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INVERSE SEMIGROUPS WITH IDEMPOTENTS DUALLY WELL-ORDERED

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Abstract

All inverse semigroups with idempotents dually well-ordered may be constructed inductively. The techniques involved are the constructions of ordinal sums, direct limits and Bruck-Reilly extensions.

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1. Introduction

We use the terminology and the results of Howie [3] and Sierpiński [12]. The axiom of choice will be assumed throughout.

A study of inverse semigroups with idempotents dually well-ordered can be motivated by the findings of Feller and Gantos [1]. They may also be studied within the context of the investigations by Megyesi and Pollák ([5], [6], [7]) concerning principal ideal semigroups. Recall that a principal ideal semigroup is a semigroup where the left, right and two-sided ideals are all principal, or equivalently, it is a semigroup for which the posets of left, right and two-sided ideals are dually well-ordered chains. In a [simple] principal ideal semigroup, the set of regular elements—if non-empty—constitutes a [simple] inverse semigroup with idempotents dually well-ordered. In fact, a principal ideal semigroup is regular if and only if it is an inverse semigroup with idempotents dually well-ordered.

Particular structure theorems for inverse semigroups with idempotents dually well-ordered were given by Hogan [2], Kočin [4], Munn [8], Reilly [11], and Warne ([13], [14]).

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2. Main results

Let δ be any ordinal. An inverse semigroup S will be called a δ-regular semigroup if the set E_S of idempotents of S constitutes a chain whose order type is $\bar{E}_S = \delta^*$.

Recall that an inverse semigroup S is called a fundamental inverse semigroup if the greatest idempotent-separating congruence on S is the identity relation. Let us consider a fundamental inverse semigroup S whose idempotents form a chain E_S. Then Green's equivalence relation $\mathcal{J}$ is the least semilattice congruence on S, and S is a chain $S/\mathcal{J}$ of its $\mathcal{J}$-classes, which are all simple inverse semigroups. Each $\mathcal{J}$-class is the disjoint union of $\mathcal{D}$-classes which all constitute bisimple inverse semigroups. In general it is not straightforward how to describe S as a chain composition of its $\mathcal{J}$-classes. There is however an important instance in which things simplify. Indeed, let us consider the case where the principal ideals of E_S each have a trivial automorphism group. Since S is fundamental, one can embed S isomorphically into the Munn semigroup T_{E_S} (see, for example, Howie [3]). It follows that S must be combinatorial ($= \mathcal{H}$-trivial), and for any $a, b \in S$, with $J_a < J_b$ in $S/\mathcal{J}$, we have $ab = ba = a$. The situation described here is satisfied whenever E_S is a dually well-ordered chain. We thus have the following.

THEOREM 1. *Let $\delta = \sum_{\xi<\alpha} \alpha_\xi$ be an ordinal such that for each $\xi < \alpha$, S_ξ is a combinatorial simple α_ξ-regular semigroup, with $S_\xi \cap S_\eta = \varnothing$ if $\xi \neq \eta$. On $S = \bigcup_{\xi<\alpha} S_\xi$ define a multiplication by the following. If $a \in S_\xi$, $b \in S_\eta$, then ab coincides with the product of a and b already defined in S_ξ if $\xi = \eta$, whereas $ab = a$ if $\xi > \eta$ and $ab = b$ if $\eta > \xi$. Then S is a fundamental δ-regular semigroup.*

Conversely, every fundamental δ-regular semigroup can be so obtained.

COROLLARY 2. *Let $\delta = \sum_{\xi<\alpha} \alpha_\xi$ be an ordinal such that for each $\xi < \alpha$, S_ξ is a simple α_ξ-regular semigroup, with $S_\xi \cap S_\eta = \varnothing$ if $\xi \neq \eta$. For every $\xi < \eta < \alpha$, let $\phi_{\xi,\eta}$ be a homomorphism of S_ξ into the group of units of S_η, such that $\phi_{\xi,\eta}\phi_{\eta,\zeta} = \phi_{\xi,\zeta}$ whenever $\xi < \eta < \zeta < \alpha$. For each $\xi < \alpha$, let $\phi_{\xi,\xi}$ be the identity transformation on S_ξ. Let S be the strong chain of the semigroups in the system*

$$(1) \qquad \left(\alpha; \{S_\xi \mid \xi < \alpha\}; \{\phi_{\xi,\eta} \mid \xi \leqslant \eta < \alpha\}\right).$$

Then S is a δ-regular semigroup.

Conversely, every δ-regular semigroup can be so obtained.

PROOF. The direct part can be verified without difficulty.

Let us conversely suppose that S is a δ-regular semigroup. Since the principal ideals of E_S have trivial automorphism groups, Green's relation $\mathcal{H}$ is a congruence relation on S (see also Theorem 5 in Megyesi and Pollák [5]). Therefore $S/\mathcal{H}$ is a fundamental δ-regular semigroup, and we can apply Theorem 1. The results of Theorem 1 guarantee that we can write S as a chain α of simple α_ξ-regular semigroups S_ξ, $\xi < \alpha$, with $\delta = \Sigma_{\xi<\alpha}\alpha_\xi$. Further, if $\xi < \eta$, and if 1_η denotes the identity element of S_η, then the mapping

$$\phi_{\xi,\eta}\colon S_\xi \to S_\eta,\ a \to a1_\eta \tag{2}$$

is a homomorphism of S_ξ into the group of units of S_η. As a result we obtain a system (1), and one easily shows that S is the sum of this system.

