta, \eta)h(\eta, \zeta, k)}{|\eta|^2 - k^2 - i0} d\eta = -q_0(\xi - \zeta, \zeta), \tag{10}$$

where

$$q_0(\xi, \zeta) = 2 \sum_{j=1}^n \hat{A}_j(\xi) \zeta_j + \hat{q}(\xi). \tag{11}$$

Note that (3) implies that $q_0(\xi - \zeta, \zeta)$ is analytic in (ξ, ζ) for $|\text{Im } \xi| < \delta/2, |\text{Im } \zeta| < \delta/2$. For fixed λ , the integral operator

$$T_\lambda w = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{q_0(\xi - \eta, \eta)w(\eta)}{|\eta|^2 - \lambda - i0} d\eta \tag{12}$$

is compact in the space $H_{\alpha, N}, 0 < \alpha < 1, n - 1 < N < n + 4$. Here $H_{\alpha, N}$ is the weighted Hölder space used in [2]: let $\|f\|_{\alpha, N} = \|(1 + |\xi|^2)^{N/2} f\|_{\alpha}$, where $\|\cdot\|_{\alpha}$ is the standard Hölder norm, and define $H_{\alpha, N}$ as the completion of $C_0^\infty(\mathbb{R}^n)$ is $\|\cdot\|_{\alpha, N}$. Moreover, T_λ depends analytically on λ for $\text{Im } \lambda > 0$ and extends continuously to the positive real axis, $\lambda > 0$. In the same way that Theorem 5.2 of [4] showed that the homogeneous equation corresponding to (5) had no nontrivial square-integrable solutions, it can be used here to show the $I + T_{k^2}$ has no nontrivial solutions in $H_{\alpha, N}(\mathbb{R}^n)$. Hence we see that the Fredholm operator $I + T_{k^2}$ is invertible on $H_{\alpha, N}$ for $k > 0$. This will be useful in what follows.

In the case that the magnetic field B is small uniqueness results at fixed energy have been obtained previously by Henkin and Novikov [6] and by Sun [9]. Recently Nakamura, Sun and Uhlmann [5] obtained the uniqueness result analogous to Theorem 1 for the Dirichlet to Neumann map. This implies Theorem 1 for magnetic and electric potentials of compact support. In fact, when the magnetic and electric potentials have compact support, as in [9], uniqueness for inverse scattering at fixed energy and uniqueness for the Dirichlet-to-Neumann map inverse problem at fixed energy are equivalent.

For potentials without compact support the previous work which influenced us considerably was by Novikov [8]. He proved Theorem 1 in the case of zero magnetic potential, and the methods of [8] could be used to give a different proof of some of the results in Sect. 2.

Finally, we are deeply indebted to Adrian Nachman for calling our attention to a serious error in the first version of Sect. 2.

2. Faddeev-Type Scattering Amplitudes

Following Faddeev [3] and Novikov–Khenkin [6], we introduce a new scattering amplitude which will contain a large parameter. The later will be helpful in solving the inverse scattering problem.

Let v be an arbitrary unit vector, $|v| = 1$, and $E_{v, \sigma}(x)$ be the following fundamental solution to the equation $(-\Delta - k^2)u = f$:

$$E_{v, \sigma}(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{e^{ix \cdot \eta} d\eta}{\eta \cdot \eta - k^2 + i0(\eta_v - \sigma)}, \tag{13}$$

where $\eta_v = \eta \cdot v$ and $-k < \sigma < k$. Comparing $E_{v,\sigma}(x)$ with the fundamental solution

$$E_0(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{e^{ix \cdot \eta}}{\eta \cdot \eta - k^2 - i0} d\eta, \tag{14}$$

we have

$$E_{v,\sigma}(x) = E_0(x) - \frac{i\pi k^{n-2}}{(2\pi)^n} \int_{k\omega \cdot v > \sigma} e^{ix \cdot k\omega} d\omega, \tag{15}$$

where $d\omega$ is the area element of the unit sphere in R^n . Analogously to (10) consider the following integral equation

$$h_{v,\sigma}(\xi, \zeta, k) + \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{q_0(\xi - \eta, \eta) h_{v,\sigma}(\eta, \zeta, k)}{\eta \cdot \eta - k^2 + i0(\eta_v - \sigma)} d\eta = -q_0(\xi - \zeta, \zeta). \tag{16}$$

Set

$$v_{v,\sigma}(x, \zeta, k) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{h_{v,\sigma}(\xi, \zeta, k) e^{ix \cdot \xi}}{\xi \cdot \xi - k^2 + i0(\xi_v - \sigma)} d\xi, \tag{17}$$

assuming that $h_{v,\sigma}(\xi, \zeta, k)$ is the solution of (16). Then $v_{v,\sigma}(x, \zeta, k)$ is a solution of the differential equation (5) for $\zeta = k\omega$ with asymptotics at infinity that can be obtained by applying the stationary phase method to (17).

Now we shall find the relation between $h_{v,\sigma}(\xi, \zeta, k)$ and $h(\xi, \zeta, k)$. Analogously to (15) we have

$$\begin{aligned} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{q_0(\xi - \eta, \eta) h_{v,\sigma}(\eta, \zeta, k)}{\eta \cdot \eta - k^2 + i0(\eta_v - \sigma)} d\eta &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{q_0(\xi - \eta, \eta) h_{v,\sigma}(\eta, \zeta, k)}{\eta \cdot \eta + k^2 - i0} d\eta \\ &- \frac{i\pi k^{n-2}}{(2\pi)^n} \int_{k\omega \cdot v > \sigma} q_0(\xi - k\omega, k\omega) h_{v,\sigma}(k\omega, \zeta, k) d\omega. \end{aligned} \tag{18}$$

It follows from (16) and (18) that

$$\begin{aligned} h_{v,\sigma}(\xi, \zeta, k) + \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{q_0(\xi - \eta, \eta) h_{v,\sigma}(\eta, \zeta, k)}{\eta \cdot \eta - k^2 - i0} d\eta \\ = -\frac{i\pi k^{n-2}}{(2\pi)^n} \int_{k\omega \cdot v > \sigma} q_0(\xi - k\omega, k\omega) h_{v,\sigma}(k\omega, \zeta, k) d\omega - q_0(\xi - \zeta, \zeta). \end{aligned} \tag{19}$$

Set

$$A(q_0)w = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{q_0(\xi - \eta, \eta) w(\eta)}{\eta \cdot \eta - k^2 - i0} d\eta, \tag{20}$$

and

$$A(h)w = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{h(\xi, \eta, k) w(\eta)}{\eta \cdot \eta - k^2 - i0} d\eta. \tag{21}$$

That (10) has a unique solution is equivalent (cf. [2]) to the equality

$$(I + A(q_0))(I + A(h)) = I. \tag{22}$$

Since $I + A(q_0)$ has an inverse, it follows from (22) that

$$(I + A(h))(I + A(q_0)) = I \tag{23}$$

or equivalently

$$h(\xi, \zeta, k) + q_0(\xi - \zeta, \zeta) + \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{h(\xi, \eta, k)q_0(\eta - \zeta, \zeta)}{\eta \cdot \eta - k^2 - i0} d\eta = 0. \tag{23'}$$

Applying $I + A(h)$ to (19) and using (23) and (23'), we obtain (cf. [3] and [6], formula (1.7)):

$$h_{v,\sigma}(\xi, \zeta, k) = h(\xi, \zeta, k) - \frac{i\pi k^{n-2}}{(2\pi)^n} \int_{k\omega \cdot v > \sigma} h(\xi, k\omega, k)h_{v,\sigma}(k\omega, \zeta, k)d\omega. \tag{24}$$

Since $I + A(q_0)$ is invertible, Eq.(24) has a unique solution for any $h(\xi, \zeta, k)$ if and only if Eq.(16) has a unique solution. Indeed, if $\varphi(\xi)$ is a solution of the homogeneous equation corresponding to (16), i.e.

$$\varphi(\xi) + (2\pi)^{-n} \int_{\mathbb{R}^n} \frac{q_0(\xi - \eta, \eta)\varphi(\eta)}{\eta \cdot \eta - k^2 + i0(\eta_v - \sigma)} d\eta = 0, \tag{25}$$

then from (25) and (18) with h_v replaced with φ we conclude that

$$\varphi(\xi) + (2\pi)^{-n} \int_{\mathbb{R}^n} \frac{q_0(\xi - \eta, \eta)\varphi(\eta)d\eta}{\eta \cdot \eta - k^2 - i0} = \frac{i\pi k^{n-2}}{(2\pi)^n} \int_{k\omega \cdot v > \sigma} q_0(\xi - k\omega, k\omega)\varphi(k\omega)d\omega.$$

Applying $(I + A(h))$ to both sides of this, we have

$$0 = \varphi(\xi) + \frac{i\pi k^{n-2}}{(2\pi)^n} \int_{k\omega \cdot v > \sigma} h(\xi, k\omega, k)\varphi(k\omega)d\omega, \tag{26}$$

i.e. φ restricted to $|\xi| = k$ solves the homogeneous equation corresponding to (24). Conversely, suppose $\varphi(\xi)$ is a nonzero solution of the preceding equation (26) on the sphere of radius k . Then (26) extends φ to R^n , since $h(\xi, k\omega, k)$ is defined for $\xi \in R^n$. Applying $I + A(q_0)$ to both sides of (26), we see that φ satisfies (25).

Denote by $E_v(x, z)$ the following function:

$$E_v(x, z) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{e^{ix \cdot \eta} d\eta}{(\eta + zv) \cdot (\eta + zv) - k^2}, \quad \text{Im } z > 0.$$

Note that $E_v(x, z)$ is a fundamental solution for $(-i\frac{\partial}{\partial x} + zv) \cdot (-i\frac{\partial}{\partial x} + zv) - k^2$, i.e.

$$\left[\left(-i\frac{\partial}{\partial x} + zv \right) \cdot \left(-i\frac{\partial}{\partial x} + zv \right) - k^2 \right] E_v(x, z) = \delta(x).$$

Note that the distribution $[(\eta + zv) \cdot (\eta + zv) - k^2]^{-1}$ is not analytically dependent on z for $\text{Im } z > 0$. This gives rise to the $\bar{\partial}$ -equation in inverse scattering (see, for example [6]).

