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ABSTRACT

Methods are developed and defined for obtaining numerical solutions to three-dimensional, free surface, inviscid, incompressible fluid flows and three-dimensional free surface Darcian flow in porous media. Since those boundaries consisting of free surface are unknown *a priori*, a solution to the space boundary value problem resulting from a formulation in the physical space is very difficult, if not impossible, to obtain. Consequently, the methods described herein are based on a formulation in a space defined by a potential function and two mutually orthogonal stream surfaces whose intersections define the streamlines of the flow. In this space the positions of free surfaces are known. The formulation considers the magnitudes of the cartesian coordinates x, y, and z as the dependent variables.

The applicability of the methods are demonstrated by implementing them in a computer program and by obtaining solutions to four problems with slightly different geometries of three-dimensional Darcian seepage flow of water through a dam with a drain over only a portion of the toe. Isometric drawing of the space flownets display the results from these solutions. Also a number of regular flownets are given which were constructed by projecting the points of intersection of the two stream surfaces and/or equipotential surfaces onto horizontal or vertical planes.

$$\frac{\partial(G,H)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} \\ \frac{\partial H}{\partial x} & \frac{\partial H}{\partial y} \end{vmatrix} \qquad \text{and etc.}$$

The equations given by 8 can be obtained by differentiating each of the second set of the above functional relationships with respect to x, y, and z respectively, and subsequently solving each group of three equations with respect to the inverse derivatives. Thus the first line of equations is obtained from the solution of the following three equations obtained by differentiating $x=f(\Phi,\psi,\psi^*)$ with respect to x, y, and z, respectively.

$$1 = \frac{\partial x}{\partial \Phi}\frac{\partial F}{\partial x} + \frac{\partial x}{\partial \psi}\frac{\partial G}{\partial x} + \frac{\partial x}{\partial \psi}\frac{\partial H}{\partial x}$$

$$0 = \frac{\partial x}{\partial \Phi}\frac{\partial F}{\partial y} + \frac{\partial x}{\partial \psi}\frac{\partial G}{\partial y} + \frac{\partial x}{\partial \psi^*}\frac{\partial H}{\partial y}$$

$$0 = \frac{\partial x}{\partial \Phi}\frac{\partial F}{\partial z} + \frac{\partial x}{\partial \psi}\frac{\partial G}{\partial y} + \frac{\partial x}{\partial \psi^*}\frac{\partial H}{\partial z} \quad . \; . \; . \; . \; (9)$$

Likewise

$$j\frac{\partial \Phi}{\partial x} = \frac{\partial(y,z)}{\partial(\psi,\psi^*)},\; j\frac{\partial \Phi}{\partial y} = -\frac{\partial(x,z)}{\partial(\psi,\psi^*)},\; j\frac{\partial \Phi}{\partial z} = \frac{\partial(x,y)}{\partial(\psi,\psi^*)}$$

$$j\frac{\partial \psi}{\partial x} = -\frac{\partial(y,z)}{\partial(\Phi,\psi^*)},\; j\frac{\partial \psi}{\partial y} = \frac{\partial(x,z)}{\partial(\Phi,\psi^*)},\; j\frac{\partial \psi}{\partial z} = -\frac{\partial(x,y)}{\partial(\Phi,\psi^*)}$$

$$j\frac{\partial \psi^*}{\partial x} = \frac{\partial(y,z)}{\partial(\Phi,\psi)},\; j\frac{\partial \psi^*}{\partial y} = -\frac{\partial(x,z)}{\partial(\Phi,\psi)},\; j\frac{\partial \psi^*}{\partial z} = \frac{\partial(x,y)}{\partial(\Phi,\psi)}$$

$$. \; . \; . \; . \; . \; . \; . \; . \; . \; (10)$$

in which $j=1/J$ is the inverse Jacobian determinant

$$j = \begin{vmatrix} \frac{\partial f}{\partial \Phi} & \frac{\partial f}{\partial \psi} & \frac{\partial f}{\partial \psi^*} \\ \frac{\partial g}{\partial \Phi} & \frac{\partial g}{\partial \psi} & \frac{\partial g}{\partial \psi^*} \\ \frac{\partial h}{\partial \Phi} & \frac{\partial h}{\partial \psi} & \frac{\partial h}{\partial \psi^*} \end{vmatrix}$$

If the partial derivatives of F, G, and H with respect to x, y, and z are considered unknown, then Eq. 8 represents a system of 9 equations in 9 unknowns. Solving for these unknowns and substituting the results into Eqs. 1, 2, and 3 gives the basic inverse equation. It is easier, however, to substitute from both Eqs. 8 and 10 into Eqs. 1, 2, and 3. This latter procedure also yields the following inverse equations:

$$\frac{\partial x}{\partial \Phi} = \frac{\partial y}{\partial \psi}\frac{\partial z}{\partial \psi^*} - \frac{\partial y}{\partial \psi^*}\frac{\partial z}{\partial \psi} \quad . \; . \; . \; . \; . \; . \; (11)$$

$$\frac{\partial y}{\partial \Phi} = \frac{\partial x}{\partial \psi^*}\frac{\partial z}{\partial \psi} - \frac{\partial x}{\partial \psi}\frac{\partial z}{\partial \psi^*} \quad . \; . \; . \; . \; . \; (12)$$

$$\frac{\partial z}{\partial \Phi} = \frac{\partial x}{\partial \psi}\frac{\partial y}{\partial \psi^*} - \frac{\partial x}{\partial \psi^*}\frac{\partial y}{\partial \psi} \quad . \; . \; . \; . \; . \; . \; (13)$$

These last three basic equations define the inverse functions $x(\Phi,\psi,\psi^*)$, $y(\Phi,\psi,\psi^*)$, and $z(\Phi,\psi,\psi^*)$ just as Eqs. 1, 2, and 3 formed the basic equations defining the potential function and the two stream functions in the physical space. Consequently, Eqs. 11, 12, and 13, when associated with appropriate boundary conditions for a particular problem, constitute an inverse mathematical formulation of potential fluid or porous media flow. The major effort of this investigation has been devoted to the development of methods for solving Eqs. 11, 12, and 13 with appropriate associated boundary conditions.

Method of Solution

Since Eqs. 11, 12, and 13 (the basic equations for which a solution is sought) are nonlinear, and each of the equations contain all three of the unknown functions $x(\Phi,\psi,\psi^*)$, $y(\Phi,\psi,\psi^*)$, and $z(\Phi,\psi,\psi^*)$, numerical methods offer the best present approach to obtaining a solution. No effort in this investigation has been devoted to seeking transformations, etc., which might make a closed form solution possible for certain problems with idealized boundary conditions. Rather, the effort has been to examine possible numerical approaches which are workable and feasible in solving problems of a general nature.

A number of variations of commonly used finite differences methods have been implemented in attempting to obtain such a solution. The method described as an integral part of this report does provide such a solution capability provided its implementation is adapted to certain features of the particular problem being solved. Two alternate methods of solution are also being studied further. With the exception of these three approaches the attempts at solution by common methods meet with limited success. One of the alternative methods utilizes finite difference operators based on all possible combinations of first order forward and backward differences of the basic Eqs. 11, 12, and 13 and weights the results in proportion to the distance the grid point is from the various boundaries of the problem. This alternative is being studied by a Ph.D. candidate who is attempting to obtain the solution to the problem of potential flow at the free overfall end of an open channel. In this alternative the successive overrelaxation method has been modified by using Newton's method to simultaneously obtain solutions to the various finite difference operators obtained from Eqs. 11, 12, and 13 as well as from the boundary conditions at each mesh point. The details of this method, as well as the results of solutions, will be forthcoming.

The second alternative which combines direct methods for solving finite difference equations with iterative

Table 1. Second order equations obtained from Eqs. 11, 12, and 13 under the assumption that variables are known from adjacent planes.

Eq. No.	Derived from Equations	Plane of Equations	Second Order Partial Differential Equation
18	11 & 12	$\Phi\psi$	$\frac{\partial^2 x}{\partial\Phi^2} + e^2\frac{\partial^2 x}{\partial\psi^2} + e\frac{\partial e}{\partial\psi}\frac{\partial x}{\partial\psi} - \frac{1}{e}\frac{\partial e}{\partial\Phi}\left[\overset{e\frac{\partial y}{\partial\psi}}{\frac{\partial x}{\partial\Phi}} + fg\right] - e\frac{\partial(gh)}{\partial\psi} + \frac{\partial(fg)}{\partial\Phi} = 0$
19	11 & 12	$\Phi\psi$	$\frac{\partial^2 y}{\partial\Phi^2} + e^2\frac{\partial^2 y}{\partial\psi^2} + e\frac{\partial e}{\partial\psi}\frac{\partial y}{\partial\psi} - \frac{1}{e}\frac{\partial e}{\partial\Phi}\left[\overset{-e\frac{\partial z}{\partial\psi}}{\frac{\partial y}{\partial\Phi}} - gh\right] - e\frac{\partial(fg)}{\partial\psi} - \frac{\partial(gh)}{\partial\Phi} = 0$
20	11 & 13	$\Phi\psi$	$\frac{\partial^2 x}{\partial\Phi^2} + f^2\frac{\partial^2 x}{\partial\psi^2} + f\frac{\partial f}{\partial\psi}\frac{\partial x}{\partial\psi} - \frac{1}{f}\frac{\partial f}{\partial\Phi}\left[\overset{-f\frac{\partial z}{\partial\psi}}{\frac{\partial f}{\partial\Phi}} - de\right] - f\frac{\partial(hd)}{\partial\psi} - \frac{\partial(de)}{\partial\Phi} = 0$
21	11 & 13	$\Phi\psi$	$\frac{\partial^2 z}{\partial\Phi^2} + f^2\frac{\partial^2 z}{\partial\psi^2} + f\frac{\partial f}{\partial\psi}\frac{\partial z}{\partial\psi} - \frac{1}{f}\frac{\partial f}{\partial\Phi}\left[\overset{f\frac{\partial x}{\partial\psi}}{\frac{\partial z}{\partial\Phi}} + hd\right] - f\frac{\partial(de)}{\partial\psi} + \frac{\partial(hd)}{\partial\Phi} = 0$
22	12 & 13	$\Phi\psi$	$\frac{\partial^2 y}{\partial\Phi^2} + h^2\frac{\partial^2 y}{\partial\psi^2} + h\frac{\partial h}{\partial\psi}\frac{\partial y}{\partial\psi} - \frac{1}{h}\frac{\partial h}{\partial\Phi}\left[\overset{\frac{1}{h}\frac{\partial z}{\partial\psi}}{\frac{\partial y}{\partial\Phi}} + ie\right] - h\frac{\partial(if)}{\partial\psi} + \frac{\partial(ie)}{\partial\Phi} = 0$
23	12 & 13	$\Phi\psi$	$\frac{\partial^2 z}{\partial\Phi^2} + h^2\frac{\partial^2 z}{\partial\psi^2} + h\frac{\partial h}{\partial\psi}\frac{\partial z}{\partial\psi} - \frac{1}{h}\frac{\partial h}{\partial\Phi}\left[\overset{-h\frac{\partial y}{\partial\psi}}{\frac{\partial z}{\partial\Phi}} - if\right] - h\frac{\partial(ie)}{\partial\psi} - \frac{\partial(if)}{\partial\Phi} = 0$
24	11 & 12	$\Phi\psi^*$	$\frac{\partial^2 x}{\partial\Phi^2} + g^2\frac{\partial^2 x}{\partial\psi^{*2}} + g\frac{\partial g}{\partial\psi^*}\frac{\partial x}{\partial\psi^*} - \frac{1}{g}\frac{\partial g}{\partial\Phi}\left[\overset{-g\frac{\partial y}{\partial\psi^*}}{\frac{\partial x}{\partial\Phi}} - de\right] - g\frac{\partial(ie)}{\partial\psi^*} - \frac{\partial(de)}{\partial\Phi} = 0$
25	11 & 12	$\Phi\psi^*$	$\frac{\partial^2 y}{\partial\Phi^2} + g^2\frac{\partial^2 y}{\partial\psi^{*2}} + g\frac{\partial g}{\partial\psi^*}\frac{\partial y}{\partial\psi^*} - \frac{1}{g}\frac{\partial g}{\partial\Phi}\left[\overset{g\frac{\partial x}{\partial\psi^*}}{\frac{\partial y}{\partial\Phi}} + ie\right] - g\frac{\partial(de)}{\partial\psi^*} + \frac{\partial(ie)}{\partial\Phi} = 0$
26	11 & 13	$\Phi\psi^*$	$\frac{\partial^2 x}{\partial\Phi^2} + d^2\frac{\partial^2 x}{\partial\psi^{*2}} + d\frac{\partial d}{\partial\psi^*}\frac{\partial x}{\partial\psi^*} - \frac{1}{d}\frac{\partial d}{\partial\Phi}\left[\overset{d\frac{\partial z}{\partial\psi^*}}{\frac{\partial x}{\partial\Phi}} + fg\right] - d\frac{\partial(if)}{\partial\psi^*} + \frac{\partial(fg)}{\partial\Phi} = 0$
27	11 & 13	$\Phi\psi^*$	$\frac{\partial^2 z}{\partial\Phi^2} + d^2\frac{\partial^2 z}{\partial\psi^{*2}} + d\frac{\partial d}{\partial\psi^*}\frac{\partial z}{\partial\psi^*} - \frac{1}{d}\frac{\partial d}{\partial\Phi}\left[\overset{-d\frac{\partial x}{\partial\psi^*}}{\frac{\partial z}{\partial\Phi}} - if\right] - d\frac{\partial(fg)}{\partial\psi^*} - \frac{\partial(fi)}{\partial\Phi} = 0$
28	12 & 13	$\Phi\psi^*$	$\frac{\partial^2 y}{\partial\Phi^2} + i^2\frac{\partial^2 y}{\partial\psi^{*2}} + i\frac{\partial i}{\partial\psi^*}\frac{\partial y}{\partial\psi^*} - \frac{1}{i}\frac{\partial i}{\partial\Phi}\left[\overset{-i\frac{\partial z}{\partial\psi^*}}{\frac{\partial y}{\partial\Phi}} - gh\right] - i\frac{\partial(hd)}{\partial\psi^*} - \frac{\partial(gh)}{\partial\Phi} = 0$
29	12 & 13	$\Phi\psi^*$	$\frac{\partial^2 z}{\partial\Phi^2} + i^2\frac{\partial^2 z}{\partial\psi^{*2}} + i\frac{\partial i}{\partial\psi^*}\frac{\partial z}{\partial\psi^*} - \frac{1}{i}\frac{\partial i}{\partial\Phi}\left[\overset{i\frac{\partial y}{\partial\psi^*}}{\frac{\partial z}{\partial\Phi}} + hd\right] + i\frac{\partial(gh)}{\partial\psi^*} - \frac{\partial(dh)}{\partial\Phi} = 0$
30	11 & 12	$\psi\psi^*$	$fh\frac{\partial^2 z}{\partial\psi^2} - id\frac{\partial^2 z}{\partial\psi^{*2}} + \left(f\frac{\partial h}{\partial\psi} + i\frac{\partial f}{\partial\psi^*}\right)\frac{\partial z}{\partial\psi} - \left(i\frac{\partial d}{\partial\psi^*} + f\frac{\partial i}{\partial\psi}\right)\frac{\partial z}{\partial\psi^*} + i\frac{\partial a}{\partial\psi^*} - f\frac{\partial b}{\partial\psi} = 0$
31	11 & 13	$\psi\psi^*$	$he\frac{\partial^2 y}{\partial\psi^2} - gi\frac{\partial^2 y}{\partial\psi^{*2}} + \left(h\frac{\partial e}{\partial\psi} + g\frac{\partial h}{\partial\psi^*}\right)\frac{\partial y}{\partial\psi} - \left(g\frac{\partial i}{\partial\psi^*} + h\frac{\partial g}{\partial\psi}\right)\frac{\partial y}{\partial\psi^*} + g\frac{\partial c}{\partial\psi^*} - h\frac{\partial a}{\partial\psi} = 0$
32	12 & 13	$\psi\psi^*$	$fe\frac{\partial^2 x}{\partial\psi^2} - gd\frac{\partial^2 x}{\partial\psi^{*2}} - \left(f\frac{\partial g}{\partial\psi} + g\frac{\partial d}{\partial\psi^*}\right)\frac{\partial x}{\partial\psi^*} + \left(f\frac{\partial e}{\partial\psi} + g\frac{\partial f}{\partial\psi^*}\right)\frac{\partial x}{\partial\psi} + f\frac{\partial b}{\partial\psi} - g\frac{\partial c}{\partial\psi^*} = 0$

(Three other equations can be obtained which apply in the $\psi\psi^*$ plane.)

terms is very small in comparison to the other. The second criteria helps insure that the equation has some resemblance to the Laplace equation upon which many numerical as well as other solutions have been based. If this criteria is poorly satisfied, a solution in each individual plane can be obtained with few iterations by solving the system along the grid line in the direction of the dependent variable with the larger coefficient. But since the problem is of the elliptic type, this would mean that a high dependency exists between the values on this plane and the values on adjacent planes. Consequently, the reduction in arithmetic calculation in obtaining tentative solutions in separate planes would be more than offset by additional cycles of calculations resulting because of the slower convergence in the cycle from plane to plane. Furthermore, the process of solution may be less likely to converge than if this criteria were maintained. The third criteria is closely associated with the above two reasons.