If the semigroup S is obtained in the way described in Corollary 2, then we shall say that S is the ordinal sum of the system (1).

We exemplify Theorem 1 by describing the Munn semigroup T_E of a chain E whose order type $\bar{E}$ is the dual δ^* of an ordinal δ. One may identify T_E with the inverse semigroup consisting of the isomorphisms among principal filters of δ (where δ stands for the well-ordered chain of ordinals that are less than δ). The latter inverse semigroup will be denoted by T_δ. Remark that T_ω is the bicyclic semigroup, whereas T_{ω^n} (n a positive integer) is Warne's n-dimensional bicyclic semigroup [13], and T_{ω^α} (α any ordinal) is the α-bicyclic semigroup in Hogan [2] and Megyesi and Pollák [7]. Let ξ and η be ordinals such that $\delta = \xi + \tau = \eta + \tau$ for some ordinal τ. Then the principal filter generated by ξ is isomorphic to the principal filter generated by η: the two filters are of order type τ. We denote by $\binom{\xi}{\eta}$ the unique isomorphism of the principal filter generated by ξ onto the principal filter generated by η. Thus,

$$(\xi + \kappa)\binom{\xi}{\eta} = \eta + \kappa, \qquad \kappa < \tau. \tag{3}$$

The inverse of $\binom{\xi}{\eta}$ in T_δ is $\binom{\eta}{\xi}$. Clearly T_δ precisely consists of the elements $\binom{\xi}{\eta}$ where $\delta = \xi + \tau = \eta + \tau$ for some ordinal τ, and the multiplication in T_δ is given by

$$\binom{\xi}{\eta}\binom{\xi'}{\eta'} = \binom{\xi + [\xi' - \eta]}{\eta' + [\eta - \xi']}, \tag{4}$$

where for any ordinals ρ, σ

$$[\rho - \sigma] = \begin{cases} \rho - \sigma & \text{if } \sigma \leqslant \rho, \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

(we use the notation of Megyesi and Pollák [7]).

Recall that for any ordinal δ there exists a unique decomposition $\delta = \delta_1 + \cdots + \delta_k$ (k a positive integer), where $\delta_1 \geqslant \cdots \geqslant \delta_k$ is a finite nonincreasing sequence of prime (= indecomposable) ordinals. This decomposition is called the normal expansion of δ.

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Theorem 3. *Let δ be an ordinal, and $\delta = \delta_1 + \cdots + \delta_k$ its normal expansion. Then T_δ is a k-chain of the bisimple combinatorial δ_i-regular semigroups T_i, $i = 1,\ldots,k$. For each $i = 1,\ldots,k$, T_i is isomorphic to T_{δ_i}.*

Proof. Let ξ, η be ordinals such that $\delta = \xi + \tau = \eta + \tau$. The ordinal τ must be of the form $\delta_i + \cdots + \delta_k$ for some $1 \leqslant i \leqslant k$. Putting

$$(6) \qquad T_i = \left\{ \begin{pmatrix} \xi \\ \eta \end{pmatrix} \mid \delta = \xi + \delta_i + \cdots + \delta_k = \eta + \delta_i + \cdots + \delta_k \right\}$$

for $i = 1,\ldots,k$, we obtain a partitioning $T_\delta = \bigcup_{1 \leqslant i \leqslant k} T_i$. Let $\binom{\xi}{\eta}$, $\binom{\xi'}{\eta'} \in T_i$ for some $1 \leqslant i \leqslant k$. Then

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} \mathcal{R} \begin{pmatrix} \xi \\ \eta' \end{pmatrix} \mathcal{L} \begin{pmatrix} \xi' \\ \eta' \end{pmatrix} \mathcal{R} \begin{pmatrix} \xi' \\ \eta \end{pmatrix} \mathcal{L} \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

in T_δ. Consequently T_i is contained in a $\mathcal{D}$-class. Further, if $\binom{\xi}{\eta} \in T_i$, $\binom{\xi'}{\eta'} \in T_j$, $i < j$, then $\binom{\xi}{\eta}\binom{\xi'}{\eta'} = \binom{\xi'}{\eta'}\binom{\xi}{\eta} = \binom{\xi'}{\eta'}$. Thus elements belonging to different components in the partitioning $\bigcup_{1 \leqslant i \leqslant k} T_i$ cannot be $\mathcal{J}$-related. We see that $\mathcal{J} = \mathcal{D}$ in T_δ, and that the T_i, $i = 1,\ldots,k$, constitute the k $\mathcal{D}$-classes of T_δ; T_δ is a k-chain of these $\mathcal{D}$-classes.