Denote by $h_v(\xi, \zeta, k, z)$ the solution of the following integral equation:

$$\begin{aligned}
 h_v(\xi, \zeta, k, z) + \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{q_0(\xi - \eta, \eta + zv)h_v(v, \zeta, k, z)}{(\eta + zv) \cdot (\eta + zv) - k^2} d\eta \\
 = -q_0(\xi - \zeta, \zeta + zv), \quad z = i\tau, \quad \tau > 0.
 \end{aligned}
 \tag{27}$$

Let $T_{i\tau}^{(1)}$ denote the operator

$$[T_{i\tau}^{(1)}f](\xi) = (2\pi)^{-n} \int_{\mathbb{R}^n} \frac{q_0(\xi - \eta, \eta + i\tau v)f(\eta)d\eta}{(\eta + i\tau v) \cdot (\eta + i\tau v) - k^2}.
 \tag{28}$$

Then (27) can be written

$$[(I + T_{i\tau}^{(1)})h_v](\xi) = -q_0(\xi - \zeta, \zeta + i\tau v)$$

and

$$h_v(\xi, \zeta, k, i\tau) = -[(I + T_{i\tau}^{(1)})^{-1}q_0(\cdot - \zeta, \zeta + i\tau v)](\xi),$$

provided $(I + T_{i\tau}^{(1)})^{-1}$ exists. The analyticity of h_v in τ will be important for us. Thus we need to study the analyticity of $T_{i\tau}^{(1)}f$ in τ when $f(\eta)$ is analytic in a strip $|\text{Im } \eta| < \varepsilon$. We will use coordinates $\eta_v = \eta \cdot v$, $\eta' = \eta - \eta_v v$, $r = |\eta'|$ and $\omega' = \eta'/|\eta'|$. For η real and $\tau = \mu + i\sigma$,

$$\text{Im}((\eta + i\tau v) \cdot (\eta + i\tau v) - k^2) = 2\mu\eta_v - 2\mu\sigma.$$

Hence, for $|\eta_v| > \varepsilon_1$, $\text{Re } \tau > 0$ and $|\text{Im } \tau| < \varepsilon_1/2$ the denominator in the integral defining $T_{i\tau}^{(1)}$ does not vanish. Thus, choosing $\chi \in C_0^\infty(\mathbb{R})$ such that $\chi(s)$ is supported in $|s| < 2\varepsilon_1$ and $1 - \chi(s)$ is supported in $|s| > \varepsilon_1$, we have

$$\begin{aligned}
 [T_{i\tau}^{(1)}f](\xi) &= (2\pi)^{-n} \int_{\mathbb{R}^n} \frac{\chi(\eta_v)q_0(\xi - \eta, \eta + i\tau v)f(\eta)d\eta}{(\eta + i\tau v) \cdot (\eta + i\tau v) - k^2} \\
 &\quad + (2\pi)^{-n} \int_{\mathbb{R}^n} \frac{(1 - \chi(\eta_v))q_0(\xi - \eta, \eta + i\tau v)f(\eta)d\eta}{(\eta + i\tau v) \cdot (\eta + i\tau v) - k^2} \\
 &\equiv [V_\tau^{(1)}f](\xi) + [V_\tau^{(2)}f](\xi),
 \end{aligned}$$

where $[V_\tau^{(2)}f](\xi)$ is analytic in (ξ, τ) in the set $|\text{Im } \xi| < \delta$, $\text{Re } \tau > 0$ and $|\text{Im } \tau| < \varepsilon_1/2$.

In our coordinates we have

$$(\eta + i\tau v) \cdot (\eta + i\tau v) - k^2 = (r - \sqrt{B})(r + \sqrt{B}),$$

where $B = k^2 + (\tau - i\eta_v)^2$. Using $\tau = \mu + i\sigma$ again, we have $\text{Re } B = k^2 + \mu^2 - \sigma^2 + 2\sigma\eta_v - \eta_v^2$, and $\text{Im } B = 2\mu\sigma - 2\mu\eta_v$. Hence for $k^2 > 8\varepsilon_1^2$, $\text{Re } B > k^2/8$ for $|\eta_v| < 2\varepsilon_1$ and $|\text{Im } \tau| < \varepsilon_1/2$, and we fix \sqrt{B} as the square root in the right half plane. We wish to define $V_\tau^{(1)}$, and hence $T_{i\tau}^{(1)}$, by analytic continuation from $\tau > 0$. When $\tau > 0$, i.e. when $\mu > 0$ and $\sigma = 0$, $r - \sqrt{B} \neq 0$ for $\eta_v \neq 0$, and we have $\text{sgn}(\text{Im } B) = -\text{sgn } \eta_v$. Therefore, we will deform the integration in r in

$$[V_\tau^{(1)}f](\xi) = \int_{S^{n-2}} d\omega' \int_{\mathbb{R}} d\eta_v \left(\int_0^\infty \frac{\chi(\eta_v)q_0(\xi - \eta, \eta + i\tau v)f(\eta)r^{n-2}}{(r - \sqrt{B})(r + \sqrt{B})} dr \right)$$

into the upper half plane for $\eta_v > 0$ and into the lower half plane for $\eta_v < 0$. We need to deform $[0, \infty)$ far enough that $r - \sqrt{B}$ will not vanish on the new contour for τ in a complex neighborhood of $[0, \tau_0]$. Note that for $\tau = \mu + i\sigma$,

$$\begin{aligned} \sqrt{B} &= \sqrt{\mu^2 + k^2 + 2i(\sigma - \eta_v)\mu - (\sigma - \eta_v)^2} \\ &= \sqrt{\mu^2 + k^2} + i(\sigma - \eta_v) \frac{\mu}{\sqrt{\mu^2 + k^2}} + O((\sigma - \eta_v)^2). \end{aligned}$$

Hence, for $|\sigma| < \varepsilon_1/2$ and $|\eta_v| < 2\varepsilon_1$, we have $|\operatorname{Re}(\sqrt{B} - \sqrt{\mu^2 + k^2})| < C\varepsilon_1^2$ and $|\operatorname{Im} \sqrt{B}| < 5\varepsilon_1/2 + C\varepsilon_1^2$. We now fix $\varepsilon_1 > 0$ such that $C\varepsilon_1^2 < k/3, 5\varepsilon_1/2 + C\varepsilon_1^2 < \varepsilon/2$ and $8\varepsilon_1^2 < k^2$. Then we may deform the r integration in $V_\tau^{(1)} f$ to the piecewise linear curve Γ from 0 to $k/2$ to $k/2 + i\varepsilon/2 \operatorname{sgn} \eta_v$ to $\sqrt{k^2 + \tau_0^2} + k/2 + i\varepsilon/2 \operatorname{sgn} \eta_v$ to $\sqrt{k^2 + \tau_0^2} + k/2$ to ∞ . With this choice of $\Gamma, r - \sqrt{B}$ will not vanish on Γ for $|\eta_v| < 2\varepsilon_1, |\sigma| < \varepsilon_1/2$ and $0 \leq \mu \leq \tau_0$. Thus we have proven:

Lemma 1. *If $f(\eta)$ is analytic in $|\operatorname{Im} \eta| < \varepsilon$, satisfying $|f(\eta)| \leq C(1 + |\eta|)^{-n-1}$ for $|\operatorname{Im} \eta| < \varepsilon$, then $[T_{it}^{(1)} f](\xi)$ has an analytic extension from $\tau > 0$ to the half strip $\{(\xi, \tau) : |\operatorname{Im} \xi| < \delta - \varepsilon, \operatorname{Re} \tau > 0, |\operatorname{Im} \tau| < \varepsilon_1/2\}$.*

Let $A_{N,r}$ denote the space of functions $f(\eta)$, analytic on $S_r = \{\eta \in \mathbb{C}^n : |\operatorname{Im} \eta| < r\}$ and continuous on \bar{S}_r , which satisfy

$$|f(\eta)| \leq C(1 + |\eta|)^{-N}$$

on S_r . $A_{N,r}$ is a Banach space in the norm

$$\|f\|_{N,r} = \sup_S (1 + |\eta|)^N |f(\eta)|.$$

Proposition 1. *For ε_1 sufficiently small $T_{it}^{(1)}$ is a family of compact operators on $A_{n+1, \delta/3}$, depending continuously on τ in the closed half strip $D = \{\tau = \mu + i\sigma : \mu \geq 0, |\sigma| \leq \varepsilon_1/2\}$ and analytically on τ in $\overset{\circ}{D}$, the interior of D .*

Remark 1. The choice $N = n + 1$ is made simply to make the Banach spaces used here compatible with those used in Sect. 3. The δ here is from (3).

Proof. For $\tau \in \overset{\circ}{D}$, $T_{it}^{(1)} f = V_\tau^{(1)} f + V_\tau^{(2)} f$ by definition. Since $r^2 + (\eta_v + i\tau)^2 - k^2$ does not vanish for $r \in \Gamma$ and $\tau \in D$, the operator $V_\tau^{(1)}$ satisfies

$$|[V_\tau^{(1)} f](\xi)| \leq C_\tau \int_{S^{n-2}} \int_{\mathbb{R}} d\omega' \int_{\Gamma} d\eta_v \frac{|q_0(\xi - \eta, \eta + i\tau)| |f(\eta)| r^{n-2} |dr|}{(1 + |\eta|)^2}, \tag{29}$$

where the constant C_τ is uniformly bounded on compact subsets of D . By hypothesis (3) for any $\delta' < \delta$,

$$|q_0(\xi - \eta, \eta + i\tau v)| \leq C_{\tau, \delta'} (1 + |\xi - \eta|)^{-n-4} (1 + |\eta|) \tag{30}$$

for $\xi \in S_{\delta' - \varepsilon}$ and $\eta \in S_\varepsilon$, where again $C_{\tau, \delta'}$ is uniformly bounded on compact subsets of D . Since $|f(\eta)| \leq (1 + |\eta|)^{-n-1} \|f\|_{n+1, \varepsilon}$ on S_ε , the integrand in (29) is bounded by

$$C_{\tau, \delta'} \frac{|r^{n-2}|}{(1 + |\xi - \eta|)^{n+2} (1 + |\eta|)^{n+2}}.$$

Since for any $p > 0$,

$$(1 + |\xi|)^p (1 + |\xi - \eta|)^{-p} (1 + |\eta|)^{-p} \leq C((1 + |\xi - \eta|)^{-p} + (1 + |\eta|)^{-p}),$$

we conclude

$$(1 + |\xi|)^{n+2} |[V_\tau^{(1)} f](\xi)| \leq C \|f\|_{n+1, \varepsilon}. \tag{31}$$

Taking $\varepsilon = \delta/3$ and $\delta' = 5\delta/6$, we have $[V_\tau^{(1)} f](\xi)$ analytic in $S_{\delta/2}$. Thus for $\tau \in D$, $V_\tau^{(1)}$ maps $A_{n+1, \delta/3}$ into $A_{n+2, \delta/2}$ with norm uniformly bounded on compact subsets of D . Hence $V_\tau^{(1)}$ is compact for $\tau \in D$.

In proving Lemma 1 we showed that for $f \in A_{n+1, \delta/3}$, $[V_\tau^{(1)} f](\xi)$ was analytic in (ξ, τ) for $\tau \in D$ and $\xi \in S_{\delta/2}$. Since the norm of $V_\tau^{(1)}$ as an operator on $A_{n+1, \delta/3}$ is uniformly bounded on compact subsets it follows by Cauchy's formula that $V_\tau^{(1)}$ is an analytic family of operators for $\tau \in D$.