To illustrate how the selection of the best equations from those listed in Table 1 might be arrived at, consider a three-dimensional potential flow with the major component of velocity throughout the majority of the region as being in the x-direction. The velocity components in the y-direction in general are also greater than those in the z-direction. Furthermore assume, because of the nature of the particular problem, that a boundary of the problem along which ψ is selected to be held constant is normal to the y-direction, and another boundary at right angles to this boundary is selected as a ψ^*=constant surface. In this problem the major changes in the flow field exist within $\Phi\psi$ planes with lesser changes occurring between adjacent $\Phi\psi$ planes. Consequently, derivatives of x and y with respect to ψ^* will generally be of smaller magnitude than those with respect to Φ and ψ. A logical choice, therefore, would be to select equations for solving for x and y which apply on $\Phi\psi$ planes and an equation for z which applies on $\Phi\psi^*$ planes. In other words the variables x, y, and z would each be obtained by use of the equation which applies in the plane where the greatest action of that variable is. Therefore either Eq. 18 or Eq. 20 should be used in solving for x. Proceeding with this selection it is clear that Eq. 18 is better suited for a solution than Eq. 20 because the magnitude of $e = \partial z/\partial\psi^*$, the coefficient for the second derivative in Eq. 18 is close to unity whereas the value of $f = \partial y/\partial\psi^*$ in Eq. 20 is much smaller. Consequently Eq. 18 would constitute a good choice for solving for x. Likewise Eq. 19 constitutes a good selection of the equation for use in solving for y. Since the equation for z is to apply in $\Phi\psi^*$ planes, the choice is between Eqs. 27 and 29. For this problem the magnitude of d is nearer unity than the magnitude of i, and Eq. 27 should be used.

For other problems the equations to be used might be different, but their selections would be based on a physical understanding of the flow situation and similar criteria. Conceivably, in certain types of problems in which the nature of the flow changes drastically in different portions of the flow field, it may actually be advantageous to use different equations in different regions of the $\Phi\psi\psi^*$ space.

In Eqs. 30, 31, and 32 opposite signs accompany the coefficients of the two second order derivatives. Since the differential equations must be of the elliptic type, it follows that the coefficients must have opposite signs associated with them, i.e. if the product of fh in Eq. 30 is positive then the product id must be negative or conversely, etc., for Eqs. 31 and 32. Should the sign of the coefficients in these equations be the same, the equations would be hyperbolic, in contradiction to physical facts of ideal incompressible fluid flow. Therefore in initializing the field values of a problem which is to use Eqs. 30, 31, or 32 in its solution, it is necessary to insure that value of the products which constitute these coefficients are of opposite sign. To do otherwise would probably cause divergence or other difficulties in attempting a numerical solution. Since in the iterative solution process it is conceivable that the coefficients to these equations may take on the same sign, it may also be necessary to add some additional constraint to prevent this from occurring. In the problem solved in this report, any such difficulties have been avoided by solving only equations that apply in the $\Phi\psi$ and $\Phi\psi^*$ planes.

Finite difference operators

In finite difference solutions to partial differential equations, the continuous variables are replaced by discrete values at the finite difference grid points placed throughout the region. For the applications which have been considered in this report, a system of grid points has been used which forms cubes throughout the region of flow. Thus in differencing the partial differential equations, $\Delta\psi^* = \Delta\psi = \Delta\Phi$. Furthermore each of the increments $\Delta\Phi$, $\Delta\psi$, and $\Delta\psi^*$ have been assumed to have a unit value. This is possible in the inverse formulation and solution methods used because the region of the problem is defined by specifying the number of $\Delta\Phi$, $\Delta\psi$, and $\Delta\psi^*$ increments in each of the inverse coordinate directions, and thus several dimensions of the physical problem become part of the solution.

The finite difference operators for the interior grid points have all been obtained by replacing the derivatives by second order central differences. Table 2 gives these finite difference operators for the equations in Table 1 with the terms contained within the square brackets. The equations which define the α's in each finite difference operator are given in the right portion of the table. Table 3 gives the equivalent operators for the equations using the replacement term above the square bracket in Table 1.

The operators may be written in a number of different forms. The form in which they are given in Tables 2 and 3 conforms to that needed to apply the line successive overrelaxation iterative (LSOR) method (see Forsythe and Wasow, 1960, or Varga, 1962) along lines

Table 2. Finite difference operators which are based on the partial differential equations in Table 1.

Eq. No.	Finite difference operator	Definition of α coefficient in operator
18	$-x_{i-1jk} + 2\left(\frac{1+\alpha_1}{1+\alpha_3}\right)x_{ijk} - \left(\frac{1-\alpha_3}{1+\alpha_3}\right)x_{i+1jk}$ $= [(\alpha_1-\alpha_2)x_{ij-1k} + (\alpha_1+\alpha_2)x_{ij+1k} + \alpha_4]/(1+\alpha_3)$	$\alpha_1 = e^2,\ \alpha_2 = \frac{e}{2}\frac{\partial e}{\partial \psi},\ \alpha_3 = \frac{1}{2e}\frac{\partial e}{\partial \Phi}$ $\alpha_4 = \frac{\partial (fg)}{\partial \Phi} - e\frac{\partial (hg)}{\partial \psi} - \frac{fg}{e}\frac{\partial e}{\partial \Phi}$
19	$-y_{i-1jk} + 2\left(\frac{1+\alpha_1}{1+\alpha_3}\right)y_{ijk} - \left(\frac{1-\alpha_3}{1+\alpha_3}\right)y_{i+1jk}$ $= [(\alpha_1-\alpha_2)y_{ij-1k} + (\alpha_1+\alpha_2)y_{ij+1k} + \alpha_4]/(1+\alpha_3)$	$\alpha_1 = e^2,\ \alpha_2 = \frac{e}{2}\frac{\partial e}{\partial \psi},\ \alpha_3 = \frac{1}{2e}\frac{\partial e}{\partial \Phi}$ $\alpha_4 = \frac{gh}{e}\frac{\partial e}{\partial \Phi} - e\frac{\partial (fg)}{\partial \psi} - \frac{\partial (hg)}{\partial \Phi}$
20	$-x_{i-1jk} + 2\left(\frac{1+\alpha_1}{1+\alpha_3}\right)x_{ijk} - \left(\frac{1-\alpha_3}{1+\alpha_3}\right)x_{i+1jk}$ $= [(\alpha_1-\alpha_2)x_{ij-1k} + (\alpha_1+\alpha_2)x_{ij+1k} + \alpha_4]/(1+\alpha_3)$	$\alpha_1 = f^2,\ \alpha_2 = \frac{f}{2}\frac{\partial f}{\partial \psi},\ \alpha_3 = \frac{1}{2f}\frac{\partial f}{\partial \Phi}$ $\alpha_4 = \frac{de}{f}\frac{\partial f}{\partial \Phi} - f\frac{\partial (hd)}{\partial \psi} - \frac{\partial (de)}{\partial \Phi}$
21	$-z_{i-1jk} + 2\left(\frac{1+\alpha_1}{1+\alpha_3}\right)z_{ijk} - \left(\frac{1-\alpha_3}{1+\alpha_3}\right)z_{i+1jk}$ $= [(\alpha_1-\alpha_2)z_{ij-1k} + (\alpha_1+\alpha_2)z_{ij+1k} + \alpha_4]/(1+\alpha_3)$	$\alpha_1 = f^2,\ \alpha_2 = \frac{f}{2}\frac{\partial f}{\partial \psi},\ \alpha_3 = \frac{1}{2f}\frac{\partial f}{\partial \Phi}$ $\alpha_4 = \frac{\partial (hd)}{\partial \Phi} - f\frac{\partial (de)}{\partial \psi} - \frac{hd}{f}\frac{\partial f}{\partial \Phi}$
22	$-y_{i-1jk} + 2\left(\frac{1+\alpha_1}{1+\alpha_3}\right)y_{ijk} - \left(\frac{1-\alpha_3}{1+\alpha_3}\right)y_{i+1jk}$ $= [(\alpha_1-\alpha_2)y_{ij-1k} + (\alpha_1+\alpha_2)y_{ij+1k} + \alpha_4]/(1+\alpha_3)$	$\alpha_1 = h^2,\ \alpha_2 = \frac{h}{2}\frac{\partial h}{\partial \psi},\ \alpha_3 = \frac{1}{2h}\frac{\partial h}{\partial \Phi}$ $\alpha_4 = \frac{\partial (ie)}{\partial \Phi} - \frac{ie}{h}\frac{\partial h}{\partial \Phi} - h\frac{\partial (if)}{\partial \psi}$
23	$-z_{i-1jk} + 2\left(\frac{1+\alpha_1}{1+\alpha_3}\right)z_{ijk} - \left(\frac{1-\alpha_3}{1+\alpha_3}\right)z_{i+1jk}$ $= [(\alpha_1-\alpha_2)z_{ij-1k} + (\alpha_1+\alpha_2)z_{ij+1k} + \alpha_4]/(1+\alpha_3)$	$\alpha_1 = h^2,\ \alpha_2 = \frac{h}{2}\frac{\partial h}{\partial \psi},\ \alpha_3 = \frac{1}{2h}\frac{\partial h}{\partial \Phi}$ $\alpha_4 = \frac{if}{h}\frac{\partial h}{\partial \Phi} - h\frac{\partial (ie)}{\partial \psi} - \frac{\partial (if)}{\partial \Phi}$
24	$-x_{i-1jk} + 2\left(\frac{1+\alpha_1}{1+\alpha_3}\right)x_{ijk} - \left(\frac{1-\alpha_3}{1+\alpha_3}\right)x_{i+1jk}$ $= [(\alpha_1-\alpha_2)x_{ijk-1} + (\alpha_1+\alpha_2)x_{ijk+1} + \alpha_4]/(1+\alpha_3)$	$\alpha_1 = g^2,\ \alpha_2 = \frac{g}{2}\frac{\partial g}{\partial \psi^*},\ \alpha_3 = \frac{1}{2g}\frac{\partial g}{\partial \Phi}$ $\alpha_4 = \frac{de}{g}\frac{\partial g}{\partial \Phi} - g\frac{\partial (ie)}{\partial \psi^*} - \frac{\partial (de)}{\partial \Phi}$

Table 2. Continued.

Eq. No.	Finite difference operator	Definition of α coefficient in operator
25	$-y_{i-1jk} + 2\left(\frac{1+\alpha_1}{1+\alpha_3}\right) y_{ijk} - \left(\frac{1-\alpha_3}{1+\alpha_3}\right) y_{i+1jk}$ $= [(\alpha_1 - \alpha_2)y_{ijk-1} + (\alpha_1 + \alpha_2)y_{ijk+1} + \alpha_4]/(1+\alpha_3)$	$\alpha_1 = g^2,\ \alpha_2 = \frac{g}{2}\frac{\partial g}{\partial \psi^*},\ \alpha_3 = \frac{1}{2g}\frac{\partial g}{\partial \Phi}$ $\alpha_4 = \frac{\partial (ie)}{\partial \Phi} - \frac{ie}{g}\frac{\partial g}{\partial \Phi} - g\frac{\partial (de)}{\partial \psi^*}$
26	$-x_{i-1jk} + 2\left(\frac{1+\alpha_1}{1+\alpha_3}\right) x_{ijk} - \left(\frac{1-\alpha_3}{1+\alpha_3}\right) x_{i+1jk}$ $= [(\alpha_1 - \alpha_2)x_{ijk-1} + (\alpha_1 + \alpha_2)x_{ijk+1} + \alpha_4]/(1+\alpha_3)$	$\alpha_1 = d^2,\ \alpha_2 = \frac{d}{2}\frac{\partial d}{\partial \psi^*},\ \alpha_3 = \frac{1}{2d}\frac{\partial d}{\partial \Phi}$ $\alpha_4 = \frac{\partial (fg)}{\partial \Phi} - \frac{fg}{d}\frac{\partial d}{\partial \Phi} - d\frac{\partial (if)}{\partial \psi^*}$
27	$-z_{i-1jk} + 2\left(\frac{1+\alpha_1}{1+\alpha_3}\right) z_{ijk} - \left(\frac{1-\alpha_3}{1+\alpha_3}\right) z_{i+1jk}$ $= [(\alpha_1 - \alpha_2)z_{ijk-1} + (\alpha_1 + \alpha_2)z_{ijk+1} + \alpha_4]/(1+\alpha_3)$	$\alpha_1 = d^2,\ \alpha_2 = \frac{d}{2}\frac{\partial d}{\partial \psi^*},\ \alpha_3 = \frac{1}{2d}\frac{\partial d}{\partial \Phi}$ $\alpha_4 = \frac{if}{d}\frac{\partial d}{\partial \Phi} - d\frac{\partial (fg)}{\partial \psi^*} - \frac{\partial (if)}{\partial \Phi}$
28	$-y_{i-1jk} + 2\left(\frac{1+\alpha_3}{1+\alpha_3}\right) y_{ijk} - \left(\frac{1-\alpha_3}{1+\alpha_3}\right) y_{i+1jk}$ $= [(\alpha_1 - \alpha_2)y_{ijk-1} + (\alpha_1 + \alpha_2)y_{ijk+1} + \alpha_4]/(1+\alpha_3)$	$\alpha_1 = i^2,\ \alpha_2 = \frac{i}{2}\frac{\partial i}{\partial \psi^*},\ \alpha_3 = \frac{1}{2i}\frac{\partial i}{\partial \Phi}$ $\alpha_4 = \frac{gh}{i}\frac{\partial i}{\partial \Phi} - i\frac{\partial (dh)}{\partial \psi^*} - \frac{\partial (gh)}{\partial \Phi}$
29	$-z_{i-1jk} + 2\left(\frac{1+\alpha_1}{1+\alpha_3}\right) z_{ijk} - \left(\frac{1-\alpha_3}{1+\alpha_3}\right) z_{i+1jk}$ $= [(\alpha_1 - \alpha_2)z_{ijk-1} + (\alpha_1 + \alpha_2)z_{ijk+1} + \alpha_4]/(1+\alpha_3)$	$\alpha_1 = i^2,\ \alpha_2 = \frac{i}{2}\frac{\partial i}{\partial \psi^*},\ \alpha_3 = \frac{1}{2i}\frac{\partial i}{\partial \Phi}$ $\alpha_4 = \frac{\partial (dh)}{\partial \Phi} - i\frac{\partial (gh)}{\partial \psi^*} - \frac{dh}{i}\frac{\partial i}{\partial \Phi}$
30	$-z_{ij-1k} + 2\left(\frac{\alpha_2-\alpha_1}{\alpha_3-\alpha_1}\right) z_{ijk} + \left(\frac{\alpha_1+\alpha_3}{\alpha_3-\alpha_1}\right) z_{ij+1k}$ $= [(\alpha_2 - \alpha_4)z_{ijk-1} + (\alpha_2 + \alpha_4)z_{ijk+1} - \alpha_5]/(\alpha_3-\alpha_1)$	$\alpha_1 = fg,\ \alpha_2 = id,\ \alpha_3 = \frac{1}{2}\left(f\frac{\partial h}{\partial \psi} + i\frac{\partial f}{\partial \psi^*}\right)$ $\alpha_4 = \frac{1}{2}\left(i\frac{\partial d}{\partial \psi^*} + f\frac{\partial i}{\partial \psi}\right),\ \alpha_5 = i\frac{\partial a}{\partial \psi^*} - f\frac{\partial b}{\partial \psi}$
31	$-y_{ij-1k} + 2\left(\frac{\alpha_2-\alpha_1}{\alpha_3-\alpha_1}\right) y_{ijk} + \left(\frac{\alpha_1+\alpha_3}{\alpha_3-\alpha_1}\right) y_{ij+1k}$ $= [(\alpha_2 - \alpha_4)y_{ijk-1} + (\alpha_2 + \alpha_4)y_{ijk+1} - \alpha_5]/(\alpha_3-\alpha_1)$	$\alpha_1 = he,\ \alpha_2 = gi,\ \alpha_3 = \frac{1}{2}\left(h\frac{\partial c}{\partial \psi} + g\frac{\partial h}{\partial \psi^*}\right)$ $\alpha_4 = \frac{1}{2}\left(g\frac{\partial i}{\partial \psi^*} + h\frac{\partial g}{\partial \psi}\right),\ \alpha_5 = g\frac{\partial c}{\partial \psi^*} - h\frac{\partial a}{\partial \psi}$
32	$-x_{ij-1k} + 2\left(\frac{\alpha_2-\alpha_1}{\alpha_3-\alpha_1}\right) x_{ijk} + \left(\frac{\alpha_1+\alpha_4}{\alpha_3-\alpha_1}\right) x_{ij+1k}$ $= [(\alpha_2 = \alpha_4)x_{ijk-1} + (\alpha_2 + \alpha_4)x_{ijk+1} - \alpha_5]/(\alpha_4-\alpha_1)$	$\alpha_1 = fe,\ \alpha_2 = gd,\ \alpha_3 = \frac{1}{2}\left(f\frac{\partial e}{\partial \psi} + g\frac{\partial f}{\partial \psi^*}\right)$ $\alpha_4 = \frac{1}{2}\left(f\frac{\partial c}{\partial \psi} + g\frac{\partial d}{\partial \psi^*}\right),\ \alpha_5 = f\frac{\partial b}{\partial \psi} - g\frac{\partial c}{\partial \psi^*}$

Table 3. Finite difference operators which are based on the alternate partial differential equation in Table 1.