The $\mathcal{D}$-classes T_i, $i = 1,\ldots,k$, form bisimple inverse semigroups (see the remark made before Theorem 1). T_δ is combinatorial since well-ordered chains have a trivial automorphism group. Thus the T_i, $i = 1,\ldots,k$ are combinatorial as well. The idempotents of T_i are of the form $\binom{\xi}{\xi}$, with $\xi < \delta_1$ if $i = 1$, or $\delta_1 + \cdots + \delta_{i-1} \leqslant \xi < \delta_1 + \cdots + \delta_i$ otherwise. Therefore T_i is a δ_i-regular semigroup.

The mapping

$$T_1 \to T_{\delta_1}, \qquad \begin{pmatrix} \xi \\ \eta \end{pmatrix} \to \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

is easily seen to be an isomorphism of T_1 onto T_{δ_1}, whereas in the case $1 < i \leqslant k$,

$$T_i \to T_{\delta_i}, \qquad \begin{pmatrix} \xi \\ \eta \end{pmatrix} \to \begin{pmatrix} \xi - (\delta_1 + \cdots + \delta_{i-1}) \\ \eta - (\delta_1 + \cdots + \delta_{i-1}) \end{pmatrix}$$

is an isomorphism of T_i onto T_{δ_i}.

Corollary 4. *Let E be a chain such that $\bar{E}^*$ is an ordinal. In the Munn semigroup T_E, $\mathcal{J}$ and $\mathcal{D}$ coincide. The number of $\mathcal{D}$-classes in T_E is finite. It is the number of terms in the normal expansion of $\bar{E}^*$.*

Corollary 5 (Hogan [2], Munn [9], White [15]). *If S is a simple δ-regular semigroup, then δ is a prime ordinal. If E is a chain such that $\bar{E}^* = \delta$ is a prime ordinal, then T_E is a bisimple δ-regular semigroup.*

Theorem 1 and Corollary 2 show that the problem of describing the structure of [fundamental] inverse semigroups with idempotents dually well-ordered can be reduced to the case of simple [fundamental] inverse semigroups with idempotents dually well-ordered. Therefore we shall from now on concentrate on simple δ-regular semigroups. From Corollary 5 we know that δ must then be a prime ordinal, that is, $\delta = \omega^\alpha$ for some ordinal α (well-defined by δ). The aim of our considerations will be to construct simple ω^α-regular semigroups in terms of ξ-regular semigroups, with $\xi < \omega^\alpha$. This will enable us to construct inductively all inverse semigroups with idempotents dually well-ordered.

If T is a δ-regular semigroup and θ an endomorphism of T into the unit group of T, then one can consider the Bruck-Reilly extension $BR(T, \theta)$ of T determined by θ. This inverse semigroup $BR(T, \theta)$ must be a simple $\delta\omega$-regular semigroup (see for example III.2 of Petrich [10]). Thus, any δ-regular semigroup can be embedded into a simple $\delta\omega$-regular semigroup. Note that $BR(T, \theta)$ is fundamental if and only if T is fundamental. If this is the case, then θ is simply the constant mapping of T onto the identity of T. The following characterizes the inverse semigroups with idempotents dually well-ordered which are obtained by considering Bruck-Reilly extensions.

THEOREM 6. *Let S be a δ-regular semigroup, with $E_S = \{e_\xi \mid \xi < \delta\}$, where $e_\xi < e_\eta$ in E_S if and only if $\eta < \xi$. Then S is a Bruck-Reilly extension $BR(T, \theta)$ if and only if the following conditions are satisfied*:

(i) $\delta = \omega^{\alpha+1}$ *for some* α,

(ii) *there exists a* $\omega^\alpha \leqslant \gamma < \omega^{\alpha+1}$ *such that* $e_0 \mathcal{D} e_\gamma$ *and such that the elements* $x \in S$ *for which* $e_\gamma < (xx^{-1})(x^{-1}x)$ *form a subsemigroup of* S.

PROOF. Let $S = BR(T, \theta)$ for some inverse semigroup T, and for some appropriate endomorphism θ of T. From the fact that S is a δ-regular semigroup it follows that T is an inverse semigroup with idempotents dually well-ordered. In other words, T is a γ-regular semigroup for some ordinal γ, where $\delta = \gamma\omega$. Let ω^α be the first term in the normal expansion of γ. Then $\delta = \omega^{\alpha+1}$, and so (i) is satisfied. Let us denote the set of idempotents of T by $\{f_\zeta \mid \zeta < \gamma\}$, where $f_\zeta < f_\tau$ in E_T if and only if $\tau < \zeta$. The idempotents of S are then of the form

$$e_{\gamma n+\zeta} = (n, f_\zeta, n), \qquad n \in N, \zeta < \gamma, \tag{7}$$

and one sees that (ii) is satisfied.

Let us conversely suppose that S satisfies (i) and (ii). Let T be the subsemigroup of S which is given by (ii). Since T is clearly closed with respect to the taking of inverses, we have that T is an inverse subsemigroup of S. Consequently, T is a γ-regular subsemigroup of S. Let a be an element of S for which $aa^{-1} = e_0$ and $a^{-1}a = e_\gamma$.