For $\tau \in \overset{\circ}{D}$ the preceding arguments apply equally well to $V_\tau^{(2)}$, and we may conclude that $T_{i\tau}^{(1)}$ is an analytic family of compact operators in $\overset{\circ}{D}$. However, since

$$\begin{aligned} [V_{\mu+i\sigma}^{(2)} f](\xi) &= (2\pi)^{-n} \int_{\mathbb{R}^n} \frac{(1 - \chi(\eta_\nu)) q_0(\xi - \eta, \eta - \sigma\nu + i\mu\nu) f(\eta)}{|\eta - \sigma\nu|^2 - k^2 - \mu^2 + 2i\mu(\eta_\nu - \sigma)} d\eta \\ &= (2\pi)^{-n} \int_{\mathbb{R}^n} \frac{(1 - \chi(\eta_\nu + \sigma)) q_0(\xi - \eta - \sigma\nu, \eta + i\mu\nu) f(\eta + \sigma\nu)}{|\eta|^2 - k^2 - \mu^2 + 2i\mu\eta_\nu} d\eta, \end{aligned}$$

we need to show that $V_{\mu+i\sigma}^{(2)}$ extends continuously to $\mu = 0$ from $\mu > 0$. Since η_ν does not vanish on the support of $(1 - \chi(\eta_\nu + \sigma))$ for $|\sigma| < \varepsilon_1/2$, we can again deform the integration in r into $\text{Im } r > 0$ for $\eta_\nu > 0$ and into $\text{Im } r < 0$ for $\eta_\nu < 0$, using the piecewise linear contour Γ' connecting 0 to $\varepsilon/2 + i\varepsilon/2 \text{sgn } \eta_\nu$ to $3k/2 + i\varepsilon/2 \text{sgn } \eta_\nu$ to $3k/2$ to ∞ . Then for $r \in \Gamma'$ and $0 \leq \mu \leq \varepsilon_1/2$,

$$\begin{aligned} |\eta \cdot \eta - k^2 - \mu^2 + 2i\mu\eta_\nu|^{-1} &= |r^2 + \eta_\nu^2 - k^2 - \mu^2 + 2i\mu\eta_\nu|^{-1} \\ &\leq C_{k, \varepsilon/2} (|r|^2 + |\eta_\nu - (\text{sgn } \eta_\nu)k|)^{-1}, \end{aligned}$$

because $r = (1 + i \text{sgn } \eta_\nu)t$ on the first segment of Γ' and $r^2 = 2i(\text{sgn } \eta_\nu)t^2$. Since $(|r|^2 + |\eta_\nu - (\text{sgn } \eta_\nu)k|)^{-1}$ is locally integrable with respect to $|r|^{n-2} d|r| d\eta_\nu$, we may argue as follows. Removing small disks about $(r, \eta_\nu) = (0, \pm k)$ in the integral defining $V_{\mu+i\sigma}^{(2)} f$, we get an operator to which our previous arguments apply. Since this operator differs in norm from $V_{\mu+i\sigma}^{(2)}$ by an amount which goes to zero with the radius of disks, uniformly for $0 \leq \mu \leq \varepsilon_1/2$, we conclude that $V_{\mu+i\sigma}^{(2)}$ extends continuously to a compact operator on $\mu = 0$. \square

In Sect. 3 we will show that $I + T_{it}^{(1)}$ is invertible on $H_{0, n+1}$ for $\tau \gg 0$. This implies immediately that it is invertible on $A_{n+1, \delta/3}$, since the null space of $I + T_{it}^{(1)}$ on $A_{n+1, \delta/3}$ is a subspace of its nullspace on $H_{0, n+1}$. Therefore, by Proposition 1 the set Z where $I + T_{it}^{(1)}$ is not invertible is discrete in $\overset{\circ}{D}$ and closed of measure zero in $D \cap \{\text{Re } \tau = 0\}$. In particular, there is an open interval $I = (\sigma_1, \sigma_2) \subset (-\varepsilon_1/2, \varepsilon_1/2)$ such that $I + T_{it}^{(1)}$ is invertible for $\tau = -i\sigma, \sigma \in I$. Hence

$$h_\nu(\xi, \zeta, k, i\tau) = [(I + T_{it}^{(1)})^{-1} q_0(\cdot - \zeta, \zeta + i\tau\nu)](\xi)$$

exists for $\tau \in D \setminus Z$ and is analytic in (ξ, ζ, τ) on $S_{\delta/2} \times S_{\delta/2} \times \overset{\circ}{D} \setminus Z$.

Our goal is to recover $h_\nu(\xi, \zeta, k, i\tau)$ from the scattering data. To make the connection with scattering data we will need to use $\tau = -i\sigma$ and identify h_ν with a translate of $h_{\nu, \sigma}$. Since denominator $(\eta + i\tau\nu) \cdot (\eta + i\tau\nu) - k^2$ with $\tau = \mu - i\sigma$ goes to $\eta \cdot \eta + 2\sigma\eta_\nu + \sigma^2 - k^2$ as $\mu \downarrow 0$, we can remove the contour deformation in the definition of $V_\tau^{(1)} f$. However, since the integration in r is deformed into the upper half-plane when $\eta_\nu > 0$ and the lower half-plane when $\eta_\nu < 0$, we have

$$[T_\sigma^{(1)} f](\xi) = (2\pi)^{-n} \int_{\mathbb{R}^n} \frac{q_0(\xi - \eta, \eta + \sigma\nu) f(\eta)}{\eta \cdot \eta + 2\sigma\eta_\nu + \sigma^2 - k^2 + i0\eta_\nu} d\eta,$$

and for $\sigma \in I, h_\nu(\xi, \zeta, k, \sigma)$ is the unique solution in $A_{n+1, \delta/3}$ to

$$f(\xi, \zeta) + (2\pi)^{-n} \int_{\mathbb{R}^n} \frac{q_0(\xi - \eta, \eta + \sigma\nu) f(\eta, \zeta)}{\eta \cdot \eta + 2\sigma\eta_\nu + \sigma^2 - k^2 + i0\eta_\nu} d\eta = -q_0(\xi - \zeta, \zeta + \sigma\nu). \quad (32)$$

Since the changes of variables $\eta \rightarrow \eta - \sigma\nu, \xi \rightarrow \xi - \sigma\nu$ and $\zeta \rightarrow \zeta - \sigma\nu$, transform Eq. (32) to (16), we conclude that $h_\nu(\xi - \sigma\nu, \zeta - \sigma\nu, k, \sigma)$ is the unique solution of (16) in $A_{n+1, \delta/3}$ and hence for $\sigma \in I$,

$$h_\nu(\xi - \sigma\nu, \zeta - \sigma\nu, k, \sigma) = h_{\nu, \sigma}(\xi, \zeta, k). \quad (33)$$

Therefore, assuming the results of Sect. 3, we have proven the following theorem:

Theorem 2. *The solution $h_\nu(\xi, \zeta, k, i\tau)$ of (27) exists for $\tau \in D \setminus Z$ and is analytic in (ξ, ζ, τ) on $S_{\delta/3} \times S_{\delta/3} \times (\overset{\circ}{D} \setminus Z)$. The limiting values of $h_\nu(\xi, \zeta, k, i\tau)$ when $\tau \rightarrow -i\sigma$ satisfy (33), where $h_{\nu, \sigma}(\xi, \zeta, k)$ is the solution of (16).*

Since the unique solvability of (16) in $A_{n+1, \delta/3}$ implies the unique solvability of (24) in $C(S^{n-1})$, we know that (24) has a unique solution for $\sigma \in I$. Hence, knowing the scattering amplitude $h(\xi, \zeta, k)$ for $|\xi|^2 = |\zeta|^2 = k^2$, we can find $h_{\nu, \sigma}(\xi, \zeta, k)$ for $|\xi|^2 = |\zeta|^2 = k^2$ and $\sigma \in I$, which translates (by (33)) to knowing $h_\nu(\xi, \zeta, k, \sigma)$ for $|\xi + \sigma\nu|^2 = |\zeta + \sigma\nu|^2 = k^2$, for $\sigma \in I$. Since $h_\nu(\xi, \zeta, k, i\tau)$ is analytic for $(\xi, \zeta, \tau) \in S_{\delta/3} \times S_{\delta/3} \times (\overset{\circ}{D} \setminus Z)$ with a continuous extension to $S_{\delta/3} \times S_{\delta/3} \times (-iI)$, we can determine it on the variety

$$(\xi + i\tau\nu) \cdot (\xi + i\tau\nu) = (\zeta + i\tau\nu) \cdot (\zeta + i\tau\nu) = k^2$$

for $(\xi, \zeta, \tau) \in S_{\delta/3} \times S_{\delta/3} \times (\overset{\circ}{D} \setminus Z)$ by analytic continuation.

Fix $l \in R^n, \mu \in R^n, n \geq 3$, such that

$$l \cdot v = 0, \quad \mu \cdot v = 0, \quad l \cdot \mu = 0, \quad \mu \cdot \mu = 1, \tag{34}$$

and put

$$\begin{aligned} \xi(s) &= \frac{1}{2}l + s\mu, \\ \zeta(s) &= \frac{-1}{2}l + s\mu, \\ z(s) = i\tau(s) &= i\sqrt{s^2 + \frac{1}{4}l \cdot l - k^2}, \end{aligned} \tag{35}$$

$s \geq s_0, s_0$ large. We have that $h_v(\xi(s), \zeta(s), k, z(s))$ is analytic in s for $s > s_0$ and

$$(\xi(s) + i\tau(s)v) \cdot (\zeta(s) + i\tau(s)v) = (\zeta(s) + i\tau(s)v) \cdot (\zeta(s) + i\tau(s)v) = k^2.$$

Hence $h_v(\xi(s), \zeta(s), k, z(s))$ is known for $s > s_0$.

Remark 1. In the case $A(x) = 0$ the operator $T_{i\tau}^{(1)}$ has a small norm in $H_{\zeta, n+1}$ (see Proposition 4) when $\tau > 0$ is large. Substituting $\xi = \xi(s), \zeta = \zeta(s), z = z(s) = i\tau(s)$ in (27) and passing to the limit when $s \rightarrow +\infty$, we obtain that the integral in (27) tends to zero, and we can recover

$$\hat{V}(l) = \lim_{s \rightarrow \infty} h_v(\xi(s), \zeta(s), k, z(s)).$$

Thus we obtain an alternate proof of R. Novikov’s result [8].

3. Solution of an Integral Equation

In this section we set $z = i\tau$ and only consider τ real and positive.

In order to solve the integral equation (27) when τ is large and positive we will pass to an equivalent differential equation. Let

$$v_v(x, \zeta, k, z) = (2\pi)^{-n} \int_{\mathbb{R}^n} \frac{h_v(\eta, \xi, k, z)e^{ix \cdot \eta}}{(\eta + zv) \cdot (\eta + zv) - k^2} d\eta, \quad z = i\tau, \tau > 0. \tag{36}$$

Then v_v satisfies the differential equation

$$\begin{aligned} & [(-i\partial/\partial x + zv)^2 - k^2 + 2A(x) \cdot (-i\partial/\partial x + zv) + q(x)]v_v \\ &= -2(\zeta + zv) \cdot A(x)e^{ix \cdot \zeta} - q(x)e^{ix \cdot \zeta}. \end{aligned} \tag{37}$$

Our strategy will be to construct solutions of the equation

$$[(-i\partial/\partial x + zv)^2 - k^2 + 2A(x) \cdot (-i\partial/\partial x + zv) + q(x)]v = f \tag{37'}$$

for all f in the Banach space $H_{0, n+1}(R^n)$, where $H_{0, N}(R^n)$ is defined as the closure of $C_0^\infty(R^n)$ in the norm, $\|f\|_{0, N} = \sup_{\xi} (1 + |\xi|)^N |\hat{f}(\xi)|$, i.e. $H_{0, N}$ is the Fourier transform of $H_{0, N}$. Then

$$h(\xi) = \int_{\mathbb{R}^n} ((-i\partial/\partial x + zv)^2 - k^2)v(x)e^{-ix \cdot \xi} dx$$

will be a solution of (27) with the inhomogeneous term replaced by $\hat{f}(\xi)$, i.e.

$$h(\xi) + (2\pi)^{-n} \int_{\mathbb{R}^n} \frac{q_0(\xi - \eta, \eta + z\nu)h(\eta)}{(\eta + z\nu) \cdot (\eta + z\nu) - k^2} d\eta = \hat{f}(\xi), \quad (38)$$

and we will show that $h \in H_{0,n+1}$. Thus we can conclude that $I + T_{i\tau}^{(1)}$ (see (28)) maps $H_{0,n+1}$ onto $H_{0,n+1}$ for $\tau \gg 0$. Since $T_{i\tau}^{(1)}$ is also compact on $H_{0,n+1}$ for $\tau > 0$, it follows that $I + T_{i\tau}^{(1)}$ is invertible on $H_{0,n+1}$ for $\tau \gg 0$, and (27) is uniquely solvable in $H_{0,n+1}$, when τ is sufficiently large positive.