Eq. No.	Finite difference operator	Definition of α coefficients in operator
18	$-x_{i-1jk}+(2+2\alpha_1)x_{ijk}-x_{i+1jk}=(\alpha_1-\alpha_2)x_{ij-1k}+(\alpha_1+\alpha_2)x_{ij+1k}+\alpha_3$	$\alpha_1=e^2, \alpha_2=\frac{e}{2}\frac{\partial e}{\partial\psi}, \alpha_3=-\frac{\partial e}{\partial\Phi}\frac{\partial y}{\partial\psi}-e\frac{\partial(gh)}{\partial\psi}+\frac{\partial(fg)}{\partial\Phi}$
19	$-y_{i-1jk}+(2+2\alpha_1)y_{ijk}-y_{i+1jk}=(\alpha_1-\alpha_2)y_{ij-1k}+(\alpha_1+\alpha_2)y_{ij+1k}+\alpha_3$	$\alpha_1=e^2, \alpha_2=\frac{e}{2}\frac{\partial e}{\partial\psi}, \alpha_3=\frac{\partial e}{\partial\Phi}\frac{\partial x}{\partial\psi}-e\frac{\partial(fg)}{\partial\psi}-\frac{\partial(gh)}{\partial\Phi}$
20	$-x_{i-1jk}+(2+2\alpha_1)x_{ijk}-x_{i+1jk}=(\alpha_1-\alpha_2)x_{ij-1k}+(\alpha_1+\alpha_2)x_{ij+1k}+\alpha_3$	$\alpha_1=f^2, \alpha_2=\frac{f}{2}\frac{\partial f}{\partial\psi}, \alpha_3=\frac{\partial f}{\partial\Phi}\frac{\partial z}{\partial\psi}-f\frac{\partial(dh)}{\partial\psi}-\frac{\partial(de)}{\partial\Phi}$
21	$-z_{i-1jk}+(2+2\alpha_1)z_{ijh}-z_{i+1jh}=(\alpha_1-\alpha_2)x_{ij-1k}+(\alpha_1+\alpha_2)z_{ij+1k}+\alpha_3$	$\alpha_1=f^2, \alpha_2=\frac{f}{2}\frac{\partial f}{\partial\psi}, \alpha_3=f\frac{\partial(de)}{\partial\psi}+\frac{\partial(dh)}{\partial\Phi}-\frac{\partial f}{\partial\Phi}\frac{\partial x}{\partial\psi}$
22	$-y_{i-1jh}+(2+2\alpha_1)y_{ijh}-y_{i+1jk}=(\alpha_1-\alpha_2)y_{ij-1k}+(\alpha_1+\alpha_2)y_{ij+1h}+\alpha_3$	$\alpha_1=h^2, \alpha_2=\frac{h}{2}\frac{\partial h}{\partial\psi}, \alpha_3=\frac{\partial(ie)}{\partial\Phi}-h\frac{\partial(if)}{\partial\psi}-\frac{\partial h}{\partial\Phi}\frac{\partial z}{\partial\psi}$
23	$-z_{i-1jk}+(2+2\alpha_1)z_{ijk}-z_{i+1jk}=(\alpha_1-\alpha_2)z_{ij-1k}+(\alpha_1+\alpha_2)z_{ij+1h}+\alpha_3$	$\alpha_1=h^2, \alpha_2=\frac{h}{2}\frac{\partial h}{\partial\psi}, \alpha_3=\frac{\partial h}{\partial\Phi}\frac{\partial y}{\partial\psi}-h\frac{\partial(ie)}{\partial\psi}-\frac{\partial(if)}{\partial\Phi}$
24	$-x_{i-1jk}+(2+2\alpha_1)x_{ijk}-x_{i+1jk}=(\alpha_1-\alpha_2)x_{ijk-1}+(\alpha_1+\alpha_2)x_{ijk+1}+\alpha_3$	$\alpha_1=g^2, \alpha_2=\frac{g}{2}\frac{\partial g}{\partial\psi^*}, \alpha_3=\frac{\partial g}{\partial\Phi}\frac{\partial y}{\partial\psi^*}-g\frac{\partial(ie)}{\partial\psi^*}-\frac{\partial(de)}{\partial\Phi}$
25	$-y_{i-1jk}+(2+2\alpha_1)y_{ijk}-y_{i+1jk}=(\alpha_1-\alpha_2)y_{ijk-1}+(\alpha_1+\alpha_2)y_{ijk+1}+\alpha_3$	$\alpha_1=g^2, \alpha_2=\frac{g}{2}\frac{\partial g}{\partial\psi^*}, \alpha_3=\frac{\partial(ie)}{\partial\Phi}-\frac{\partial g}{\partial\Phi}\frac{\partial x}{\partial\psi^*}-g\frac{\partial(de)}{\partial\psi^*}$
26	$-x_{i-1jk}+(2+2\alpha_1)x_{ijk}-x_{i+1jk}=(\alpha_1-\alpha_2)x_{ijk-1}+(\alpha_1+\alpha_2)x_{ijk+1}+\alpha_3$	$\alpha_1=d^2, \alpha_2=\frac{d}{2}\frac{\partial d}{\partial\psi^*}, \alpha_3=\frac{\partial(fg)}{\partial\Phi}-\frac{\partial d}{\partial\Phi}\frac{\partial z}{\partial\psi^*}-d\frac{\partial(if)}{\partial\psi^*}$
27	$-z_{i-1jk}+(2+2\alpha_1)z_{ijk}-z_{i+1jk}=(\alpha_1-\alpha_2)z_{ijk-1}+(\alpha_1+\alpha_2)z_{ijk+1}+\alpha_3$	$\alpha_1=d^2, \alpha_2=\frac{d}{2}\frac{\partial d}{\partial\psi^*}, \alpha_3=\frac{\partial d}{\partial\Phi}\frac{\partial x}{\partial\psi^*}-d\frac{\partial(fg)}{\partial\psi^*}-\frac{\partial(if)}{\partial\Phi}$
28	$-y_{i-1jk}+(2+2\alpha_1)y_{ijk}-y_{i+1jk}=(\alpha_1-\alpha_2)y_{ijk-1}+(\alpha_1+\alpha_2)y_{ijk+1}+\alpha_3$	$\alpha_1=i^2, \alpha_2=\frac{i}{2}\frac{\partial i}{\partial\psi^*}, \alpha_3=\frac{\partial i}{\partial\Phi}\frac{\partial z}{\partial\psi^*}-i\frac{\partial(hd)}{\partial\psi^*}-\frac{\partial(gh)}{\partial\Phi}$
29	$-z_{i-1jk}+(2+2\alpha_1)z_{ijh}-z_{i+1jk}=(\alpha_1-\alpha_2)z_{ijk-1}+(\alpha_1+\alpha_2)z_{ijk+1}+\alpha_3$	$\alpha_1=i^2, \alpha_2=\frac{i}{2}\frac{\partial i}{\partial\psi^*}, \alpha_3=\frac{\partial(dh)}{\partial\Phi}-\frac{\partial i}{\partial\Phi}\frac{\partial y}{\partial\psi^*}-i\frac{\partial(gh)}{\partial\psi^*}$

defined by the incremented subscripts on the left side of the equation within the plane to which the particular equation applies. The operators can readily be modified to represent the form needed to apply them in the LSOR method in the other coordinate direction by solving for the present operator in terms of the value of the variables presently on the right side of the equation and the values of the variable at the grid point in question (ijk). The operator could be made suitable for direct application in the successive overrelaxation method (SOR) by writing the equation so that only the value of the variable at the grid point in question (ijk) is on the left side of the equation.

In order to evaluate which method would minimize the amount of computer execution time required to obtain a tentative solution in individual planes, FORTRAN subroutines were written to implement both the LSOR and the SOR methods (each with the over-relaxation factor equal to 1.4) for each operator in Tables 2 and 3, to solve a Dirichlet type boundary value problem in a plane. The comparison was close but did favor the LSOR method. Since the total execution time depends to some extent on the efficiency of the computer system in handling triple subscripted arrays, the comparison of the two methods may be different on different computer systems even though the identical programs were used in the comparison. In the LSOR method more computations are involved per iteration but fewer iterations are required for a solution than with the SOR method. Since the additional computations per iteration are primarily with nonsubscripted variables or single arrayed variables, the LSOR method requires fewer operations with the triple subscripted arrays. The comparisons referred to above were made on the UNIVAC 1108 system, under EXEC 8, at the University of Utah. Coefficients, which are composed of the α's in the operators, were in each case computed only during the first iteration and stored for use during subsequent iterations. Doing this is justified because the α's are either determined from values on adjacent planes which do not vary or they are of relatively small magnitudes, and their values will be adjusted during the next cycle of solutions anyway.

Line successive overrelaxation method

While the LSOR method is well documented in a number of references, two of which are given previously, a brief description of this method is given here for completeness.

To understand the LSOR method it should be noted that when any one of the operators in Tables 2 or 3 is applied across all interior grid points of a line, a system of linear algebraic equations results, with the following form

$$A\vec{X} = \vec{B} \quad \ldots\ldots\ldots\ldots (33)$$

in which $\vec{X}$ and $\vec{B}$ are column vectors and A is the (nxn) matrix given by

$$A = \begin{bmatrix} a_{11} & a_{12} & & & & \\ a_{21} & a_{22} & a_{23} & & & \\ & a_{32} & a_{33} & a_{34} & & \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & a_{mm-1} & a_{mm} & a_{mm+1} & \\ & & & \cdot & \cdot & \cdot \\ & & & & a_{nn-1} & a_{nn} \end{bmatrix} \quad \ldots (34)$$

and n represents the number of grid points at which the operator has been applied. If Dirichlet boundary conditions exist at both ends of the grid line, this number is two less than the number of grid points on the line including the boundary points. For non-Dirichlet boundary conditions a finite difference operator from that condition gives the first, the last, or both the first and last equations of the system, Eq. 33.

The simple tridiagonal coefficient matrix A is an important feature of the method from a computational viewpoint, since the system of equations with such a coefficient matrix can be solved by a single pass through the rows with a Gaussian elimination. The solution is subsequently obtained by back substitution. The method for accomplishing this has been referred to as the Thomas algorithm (Thomas, 1949) by some writers. This method defines the following elements, along and above the diagonal by the vectors $\vec{q}$, $\vec{r}$, and $\vec{s}$ respectively, and also defines elements of the vectors f and g by

$$f_1 = \frac{s_1}{r_1}, \quad g_1 = \frac{b_1}{r_1}$$

$$\left.\begin{aligned} f_m &= \frac{s_m}{r_m - f_{m-1}\, q_m} \\ g_m &= f_m (b_m - q_m g_{m-1})/s_m \end{aligned}\right\} 2 \le m \le n \cdot \quad (35a)$$

in which the b's are the elements of $\vec{B}$. Then the solution vector $\vec{X}$ can be obtained from

$$x_n = g_n$$

$$x_m = g_m - f_m x_{m+1}, \quad n-1 \ge m \ge 1 \quad \ldots (35b)$$

Should the elements of all off-diagonal terms in A be equal to -1, as is the case for many of the operators in Table 3, then the algorithm becomes

$$f_1 = \frac{1}{r_1}, \quad g_1 = f_1 b_1$$

$$\left.\begin{aligned} f_m &= \frac{1}{r_m - f_{m-1}} \\ g_m &= f_m (b_m + g_{m-1}) \end{aligned}\right\} 2 \le m \le n \quad \ldots (36a)$$

and

$$x_n = g_n$$

$$x_m = g_m + f_m x_{m+1}, \quad n-1 \geq m \geq 1 \quad . \quad . \quad (36b)$$

In executing the algorithm given by either Eq. 35 or Eq. 36, it is not necessary to set aside storage for a new array f. Rather, since the values of r need not be retained, the values of f may be stored in the former array positions for r.

Upon obtaining the solution vector $\vec{X}$ which represents the values of the variable of the finite difference operator across an entire grid line, they are immediately adjusted by the formula

$$x^{p+1}_{ijk} = x_i + W_1 (x_i - x^p_{ijk}) \quad . \quad . \quad . \quad . \quad . (37)$$

in which x_i, with the single subscript, represents the solution as described and x_{ijk} with the triple subscript, is the value of the variable at the grid point in question. The superscript p represents the number of the iteration, and W_1 is the overrelaxation factor with a value between zero and unity. It should be noted that Eq. 37 is not the usual form of the overrelaxation equation, which is

$$x^{p+1}_{ijk} = x^p_{ijk} + W(x_i - x^p_{ijk}) \quad . \quad . \quad . \quad . \quad . \quad . (38)$$

in which $W = W_1 + 1$.

It is easy to demonstrate that Eqs. 37 and 38 are identical. In a computer solution it is more efficient to use Eq. 37 since core positions from the triple subscripted array x_{ijk} need only be located once instead of twice as required by Eq. 38, and since x_i may be a non-subscripted variable.

The LSOR method proceeds from line to line until the value of the variables across all lines within the plane have been adjusted. Upon completing the last line the entire process is repeated as the next iteration. The iterations are continued until changes of the variables between consecutive iterations are smaller than some prescribed error criteria. An often used and easily applied criterion is to continue the iterations until the sum, over all grid points, of the absolute values (or sum of squares) of the quantity within the parentheses in Eq. 37 is less than a small specified error parameter.

Three-dimensional Seepage Through Dam with Partial Toe Drain

Formulation

The inverse formulation and method of solution described earlier have been applied to a relatively simple three-dimensional problem in order to obtain a numerical solution and verify the applicability of the methods. The problem selected for this initial application consists of the saturated seepage flow through a dam with a vertical face which rests on an impervious horizontal base with a partial opening on the base through which water can drain. Furthermore, the dam lies between two vertical side walls. An isometric sketch of this problem is shown in the upper portion of Fig. 1. This problem was selected because it represents a number of possible real situations and has relatively simple boundary conditions associated with it. Unfortunately it also contains mathematical singularities where the theoretical velocity becomes either infinite or zero.

The problem illustrated in the upper portion of Fig. 1 is also sketched in the $\Phi\psi\psi^*$ space in the lower portion of the figure. This space has been selected such that the impervious bottom defines the $\psi=0$ stream surface and the left vertical wall (when facing downstream) defines the $\psi^*=0$ stream surface. The right vertical wall then defines the final ψ^*=constant surface, i.e. $\psi_f{}^*$, and the phreatic surface (i.e. the surface between the saturated flow and zero flow regions at atmospheric pressure) becomes the final ψ=constant surface, i.e. ψ_f. The front face through which the water enters has a constant hydraulic head and consequently represents an equipotential surface, as does the drain surface through which water leaves the region of interest. Through Darcy's Law, given by

$$\vec{V} = -K\nabla h \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . (39)$$

in which $\vec{V}$ is the velocity vector, K (assumed constant) is the hydraulic conductivity with dimensions of velocity and h is the hydraulic head, the potential function is given by

$$\Phi = -Kh + \text{Constant} \quad . \quad . \quad . \quad . \quad . \quad . (40)$$

Consequently the face through which water enters is an equipotential surface which can be defined by Φ=0 and the drain surface coincides with the final Φ=constant surface, i.e. Φ_f.

With this definition of the $\Phi\psi\psi^*$ space, the following boundary conditions can be developed. (Obvious boundary conditions are also shown by an equation by that boundary in the lower portion of Fig. 1 to help identify that boundary in the $\Phi\psi\psi^*$ with the physical problem.)

A. Bottom 4, 5, 9, 10

$$y(\Phi, 0, \psi^*) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . (41)$$

$$x(\Phi, 0, \psi^*) = \int_{\psi\psi^*} \left(\frac{\partial y}{\partial \psi}\right) \left(\frac{\partial z}{\partial \psi^*}\right) d\Phi \quad . \quad . \quad . (42)$$

The finite difference operator for either Eq. 43 or 46 is

$$-z_{i-1jk}+2\left(\frac{1+\alpha_1}{1+\alpha_3}\right)z_{ijk}+\left(\frac{\alpha_3-1}{1+\alpha_3}\right)=\left(\frac{\alpha_1-\alpha_2}{1+\alpha_3}\right)z_{ijk-1}+\left(\frac{\alpha_1+\alpha_2}{1+\alpha_3}\right)z_{ijk+1} \quad \ldots\ldots\ldots\ldots (67)$$

in which

$$\alpha_1=\left(\frac{\partial y}{\partial \psi}\right)^2,\ \alpha_2=\frac{1}{2}\frac{\partial y}{\partial \psi}\frac{\partial^2 y}{\partial \psi \partial \psi^*},\ \alpha_3=\frac{1}{2}\frac{\frac{\partial^2 y}{\partial \Phi \partial \psi}}{\partial y/\partial \psi}$$

and j=1 or j=M respectively depending upon whether Eq. 67 applies to the bottom or to the phreatic surface. Prior to adjusting the values along the interior lines on the bottom and on the phreatic surface by means of the operator Eq. 67, the values of z along the Φ=0 lines of these surfaces are obtained from the equation

$$z=\int_{\Phi\psi}\frac{\frac{\partial x}{\partial \Phi}}{\frac{\partial y}{\partial \psi}}\,d\psi^* \quad \ldots\ldots (68)$$

The solution of Eq. 68 follows the approach described for solving Eq. 42.

The boundary conditions, Eqs. 48, for y on the upstream face, are obtained by noting that since this vertical face is an equipotential surface, the streamlines intersect with it normally. Therefore, $\partial y/\partial \Phi=0$, $\partial x/\partial \psi=0$, and $h=\partial x/\partial \psi^*=0$ and Eq. 19 reduces to Eq. 48a. The finite difference operator for y on the front face is obtained by first differencing Eq. 48a and then noting from Eq. 48b that $y_{i-1jk}=y_{i+1jk}$. The resulting operator is

$$y_{ijk}=\{y_{i+1jk}+.5[(\alpha_1+\alpha_2)y_{ij+1k}+(\alpha_1-\alpha_2)y_{ij-1k}+\alpha_3]\}/(1+\alpha_1) \quad \ldots\ldots\ldots\ldots (69)$$

in which

$$\alpha_1=e^2,\ \alpha_2=\frac{1}{2}e\frac{\partial e}{\partial \psi}\ \text{and}\ \alpha_3=e\frac{\partial (fg)}{\partial \psi}$$

The boundary condition Eqs. 51 and 54, for x along the left and right sides of the problem, are obtained from Eqs. 11 and 12 and from the fact that along the sides z is constant. The latter fact leads to $g=\partial z/\partial \psi=0$ and $\partial z/\partial \Phi=0$, and consequently

$$\frac{\partial x}{\partial \Phi}=\frac{\partial y}{\partial \psi}\frac{\partial z}{\partial \psi^*} \quad \ldots\ldots\ldots (70)$$

and

$$\frac{\partial y}{\partial \Phi}=-\frac{\partial x}{\partial \psi}\frac{\partial z}{\partial \psi^*} \quad \ldots\ldots\ldots (71)$$

Differentiating Eq. 70 with respect to Φ and Eq. 71 with respect to ψ and combining leads to Eqs. 51 and 54. Alternatively Eqs. 51 and 54 can be derived by setting g equal to zero in Eq. 18.