Let $x \in S$, with $xx^{-1} = e_\xi$ and $x^{-1}x = e_\eta$. Then

$$S \to T_\delta, \qquad x \to \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

is a representation of S which is equivalent to the Munn representation. In particular, if $x \in T$, then $\binom{\xi}{\eta}$ must fix γ since T forms a subsemigroup, and since T_δ is combinatorial. In this case we must have $\binom{\xi}{\eta}\binom{\gamma}{\gamma} = \binom{\gamma}{\gamma}\binom{\xi}{\eta} = \binom{\gamma}{\gamma}$, and consequently

(8) $$e_\gamma x \mathcal{H} e_\gamma \mathcal{H} x e_\gamma \quad \text{for all } x \in T.$$

It follows that

(9) $$\theta\colon T \to H_{e_0}, \qquad x \to axa^{-1}$$

is an endomorphism of T into its group of units.

For $m, n \in N$, let $S_{m,n}$ consist of the elements x of S for which $e_{\gamma m} \geqslant xx^{-1} > e_{\gamma(m+1)}$ and $e_{\gamma n} \geqslant x^{-1}x > e_{\gamma(n+1)}$. Then $S = \bigcup_{m,n \in N} S_{m,n}$ yields a partitioning of S. Remark that $T = S_{0,0}$. The mapping $T \to S_{m,n}$, $x \to a^{-m}xa^n$ is a bijection of T onto $S_{m,n}$, and the mapping $S_{m,n} \to T$, $y \to a^m y a^{-n}$ is its inverse. For this reason

(10) $$\psi\colon S \to BR(T,\theta), \quad a^{-m}xa^n \to (m, x, n), \qquad m, n \in N, x \in T,$$

is a well-defined bijection of S onto $BR(T, \theta)$. It is easy to show that ψ is in fact an isomorphism.

THEOREM 7. *Let S be a simple ω^α-regular semigroup, with α a limit ordinal. Then there exists a well-ordered system*

(11) $$\big(\beta; \{S_\xi \mid \xi < \beta\}; \{\phi_{\xi,\eta} \mid \xi \leqslant \eta < \beta\}\big)$$

of simple $\omega^{\alpha_\xi+1}$-semigroups S_ξ, $\xi < \beta$, where

(i) *for $\xi < \beta$, $S_\xi = BR(T_\xi, \theta_\xi)$ is a Bruck-Reilly extension of a $(\omega^{\alpha_\xi} + \delta_\xi)$-regular semigroup T_ξ, with $\delta_\xi < \omega^{\alpha_\xi+1}$,*

(ii) $\alpha = \lim_{\xi<\beta}(\alpha_\xi + 1)$,

(iii) *for $\xi \leqslant \eta < \beta$, $\phi_{\xi,\eta}$ is a monomorphism of S_ξ into S_η, such that S is the direct limit of the system* (11).

Conversely, if the well-ordered system (11) *satisfies the above conditions* (i), (ii) *and* (iii), *then its direct limit is a simple ω^α-regular semigroup.*

PROOF. The proof of the converse part is routine, and is left to the reader. We now proceed to show the direct part.

Let $\{e_\zeta \mid \zeta < \omega^\alpha\}$ be the set of idempotents of S, where $e_\zeta < e_\eta$ in E_S if $\eta < \zeta$. Let A be a set of ordinals, where $\kappa \in A$ if and only if there exists a $\omega^\kappa \leqslant \gamma < \omega^{\kappa+1}$ such that $e_0 \mathcal{D} e_\gamma$ in S. Let β be the order type of the chain A. We shall denote the chain A by $A = \{\alpha_\xi \mid \xi < \beta\}$. We have $\lim_{\xi<\beta}\alpha_\xi = \lim_{\xi<\beta}(\alpha_\xi + 1) = \alpha$ since S is simple. Therefore (ii) is satisfied.

For $\xi < \beta$, let γ_ξ be an ordinal such that $e_0 \mathcal{D} e_{\gamma_\xi}$, and $\omega^{\alpha_\xi} \leqslant \gamma_\xi < \omega^{\alpha_\xi+1}$, and let a_ξ be an element of S such that $a_\xi a_\xi^{-1} = e_0$ and $a_\xi^{-1}a_\xi = e_{\gamma_\xi}$. Let T_ξ be the subset of S consisting of the elements $x \in S$ for which $e_{\omega^{\alpha_\xi}} < (xx^{-1})(x^{-1}x)$, together with the elements of the maximal subgroups containing the idempotents e_ζ, $\zeta < \gamma_\xi$. If θ: $S \to T_{\omega^\alpha}$, $x \to \binom{\mu}{\nu}$, with $xx^{-1} = e_\mu$ and $x^{-1}x = e_\nu$ in S, stands for the canonical homomorphism of S into the Munn semigroup T_{ω^α}, then $T_\xi\theta$ consists of the elements

$$\binom{\mu}{\nu} = x\theta, \qquad \text{with } xx^{-1} = e_\mu, \quad x^{-1}x = e_\nu, \quad \mu, \nu < \omega^{\alpha_\xi},$$

and

$$\binom{\zeta}{\zeta}, \qquad \zeta < \gamma_\xi.$$

Obviously $T_\xi\theta$ forms an inverse subsemigroup of the Munn semigroup $T_{\omega^{\alpha_\xi}}$. Further, since $T_\xi = T_\xi\theta\theta^{-1}$, we deduce that T_ξ forms an inverse subsemigroup of S. Let S_ξ be the inverse subsemigroup of S which is generated by a_ξ and by the elements of T_ξ. Using Theorem 6, we deduce that S_ξ is (isomorphic to) a Bruck-Reilly extension of the γ_ξ-regular semigroup T_ξ, where $\gamma_\xi = \omega^{\alpha_\xi} + \delta_\xi$, with $\delta_\xi < \omega^{\alpha_\xi+1}$. Thus (i) is satisfied, and S_ξ is a simple $\omega^{\alpha_\xi+1}$-regular semigroup (since $\gamma_\xi\omega = \omega^{\alpha_\xi+1}$).