We will look for a solution of (37') in the form

$$v(x, \zeta, z) = (2\pi)^{-n} \int_{\mathbb{R}^n} \frac{c(x, \eta, z) \tilde{g}(\eta, \zeta, z) e^{ix \cdot \eta}}{(\eta + z\nu) \cdot (\eta + z\nu) - k^2} d\eta, \quad (39)$$

where $z = i\tau, \tau > 0$. Here $g(x, \zeta, z)$ is the new unknown and $\tilde{g}(\eta, \zeta, z)$ is its Fourier transform in the first variable. The factor $c(x, \eta, z)$ will be chosen so that the analogue of Eq. (27) for \tilde{g} will not have the unbounded terms in $q_0(\xi - \eta, \eta + z\nu)$. For this reason we choose $c(x, \eta, z)$ as a solution of the transport equation

$$-2i \frac{\partial c}{\partial x} \cdot (\eta + z\nu) + 2A(x) \cdot (\eta + z\nu) \chi_1(\eta, z) c = 0 \quad (40)$$

of the form $c = \exp(-i\chi_1 \varphi)$. Thus φ must satisfy

$$(\eta + z\nu) \cdot \frac{\partial \varphi}{\partial x} = A(x) \cdot (\eta + z\nu), \quad (40')$$

and we choose

$$\varphi = (2\pi)^{-n} \int_{\mathbb{R}^n} \frac{\hat{A}(\xi) \cdot (\eta + z\nu) e^{ix \cdot \xi}}{i\xi \cdot (\eta + z\nu)} d\xi. \quad (41)$$

The function $\chi_1(\eta, z)$ in (40) is a cutoff to a neighborhood of $(\eta + z\nu) \cdot (\eta + z\nu) = k^2$. The cancellation of unbounded terms is not needed outside this neighborhood, and it is convenient to have $c \equiv 1$ there. We choose $\chi(t) \in C_0^\infty(\mathbb{R})$ such that $\chi(t) \geq 0, \chi(t) = 1$ on $|t| < \varepsilon/2$ and $\chi(t) = 0$ on $|t| > \varepsilon$, and define

$$\chi_1(\eta, z) = \chi \left(\frac{|(\eta + z\nu) \cdot (\eta + z\nu) - k^2|}{|\eta|^2 + \tau^2 + k^2} \right).$$

Since, setting $\eta_\nu = \eta \cdot \nu$,

$$|(\eta + z\nu) \cdot (\eta + z\nu) - k^2| = ((|\eta|^2 - \tau^2 - k^2)^2 + 4\tau^2 \eta_\nu^2)^{1/2}, \quad (42)$$

it follows that on the support of χ_1

$$\varepsilon(|\eta|^2 + \tau^2 + k^2) \geq ||\eta|^2 - (\tau^2 + k^2)|,$$

and hence

$$\left(\frac{1 - \varepsilon}{1 + \varepsilon} \right) |\eta|^2 < \tau^2 + k^2 < \left(\frac{1 + \varepsilon}{1 - \varepsilon} \right) |\eta|^2. \quad (43)$$

Setting $\eta' = \eta - (\eta \cdot v)v$, (42) also implies that on the support of χ_1 ,

$$2\varepsilon(|\eta'|^2 + \eta_v^2 + \tau^2 + k^2) \geq \|\eta'\|^2 + \eta_v^2 - \tau^2 - k^2 + 2\tau|\eta_v|,$$

and hence, using (43),

$$\begin{aligned} (1 + 2\varepsilon)|\eta'|^2 &\geq (1 - 2\varepsilon)(\tau^2 + k^2) - (1 + 2\varepsilon)\eta_v^2 + 2\tau|\eta_v| \\ &\geq (1 - 2\varepsilon)(\tau^2 + k^2) + \left(2 \left(\frac{1 - \varepsilon}{1 + \varepsilon}\right)^{1/2} \frac{\tau}{(\tau^2 + k^2)^{1/2}} - (1 + 2\varepsilon)\right) \eta_v^2. \end{aligned}$$

Thus, choosing ε sufficiently small and τ_0 sufficiently large, we have for $\tau \geq \tau_0$,

$$(\tau^2 + k^2) + \eta_v^2 \leq C_\varepsilon |\eta'|^2 \tag{44}$$

on support χ_1 .

We will need some detailed estimates on φ . The behavior of φ in the x -variables is strongly dependent on η . We introduce $\mu = \eta'/|\eta'|$, and use the orthogonal expansion $x = x_1v + x_2\mu + x_\perp$, where x_\perp is the projection of x on the orthogonal complement of span $\{v, \eta\}$.

Proposition 2. *Assume that $B(x)$ is a vector-valued function satisfying (3) and define*

$$\psi(x, \eta + zv) = (2\pi)^{-n} \int_{\mathbb{R}^n} \frac{\hat{B}(\xi) \cdot (\eta + zv)}{\xi \cdot (\eta + zv)} e^{ix \cdot \xi} d\xi.$$

Then for $(\eta, z) \in \text{supp } \chi_1, \tau \geq \tau_0$ and $|\alpha| + |\beta| \leq P$ in (3') one has

$$\left| \frac{\partial^{|\alpha|+|\beta|} \psi}{\partial x^\alpha \partial \eta^\beta} \right| \leq C_{\alpha\beta} \tau^{-|\beta|} e^{-\frac{\delta}{2}|x_\perp|}. \tag{45}$$

Proof. By contour integration one computes

$$(2\pi)^{-2} \int_{\mathbb{R}^2} \frac{e^{i(x_1\xi_1 + x_2\xi_2)}}{\xi \cdot (\eta + zv)} d\xi_1 d\xi_2 = \frac{1}{2\pi} \frac{1}{|\eta'|x_1 - (\eta_v + z)x_2}.$$

Thus

$$\psi(x, \eta + zv) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{B(x - y_1v - y_2\mu) \cdot (\eta + zv)}{|\eta'|y_1 - (\eta_v + z)y_2} dy, \tag{46}$$

and, using (3'), for $|\alpha| \leq P$,

$$\left| \frac{\partial^{|\alpha|} \psi}{\partial x^\alpha}(x, \eta + zv) \right| \leq \int_{\mathbb{R}^2} \frac{C e^{-\delta|(x_1 - y_1)v + (x_2 - y_2)\mu + x_\perp|} |\eta + zv|}{\|\eta'\|y_1 - (\eta_v + z)y_2} dy. \tag{47}$$

Since (43) and (44) imply that

$$\begin{aligned} \|\eta'\|y_1 - (\eta_v + z)y_2 &= ((|\eta'|y_1 - \eta_v y_2)^2 + \tau^2 y_2^2)^{1/2} \\ &\geq C\tau(y_1^2 + y_2^2)^{1/2} = C\tau|y|, \end{aligned} \tag{48}$$

it follows from (43) and (47) that

$$\left| \frac{\partial^{|\alpha|} \psi}{\partial x^\alpha}(x, \eta + zv) \right| \leq C_x e^{-\frac{\delta}{2}|x_\perp|}$$

for $|\alpha| \leq P$, where C_x is independent of η and z .

To estimate η derivatives of ψ we first observe that (48) implies

$$\left| \frac{\partial}{\partial \eta_j} \left(\frac{1}{|\eta'|y_1 - (\eta_v + z)y_2} \right) \right| = \left| \frac{\frac{\partial |\eta'|}{\partial \eta_j} y_1 - \frac{\partial \eta_v}{\partial \eta_j} y_2}{(|\eta'|y_1 - (\eta_v + z)y_2)^2} \right| \leq \frac{C}{\tau^2 |y|}.$$

Thus, differentiating (46),

$$\begin{aligned} \left| \frac{\partial \psi}{\partial \eta_j} \right| &\leq \frac{C}{\tau} \int_{\mathbb{R}^2} \frac{|B(x - y_1 v - y_2 \mu)| dy}{|y|} + \frac{C}{\tau} \int_{\mathbb{R}^2} \left| \frac{\partial B}{\partial x}(x - y_1 v - y_2 \mu) \right| dy \\ &\leq \frac{C}{\tau} e^{-\frac{\delta}{2}|x_\perp|}. \end{aligned}$$

Repeating the same argument and noting that $\partial^{|\gamma|} / \partial \eta^\gamma (|\eta'|y_1 - (\eta_v + z)y_2)^{-1}$ is homogeneous of degree -1 in y for any γ , one concludes

$$\left| \frac{\partial^{|\alpha|+|\beta|} \psi}{\partial x^\alpha \partial \eta^\beta} \right| \leq \frac{C_{\alpha\beta}}{\tau^{|\beta|}} e^{-\frac{\delta}{2}|x_\perp|} \quad (49)$$

for $|\alpha| + |\beta| \leq P$ and $\tau \geq \tau_0$ on the support of χ_1 . \square

To study φ in (41) we will use Proposition 2. We introduce

$$w = x_1 - (\eta_v + z)|\eta'|^{-1}x_2 \quad \text{and} \quad w' = y_1 - (\eta_v + z)|\eta'|^{-1}y_2$$

and observe that

$$\frac{1}{w - w'} = \frac{1}{w(1 - \frac{w'}{w})} = \sum_{k=0}^N \frac{(w')^k}{w^{k+1}} + \frac{(w')^{N+1}}{w^{N+1}(w - w')}. \quad (50)$$

Then we can write (46) with B replaced by A/i in the form

$$\begin{aligned} \varphi(x, \eta + zv) &= \frac{1}{2\pi i} \int_{\mathbb{R}^2} \frac{A(y_1 v + y_2 \mu + x_\perp) \cdot (\eta + zv)}{|\eta'| (x_1 - y_1) - (\eta_v + z)(x_2 - y_2)} dy \\ &= \frac{1}{2\pi |\eta'| i} \int_{\mathbb{R}^2} \frac{A(y_1 v + y_2 \mu + x_\perp) \cdot (\eta + zv)}{w - w'} dy. \end{aligned} \quad (51)$$

Using (50) to expand (51), the remainder term in (50) contributes a term to φ of the form

$$\frac{1}{2\pi i} \frac{1}{w^{N+1}} \int_{\mathbb{R}^2} \frac{B_N(x - y_1 v - y_2 \mu, \eta, z) \cdot (\eta + zv)}{|\eta'| y_1 - (\eta_v + z)y_2} dy,$$

where $B_N(x, \eta, z) = (x_1 - (\eta_v + z)|\eta'|^{-1}x_2)^{N+1} A(x)$ satisfies (3) uniformly in (η, z) on the support of χ_1 for $\tau \geq \tau_0$. The other terms in (50) contribute terms to φ of the form

$$\frac{1}{2\pi i} \frac{1}{w^{k+1}} \int_{\mathbb{R}^2} |\eta'|^{-1} A(x_\perp + y_1 v + y_2 \mu) \cdot (\eta + zv) (w')^k dy.$$

Thus we see that for any $N \geq 0$, when (η, z) is in the support of χ_1 and $\tau \geq \tau_0$,

$$\varphi = \sum_{k=1}^{N-1} w^{-k} b_k(x_\perp, \eta, z) + w^{-N} b_N, \tag{52}$$

where $\psi = b_N$ satisfies (45) and $b_k(x_\perp, \eta, z)$ is exponentially decreasing in x_\perp together with its derivatives up to order P uniformly in (η, z) .