The finite difference operator for Eqs. 51 and 54 is

$$-x_{i-1jk}+2\left(\frac{1+\alpha_1}{1+\alpha_3}\right)x_{ijk}+\left(\frac{\alpha_3-1}{\alpha_3+1}\right)x_{i+1jk}=[(\alpha_1-\alpha_2)x_{ij-1k}+(\alpha_1+\alpha_2)x_{ij+1k}]/(\alpha_3+1) \quad \ldots\ldots\ldots\ldots (72)$$

in which

$$\alpha_1=e^2,\ \alpha_2=\frac{e}{2}\frac{\partial e}{\partial \psi},\ \alpha_3=\frac{1}{2e}\frac{\partial e}{\partial \Phi}$$

and the subscript k equals either 1 or N respectively, depending upon whether the operator Eq. 72 is applied to the left or right side of the problem.

The boundary condition Eqs. 52 and 55 for y along the sides are also obtained from Eqs. 70 and 71. Differentiating Eq. 70 with respect to ψ and Eq. 71 with respect to Φ leads to these boundary conditions. Alternative Eqs. 52 and 55 can be obtained from Eq. 19 with g set equal to zero. The finite difference operator for Eqs. 52 and 53 is,

$$-y_{i-1jk}+2\left(\frac{1+\alpha_1}{\alpha_3+1}\right)y_{ijk}+\left(\frac{\alpha_3-1}{\alpha_3+1}\right)y_{i+1jk}=[(\alpha_1-\alpha_2)y_{ij-1k}+(\alpha_1+\alpha_2)y_{ij+1k}]/(\alpha_3+1) \quad \ldots\ldots\ldots\ldots (73)$$

in which

$$\alpha_1=e^2,\ \alpha_2=\frac{e}{2}\frac{\partial e}{\partial \psi},\ \alpha_3=\frac{1}{2e}\frac{\partial e}{\partial \Phi}$$

and as before k=1 or N depending respectively upon which of the two sides is being considered.

Method of solution

An examination of the problem as defined earlier, indicates that the flow is principally in the x direction in the vicinity of the vertical front face and in the y direction upon leaving the dam. Consequently in general the variables x and y vary more within $\Phi\psi$ planes than within $\Phi\psi^*$ or $\psi\psi$ planes of the $\Phi\psi\psi^*$ space. The regions of the problem in which an exception to this occurs are the small spaces in the vicinity of the bottom and vertical sides at the drain end of the flow. In these regions singularities (i.e. stagnation lines) exist which should be given special consideration in order to improve the accuracy of the finite difference solution. Since the primary purpose of this study was to examine inverse formulations for three-dimensional problems and methods of solution, this special consideration has not been given to these regions, with the consequence that details of the solution in these regions must be accepted with reservation. In addition this examination of the problem indicates little change of the variable z occurs within $\Phi\psi$ planes and much more variation of z occurs within $\Phi\, \psi^*$ or $\psi\psi^*$ planes.

Therefore, by using the criteria outlined earlier for selecting the planes on which tentative solutions for each variable will be obtained, it is obvious that x and y should be solved for on $\Phi\psi$ planes, and that tentative solutions for z should be either on individual $\Phi\psi^*$ or $\psi\,\psi^*$ planes. Since the equations on the $\Phi\psi^*$ planes generally require less arithmetic to obtain a solution than do the equations which apply on the $\psi\psi^*$ plane as well as the possible difficulties which result if the coefficients of the second derivative terms should take on like signs during the solution process, z will be solved for on $\Phi\psi^*$ planes. An examination of the equations in Table 1, which may be used for solutions of x, y, and z on the planes as indicated previously reveals the following:

1. Eq. <u>18</u> or Eq. 20 can be used for the solutions of x.
2. Eq. <u>19</u> or Eq. 22 can be used for the solutions of y.
3. Eq. <u>27</u> or Eq. 29 can be used for the solutions of z.

Further examination of Eqs. 18 and 20 indicates that the coefficient, e^2, for $\partial^2 x/\partial\psi^2$, is close to unity for this particular problem throughout most of the region in Eq. 18 whereas this same coefficient f^2 for Eq. 20 is relatively small. Based on the second criteria given earlier, Eq. 18 will be used to solve for x. Based on the same criteria the solutions for y will be based on Eq. 19. Comparing the magnitudes of the coefficients d^2 and i^2 for the term $\partial^2 z/\partial\psi^{*2}$ in Eqs. 27 and 29 shows that d^2 in general will be larger than i^2 and therefore Eq. 27 will be used to obtain the solutions for z. The equation numbers to be used in solving for x, y, and z have been underlined in the previous paragraph.

To help describe the procedure used in obtaining the finite difference solution to the three-dimensional seepage flow through a dam, the following terminology will be used.

(a) *Tentative solution*—refers to a solution based on any of the finite difference operators in Tables 2 and 3 or any of the operators for a boundary condition. These solutions are obtained on a specified plane within the $\Phi\,\psi\,\psi^*$ space and are based on the assumption that certain quantities, which are given by single letters in the finite difference operator, are known. Actually these quantities will have their values adjusted as the solution proceeds. All of these tentative solutions in the current computer program are obtained by the line successive overrelaxation (LSOR) method as described earlier.

(b) *Iteration number*—refers to the number of times the LSOR method adjusts the values of x, y, or z on any single plane in obtaining a particular tentative solution.

(c) *Cycle number*—refers to the number of times all tentative solutions are obtained. Thus during the first cycle all tentative solutions for x, y, and z will be obtained as well as tentative solutions for these variables on the boundary plane which are not of the Dirichlet type. The same process is repeated for the second cycle, etc.

The procedure followed in the computer program for obtaining the solutions is illustrated in the gross flow chart given in Fig. 2. There is no reason why this exact procedure needs to be followed, and it would be of interest to study whether solutions can be obtained in fewer cycles by altering the order in which tentative solutions are obtained. The present program first obtains all the tentative solutions for x, then obtains all the tentative solutions for y, and finally obtains all the tentative solutions for z. Furthermore, the tentative solutions for the boundaries are obtained in general prior to obtaining the tentative solutions for that variable on planes within the interior of the region. The additional feature has been incorporated within the program, so that any of the tentative solutions on boundary planes need not be obtained during any cycle number. This feature provides a means for settling the values at interior grid points prior to adjusting the values on the boundary planes, and also permits control to be exercised in not adjusting certain boundary values during some cycles should one decide to do this. Boundary values, for which this latter feature may be used, are the x values on the bottom and phreatic surfaces, or the x and z values on the drain surface. These values are obtained by a numerical differentiation and integration process. For x on the bottom this integration starts at the upstream face and proceeds toward the drain, and consequently any error is accumulative. This accumulative error, if it changes the x values each cycle, in turn prevents the interior values from settling very fast. By permitting the interior values to become fairly well settled before adjusting the x values on the bottom, phreatic surface, and drain, and then adjusting these values only during part of the cycle numbers, results in more rapid convergence to the final solution.

The manner in which each of the tentative solutions is obtained is slightly different depending upon which variable is involved and depending upon whether the plane is interior or a boundary plane. For those tentative solutions which are obtained by the LSOR method, the flow chart in Fig. 3 outlines the procedure used. The LSOR method is used for all the solutions on interior planes and on all boundary planes with the exception of the following five: (1) The x-values on the bottom, (2) the x-values on the phreatic surface, (3) the x-values on the drain, (4) the z-values on the drain, and (5) the z-values on the upstream face. These five boundary values are

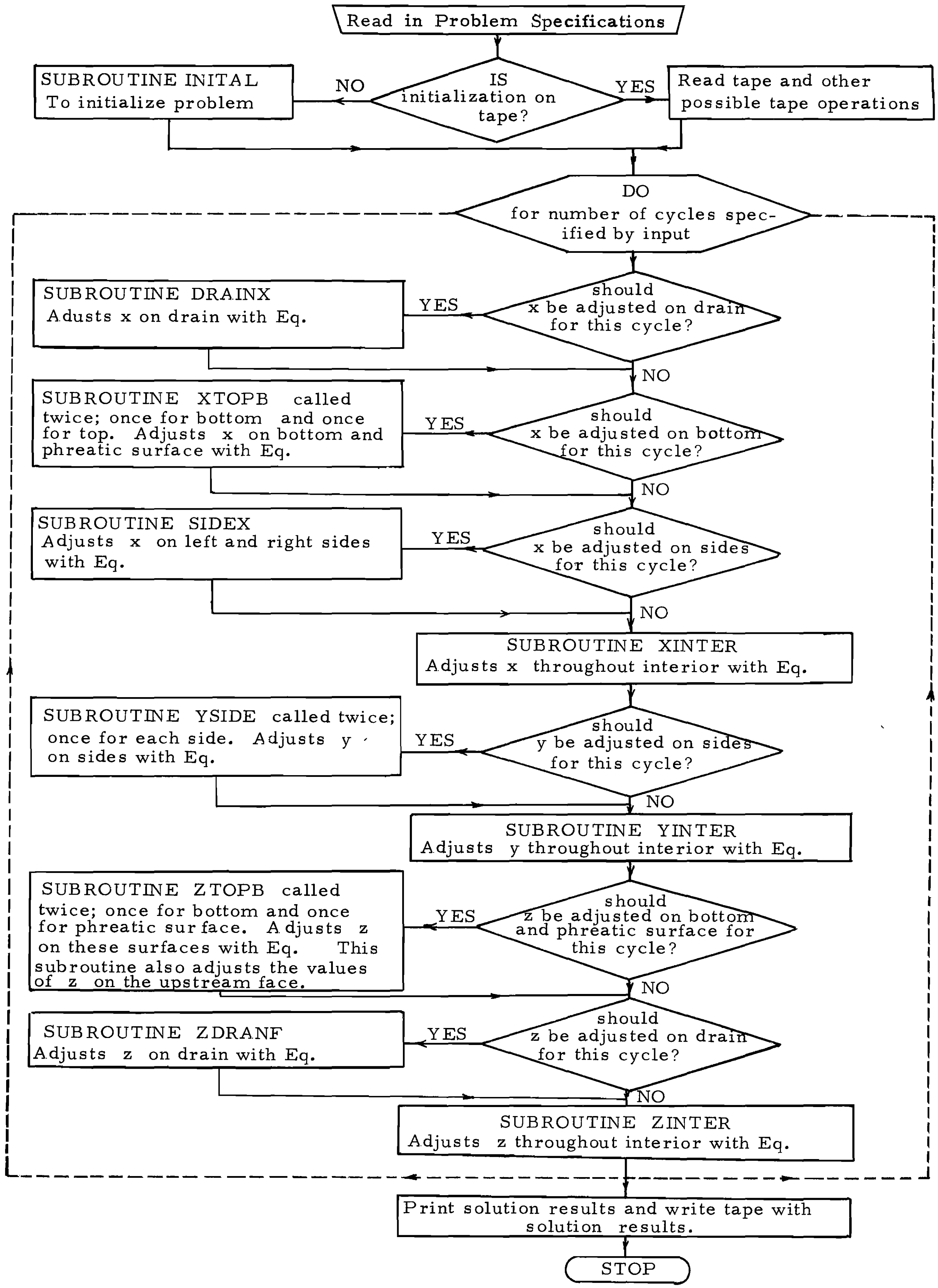

Fig. 2. Flow chart of computer program for solving problems of three-dimensional flow through a dam.

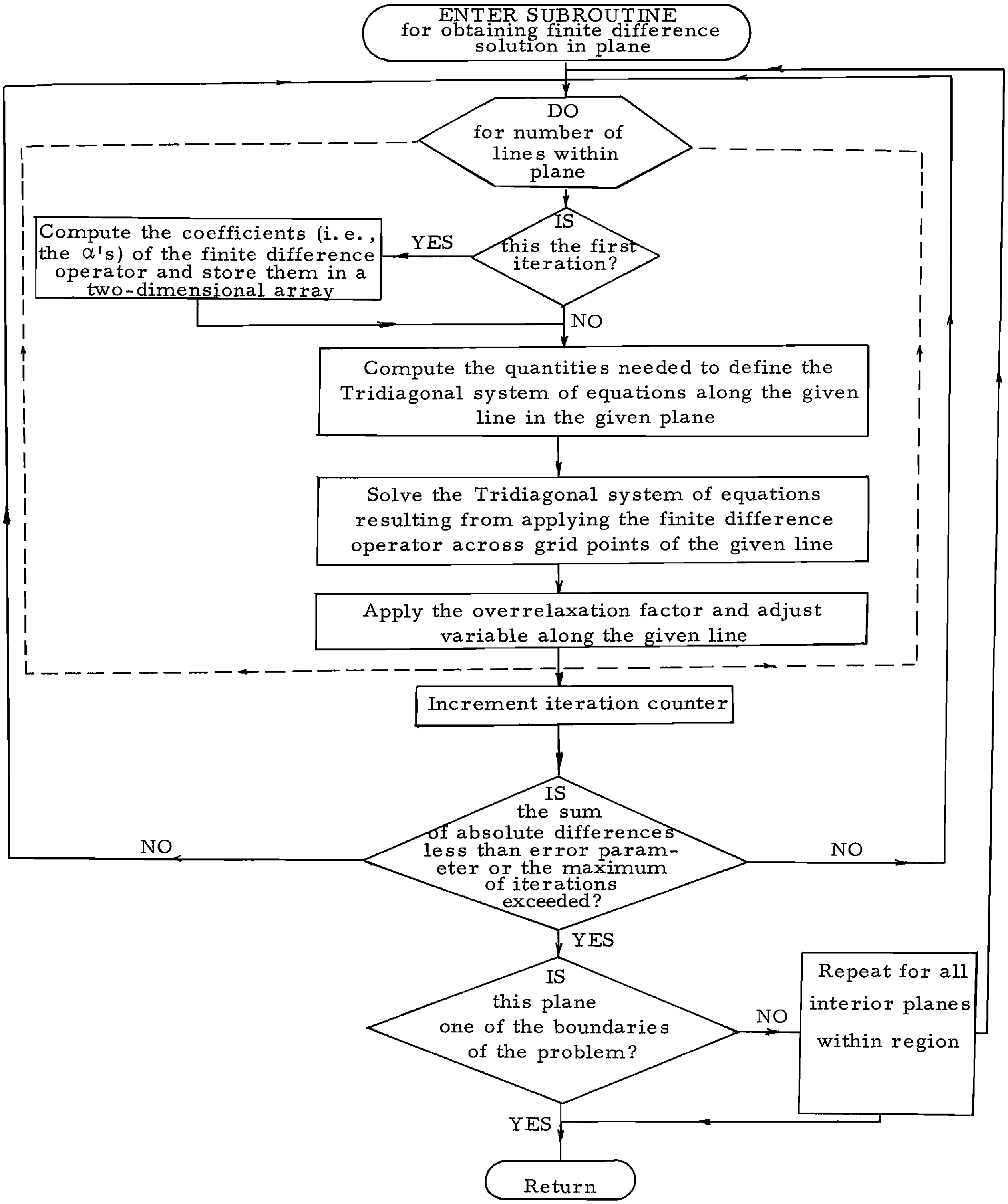

Fig. 3. Flow chart of the computer program subroutines which obtain the tentative solutions by the LSOR-Method.

obtained by the numerical differentiation integration process which has been described earlier.

The procedure followed in obtaining each of the tentative solutions is first to establish the system of equations which results from simultaneous application of the finite difference operator across the particular line being considered. The coefficients which are needed to do this are actually only computed during the first iteration of any tentative solution. Next the tridiagonal system is solved and the values of that variable along the line are adjusted by the overrelaxation factor. The procedure is repeated for subsequent lines until all lines within the plane have been adjusted. This entire process constitutes one iteration. Iterations are continued until either the sum (across all grid points within the plane) of absolute differences in values between consecutive iterations is less than the prescribed error parameter or the iteration number exceeds the maximum specified as allowable. In the case of the subroutines which obtain the tentative solution on interior planes, the program allows that this process of obtaining tentative solutions on consecutive interior planes will be repeated either a limited number of times or until the maximum iteration number required to obtain the tentative solutions is less than a specified number. By allowing this to occur, it is possible to obtain tentative solutions before leaving that subroutine, such that the variable being solved for will no longer change because of changes in its values on consecutive planes.

All lines in the LSOR method have been taken in the direction of Φ. In other words, the lines are defined by either holding ψ or ψ^* constant depending, respectively, on whether the solution is obtained on a $\Phi\psi$ or a $\Phi\psi^*$ plane. An earlier version of the program used lines taken in the direction of ψ^* when using the LSOR method to adjust the values of z in the bottom and phreatic surface planes. This choice was arbitrary and was not based upon considerations of increasing convergence rates, etc. Since the performance of the program showed no appreciable difference with direction of the lines, the final version of the program which is given in Appendix B uses lines in the direction of Φ for computing z on these planes.

Solution Results

The results from the inverse formulation given in this paper are in terms of the magnitude of x, y, and z at the intersection points of the potential surfaces with the two stream surfaces defined by holding ψ and ψ^* equal to constants. As a consequence a three-dimensional flownet can readily be plotted by simply connecting the points defined by the x, y, and z coordinates given at each grid point throughout the $\Phi\psi\psi^*$ space by lines and visualizing the small planes defined by these lines as representing sides of the parallelepiped elements of the space flownet.