We consider the system (11), where for $\xi \leqslant \eta < \beta$, $\phi_{\xi,\eta}$: $S_\xi \to S_\eta$ is just the inclusion mapping. We must show that S is the direct limit of (11). Therefore, let x be any element of S. Since S is simple, there exists a γ such that $e_0 \mathcal{D} e_\gamma < (xx^{-1})(x^{-1}x)$. Let us suppose $\omega^{\alpha_\xi} \leqslant \gamma < \omega^{\alpha_\xi+1}$, with $\alpha_\xi \in A$. Then $x \in T_{\xi+1}$, and thus also $x \in S_{\xi+1}$. We conclude $S = \bigcup_{\xi<\beta} S_\xi$.

THEOREM 8. *Let T be a δ-regular semigroup where $\omega^\alpha \leqslant \delta < \omega^{\alpha+1}$, and let $BR(T,\theta)$ be a Bruck-Reilly extension of T. Let e be an idempotent of $BR(T,\theta)$. Then $eBR(T,\theta)e$ is a simple $\omega^{\alpha+1}$-regular semigroup.*

Conversely, every simple $\omega^{\alpha+1}$-regular semigroup can be obtained in this way.

PROOF. If S is a simple regular semigroup, and $e \in E_S$, then eSe is a simple regular subsemigroup of S. From this well-known fact follows the direct part of our theorem.

Let us conversely suppose that S is a simple $\omega^{\alpha+1}$-regular semigroup. Let $E_S = \{e_\xi \mid \xi < \omega^{\alpha+1}\}$, where $e_\xi < e_\eta$ in E_S if $\eta < \xi$. Let D be the set of ordinals

$$D = \left\{\xi \mid \xi = \eta - \zeta \geqslant \omega^\alpha, \omega^{\alpha+1} > \eta > \zeta, e_\eta \mathcal{D} e_\zeta \text{ in } S\right\},$$

and let δ be the least ordinal in D. We have $\delta = \omega^\alpha n + \mu$, with $\mu < \omega^\alpha$. Let ζ and η be any ordinals, with $\zeta < \eta < \omega^{\alpha+1}$, such that $e_\eta \mathcal{D} e_\zeta$ in S and $\eta - \zeta = \omega^\alpha n + \mu$. Putting $\zeta = \omega^\alpha m + \mu'$, with $\mu' < \omega^\alpha$, we have $\eta = \omega^\alpha(m+n) + \mu$. Let us investigate $S' = e_\zeta S e_\zeta$.

S' is of course a $\omega^{\alpha+1}$-regular semigroup which is simple. Let T be the subset of S' which consists of the elements of S for which $e_\eta < xx^{-1}, x^{-1}x \leqslant e_\zeta$. Due to the minimality of δ in D, we have either

(12) $$e_{\zeta+\omega^\alpha(i-1)} < xx^{-1}, x^{-1}x \leqslant e_{\zeta+\omega^\alpha i}$$

for some $i = \{0,\ldots,n-1\}$, or

(13) $$e_\eta < xx^{-1}, x^{-1}x \leqslant e_{\omega^\alpha(m+n)}.$$

Take any other $y \in T$. Again, either

(14) $$e_{\zeta+\omega^\alpha(j+1)} < yy^{-1}, y^{-1}y \leqslant e_{\zeta+\omega^\alpha j}$$

for some $j \in \{0,\ldots,n-1\}$, or

(15) $$e_\eta < yy^{-1}, y^{-1}y \leqslant e_{\omega^\alpha(m+n)}.$$

If x and y are elements of T such that (12) and (14) or (15) hold, with $j > i$, then $xy\mathcal{H}y$, and so $xy \in T$. Similarly, if (14) and (12) or (13) hold, with $i > j$, then $xy\mathcal{H}x$, and thus $xy \in T$. Further, if (12) and (14) hold, with $i = j$, then

$$e_{\zeta+\omega^\alpha(i+1)} < (xy)(xy)^{-1}, (xy)^{-1}(xy) \leqslant e_{\zeta+\omega^\alpha i}$$

and again $xy \in T$. Finally, let $x, y \in T$ such that both (13) and (15) hold. Let us suppose that $xy \notin T$, that is,

$$e_{\omega^\alpha(n+m+1)} < \big((xy)(xy)^{-1}\big)\big((xy)^{-1}(xy)\big) = e_\nu \leqslant e_\eta.$$