Substituting (39) into (37') and using (40), we obtain

$$C(x, D, z)g + T_1g + T_2g + T_3g = f, \tag{53}$$

where

$$\begin{aligned} [T_1g](x) &= (2\pi)^{-n} \int_{\mathbb{R}^n} \frac{(-2iA \cdot \frac{\hat{c}c}{\partial x} + qc)\hat{g}(\eta)e^{ix \cdot \eta}}{(\eta + zv) \cdot (\eta + zv) - k^2} d\eta, \\ [T_2g](x) &= (2\pi)^{-n} \int_{\mathbb{R}^n} \frac{(-\Delta c)\hat{g}(\eta)e^{ix \cdot \eta} d\eta}{(\eta + zv) \cdot (\eta + zv) - k^2}, \\ [T_3g](x) &= (2\pi)^{-n} \int_{\mathbb{R}^n} \frac{2(1 - \chi_1)A \cdot (\eta + zv)c\hat{g}(\eta)e^{ix \cdot \eta}}{(\eta + zv) \cdot (\eta + zv) - k^2} d\eta, \end{aligned}$$

and $C(x, D, z)$ is a pseudo-differential operator with symbol $c(x, \eta, z)$.

In Sects. 4 and 5 we will need uniform estimates on the norms of the operators $e^{-ix \cdot \zeta} T_j e^{ix \cdot \zeta}, j = 1, 2, 3$, and $e^{-ix \cdot \zeta} C e^{ix \cdot \zeta}$. Since multiplication by $e^{ix \cdot \zeta}$ is not bounded on $H_{0,N}$ (for $N > 0$) and $\zeta \rightarrow \infty$, these estimates do not follow from estimates on the norms of the $T_j, j = 1, 2, 3$ and C on $H_{0,N}$. To prove what we will use later efficiently we are going to equip $H_{0,N}$ with a family of norms, $\| \cdot \|_{\zeta, N}$ so that estimates in these norms *uniform in ζ* will imply the needed estimates for Sects. 4 and 5. We will refer to $H_{0,N}$ with the norm $\| \cdot \|_{\zeta, N}$ as " $H_{\zeta, N}$."

Proposition 3. *Let $H_{\zeta, N}(R^n)$ be the closure of $C_0^\infty(R^n)$ in the norm $\|f\|_{\zeta, N} = \sup_{R^n} (1 + |\zeta - \zeta|)^N |\hat{f}(\zeta)|$. Then $C(x, D, z)$ is invertible as an operator on $H_{\zeta, n+1}(R^n)$ for τ sufficiently large.*

Proof. Our approach here will be to show that $C(x, D)$ and the operator $C^{(-1)}(x, D)$ with the reciprocal symbol $e^{ix \cdot \zeta} \varphi$ are bounded on $H_{\zeta, n+1}$. Then the composition formula for pseudo-differential operators and Proposition 2 will be used to show

$$C^{(-1)}C = I + T, \tag{54}$$

where the norm of T on $H_{\zeta, n+1}$ goes to zero as $\tau \rightarrow \infty$ uniformly in ζ .

The proof that C and $C^{(-1)}$ are uniformly bounded on $H_{\zeta, n+1}$ uses only (52). Expanding $c(x, \eta, z) = \exp(-i\varphi\chi_1)$ in a Taylor series in $\varphi\chi_1$, it is clear that $c - 1$ also has an expansion of the form (52) for $\tau \geq \tau_0$. A linear transformation of R^n takes w in (52) to the standard complex variable $z = s + it$. Hence analytic functions of w are annihilated by the pull-back of $\partial/\partial\bar{z}$ under this transformation which is

$\frac{\partial}{\partial \bar{w}} = \frac{1}{2}(\frac{\partial}{\partial x_2} + (\eta_v + z)|\eta'|^{-1} \frac{\partial}{\partial x_1})$. From (52) we have $\|(\partial^{|\alpha|}/\partial x^\alpha)\partial c/\partial \bar{w}\|_{L^1(\mathbb{R}^n)} \leq C$ for $|\alpha| < P$ uniformly on support χ_1 for $\tau > \tau_0$. Thus setting $v_0 = \partial c/\partial \bar{w}$,

$$|\hat{v}_0(\xi, \eta, z)| \leq C(1 + |\xi|)^{-P+1}. \tag{55}$$

Thus, since $P \geq n + 2$, the inverse Fourier transform of $v_0(\xi)(\xi_2 + (\eta_v + z)|\eta'|^{-1}\xi_1)^{-1}$ is continuous, tending to zero as $|x| \rightarrow 0$. Since c is bounded, we conclude (by Liouville's theorem)

$$\begin{aligned} c(x, \eta, z) &= 1 + (2\pi)^{-n} \int_{\mathbb{R}^n} \frac{2\hat{v}_0(\xi)e^{ix \cdot \xi}}{i(\xi_2 + (\eta_v + z)|\eta'|^{-1}\xi_1)} d\xi \\ &= 1 + (2\pi)^{-n} \int_{\mathbb{R}^n} \frac{2\hat{v}_0(\xi)|\eta'|e^{ix \cdot \xi}}{i\xi \cdot (\eta + zv)} d\xi. \end{aligned} \tag{56}$$

Using (55) and (56), given $C(x, D, z)g = h$, we have, setting $c_1 = c - 1$,

$$\hat{h}(\xi) = \hat{g}(\xi) + \int_{\mathbb{R}^n} \tilde{c}_1(\xi - \eta, \eta, z)\hat{g}(\eta) d\eta,$$

where $\tilde{c}_1(\xi, \eta, \zeta)$ has support in the support of χ_1 and satisfies

$$|\tilde{c}_1(\xi, \eta, z)| \leq C|\eta'| (1 + |\xi|)^{-n-1} |\xi \cdot (\eta + zv)|^{-1}. \tag{57}$$

Hence

$$\begin{aligned} \sup_{\xi} (1 + |\xi - \zeta|)^{n+1} |\hat{h}(\xi)| &\leq (1 + \sup_{\xi, \zeta} \int_{\mathbb{R}^n} (1 + |\xi - \zeta|)^{n+1} |\hat{c}_1(\xi - \eta, \eta, z)| \\ &\quad (1 + |\eta - \zeta|)^{-n-1} d\eta) \sup_{\xi} (1 + |\xi - \zeta|)^{n+1} |\hat{g}(\xi)|, \end{aligned}$$

and the boundness of $C(x, D, z)$ on $H_{\zeta, n+1}(\mathbb{R}^n)$ uniformly in (ζ, z) for $\tau \geq \tau_0$ follows from (57) and the estimate

$$\begin{aligned} (1 + |\xi - \zeta|)^{n+1} (1 + |\xi - \eta|)^{-n-1} (1 + |\eta - \zeta|)^{-n-1} \\ \leq C((1 + |\xi - \eta|)^{-n-1} + (1 + |\eta - \zeta|)^{-n-1}). \end{aligned} \tag{58}$$

To see that C is invertible on $H_{\zeta, n+1}$ when τ is large, we recall that the integral remainder formula for Taylor series implies that the symbol of $C^{(-1)}(x, D, z)C(x, D, z) - I$ is given by

$$r(x, \eta, z) = \sum_{|\alpha|=1} (2\pi)^{-n} \int_{\mathbb{R}^n} \left(\int_0^1 e^{ix \cdot \zeta} \frac{\partial c^{-1}}{\partial \eta^\alpha}(x, \eta + i\zeta)\zeta^\alpha dt \right) \tilde{c}_1(\zeta, \eta) d\zeta.$$

The analogue of (57) for $\partial c^{-1}/\partial \eta^\alpha$, $|\alpha| = 1$, is

$$\left| \frac{\partial \bar{c}^{-1}}{\partial \eta^\alpha}(\xi, \eta, z) \right| \leq C \frac{|\eta'|}{\tau} (1 + |\xi|)^{-n-1} |\xi \cdot (\eta + zv)|^{-1}.$$

We can now apply the argument, used above to show that $C(x, D, z)$ is bounded on $H_{\zeta, n+1}$, to $R(x, D, z)$. The superpositions in ζ and τ produce no new difficulties and

the factor of $1/\tau$ in the estimate for $\partial c^{-1}/\partial \eta^\alpha$ above makes $\|R(x, D)\|$ go to zero as $\tau \rightarrow \infty$. Thus C is invertible for τ sufficiently large. \square

Proposition 4. *The norms of the operators $T_1(\tau), T_2(\tau)$ and $T_3(\tau)$ on $H_{\zeta, n+1}(R^n)$ tend to zero as $\tau \rightarrow \infty$ uniformly in ζ .*

Proof. Let $\tilde{T}_k(\xi - \eta, \eta, z)$ be the kernel of the Fourier transform of $T_k, k = 1, 2, 3$, i.e.

$$\widehat{T_k g}(\xi) = \int_{\mathbb{R}^n} \tilde{T}_k(\xi - \eta, \eta, z) \hat{g}(\eta) d\eta.$$

In order to show that the norm of T_k on $H_{\zeta, n+1}(R^n)$, is arbitrarily small for τ large uniformly in ζ , it suffices to prove that

$$\sup_{\xi, \zeta} \int_{\mathbb{R}^n} (1 + |\xi - \zeta|)^{n+1} |\tilde{T}_k(\xi - \eta, \eta, z)| (1 + |\eta - \zeta|)^{-n-1} d\eta \leq \frac{C}{\tau} \log \tau. \tag{59}$$

On the support of $1 - \chi_1$ we have $|(\eta + zv) \cdot (\eta + zv) - k^2| \geq \frac{\epsilon}{2} (|\eta|^2 + \tau^2 + k^2)$. Hence

$$|\tilde{T}_3(\xi - \eta, \eta, z)| \leq C(1 + |\xi - \eta|)^{-n-1} \frac{|\eta + zv|}{|\eta|^2 + \tau^2 + k^2} \leq \frac{C}{\tau} (1 + |\xi - \eta|)^{-n-1},$$

and (59) for $k = 3$ follows from (58).