The solution results can also readily be used to obtain other quantities of interest about the flow. For instance the velocity at any point within the flow space is given by the following equation

$$V = \frac{1}{\left[\left(\frac{\partial x}{\partial \Phi}\right)^2 + \left(\frac{\partial y}{\partial \Phi}\right)^2 + \left(\frac{\partial z}{\partial \Phi}\right)^2\right]^{1/2}} \quad . . (74)$$

To derive Eq. 74 first note (from the definition of the Jacobian determinant J given below Eq. 8) that upon expanding the determinant for J at the top row results in

$$J = u^2 + v^2 + w^2 = V^2 \quad . . . . . (75)$$

In other words the Jacobian determinant equals the velocity of the flow squared. Next note that the first equation from each set of three equations in Eq. 8 can be written as follows

$$V^2 \frac{\partial x}{\partial \Phi} = u \quad . . . . (76a)$$

$$V^2 \frac{\partial y}{\partial \Phi} = -v \quad . . . . (76b)$$

$$V^2 \frac{\partial z}{\partial \psi} = w \quad . . . . (76c)$$

Upon squaring Eqs. 76a, 76b, and 76c and subsequently adding the resulting squares and finally solving for the magnitude of the total velocity leads to Eq. 74.

By using the inverse relationships given earlier or the inverse Eqs. 11, 12, and 13 a number of other possible equations for computing the velocity can be derived.

From the numerical solution results, the velocity is actually computed by approximating the derivatives by differences. Using second order central differences gives the following equation for the velocity at a grid point (ijk).

$$V_{ijk} = \frac{2\Delta\Phi}{[(x_{i+1jk}-x_{i-1jk})^2 + (y_{i+1jk}-y_{i-1jk})^2 + (z_{i+1jk}-z_{i-1jk})^2]^{\frac{1}{2}}} \quad . . . . (77)$$

To obtain the direction of the velocity vector at any point within the flow space, it is necessary to use Eqs. 76a, 76b, and 76c in conjunction with Eq. 74. The angles α, β, and γ which the velocity vector makes with the positive direction of the physical coordinate axes x, y, and z are:

$$\alpha = \cos^{-1}\left(V\frac{\partial x}{\partial \Phi}\right) \quad . . . . . . (78a)$$

$$\beta = \cos^{-1}\left(V\frac{\partial y}{\partial \Phi}\right) \quad . . . . . . (78b)$$

$$\gamma = \cos^{-1}\left(V\frac{\partial z}{\partial \Phi}\right) \quad . . . . . . (78c)$$

In other words the quantities $V\partial x/\partial \Phi$, $V\partial y/\partial\Phi$ and $V\partial z/\partial\Phi$ are the direction cosines of the velocity vector.

By varying the specifications which define variations of the basic problem described earlier, four separate solutions have been obtained to three-dimensional seepage through a dam with a vertical face and a drain over a portion of the horizontal bottom. A summary of the essential specifications for these problems is given in Table 4. In all four of the problems, 21 potential lines were specified, and in each problem the same number of ψ=constant and ψ^*=constant grid lines were specified. This latter specification requires, in each problem, that the width between vertical walls equals the depth of water on the upstream face, since as implemented in the computer program $\Delta\psi = \Delta\psi^* = \Delta\Phi$. An example of the final solution for Problem No. 2 showing the magnitudes of x, y, and z at each grid point within the $\Phi\psi\psi^*$ space is given as Appendix C.

Table 4. Summary of problem specifications.

Specification	Problem Number			
	1	2	3	4[a]
No. of Φ-grid lines	21	21	21	21
No. of ψ-grid lines	11	10	9	11
No. of ψ^*-grid lines	11	10	9	'11
Height of water on dam face	10.0	9.0	10.0	10.0
Width of dam	10.0	9.0	10.0	10.0
Dist. z_1 to beginning of drain	1.0	1.0	0.5	1.0
Dist. z_2 from drain to right side	1.0	0.0	1.0	1.0

[a]The specifications to this problem were the same as Problem No. 1 with the exception the drain was not specified as rectangular in shape.

In the first problem the distance z_1 was specified equal to $z_2 = 1$; thus resulting in symmetry about the center ψ^*=constant grid line. Clearly a symmetric problem such as this first one should be solved using only one-half of the region in the formulation. However, during the process of developing and debugging the computer program it was desirable to use the entire region in order to provide for an internal check on the solution capability. Should a nonsymmetric flow field result as the solution to a symmetrically specified problem, an error must exist in the computer program, in one of the equations, or the method of approach.

The final solution to this first problem was obtained in a piecemeal manner as the computer program was being developed. That is, as the various components of the program were completed they were used to adjust the value of the variables through a few iterations or cycles, but the results were always stored on tape and used to initialize the problem for the next phase in the program development and debugging. During this process, since there were mistakes in some of the program components to begin with, the values of x, y, and z showed considerable deviation from symmetry. As the mistakes in the program were eliminated, the variables took on magnitudes which represented the proper symmetry. The final results are not symmetric to all three digits beyond the decimal point which are printed out, but in general symmetry does exist in the final solution to at least two digits beyond the decimal point.

There are a number of other items from the solutions which indicate the accuracy of the numerical approach. For example in implementing Eq. 49 in obtaining the z values on the upstream face of the dam by a numerical integration, the final values of z on each of the ψ=constant grid lines on the face should equal the width of dam between the side walls. During the last cycles in the solutions these values showed agreement to within a few percent of the specified values. For the first solution, the average value computed from the integration during the latter cycles settled to a value of 10.021 instead of 10.000. The largest difference in the z on the right side occurred at the bottom where the final z from the integration was 10.044 and the smallest occurred on the phreatic surface and was equal to 9.999.

For the second solution the final average value for z from integrating along the ψ=constant lines on the upstream face from the latter cycles was 9.008 instead of 9.000, with the largest value equaling 9.013 just above the bottom and the smallest equaling 9.005 near the phreatic surface.

The same comparison for the third solution shows the largest value occurring at the phreatic surface equal to 10.026 and the smallest at the bottom equal to 10.005. The average width between sides was computed equal to 10.013 instead of 10.000 units. The largest difference at the phreatic surface represents a .26 percent error. Errors in this integration of the fourth solution were of similar magnitude with the average computed value between side walls equaling 9.932 instead of 10.0.

With one exception, in all four of the solutions, the greatest difference in these values of z either occurred on the bottom or the phreatic surface where the evaluation of the derivatives, needed in the numerical integration, had to be based on forward or backward differences.

The accuracy in evaluating x along the bottom and phreatic surface would be expected in general to be subject to error of about the same magnitude as that for z on the upstream face, with the exception that at the drain end near the side walls in the vicinity of the line singularity the error may be larger.

The writer is the least satisfied with the methods for determining the non-Dirichlet boundary values of x and z on the drain by means of Eqs. 57 and 58 respectively. This lack of satisfaction occurs 1) because both Eq. 57 and Eq. 58 come directly from Eq. 12 making the values of z strongly dependent upon the values of x as determined on the drain surface and x in turn dependent upon the values determined for z, and 2) because in the process of obtaining the solutions the equation for z produced greatly erroneous values when used prior to having the interior values (particularly for x) adjusted by their operators. Future study needs to be directed toward better procedures or formulations of boundary conditions for evaluating x and z on surfaces such as the drain surface of the dam problem.

In the following four subsections flownets for the four solutions are presented and certain features of the flows, as given by the solution results, are discussed. To fully understand the flow field from a three-dimensional problem, the reader must visualize the flow in space from flownets on the plane of a printed page. To help in this visualization process, a complete isometric flownet showing all interior cubes of the flow field was initially prepared by use of a computer driven plotter. All of this detail resulted in too many lines to give a clear visual picture of the space flow field. Subsequently isometric drawings were prepared which show only the top and right sides of the three-dimensional flownets along with the container of the dam which is assumed to be transparent so that the imaginary seepage lines are visible. These latter isometric drawings are given for Problems Nos. 1, 2, and 3. In addition, flownets from each of the solutions are given which result from projecting the flow patterns upon planes either parallel to the sides of the dam, parallel to the bottom of the dam, or parallel to the front face of the dam. These individual flownets are obtained from plotting the magnitudes of x and y, the magnitudes of x and z, or the magnitudes of y and z from individual $\Phi\psi$ planes, $\Phi\psi^*$ planes or $\psi\psi^*$ planes within the $\Phi\psi\psi^*$ space respectively.

Problem No. 1

An isometric flownet drawn from the solution for Problem No. 1 is given in Fig. 4. Figures 5 through 10 are flownets from separate $\Phi\psi$ planes for j = 1,2,...6 respectively. Since Problem No. 1 is symmetric about the center j = 6 $\Phi\psi$ plane, the remaining 5 flownets from $\Phi\psi$ planes which could be plotted from the solution are identical to those given in Figs. 5 through 10. The flownet from j = 7 is identical to that from j = 5; the flownet from j = 8 is identical to that from j = 4; etc., until the flownet from j = 11 is identical to that from j = 1.

These flownets (Figs. 5 through 10) represent plottings of the magnitudes of x and y onto a vertical plane parallel to the sides of the dam, and as such do not show any of the change in the z-direction of the streamlines and stream surfaces of the flownet. In the outer portions of the dam in the vicinity of the drain the seepage velocity has a sizable component in the z-direction. To illustrate changes of the stream surfaces in the z-direction, flownets have also been plotted from the solution to Problem No. 1 which result by projecting the magnitudes of x and z from separate $\Phi\psi^*$ planes within the $\Phi\psi\psi^*$ space upon a horizontal plane. Figures 11, 12, and 13 are such flownets obtained from the $\Phi\psi^*$ planes with k=1, 6 and 11 respectively. Figure 11, therefore, represents the flownet on the bottom of the dam, and Fig. 13 the plan view of the flownet of the phreatic surface. In these last three flownets a dashed line is shown entering the outer edge of the drain. This line has been drawn to point out that the outside stream surfaces move inward abruptly and actually leave the region of interest within the area of the drain. The very outside stream surface actually abruptly changes from vertical to horizontal and then to vertical again at the bottom and sides of the drain. The resulting discontinuities in the variables of the problem are poorly approximated in this region by the polynomials resulting from the finite differences. Consequently details of the solutions, particularly in the immediate vicinity of these singularities, cannot be considered very accurate.

Flownets obtained by plotting the z and y coordinates from individual $\psi\psi^*$ planes are given in Figs. 14, 15, and 16. The $\psi\psi^*$ plane of Fig. 15 is associated with i=11; and the $\psi\psi^*$ plane of Fig. 16 with i=1.

The flownets taken from the various planes illustrate various features of this flow. From them it can be noted that 60.5 percent of the total flux entering the face of the dam enters through the lower half of the face. At approximately two-thirds the distance from the face to the drain or at the i=11 $\psi\psi^*$ plane, on the other hand, 50.8 percent of the flux crosses the lower half of the seepage region between the bottom of the dam and the phreatic surface. At the front face 52.0 percent of the flux enters through the center half midway between the two side walls and 48 percent enter through the two outside quarters of the front face. At the i=11 $\psi\psi^*$ plane, 52.4 percent of the flux is still moving through the center half of this equipotential surface. Through the $\psi\psi^*$ plane with i=18 on the other hand, 56 percent passes through the center half and 44 percent through the outer halves.

An examination of the length of flow leaving through the drain reveals that the length at the outside of the drain opening is 3.591 units, and the length at the

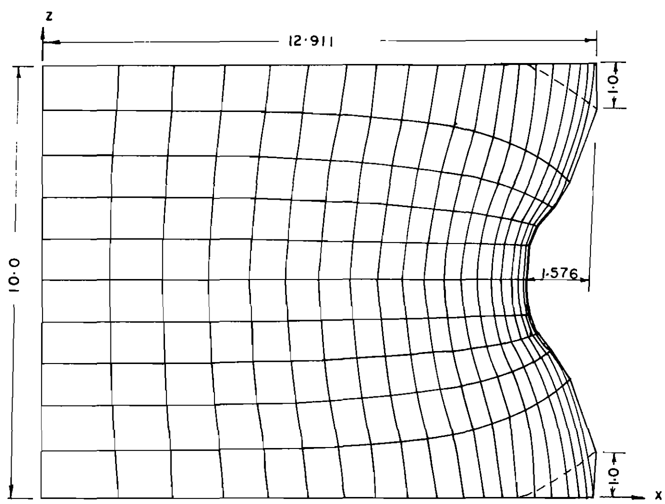

Fig. 13. Flownet from the $\Phi\psi^*$ plane coincident with the phreatic surface (i.e. j=11) obtained by plotting the magnitudes of x and z from the solution to Problem No. 1 onto a plane parallel to the bottom of the dam.

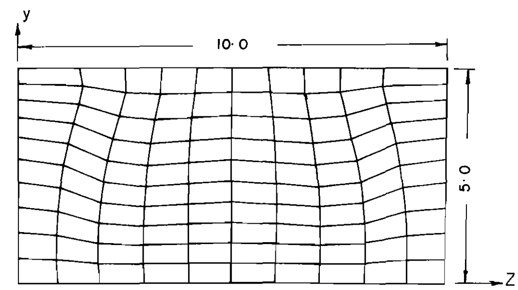

Fig. 15. Flownet from the $\psi\psi^*$ plane associated with i=11 obtained by plotting the magnitudes of z and y from the solution to Problem No. 1 onto a vertical plane parallel to the face of the dam.

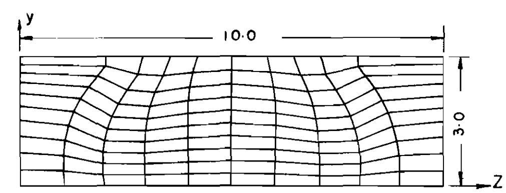

Fig. 14. Flownet from the $\psi\psi^*$ plane associated with i=15 obtained by plotting the magnitudes of z and y from the solution to Problem No. 1 onto a vertical plane parallel to the face of the dam.

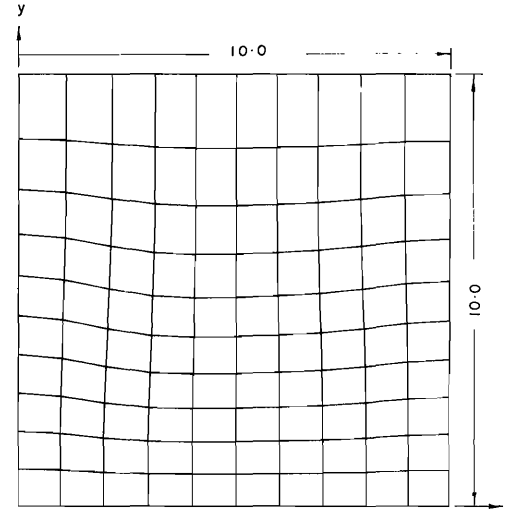

Fig. 16. Flownet from the $\psi\psi^*$ plane coincident with the face of the dam (i.e. i=1) obtained by plotting the magnitudes of z and y from the solution to Problem No. 1

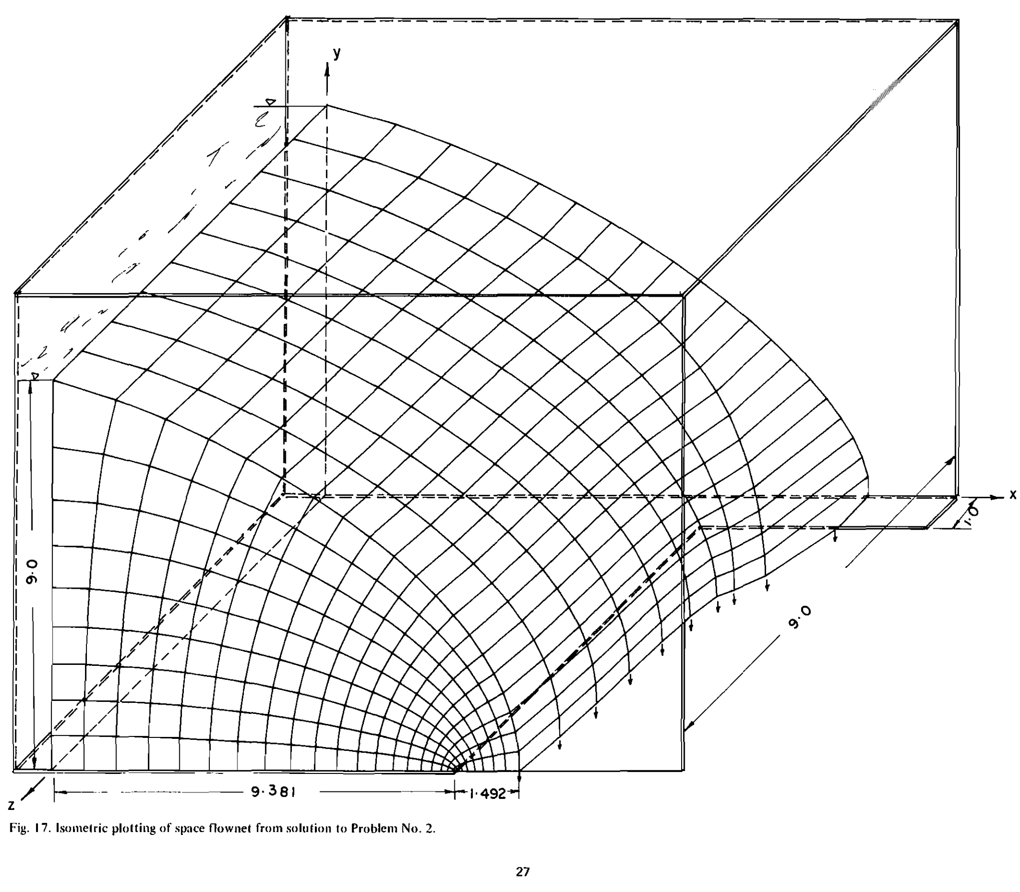

Fig. 17. Isometric plotting of space flownet from solution to Problem No. 2.