Anyway, $xy\mathcal{R}e_\nu$ or $xy\mathcal{L}e_\nu$, and $xy\mathcal{L}y$ or $xy\mathcal{R}x$, since E_S is a chain. Since both (13) and (15) hold, we conclude that there exists an idempotent $e_\tau \in E_S$, with $e_\eta < e_\tau \leqslant e_{\omega^\alpha(m+n)}$, such that $e_\tau \mathcal{D} e_\nu$. Let $\kappa = \nu - \eta$. Then $\kappa < \omega^\alpha$. If a is any element of S such that $e_\zeta \mathcal{R} a \mathcal{L} e_\eta$, then $e_{\zeta+\kappa} \mathcal{R} e_{\zeta+\kappa} a \mathcal{L} e_{\eta+\kappa} = e_\nu \mathcal{D} e_\tau$, from which $e_{\zeta+\kappa} \mathcal{D} e_\tau$. Yet, $\tau - (\zeta + \kappa) < \delta$, since $\omega^\alpha(m+n) \leqslant \tau < \eta$, and this contradicts the minimality of δ. Hence, also in this case $xy \in T$. We conclude that T is a subsemigroup of S'. It follows from Theorem 6 that S' is (isomorphic to) a Bruck-Reilly extension of T.

The identity e_0 of S is $\mathcal{D}$-related to an idempotent $e_\lambda < e_\zeta$ since S is simple. Let b be any element of S such that $bb^{-1} = e_0$ and $b^{-1}b = e_\lambda$. The mapping

$$S \to e_\lambda S e_\lambda, \qquad x \to b^{-1}xb,$$

is an isomorphism of S onto $e_\lambda S e_\lambda$. Yet $e_\lambda S e_\lambda = e_\lambda S' e_\lambda$, where S' is (isomorphic to) a Bruck-Reilly extension $BR(T,\theta)$ of the δ-regular semigroup T, with $\omega^\alpha \leqslant \delta < \omega^{\alpha+1}$. From this follows the converse part of our theorem.

Corollary 9 (Koč [4], Munn [8]). *An inverse semigroup S is a simple ω-regular semigroup if and only if S is a Bruck-Reilly extension of a finite chain of groups.*

Not every simple $\omega^{\alpha+1}$-regular semigroup needs to be a Bruck-Reilly extension of a δ-regular semigroup, with $\omega^{\alpha} \leqslant \delta < \omega^{\alpha+1}$. We depict a counterexample in Figure 1. Indeed, if a is the element of the semigroup depicted in Figure 1 for which $e_1 \mathcal{R} a \mathcal{L} e_{\omega}$, then $a^n a^{-n} = e_1$, and $a^{-n} a^n = e_{\omega n}$, $n \in N$, and it follows that the subsemigroup requirement of Theorem 6(ii) cannot be satisfied. The inverse semigroup under consideration is a combinatorial simple ω^2-regular semigroup. Remark however, that every bisimple $\omega^{\alpha+1}$-regular semigroup is a Bruck-Reilly extension of a ω^{α}-regular semigroup which is bisimple.

3. Conclusion

We note that we are now able to construct inductively all inverse semigroups with idempotents dually well-ordered. The process for doing so is based on Corollary 2, Theorem 7 and Theorem 8. The techniques involved are the constructions of ordinal sums, direct limits and Bruck-Reilly extensions.

4. The combinatorial case

We conclude with some remarks concerning combinatorial inverse semigroups with idempotents dually well-ordered.

Lemma 10. *For any prime ordinal ω^{β}, let $n(\omega^{\beta})$ denote the number of pairwise non-isomorphic combinatorial simple ω^{β}-regular semigroups. Then $\alpha < \beta$ implies $n(\omega^{\alpha}) \leqslant n(\omega^{\beta})$.*

Proof. Let S be a combinatorial simple ω^{α}-regular semigroup, where $\alpha < \beta$. We may suppose that S is a full inverse subsemigroup of $T_{\omega^{\alpha}}$. The mapping $T_{\omega^{\alpha}} \to T_{\omega^{\beta}}$, $\binom{\xi}{\eta} \to \binom{\xi}{\eta}$ is an embedding of $T_{\omega^{\alpha}}$ into $T_{\omega^{\beta}}$. Hence, we may suppose that S is a subsemigroup of $T_{\omega^{\beta}}$, where S consists of elements $\binom{\xi}{\eta}$, with $\xi, \eta < \omega^{\alpha}$. Let S' be the inverse subsemigroup of $T_{\omega^{\beta}}$ generated by the elements of S and by the elements $\binom{0}{\omega^{\nu}}$, where $\alpha \leqslant \nu < \beta$. Clearly S' is a combinatorial simple ω^{β}-regular semigroup.