To estimate \tilde{T}_1 we note that (42) implies that for all (η, z) ,

$$\begin{aligned} |\eta + zv) \cdot (\eta + zv) - k^2| &\geq \frac{1}{2} (|\eta|^2 - (\tau^2 + k^2) + 2\tau|\eta_v|) \\ &= \frac{1}{2} (|\eta| - (\tau^2 + k^2)^{1/2}) (|\eta| + (\tau^2 + k^2)^{1/2}) + 2\tau|\eta_v| \\ &\geq \frac{\tau}{2} (|\eta| - (\tau^2 + k^2)^{1/2} + |\eta_v|). \end{aligned} \tag{60}$$

Since $c - 1$ has an expansion of the form (52), qc and $A \cdot \frac{\partial c}{\partial x}$ satisfy (3) with constants uniform in (η, z) for $\tau > \tau_0$. Thus, from (58) and (60),

$$\begin{aligned} &\sup_{\xi, \zeta} \int_{\mathbb{R}^n} (1 + |\xi - \zeta|)^{n+1} |\tilde{T}_1(\xi - \eta, \eta, z)| (1 + |\eta - \zeta|)^{-n-1} d\eta \\ &\leq \frac{C}{\tau} \sup_{\xi, \zeta} \int_{\mathbb{R}^n} \frac{(1 + |\xi - \eta|)^{-n-1} + (1 + |\eta - \zeta|)^{-n-1}}{|\eta| - (\tau^2 + k^2)^{1/2} + |\eta_v|} d\eta \\ &\leq \frac{2C}{\tau} \sup_{\xi} \int_{\mathbb{R}^n} \frac{(1 + |\xi - \eta|)^{-n-1}}{|\eta| - (\tau^2 + k^2)^{1/2} + |\eta_v|} d\eta. \end{aligned} \tag{61}$$

Setting $R = (\tau^2 + k^2)^{1/2}$, $\eta = R\zeta$ and $I(\zeta) = ((|\zeta| - 1)^2 + \zeta_v^2)^{1/2}$ in the last line of (61), this gives

$$\begin{aligned} & \sup_{\xi, \zeta \in \mathbb{R}^n} \int (1 + |\xi - \zeta|)^{n+1} |\tilde{T}_1(\xi - \eta, \eta, z)|(1 + |\eta - \zeta|)^{-n-1} d\eta \\ & \leq \frac{C}{\tau} \sup_{\xi \in \mathbb{R}^n} \int (1 + |\xi - R\zeta|)^{-n-1} (I(\zeta))^{-1} R^{n-1} d\zeta \\ & \leq \frac{C}{\tau} \left[\sup_{\xi \in \mathbb{R}^n} \int (1 + |\xi - R\zeta|)^{-n-1} R^{n-1} d\zeta \right. \\ & \quad \left. + \sup_{\xi \in I(\zeta) < \varepsilon_0} \int (1 + |\xi - R\zeta|)^{-n-1} (I(\zeta))^{-1} R^{n-1} d\zeta \right]. \\ & \leq \frac{C}{\tau} \left[\frac{1}{R} + R^{n-1} \sup_{\xi \in I(\zeta) < \varepsilon_0} \int (1 + |\xi - R\zeta|)^{-n-1} (I(\zeta))^{-1} d\zeta \right]. \end{aligned}$$

Here ε_0 is any fixed constant, and we assume $\varepsilon_0 \ll 1$. Since $\tau \sim R$ for $\tau > \tau_0$, it suffices to show

$$\tau^{n-1} \sup_{\xi \in I(\zeta) < \varepsilon_0} \int (1 + \tau|\xi - \zeta|)^{-n-1} (I(\zeta))^{-1} d\zeta < C \tag{62}$$

for $\tau > \tau_0$ to conclude that (59) holds for $k = 1$.

To prove (62) we note first that when $|\xi'| < \frac{1}{2}$,

$$\int_{I(\zeta) < \varepsilon_0} (1 + \tau|\xi - \zeta|)^{-n-1} (I(\zeta))^{-1} d\zeta \leq \int_{I(\zeta) < \varepsilon_0} (1 + c_0\tau)^{-n-1} (I(\zeta))^{-1} d\zeta,$$

where $c_0 = \min_{I(\zeta) < \varepsilon_0} |\xi - \zeta| > 0$, and (62) holds.

To establish (62) for $|\xi'| > \frac{1}{2}$ we will use spherical coordinates in the hyperplane $\zeta \cdot v = 0$ with $r = |\zeta'|$ and polar angle $\theta = \cos^{-1}(\frac{\zeta'_v}{|\zeta'|} \cdot \frac{\xi'_v}{|\xi'|})$. Then we have $d\zeta = r^{n-2} dr d\omega d\zeta_v$, where $d\omega$ is the volume form on S^{n-2} , and we also have

$$\begin{aligned} |\zeta - \xi| &= (r^2 - 2|\xi'_v|r \cos \theta + |\xi'_v|^2 + (\zeta_v - \xi_v)^2)^{1/2} \\ &\geq \frac{1}{2}(((r - |\xi'_v| \cos \theta)^2 + (\zeta_v - \xi_v)^2)^{1/2} + |\xi'_v| \sin \theta). \end{aligned} \tag{63}$$

Likewise, there is $c > 0$ such that

$$I(\zeta) \geq c((r - 1)^2 + \zeta_v^2)^{1/2}. \tag{64}$$

Now we consider $v = (r - 1, \zeta_v)$ and $v_0 = (|\xi'_v| \cos \theta - 1, \xi_v)$ as vectors in R^2 and use $\| \cdot \|$ to denote the norm on R^2 . From (63) and (64) we have

$$\begin{aligned} & \int_{I(\zeta) < \varepsilon_0} (1 + \tau|\xi - \zeta|)^{-n-1} (I(\zeta))^{-1} d\zeta \\ & \leq C \int_{\mathbb{R}^2 \times S^{n-2}} \frac{(1 + \tau(\|v - v_0\| + |\sin \theta|))^{-n-1}}{\|v\|} dr d\zeta_v d\omega. \end{aligned}$$

We split the integral over $R^2 \times S^{n-2}$ into an integral over $\{\zeta : \|v\| \geq \|v - v_0\|\}$ in which we replace $\|v\|$ by $\|v - v_0\|$ and an integral over $\{\zeta : \|v\| < \|v - v_0\|\}$ in which we replace $\|v - v_0\|$ by $\|v\|$. Since the two integrands that are produced this way differ only by a translation in the (r, ζ_v) -plane, we have the estimate

$$\begin{aligned} & \int_{l(\zeta) < c_0} (1 + \tau|\xi - \zeta|)^{-n-1} (l(\zeta))^{-1} d\zeta \\ & \leq C \int_{\mathbb{R}^2 \times S^{n-2}} \frac{(1 + \tau((s^2 + t^2)^{1/2} + |\sin \theta|))^{-n-1}}{(s^2 + t^2)^{1/2}} ds dt d\omega \\ & \leq C \int_0^\infty \int_{S^{n-2}} (1 + \tau(u + |\sin \theta|))^{-n-1} dud\omega \\ & \leq C \int_0^{\infty \pi/2} \int_0^\infty (1 + \tau(u + \theta))^{-n-1} \theta^{n-3} dud\theta \end{aligned}$$

and, setting $\tau u = r, \tau \theta = s$, we have

$$\int_{l(\zeta) < c_0} (1 + \tau|\xi - \zeta|)^{-n-1} (l(\zeta))^{-1} d\zeta \leq \tau^{-n+1} C \int_0^\infty \int_0^\infty (1 + r + s)^{-n-1} s^{n-3} dr ds.$$

Thus, since the integral is finite, we have (62), and (59) holds for $k = 1$, in the stronger form

$$\sup_{\xi, \zeta} \int_{\mathbb{R}^n} (1 + |\xi - \zeta|)^{n+1} |\tilde{T}_1(\xi - \eta, \eta, z)| (1 + |\eta - \zeta|)^{-n-1} d\eta \leq \frac{C}{\tau}. \tag{64'}$$

From (56) one sees that

$$|\tilde{\Delta}c(\xi - \eta, \eta)| \leq C(1 + |\xi - \eta|)^{-P+3} |\eta'| |(\xi - \eta) \cdot (\eta + zv)|^{-1},$$

and hence

$$|\tilde{T}_2(\xi - \eta, \eta, z)| \leq \frac{C(1 + |\xi - \eta|)^{-P+3} |\eta'|}{|(\xi - \eta) \cdot (\eta + zv)| (\eta + zv) \cdot (\eta + zv) - k^2|},$$

and by the reasoning that leads to (61), we have (note $P \geq n + 4$ is needed):

$$\begin{aligned} & \sup_{\xi, \zeta} \int_{\mathbb{R}^n} (1 + |\xi - \zeta|)^{n+1} |\tilde{T}_2(\xi - \eta, \eta, z)| (1 + |\eta - \zeta|)^{-n-1} d\eta \\ & \leq \frac{C}{\tau} \sup_{\xi} \int_{\mathbb{R}^n} \frac{(1 + |\xi - \eta|)^{-n-1} |\eta'| d\eta}{|(\xi - \eta) \cdot (\eta + i\tau v)| (|\eta| - (\tau^2 + k^2)^{1/2}) + |\eta_v|}. \end{aligned} \tag{65}$$

Setting $R = (\tau^2 + k^2)^{1/2}$, $\beta = \tau(\tau^2 + k^2)^{-1/2}$, $\eta = R\zeta$ and $l(\zeta) = ((|\zeta| - 1)^2 + \zeta_v^2)^{1/2}$, (65) becomes

$$\begin{aligned} & \sup_{\xi, \zeta} \int_{\mathbb{R}^n} (1 + |\xi - \zeta|)^{n+1} |\tilde{T}_2(\xi - \eta, \eta, z)| (1 + |\eta - \zeta|)^{-n-1} d\eta \\ & \leq \frac{C}{\tau} R^{n-1} \sup_{\xi} \int_{\mathbb{R}^n} \frac{(1 + |\xi - R\zeta|)^{-n-1} |\zeta'| d\zeta}{|(\xi - R\zeta) \cdot (\zeta + i\beta v)| l(\zeta)} \\ & = \frac{C}{\tau} R^{n-2} \sup_{\xi} \int_{\mathbb{R}^n} \frac{(1 + R|\xi - \zeta|)^{-n-1} |\zeta'| d\zeta}{((\xi - \zeta) \cdot \zeta)^2 + \beta^2 (\zeta_v - \zeta_v)^2)^{1/2} l(\zeta)}. \end{aligned}$$

Since $\beta \rightarrow 1$ as $\tau \rightarrow \infty$ and $\beta R = \tau$, to show $\|T_2\| \rightarrow 0$ as $\tau \rightarrow \infty$, it suffices to show for $\tau > \tau_0$ that

$$\tau^{n-2} \sup_{\xi} \int_{\mathbb{R}^n} \frac{(1 + \tau|\xi - \zeta|)^{-n-1} |\zeta'| d\zeta}{(((\xi - \zeta) \cdot \zeta)^2 + (\xi_v - \zeta_v)^2)^{1/2} I(\zeta)} \leq C \log \tau. \tag{66}$$

When $I(\zeta) > \varepsilon_0$, the integrand in (66) is essentially the same as the one we considered for T_1 : note that $(\xi - \zeta) \cdot \zeta = |\zeta - \xi/2|^2 - |\xi/2|^2$. Thus we again assume that $I(\zeta) < \varepsilon_0 \ll 1$. We have

$$\begin{aligned} &(((\xi - \zeta) \cdot \zeta)^2 + (\xi_v - \zeta_v)^2)^{1/2} > \frac{1}{2} (|(\xi - \zeta) \cdot \zeta| + |\xi_v - \zeta_v|) \\ &= \frac{1}{2} (|\zeta' - \xi'/2|^2 - |\xi'/2|^2 + \zeta_v(\zeta_v - \xi_v) + |\xi_v - \zeta_v|) \\ &\geq \frac{1}{2} (|\zeta' - \xi'/2|^2 - |\xi'/2|^2 + (1 - \varepsilon_0)|\xi_v - \zeta_v|). \end{aligned}$$

Again using the coordinates $r = |\zeta'|, \theta = \cos^{-1}(\zeta'/|\zeta'| \cdot \xi'/|\xi'|)$, we have

$$|\zeta' - \xi'/2|^2 - |\xi'/2|^2 = r^2 - r|\xi'| \cos \theta$$

and

$$(((\xi - \zeta) \cdot \zeta)^2 + (\xi_v - \zeta_v)^2)^{1/2} \geq c((r - |\xi'| \cos \theta)^2 + (\xi_v - \zeta_v)^2)^{1/2} = c\|v - v_0\|,$$

in the notation used earlier. Thus, using (64), for $|\xi'| \leq 1/2$,

$$\int_{I(\zeta) < \varepsilon_0} \frac{(1 + \tau|\xi - \zeta|)^{-n-1} |\zeta'| d\zeta}{(((\xi - \zeta) \cdot \zeta)^2 + (\xi_v - \zeta_v)^2)^{1/2} I(\zeta)} \leq C \int_{I(\zeta) < \varepsilon_0} \frac{(1 + c_0\tau)^{-n-1} dr d\zeta_v d\omega}{\|v - v_0\| \|v\|},$$

and, since $|\xi'| \leq 1/2$ implies $\|v_0\| \geq \frac{1}{2}$, this is bounded by $C\tau^{-n-1}$. Hence we may assume that $|\xi'| > 1/2$, and in this case (63) implies

$$\begin{aligned} &\int_{I(\zeta) < \varepsilon_0} \frac{(1 + \tau|\xi - \zeta|)^{-n-1} |\zeta'| d\zeta}{(((\xi - \zeta) \cdot \zeta)^2 + (\xi_v - \zeta_v)^2)^{1/2} I(\zeta)} \\ &\leq C \int_{I(\zeta) < \varepsilon_0} \frac{(1 + \tau(\|v - v_0\| + |\sin \theta|))^{-n-1} dr d\zeta_v d\omega}{\|v - v_0\| \|v\|} \equiv I_1. \end{aligned}$$

Since (64) implies $\|v\| < \varepsilon_0$ when $I(\zeta) < \varepsilon_0$, we see that contribution to I_1 from integration over $\{\theta : \|v_0(\theta)\| \geq \frac{1}{2}\}$ is bounded by $C\tau^{-n-1}$. Thus we may replace the domain of integration in I_1 by $\{I(\zeta) < \varepsilon_0\} \cap \{\|v_0\| < \frac{1}{2}\}$.