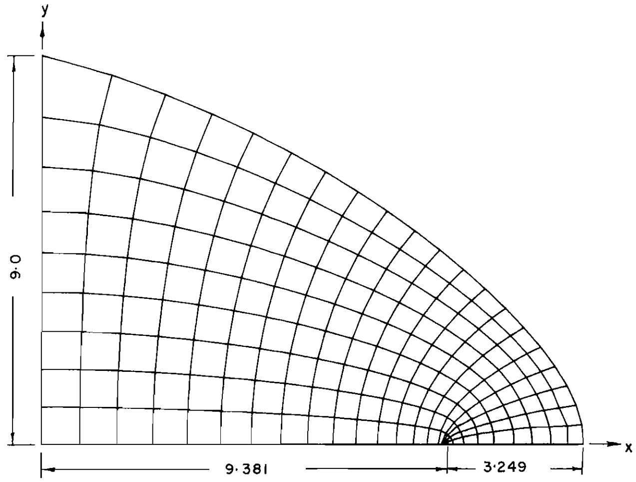

Fig. 18. Flownet from the $\Phi\psi$ plane associated with k=1 obtained by projecting the magnitudes of x and y from the solution to Problem No. 2 onto a vertical plane parallel to the sides of the dam.

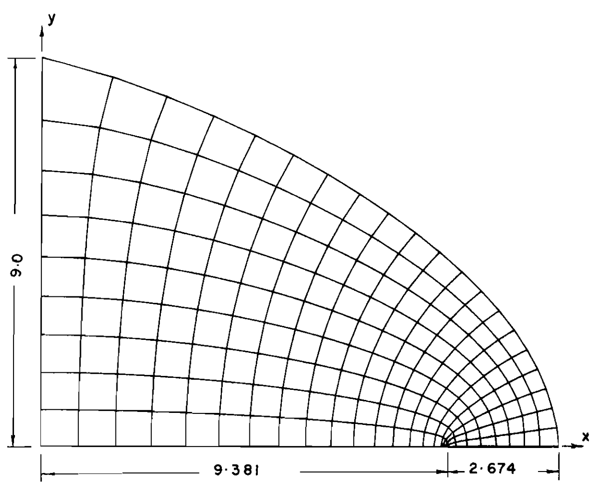

Fig. 19. Flownet from the $\Phi\psi$ plane associated with k=2 obtained by projecting the magnitudes of x and y from the solution to Problem No. 2 onto a vertical plane parallel to the sides of the dam.

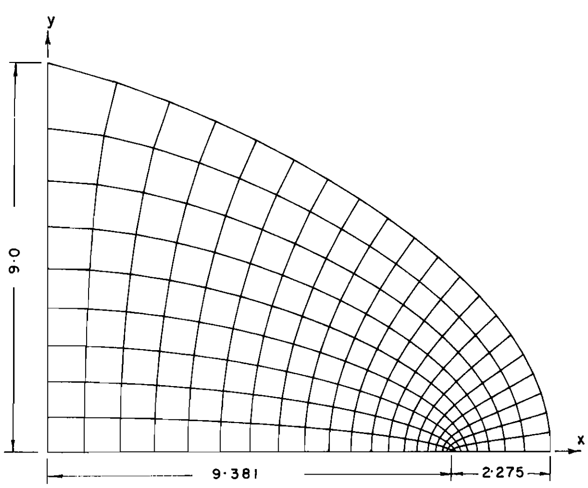

Fig. 20. Flownet from the $\Phi\psi$ plane associated with k=3 obtained by projecting the magnitudes of x and y from the solution to Problem No. 2 onto a vertical plane parallel to the sides of the dam.

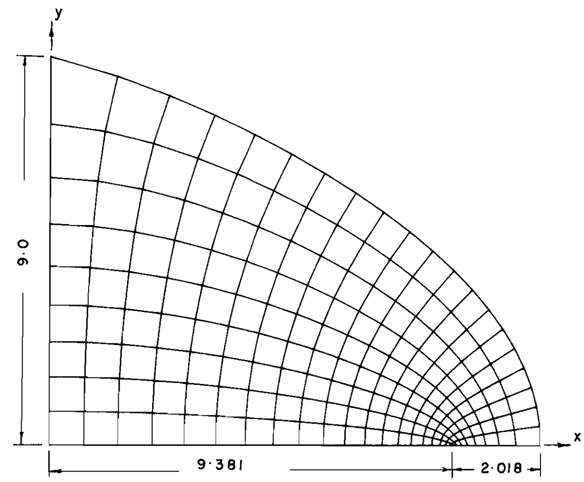

Fig. 21. Flownet from the $\Phi\psi$ plane associated with k=4 obtained by projecting the magnitudes of x and y from the solution to Problem No. 2 onto a vertical plane parallel to the sides of the dam.

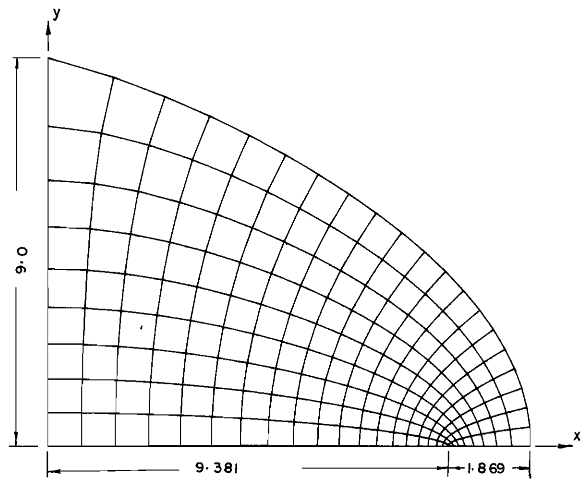

Fig. 22. Flownet from the $\Phi\psi$ plane associated with k=5 obtained by projecting the magnitudes of x and y from the solution to Problem No. 2 onto a vertical plane parallel to the sides of the dam.

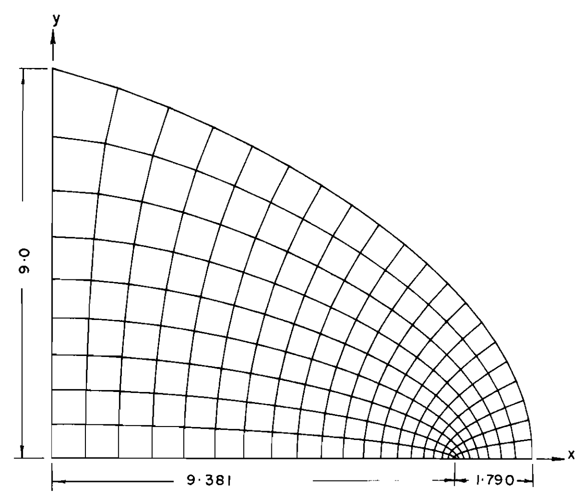

Fig. 23. Flownet from the $\Phi\psi$ plane associated with k=6 obtained by projecting the magnitudes of x and y from the solution to Problem No. 2 onto a vertical plane parallel to the sides of the dam.

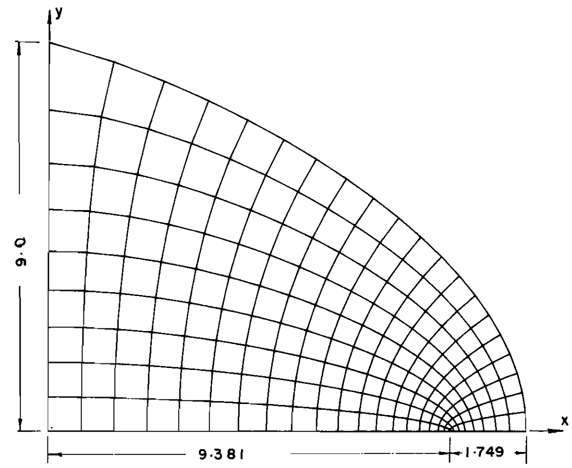

Fig. 24. Flownet from the $\Phi\psi$ plane associated with k=7 obtained by projecting the magnitudes of x and y from the solution to Problem No. 2 onto a vertical plane parallel to the sides of the dam.

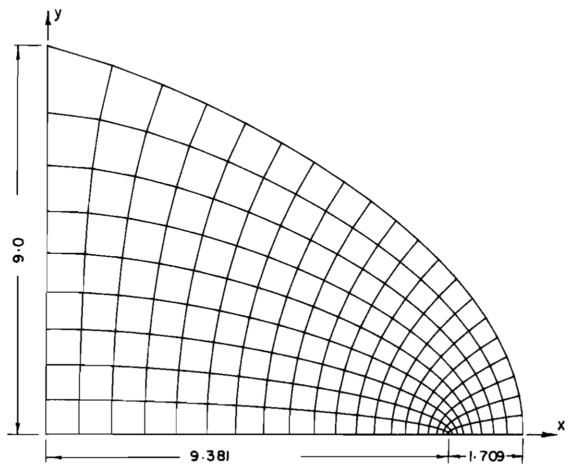

Fig. 25. Flownet from the $\Phi\psi$ plane associated with k=8 obtained by projecting the magnitudes of x and y from the solution to Problem No. 2 onto a vertical plane parallel to the sides of the dam.

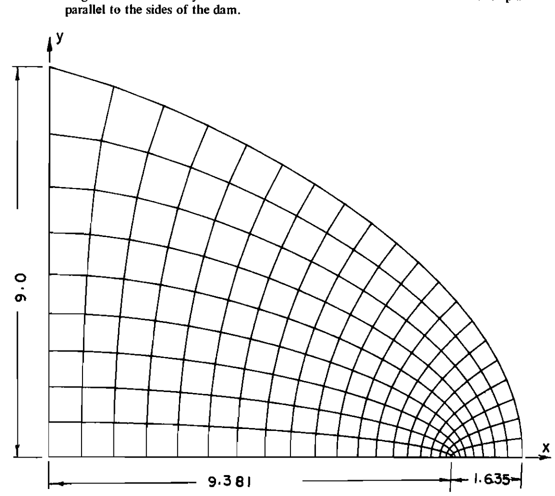

Fig. 26. Flownet from the $\Phi\psi$ plane associated with k=9 obtained by projecting the magnitudes of x and y from the solution to Problem No. 2 onto a vertical plane parallel to the sides of the dam.

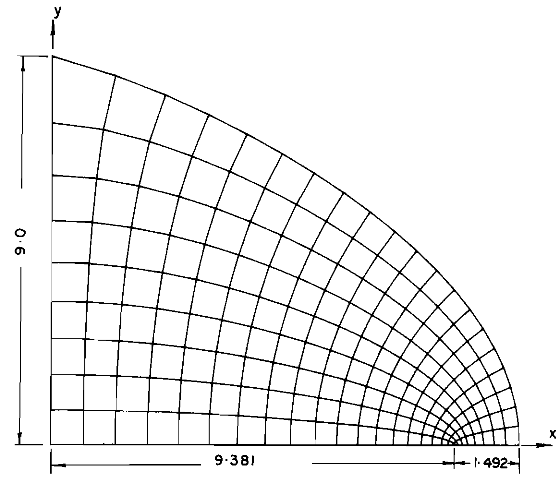

Fig. 27. Flownet from the $\Phi\psi$ plane associated with k=10 obtained by projecting the magnitudes of x and y from the solution to Problem No. 2 onto a vertical plane parallel to the sides of the dam.

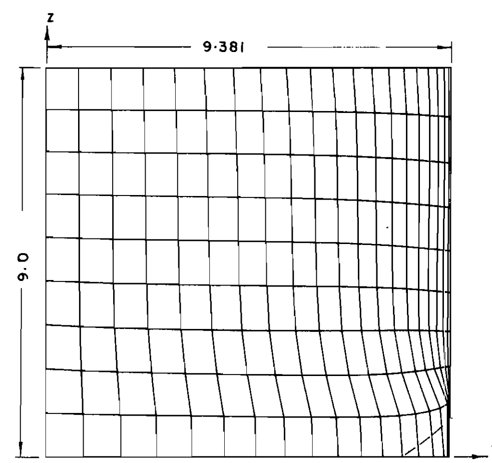

Fig. 28. Flownet from the $\Phi\psi^*$ plane coincident with the bottom of the dam (i.e. j=1) obtained by plotting the magnitudes of x and z from the solution to Problem No. 2.

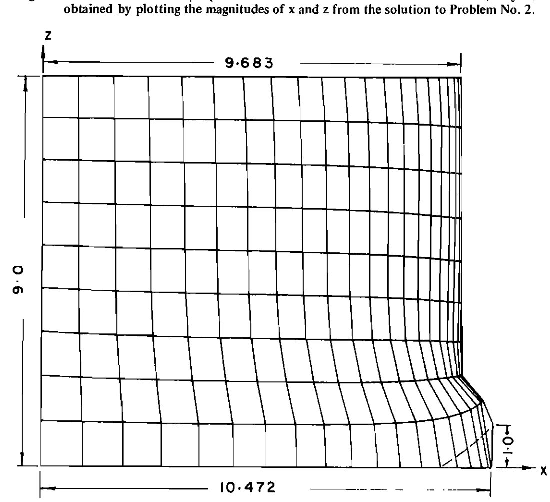

Fig. 29. Flownet from the $\psi\psi^*$ plane associated with j=5 obtained by plotting the magnitudes of x and z from the solution to Problem No. 2 onto a plane parallel to the bottom of the dam.

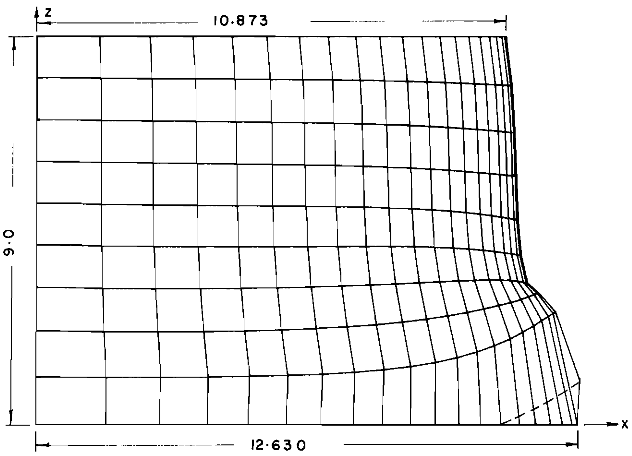

Fig. 30. Flownet from the $\Phi\psi^*$ plane coincident with the phreatic surface (i.e. j=10) obtained by plotting the magnitudes of x and z from the solution to Problem No. 2 onto a plane parallel to the bottom of the dam.

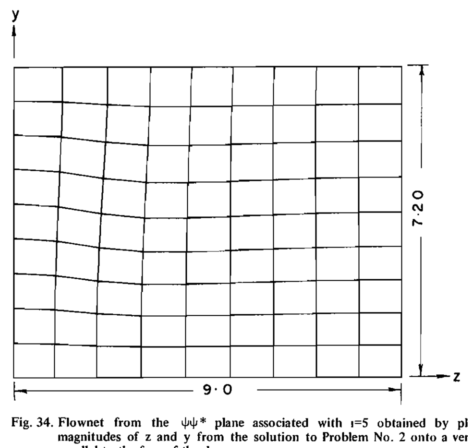

Fig. 34. Flownet from the $\psi\psi^*$ plane associated with i=5 obtained by plotting the magnitudes of z and y from the solution to Problem No. 2 onto a vertical plane parallel to the face of the dam.

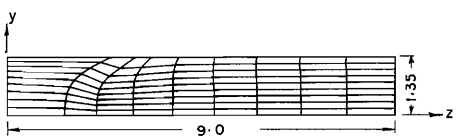

Fig. 31. Flownet from the $\psi\psi^*$ plane associated with i=18 obtained by plotting the magnitudes of z and y from the solution to Problem No. 2 onto a vertical plane parallel to the face of the dam.

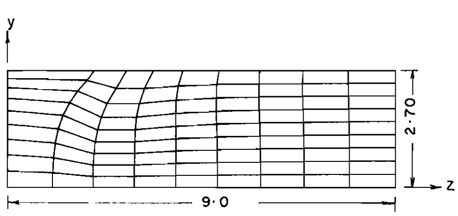

Fig. 32. Flownet from the $\psi\psi^*$ plane associate; with i=15 obtained by plotting the magnitudes of z and y from the solution to Problem No. 2 onto a vertical plane parallel to the face of the dam.

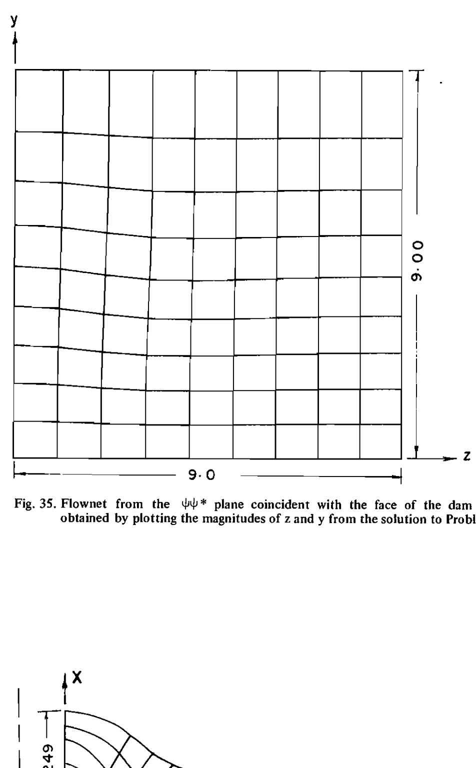

Fig. 35. Flownet from the $\psi\psi^*$ plane coincident with the face of the dam (i.e. i=1) obtained by plotting the magnitudes of z and y from the solution to Problem No. 2.