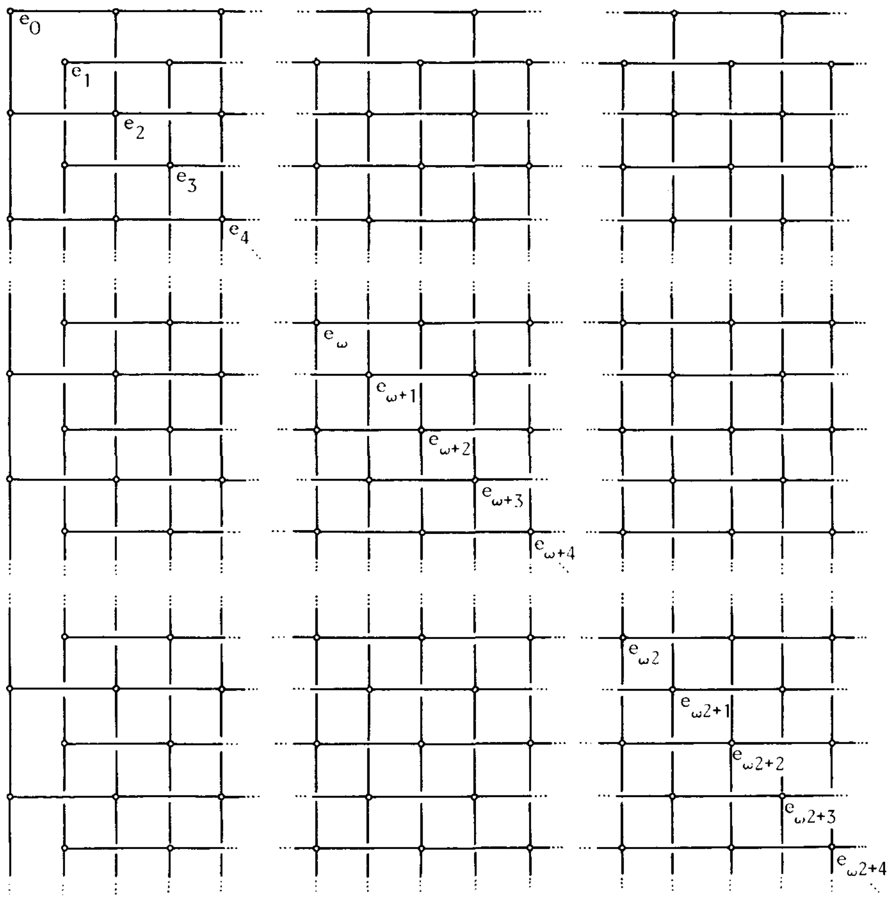

Figure 1

If S_1 and S_2 are two non-isomorphic combinatorial simple ω^α-regular semigroups, then S_1' and S_2' are non-isomorphic combinatorial simple ω^β-regular semigroups. In other words, if we start off with a set of $n(\omega^\alpha)$ pairwise non-isomorphic combinatorial simple ω^α-regular semigroups, we obtain a set of pairwise non-isomorphic combinatorial simple ω^β-regular semigroups. Thus $n(\omega^\alpha) \leqslant n(\omega^\beta)$.

For any ordinal α, ω_α will denote an initial ordinal. In the following we assume the generalized continuum hypothesis.

THEOREM 11. *Let ω^β be a prime ordinal, and let $n(\omega^\beta)$ denote the number of pairwise non-isomorphic combinatorial simple ω^β-regular semigroups. Then*

$$n(\omega) = n(\omega^2) = \aleph_0,$$
$$n(\omega^\beta) = \aleph_1 \quad \text{if } \omega^3 \leqslant \omega^\beta < \omega_1,$$
$$n(\omega^\beta) = \aleph_{\alpha+1} \quad \text{if } \omega_\alpha \leqslant \omega^\beta < \omega_{\alpha+1}, \alpha \geqslant 1.$$

PROOF. The result $n(\omega) = \aleph_0$ follows easily from the results by Kočin [4] and Munn [8] (see also Petrich [10]). In fact one shows that the number of pairwise non-isomorphic combinatorial ω-semigroups is $\aleph_0$. Therefore also, if $\omega \leqslant \delta < \omega^2$, then there are only $\aleph_0$ pairwise non-isomorphic combinatorial δ-regular semigroups. From Theorem 8 one now deduces $n(\omega^2) = \aleph_0$.

Every combinatorial ω^β-regular semigroup can be embedded as a full inverse subsemigroup in T_{ω^β}. If $\omega_\alpha \leqslant \omega^\beta < \omega_{\alpha+1}$, then $|T_{\omega^\beta}| = \aleph_\alpha$, thus also

$$n(\omega^\beta) \leqslant \aleph_{\alpha+1} \quad \text{if } \omega_\alpha \leqslant \omega^\beta < \omega_{\alpha+1}. \tag{16}$$

Let us consider a mapping $f\colon N \to \{0, 1\}$. Let us consider the system

$$\left(\omega;\ \{S_\xi \mid \xi < \omega\};\ \{\phi_{\xi,\eta} \mid \xi \leqslant \eta < \omega\}\right) \tag{17}$$

where

(i) $S_\xi \cap S_\eta = \varnothing$ whenever $\xi \neq \eta$,

(ii) S_ξ is a copy of the bicyclic semigroup whenever $f(\xi) = 1$, and S_ξ is a chain of order type ω^* whenever $f(\xi) = 0$,

(iii) $\phi_{\xi,\eta}$ maps S_ξ onto the identity of S_η if $\xi < \eta < \omega$,

(iv) $\phi_{\xi,\xi}$ is the identity transformation on S_ξ for $\xi < \omega$.