At this point the argument used for T_1 leads to divergent integrals, and we need to use the fact that the factors in the denominator only vanish simultaneously when

$|\zeta'| \cos \theta = 1$. To bound I_1 , we set $z = (\|v_0(\theta)\|^{-1})v$. Then

$$\begin{aligned}
 I_1 &\leq C \int_{S^{n-2} \times \{\|z\| \leq \|v_0\|^{-1}\} \cap \{\|v_0\| < 1/2\}} \frac{(1 + \tau |\sin \theta|)^{-n-1}}{\|z - v_0/\|v_0\| \| \|z\|} dz d\omega \\
 &\leq C \int_{S^{n-2} \cap \{\|v_0\| < 1/2\}} (1 + \tau |\sin \theta|)^{-n-1} \log (\|v_0(\theta)\|^{-1}) d\omega \\
 &\leq C \int_0^{\pi/2} (1 + \tau \theta)^{-n-1} \max\{\log 2, -\log \|v_0(\theta)\|\} \theta^{n-3} d\theta \\
 &\leq C \tau^{2-n} \int_0^{\pi/2} (1 + \beta)^{-n-1} \beta^{n-3} \max\left\{\log 2, -\log \left\|v_0\left(\frac{\beta}{\tau}\right)\right\|\right\} d\beta \\
 I_1 &\leq c \tau^{2-n} \int_0^{\pi/2} (1 + \beta)^{-n-1} \beta^{n-3} \max\left\{\log 2, -\log \left|1 - |\zeta'| \cos \frac{\beta}{\tau}\right|\right\} d\beta. \quad (67)
 \end{aligned}$$

If $1/2 \leq |\zeta'| \leq 1$, then $|1 - |\zeta'| \cos \beta/\tau| \geq c_0^2 \beta^2 \tau^{-2}$ with c_0 independent of $|\zeta'|$. Hence, in this case $I_1 \leq c \tau^{2-n} \log \tau$ for τ large. If $|\zeta'| > 1$, then $1 - |\zeta'| \cos \theta = 0$ has a unique solution θ_0 in the interval $[0, \pi/2]$ and we have

$$|1 - |\zeta'| \cos \theta| \geq c_0^2 (\theta - \theta_0)^2$$

with $0 < c_0 < 1$ and c_0 independent of $|\zeta'|$. Thus

$$|1 - |\zeta'| \cos \beta/\tau| \geq \frac{c_0^2}{\tau^2} (\beta - \beta_0)^2,$$

where $\beta_0 = \tau \theta_0$. Thus for $\tau > 1$,

$$\begin{aligned}
 &\max\{\log 2, -\log |1 - |\zeta'| \cos \frac{\beta}{\tau}|\} \\
 &\leq \log 2 + 2 \log \tau - 2 \log c_0 + 2(-\log |\beta - \beta_0|)_+. \quad (68)
 \end{aligned}$$

Combining (68) with (67) we see that $I_1 \leq C \tau^{2-n} \log \tau$ for τ large in this case also. Thus (66) holds and the proof of Proposition 4 is complete. \square

It follows from Propositions 3 and 4 that for $\tau \gg 0$ there exists a unique solution g in $H_{0,n+1}$ of the integral equation (53), given by

$$g = (I + (I + T)^{-1} C^{(-1)} (T_1 + T_2 + T_3))^{-1} (I + T)^{-1} C^{(-1)} f, \quad (69)$$

where T is the operator in (54). Thus v , given by (39) with this choice of g , is a solution of (37'). Thus to complete the proof that (27) has a unique solution in $H_{0,n+1}(R^n)$ when $\tau \gg 0$, we need only show that \check{h} given by

$$\check{h}(x) = ((-i\partial/\partial x + zv)^2 - k^2)v$$

is in $H_{0,n+1}$. From (39) we see that

$$\check{h} = Cg + T_2g + Sg,$$

where T_2 is the operator in (53) and

$$\begin{aligned} Sg &= (2\pi)^{-n} \int_{\mathbb{R}^n} \frac{-2i \frac{\partial \zeta}{\partial x} \cdot (\eta + zv) \hat{g}(\eta) e^{ix \cdot \eta}}{(\eta + zv) \cdot (\eta + zv) - k^2} d\eta \\ &= (2\pi)^{-n} \int_{\mathbb{R}^n} \frac{-2A(x) \cdot (\eta + zv) \chi_1(\eta, z) c \hat{g}(\eta) e^{ix \cdot \eta}}{(\eta + zv) \cdot (\eta + zv) - k^2} d\eta \end{aligned} \tag{70}$$

by (40). From (70) one sees that S is an operator of the same type as T_1 in (53) with an additional factor of $\eta + zv$ in the numerator. However, since we showed that the norm of T_1 on $H_{\zeta, n+1}$ was $O(\tau^{-1})$ uniformly in ζ for $\tau \rightarrow \infty$, and $|\eta + zv| \leq c\tau$ on support χ_1 (see (43)), it follows that S is bounded on $H_{\zeta, n+1}$, uniformly in (ζ, τ) for $\tau > \tau_0$. This completes the verification that $h_\nu(\xi, \zeta, k, i\tau) \in H_{\zeta, n+1}$.

4. Recovering the Magnetic Field

Proposition 5. *Let $h_\nu(\xi, \zeta, k, z)$ be the unique solution of (27) in $H_{0, n+1}$ for $\tau \gg 0$, and let $g_\nu(x, \zeta, k, z)$ be the unique solution in $H_{0, n+1}$ of (53) with $f = -(q(x) + 2(\zeta + zv) \cdot A(x)) \exp(ix \cdot \zeta)$ for $\tau \gg 0$. Then*

$$h_\nu(\xi, \zeta, k, z) = \tilde{g}_\nu(\xi, \zeta, k, z) \tag{71}$$

when $(\xi + zv) \cdot (\xi + zv) - k^2 = 0$.

Proof. We have

$$\begin{aligned} v_\nu(x, \zeta, k, z) &= (2\pi)^{-n} \int_{\mathbb{R}^n} \frac{h_\nu(\eta, \zeta, k, z) e^{ix \cdot \eta} d\eta}{(\eta + zv) \cdot (\eta + zv) - k^2} \\ &= (2\pi)^{-n} \int_{\mathbb{R}^n} \frac{c(x, \eta, z) \tilde{g}_\nu(\eta, \zeta, k, z) e^{ix \cdot \eta} d\eta}{(\eta + zv) \cdot (\eta + zv) - k^2}. \end{aligned} \tag{72}$$

As we observed earlier $c_1 = c(x, \eta, z) - 1$ has an expansion of the form (52) for $\tau > \tau_0$. Thus, as in the proof of the bound on T_2 in Proposition 4, we see that

$$f(\xi, \zeta, k, z) = (2\pi)^{-n} \int_{\mathbb{R}^n} \frac{\tilde{c}_1(\xi - \eta, \eta, z) \tilde{g}_\nu(\eta, \zeta, k, z) d\eta}{(\eta + zv) \cdot (\eta + zv) - k^2}$$

belongs to $H_{0, n+1}$ as a function of ξ , and hence is continuous in ξ . Since the Fourier transform of (72) gives (a.e. in ξ)

$$\frac{h_\nu(\xi, \zeta, z)}{(\xi + zv) \cdot (\xi + zv) - k^2} = \frac{\tilde{g}_\nu(\xi, \zeta, k, z)}{(\xi + zv) \cdot (\xi + zv) - k^2} + f(\xi, \zeta, k, z),$$

where h_ν and \tilde{g}_ν are also continuous in ξ , (71) follows immediately. \square

By Proposition 1 and the discussion following it we can recover $h_\nu(\xi(s), \zeta(s), k, z(s))$ from the scattering amplitude $h(k\theta, k\omega, k)$. Recall (see (34), (35)) that given the orthogonal frame $\{v, \mu, l\}$ with $|\mu| = |v| = 1$,

$$\begin{aligned} \xi(s) &= \frac{1}{2}l + s\mu, \\ \zeta(s) &= -\frac{1}{2}l + s\mu, \\ z(s) &= i\tau(s) = i\sqrt{s^2 + |l|^2/4 - k^2} \end{aligned} \tag{73}$$

for $s > s_0$. Since $(\xi(s) + z(s)v) \cdot (\xi(s) + z(s)v) - k^2 = 0$, it follows from Proposition 5 that $h(k\theta, k\omega, k)$ determines $\tilde{g}_\nu(\xi(s), \zeta(s), k, z(s))$ for $s > s_0$.