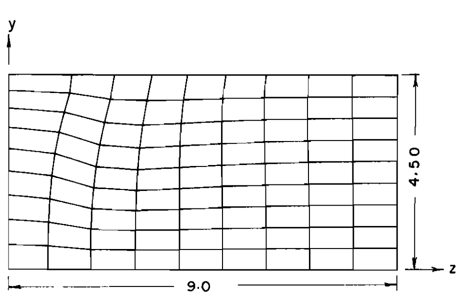

Fig. 33. Flownet from the $\psi\psi^*$ plane associated with i=11 obtained by plotting the magnitudes of z and y from the solution to Problem No. 2 onto a vertical plane parallel to the face of the dam.

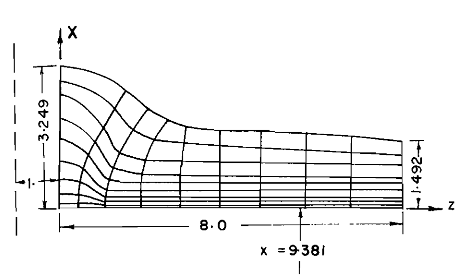

Fig. 36. Flownet from the $\psi\psi^*$ plane coincident with the drain of the dam (i.e. i=21) obtained by plotting the magnitudes of z and x from the solution to Problem No. 2 onto a horizontal plane.

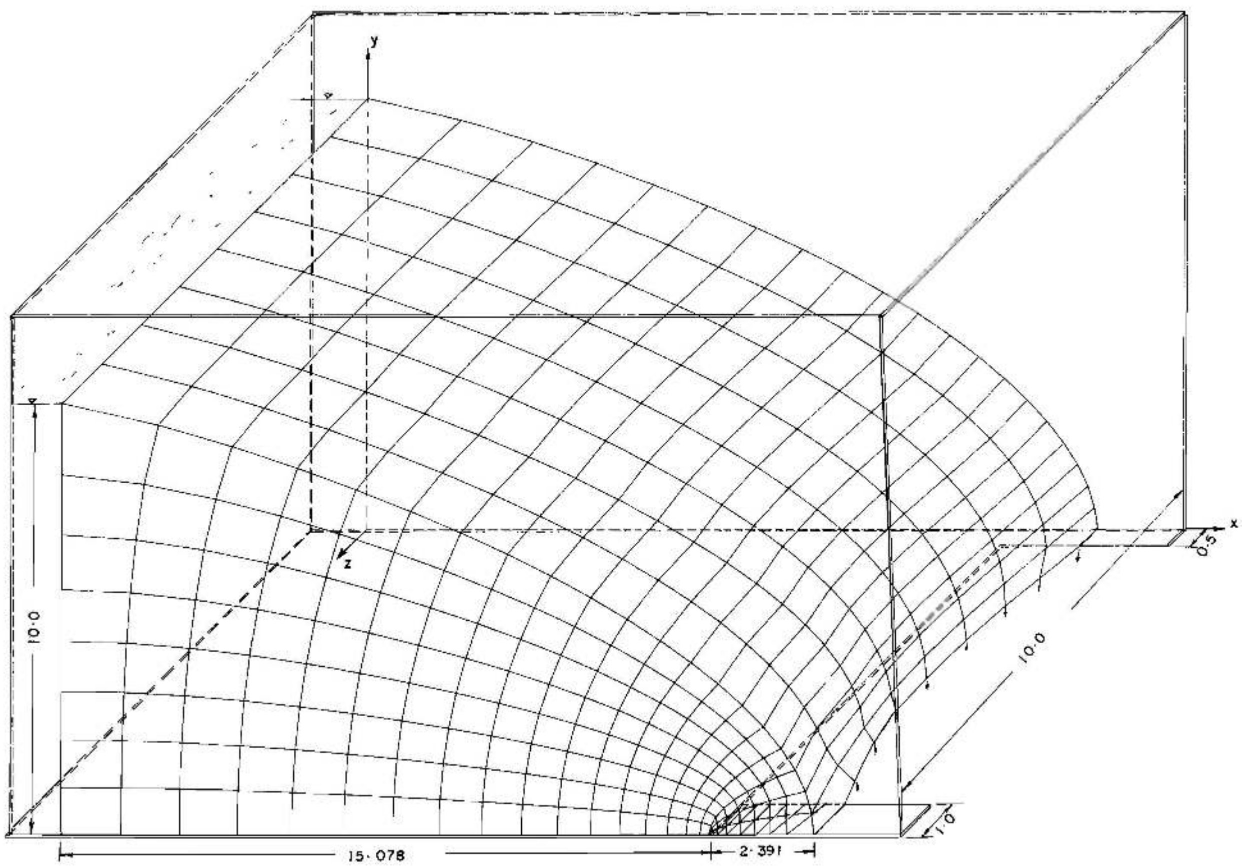

Fig. 37. Isometric plotting of space flownet from solution to Problem No. 3.

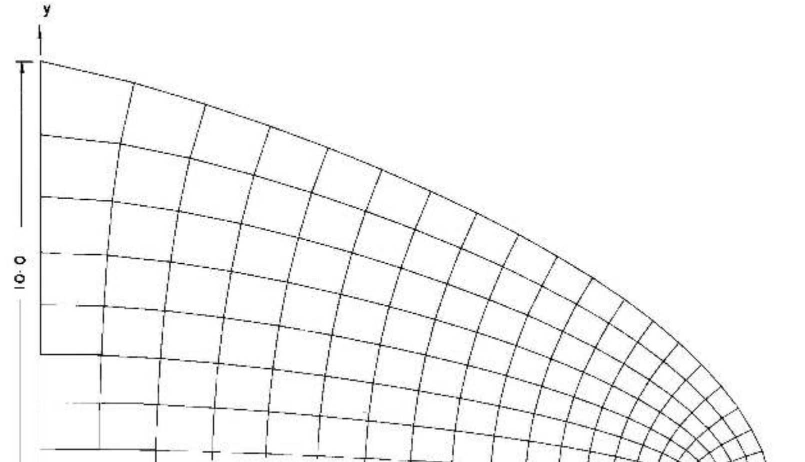

Fig. 38. Flownet from the $\Phi\psi$ plane associated with k=1 obtained by projecting the magnitudes of x and y from the solution to Problem No. 3 onto a vertical plane parallel to the sides of the dam.

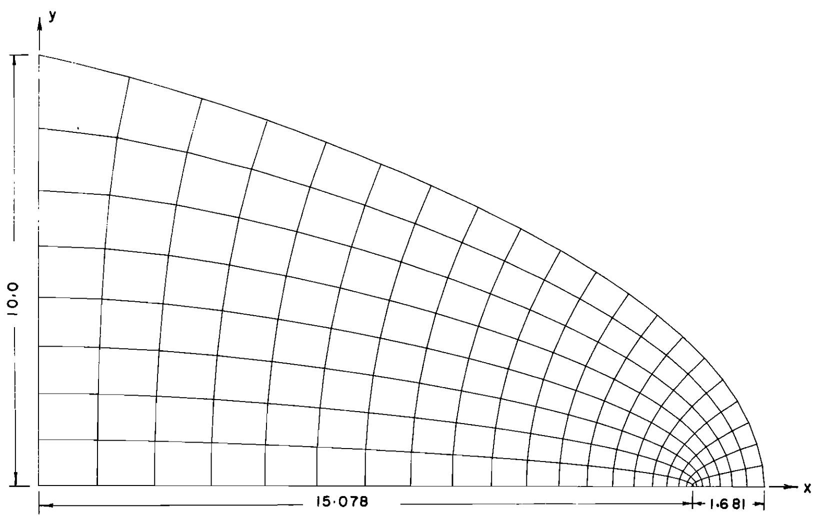

Fig. 39. Flownet from the Φψ plane associated with k=2 obtained by projecting the magnitudes of x and y from the solution to Problem No. 3 onto a vertical plane parallel to the sides of the dam.

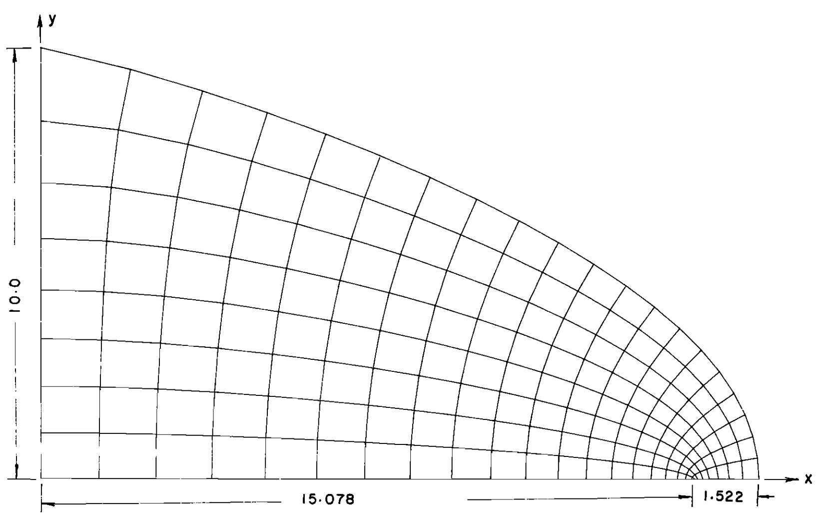

Fig. 40. Flownet from the Φψ plane associated with k=3 obtained by projecting the magnitudes of x and y from the solution to Problem No. 3 onto a vertical plane parallel to the sides of the dam.

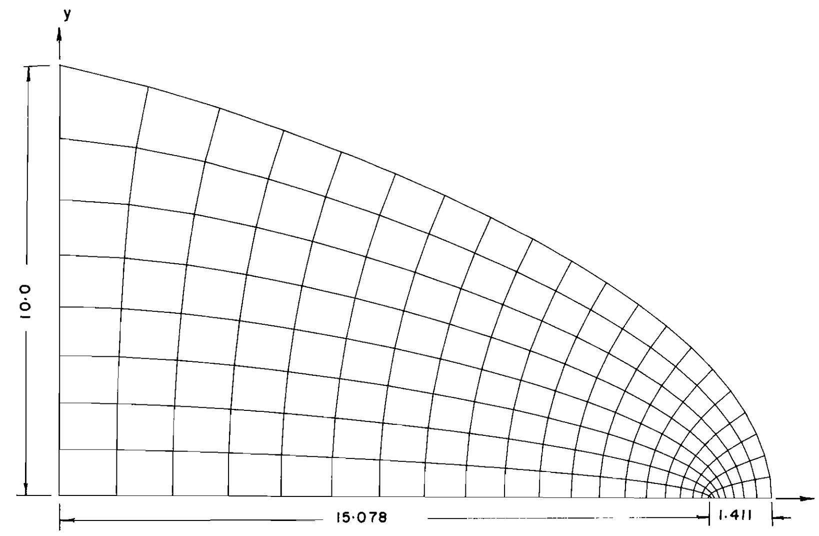

Fig. 41. Flownet from the Φψ plane associated with k=4 obtained by projecting the magnitudes of x and y from the solution to Problem No. 3 onto a vertical plane parallel to the sides of the dam.

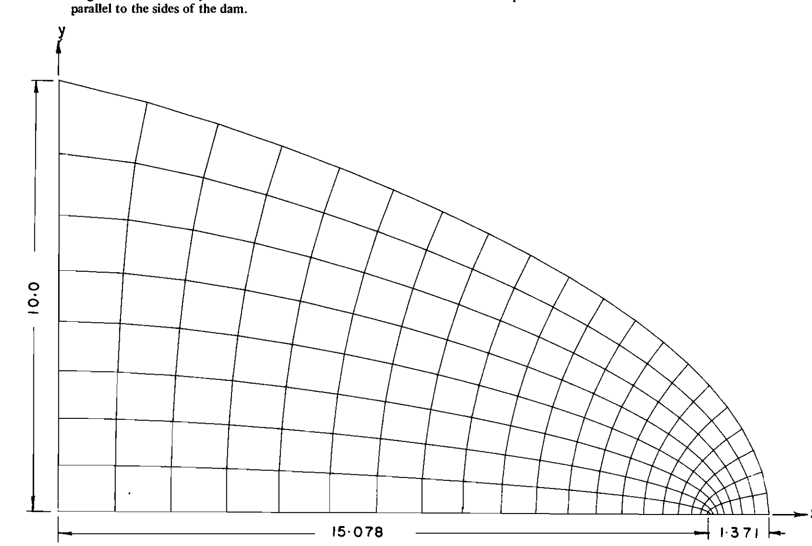

Fig. 42. Flownet from the Φψ plane associated with k=5 obtained by projecting the magnitudes of x and y from the solution to Problem No. 3 onto a vertical plane parallel to the sides of the dam.

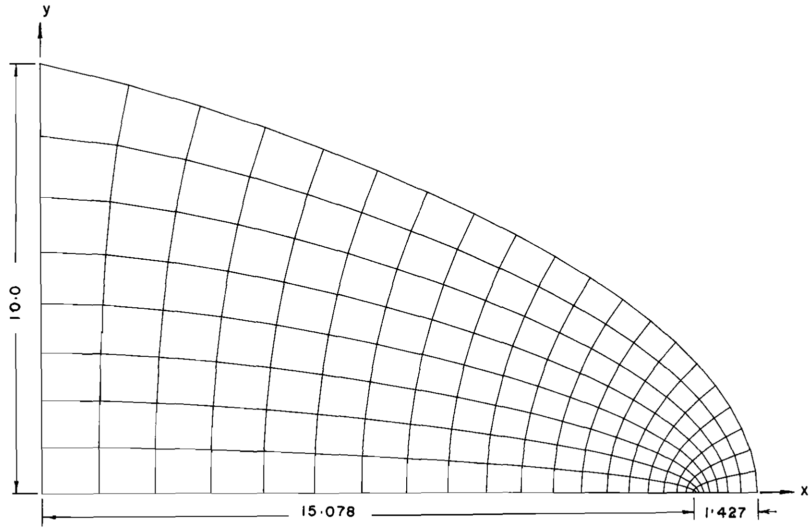

Fig. 43. Flownet from the Φψ plane associated with k=6 obtained by projecting the magnitudes of x and y from the solution to Problem No. 3 onto a vertical plane parallel to the sides of the dam.

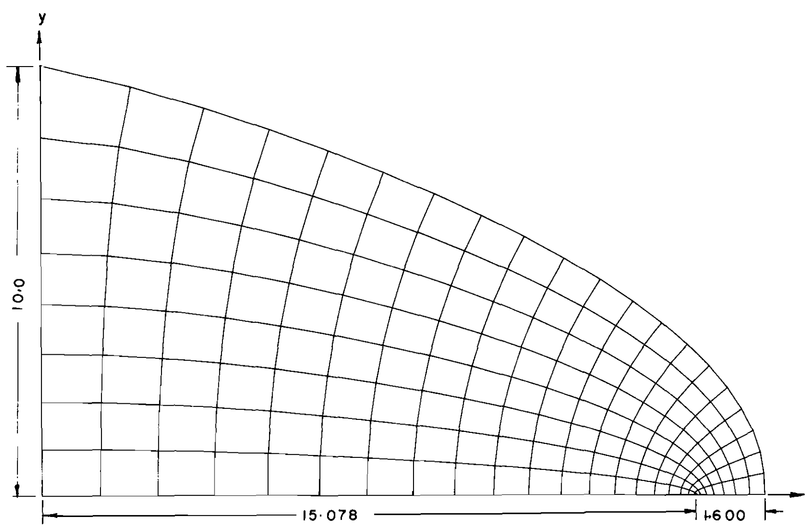

Fig. 44. Flownet from the Φψ plane associated with k=7 obtained by projecting the magnitudes of x and y from the solution to Problem No. 3 onto a vertical plane parallel to the sides of the dam.

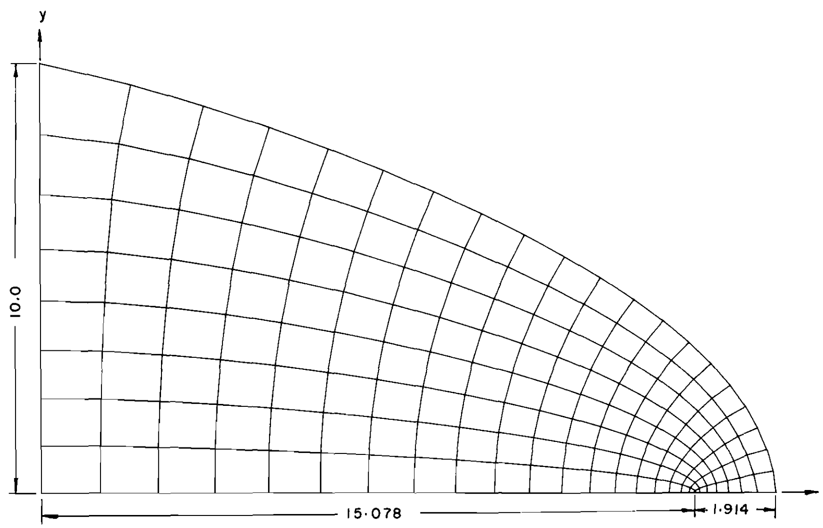

Fig. 45. Flownet from the Φψ plane associated with k=8 obtained by projecting the magnitudes of x and y from the solution to Problem No. 3 onto a vertical plane parallel to the sides of the dam.