The sum of the system (17) is denoted by S_f. If $g\colon N \to \{0, 1\}$ is any other mapping, with $f \neq g$, then S_f is not isomorphic to S_g. In other words, we are able to construct $2^{\aleph_0} = \aleph_1$ pairwise non-isomorphic combinatorial ω^2-regular semigroups. Using the method of constructing Bruck-Reilly extensions, we are able to construct $\aleph_1$ pairwise non-isomorphic combinatorial simple ω^3-regular semigroups. Thus, by Lemma 10 $n(\omega^\beta) \geqslant \aleph_1$ whenever $\omega^3 \leqslant \omega^\beta < \omega_1$. Using (16), we have $n(\omega^\beta) = \aleph_1$ whenever $\omega^3 \leqslant \omega^\beta < \omega_1$.

Let us now consider an initial ordinal ω_α $(= \omega^{\omega_\alpha})$, $\alpha \geqslant 1$. Let A_1 $[A_2]$ stand for the set of ordinals $\xi < \omega_\alpha$, which are of the form $\xi = \zeta + n$, with n odd [even], and where the least primitive remainder of ζ does not equal 1. Then $\omega_\alpha = A_1 \cup A_2$, and A_1 and A_2 both constitute well-ordered chains of order type ω_α. Let f: $A_1 \to \{0, 1\}$ by any mapping, and let S_f be the full inverse subsemigroup of T_{ω_α}

which is generated by the elements

$$\begin{pmatrix} \xi \\ \xi \end{pmatrix}, \qquad \xi < \omega_\alpha,$$

$$\begin{pmatrix} 0 \\ \omega^\xi \end{pmatrix}, \quad \text{for all } \xi \in A_2,$$

$$\begin{pmatrix} 0 \\ \omega^\xi \end{pmatrix}, \quad \text{for all } \xi \in A_1 \text{ for which } f(\xi) = 1.$$

Then S_f is a combinatorial simple ω_α-regular semigroup. Further, if $g\colon A_1 \to \{0, 1\}$ is any other mapping, with $g \neq f$, then S_f cannot be isomorphic to S_g. Thus we constructed $\aleph_{\alpha+1}$ pairwise non-isomorphic combinatorial simple ω_α-regular semigroups. Using Lemma 10, we see that for all $\omega_\alpha \leqslant \omega^\beta < \omega_{\alpha+1}$, we have $n(\omega^\beta) \geqslant \aleph_{\alpha+1}$. Yet, by (16) we also have $n(\omega^\beta) \leqslant \aleph_{\alpha+1}$ and thus the equality $n(\omega^\beta) = \aleph_{\alpha+1}$ prevails.

THEOREM 12. *Let S be a combinatorial simple ω^α-regular semigroup. The greatest group homomorphic image of S is trivial if and only if α is a limit ordinal. Otherwise the greatest group homomorphic image of S is the infinite cyclic group.*

PROOF. We may assume that S is a full inverse subsemigroup of T_{ω^α}. Assume that α is a limit ordinal and let $\binom{\xi}{\eta} \in S$. Then $\xi, \eta < \omega^\beta < \omega^\alpha$ for some $\beta < \alpha$. So $\binom{\xi}{\eta}\binom{\omega^\beta}{\omega^\beta} = \binom{\omega^\beta}{\omega^\beta}\binom{\xi}{\eta} = \binom{\omega^\beta}{\omega^\beta}$, and we see that $S \times S$ is the least group congruence on S.

If α is not a limit ordinal, then $\alpha = \beta + 1$ for some β. On S we may now introduce a relation ρ by

$$(18)\qquad \begin{pmatrix} \xi \\ \eta \end{pmatrix} \rho \begin{pmatrix} \xi' \\ \eta' \end{pmatrix} \quad \text{if and only if } \omega^\beta m \leqslant \xi, \xi' < \omega^\beta(m+1) \text{ and}$$
$$\omega^\beta n \leqslant \eta, \eta' < \omega^\beta(n+1) \text{ for some } m, n \in N.$$

One may verify that ρ is a congruence relation, and that S/ρ is a combinatorial simple ω-regular semigroup. If $\binom{\xi}{\eta}\rho\binom{\xi'}{\eta'}$ as in (18), and if $k = \max(m, n) + 1$, then

$$\begin{pmatrix} \omega^\beta k \\ \omega^\beta k \end{pmatrix}\begin{pmatrix} \xi \\ \eta \end{pmatrix}\begin{pmatrix} \omega^\beta k \\ \omega^\beta k \end{pmatrix} = \begin{pmatrix} \omega^\beta k \\ \omega^\beta k \end{pmatrix}\begin{pmatrix} \xi' \\ \eta' \end{pmatrix}\begin{pmatrix} \omega^\beta k \\ \omega^\beta k \end{pmatrix},$$

which implies that $\binom{\xi}{\eta}$ and $\binom{\xi'}{\eta'}$ are related in the least group congruence on S. Thus, the greatest group homomorphic image of S coincides with the greatest homomorphic image on S/ρ, that is, it is the infinite cyclic group.

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