To recover the magnetic field we can begin with representation for g_ν given by (69) with $f = -(q(x) + 2(\zeta + z\nu) \cdot A(x))\exp(ix \cdot \zeta)$, take the Fourier transform in x , evaluate at $\xi = \xi(s)$, $\zeta = \zeta(s)$, $z = z(s)$ as in (73), divide by $z(s)$ and take the limit as $s \rightarrow \infty$. Since the norms of T, T_1, T_2 and T_3 on $H_{\zeta(s), n+1}$ go to zero and $\frac{1}{|z(s)|} \|f\|_{\zeta(s), n+1}$ is bounded as $s \rightarrow \infty$, it follows that $h(k\theta, k\omega, k)$ determines

$$\begin{aligned} \lim_{s \rightarrow \infty} (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(-2)(\xi(s) + z(s)v)}{z(s)} \cdot \hat{A}(\eta - \zeta(s)) \\ \times e^{-ix \cdot (\xi(s) - \eta) + i\chi_1(\eta, z(s))\varphi(x, \eta + z(s)v)} d\eta dx. \end{aligned} \tag{74}$$

Replacing $\eta - \zeta(s)$ by η , (74) becomes

$$\begin{aligned} \lim_{s \rightarrow \infty} (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(-2)(\xi(s) + z(s)v)}{z(s)} \cdot \hat{A}(\eta) \\ \times e^{ix \cdot \eta - ix \cdot (\xi(s) - \zeta(s)) + i\chi_1(\eta + \zeta(s), z(s))\varphi(x, \eta + \zeta(s) + z(s)v)} d\eta dx. \end{aligned} \tag{75}$$

By (73) $\xi(s) - \zeta(s) = l$ and $\lim_{s \rightarrow \infty} (\xi(s) + z(s)v)/z(s) = v - i\mu$. Also (see definition of χ_1 before (42))

$$\lim_{s \rightarrow \infty} \chi_1(\eta + \zeta(s), z(s)) = \chi(0) = 1.$$

Finally

$$\begin{aligned} \lim_{s \rightarrow \infty} \varphi(x, \eta + \zeta(s) + z(s)v) &= \lim_{s \rightarrow \infty} (2\pi)^{-n} \int_{\mathbb{R}^n} \frac{\hat{A}(\xi) \cdot (\eta + \zeta(s) + z(s)v)e^{ix \cdot \xi}}{i\xi \cdot (\eta + \zeta(s) + z(s)v)} d\xi \\ &= (2\pi)^{-n} \int_{\mathbb{R}^n} \frac{\hat{A}(\xi) \cdot (\mu + iv)}{i\xi \cdot (\mu + iv)} e^{ix \cdot \xi} d\xi \equiv \varphi(x, \mu + iv). \end{aligned} \tag{76}$$

Hence the limit in (75) equals

$$\begin{aligned} I &\equiv -2(2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-ix \cdot l + i\varphi(x, \mu + iv) + ix \cdot \nu} (\nu - i\mu) \cdot \hat{A}(\eta) d\eta dx \\ &= 2i \int_{\mathbb{R}^n} e^{-ix \cdot l + i\varphi(x, \mu + iv)} (\mu + iv) \cdot A(x) dx. \end{aligned} \quad (77)$$

Comparing (76) with (40'), we see that

$$(\mu + iv) \cdot \frac{\partial \varphi}{\partial x} = (\mu + iv) \cdot A(x),$$

and hence, using the coordinates (x_1, x_2, x^\perp) introduced before Proposition 2, we have

$$I = 2 \int_{\mathbb{R}^{n-2}} e^{-il \cdot x^\perp} \left(\int_{\mathbb{R}^2} \left(\frac{\partial}{\partial x_2} + i \frac{\partial}{\partial x_1} \right) e^{i\varphi} dx_1 dx_2 \right) dx^\perp.$$

We have

$$\begin{aligned} \int_{\mathbb{R}^2} \left(\frac{\partial}{\partial x_2} + i \frac{\partial}{\partial x_1} \right) e^{i\varphi} dx_1 dx_2 &= \lim_{R \rightarrow \infty} \int_{x_1^2 + x_2^2 \leq R^2} \left(\frac{\partial}{\partial x_2} + i \frac{\partial}{\partial x_1} \right) e^{i\varphi} dx_1 dx_2 \\ &= \lim_{R \rightarrow \infty} \int_0^{2\pi} e^{i\varphi(R \cos \theta, R \sin \theta, x^\perp, \mu + iv)} (\sin \theta + i \cos \theta) R d\theta, \end{aligned}$$

by Green's theorem with $x_1 = r \cos \theta$ and $x_2 = r \sin \theta$. Returning to the expansion (52) for φ , we have

$$\varphi = \frac{1}{2\pi i} \frac{1}{x_1 - ix_2} \int_{\mathbb{R}^2} A(y_1 \nu + y_2 \mu + x^\perp) \cdot (\mu + iv) dy_1 dy_2 + O((x_1 - ix_2)^{-2}).$$

Thus

$$\begin{aligned} \lim_{R \rightarrow \infty} \int_0^{2\pi} e^{i\varphi(R \cos \theta, R \sin \theta, x^\perp, \mu + iv)} (\sin \theta + i \cos \theta) R d\theta \\ = i \int_{\mathbb{R}^2} A(y_1 \nu + y_2 \mu + x^\perp) \cdot (\mu + iv) dy_1 dy_2, \end{aligned}$$

and

$$\begin{aligned} I &= 2i \int_{\mathbb{R}^{n-2}} e^{-il \cdot x^\perp} \left(\int_{\mathbb{R}^2} A(y_1 \nu + y_2 \mu + x^\perp) \cdot (\mu + iv) dy_1 dy_2 \right) dx^\perp \\ &= 2i \hat{A}(l) \cdot (\mu + iv). \end{aligned}$$

Since μ and ν are a general orthonormal pair perpendicular to l , we conclude that for all $l \in \mathbb{R}^n$, I determines $\hat{A}(l) - (\hat{A}(l) \cdot l)l/|l|^2$. In other words I determines A modulo the gradient of

$$\rho(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{il \cdot x} i \hat{A}(l) \cdot l/|l|^2 dl = -\Delta^{-1}(\nabla \cdot A), \quad (78)$$

and hence I determines $\text{curl } A$.

5. Recovering the Electric Potential

To recover $V(x)$ we need to compute the next term in the asymptotic expansion of (69) which yielded (74) as the leading term. We have determined $A(x)$ modulo the gradient of a function of the form (78). Hence, we may assume that we know the scattering data for the problem with the $A(x)$ here and $q = q' \equiv A \cdot A - i\nabla \cdot A$, since the scattering data only depends on the magnetic field $B = \text{curl} A$. This scattering data determines the Fourier transform of the solution g_0 of (53) with $f = f_0 \equiv -(q' + 2(\zeta + zv) \cdot A(x))\exp(ix \cdot \zeta)$ on the set $(\zeta, \zeta, z) = (\zeta(s), \zeta(s), z(s))$ given by (73). Among the operators in (69) only T_1 is changed when we replaced g by g_0 , and we denote the new operator by $T_{1,0}$. Thus, subtracting the representation (69) for g_0 from the representation (69) for g , we may assume that we know the Fourier transform on the curve $(\xi(s), \zeta(s), z(s))$ of

$$\begin{aligned} & (I + (I + T)^{-1}C^{(-1)}(T_1 + T_2 + T_3))^{-1}(I + T)^{-1}C^{(-1)}(f - f_0) \\ & - (I + (I + T)^{-1}C^{(-1)}(T_1 + T_2 + T_3))^{-1}(I + T)^{-1}C^{(-1)}(T_1 - T_{1,0}) \\ & \cdot (I + (I + T)^{-1}C^{(-1)}(T_{1,0} + T_2 + T_3))^{-1}(I + T)^{-1}C^{(-1)}f_0. \end{aligned} \tag{79}$$

Taking the limit in the Fourier transform of (79) at $(\xi(s), \zeta(s), z(s))$ as $s \rightarrow \infty$, we recover

$$\begin{aligned} & \lim_{s \rightarrow \infty} (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} -\hat{V}(\eta - \zeta(s))e^{-ix \cdot (\zeta(s) - \eta) + i\chi_1(\eta, z(s))\phi(x, \eta + z(s)v)} d\eta dx \\ & - \lim_{s \rightarrow \infty} \mathcal{F}(C^{(-1)}(T_1 - T_{1,0})C^{(-1)}f_0)(\xi(s), \zeta(s), z(s)) \equiv J_1 - J_2. \end{aligned}$$

By the same computation that derived (77) from (75), we have

$$J_1 = - \int_{\mathbb{R}^n} e^{-ix \cdot (l + i\phi(x, \mu + iv))} V(x) dx. \tag{80}$$

To compute J_2 we argue as follows. $T_1 - T_{1,0} = VCL$, where L multiplies the Fourier transform by $((\eta + zv) \cdot (\eta + zv) - k^2)^{-1}$. Since $\{V, C\}$ goes to zero and $C^{(-1)}C$ goes to the identity as $s \rightarrow \infty$, we can conclude that

$$\begin{aligned} J_2 &= \lim_{s \rightarrow \infty} (2\pi)^{-2n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\hat{V}(\xi(s) - \eta)}{(\eta + z(s)v) \cdot (\eta + z(s)v) - k^2} \\ & \times (-2(\zeta(s) + z(s)v) \cdot \hat{A}(\delta - \zeta(s))) \\ & \times e^{ix \cdot (\delta - \eta) + i\chi_1(\delta, z(s))\phi(x, \delta + z(s)v)} d\delta dx d\eta. \end{aligned}$$

Replacing δ by $\delta + \zeta(s)$ and η by $\eta + \zeta(s)$, and arguing as before (recall $(\zeta(s) + z(s)v) \cdot (\zeta(s) + z(s)v) = k^2$), we have

$$\begin{aligned} J_2 &= (2\pi)^{-2n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\hat{V}(l - \eta)}{2\eta \cdot (\mu + iv)} (-2(\mu + iv) \cdot \hat{A}(\delta)) \\ &\quad \times e^{ix \cdot (\delta - \eta) + i\varphi(x, \mu + iv)} d\delta dx d\eta \\ &= (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} - \frac{\hat{V}(l - \eta)(\mu + iv) \cdot A(x) e^{-ix \cdot \eta + i\varphi(x, \mu + iv)}}{(\mu + iv) \cdot \eta} dx d\eta. \end{aligned}$$

Proceeding as before with $x_1 = x \cdot v$ and $x_2 = x \cdot \mu$,

$$\begin{aligned} &\int_{\mathbb{R}^n} e^{-ix \cdot \eta + i\varphi(x, \mu + iv)} (\mu + iv) \cdot A(x) dx \\ &= \int_{\mathbb{R}^{n-2}} e^{-ix^\perp \cdot \eta^\perp} dx^\perp \int_{\mathbb{R}^2} e^{-i(x_1 \eta_1 + x_2 \eta_2)} (-i) \left(\frac{\partial}{\partial x_2} + i \frac{\partial}{\partial x_1} \right) (\bar{c}(x, \mu + iv) - 1) dx_1 dx_2, \end{aligned}$$

and by Green's theorem

$$\begin{aligned} &\int_{\mathbb{R}^2} e^{-i(x_1 \eta_1 + x_2 \eta_2)} (-i) \left(\frac{\partial}{\partial x_2} + i \frac{\partial}{\partial x_1} \right) (\bar{c} - 1) dx_1 dx_2 \\ &= \lim_{R \rightarrow \infty} \left[\int_{x_1^2 + x_2^2 \leq R^2} e^{-i(x_1 \eta_2 + x_2 \eta_1)} (\eta_2 + i\eta_1) (\bar{c} - 1) dx_1 dx_2 \right. \\ &\quad \left. + \int_0^{2\pi} e^{iR(\eta_2 \cos \theta + \eta_1 \sin \theta)} R(\sin \theta + i \cos \theta) \right. \\ &\quad \left. \times (\bar{c}(R \cos \theta, R \sin \theta, x^\perp, \mu + iv) - 1) d\theta \right]. \tag{81} \end{aligned}$$

Since

$$\bar{c}(R \cos \theta, R \sin \theta, x^\perp, \mu + iv) - 1 = \frac{1}{2\pi R} \cdot \frac{f(x^\perp)}{\cos \theta - i \sin \theta} + O\left(\frac{1}{R^2}\right),$$

the second integral in the limit in (81) goes to zero as R goes to infinity when $(\eta_1, \eta_2) \neq 0$. The first integral just goes to the Fourier transform of $\bar{c} - 1$ in (x_1, x_2) multiplied by $(\eta_2 + i\eta_1) = (\mu + iv) \cdot \eta$. Thus

$$J_2 = - \int_{\mathbb{R}^n} e^{-iy \cdot l} V(y) (e^{i\varphi(y, \mu + iv)} - 1) dy.$$

Thus $J_1 - J_2 = - \int_{\mathbb{R}^n} e^{-iy \cdot l} V(y) dy$. Since l is arbitrary, we have determined the Fourier transform of V and the proof is complete.

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