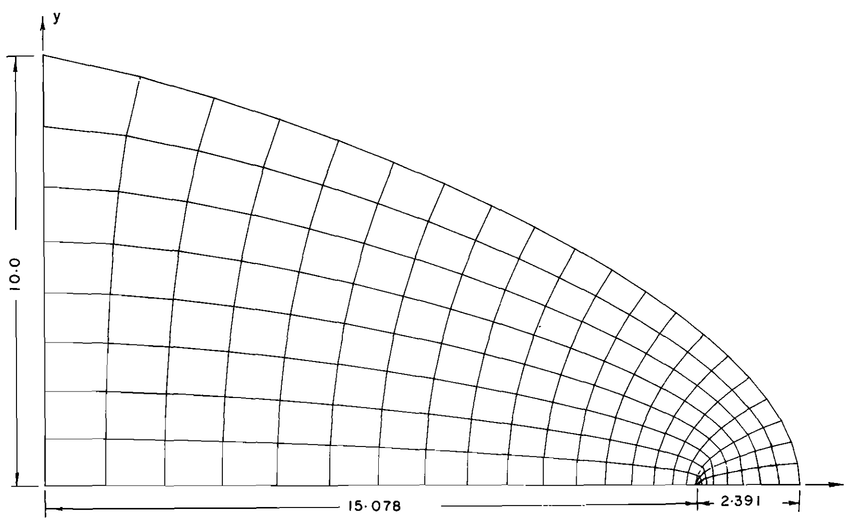

Fig. 46. Flownet from the Φψ plane associated with k=9 obtained by projecting the magnitudes of x and y from the solution to Problem No. 3 onto a vertical plane parallel to the sides of the dam.

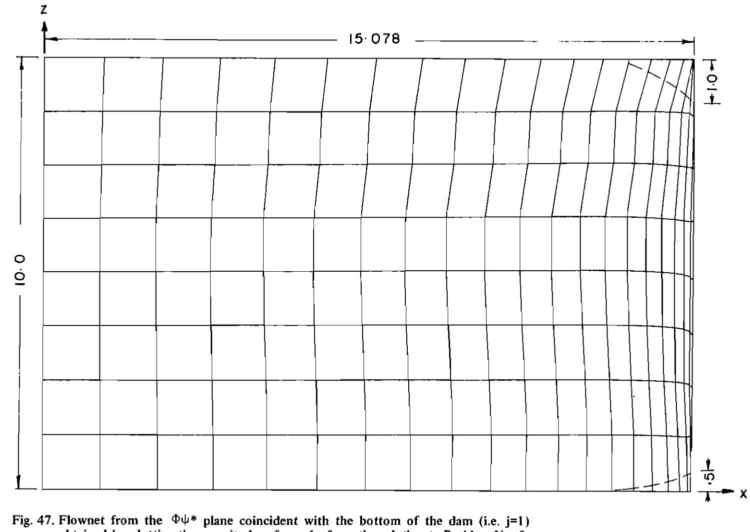

Fig. 47. Flownet from the $\Phi\psi^*$ plane coincident with the bottom of the dam (i.e. j=1) obtained by plotting the magnitudes of x and z from the solution to Problem No. 3.

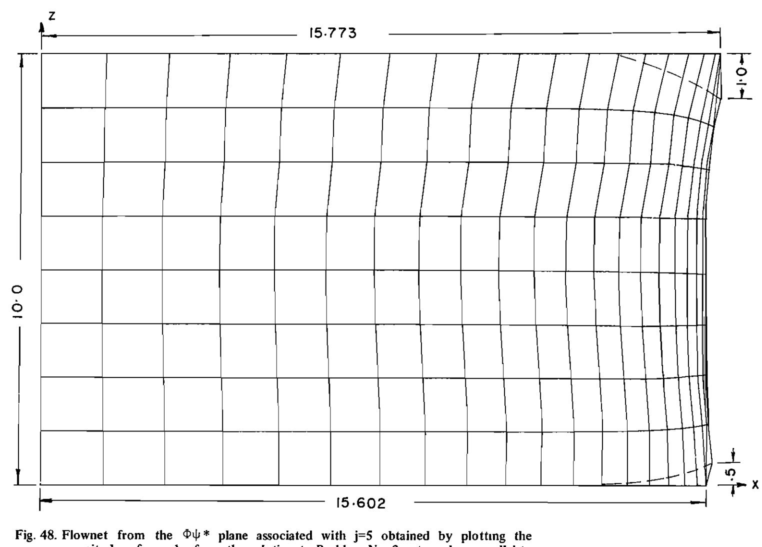

Fig. 48. Flownet from the $\Phi\psi^*$ plane associated with j=5 obtained by plotting the magnitudes of x and z from the solution to Problem No. 3 onto a plane parallel to the bottom of the dam.

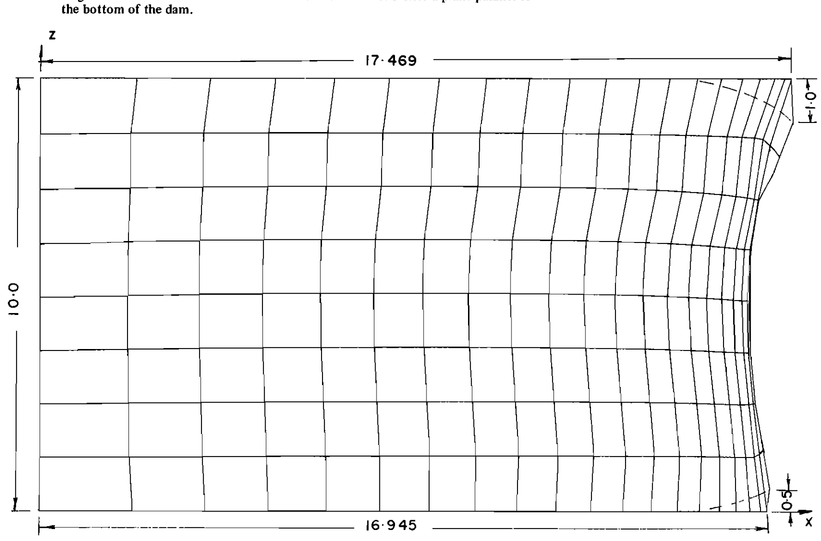

Fig. 49. Flownet from the $\Phi\psi^*$ plane coincident with the phreatic surface (i.e. j=9) obtained by plotting the magnitudes of x and z from the solution to Problem No. 3 onto a plane parallel to the bottom of the dam.

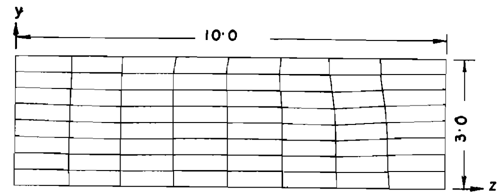

Fig. 50. Flownet from the $\psi\psi^*$ plane associated with i=15 obtained by plotting the magnitudes of z and y from the solution to Problem No. 3 onto a vertical plane parallel to the face of the dam.

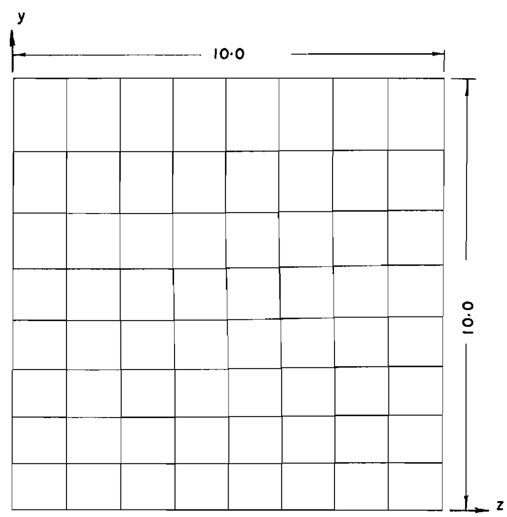

Fig. 51. Flownet from the $\psi\psi^*$ plane coincident with the face of the dam (i.e. i=1) obtained by plotting the magnitudes of z and y from the solution to Problem No. 3.

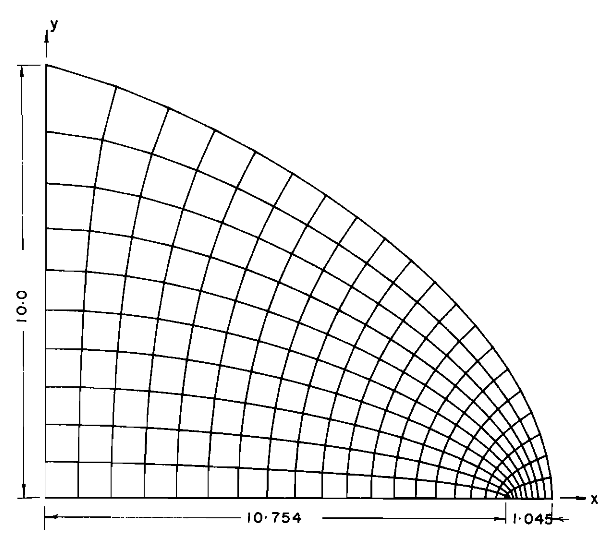

Fig. 52. Flownet from the $\Phi\psi$ plane associated with k=1 obtained by projecting the magnitudes of x and y from the solution to Problem No. 4 onto a vertical plane parallel to the sides of the dam.

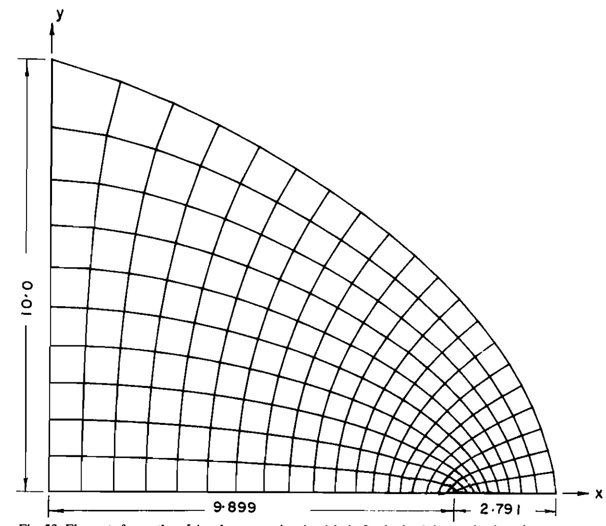

Fig. 53. Flownet from the $\Phi\psi$ plane associated with k=2 obtained by projecting the magnitudes of x and y from the solution to Problem No. 4 onto a vertical plane parallel to the sides of the dam.

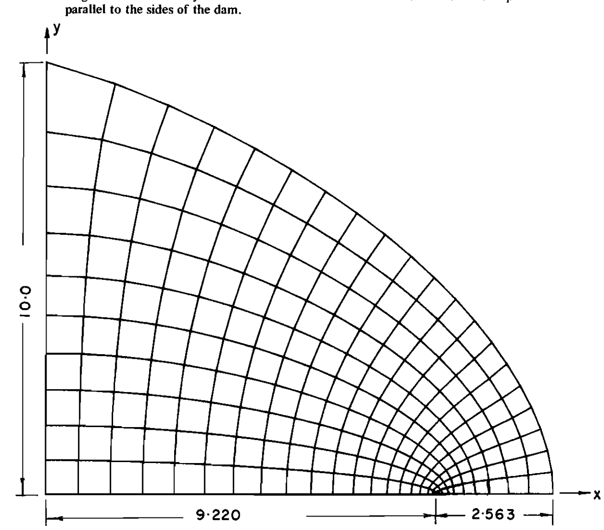

Fig. 54. Flownet from the $\Phi\psi$ plane associated with k=3 obtained by projecting the magnitudes of x and y from the solution to Problem No. 4 onto a vertical plane parallel to the sides of the dam.

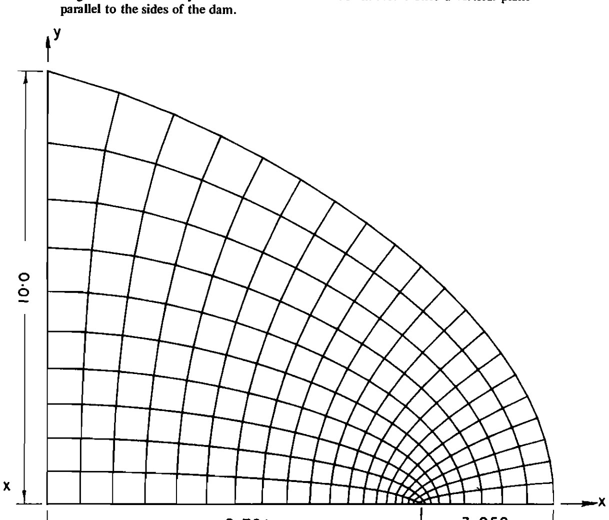

Fig. 55. Flownet from the $\Phi\psi$ plane associated with k=4 obtained by projecting the magnitudes of x and y from the solution to Problem No. 4 onto a vertical plane parallel to the sides of the dam.

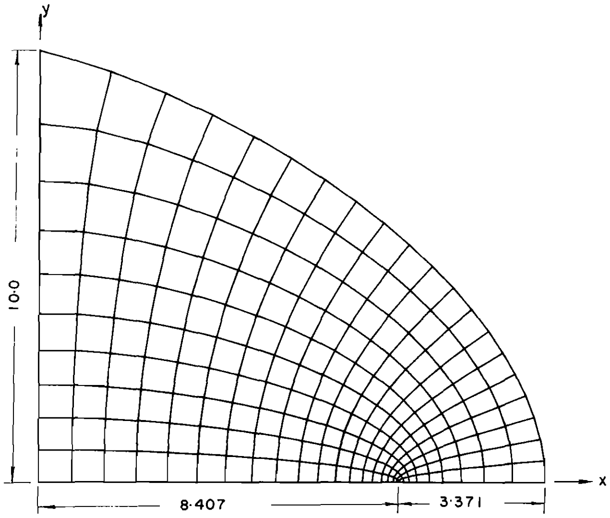

Fig. 56. Flownet from the $\Phi\psi$ plane associated with k=5 obtained by projecting the magnitudes of x and y from the solution to Problem No. 4 onto a vertical plane parallel to the sides of the dam.

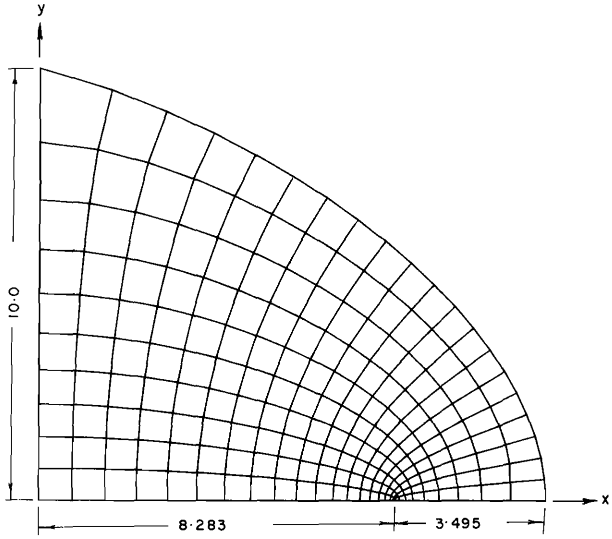

Fig. 57. Flownet from the $\Phi\psi$ plane associated with k=6 obtained by projecting the magnitudes of x and y from the solution to Problem No. 4 onto a vertical plane parallel to the sides of the dam.

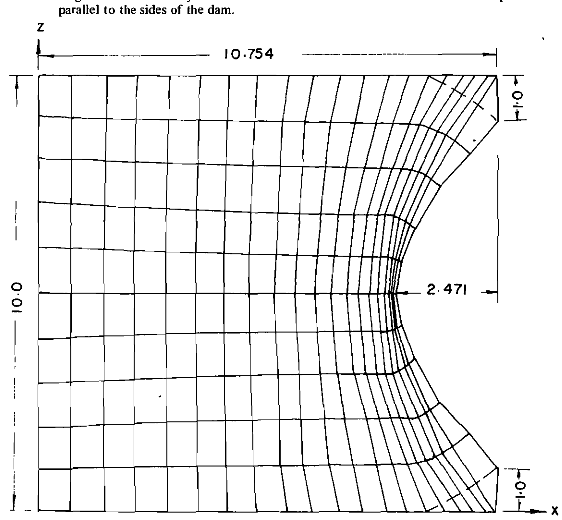

Fig. 58. Flownet from the $\Phi\psi^*$ plane coincident with the bottom of the dam (i.e. j=1) obtained by plotting the magnitudes of x and z from the solution to Problem No. 4.

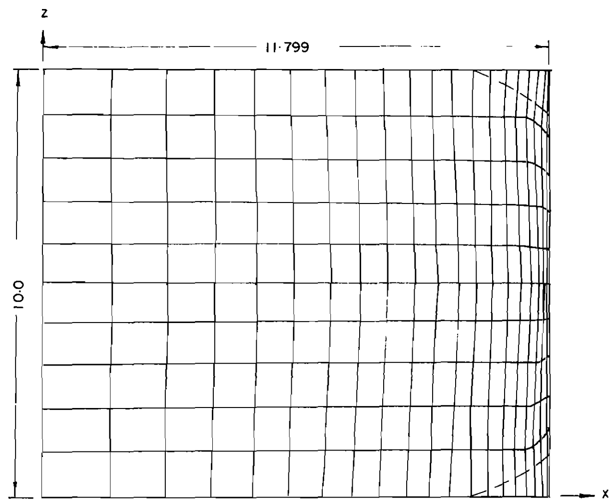

Fig. 59. Flownet from the $\Phi\psi^*$ plane coincident with the phreatic surface (i.e. j=11) obtained by plotting the magnitudes of x and z from the solution to Problem No. 4 onto a plane parallel to the bottom of the dam.

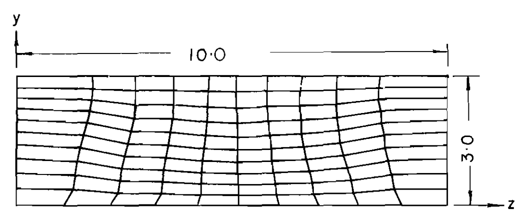

Fig. 60. Flownet from the $\psi\psi^*$ plane associated with i=15 obtained by plotting the magnitudes of z and y from the solution to Problem No. 4 onto a vertical plane parallel to the face of the dam.

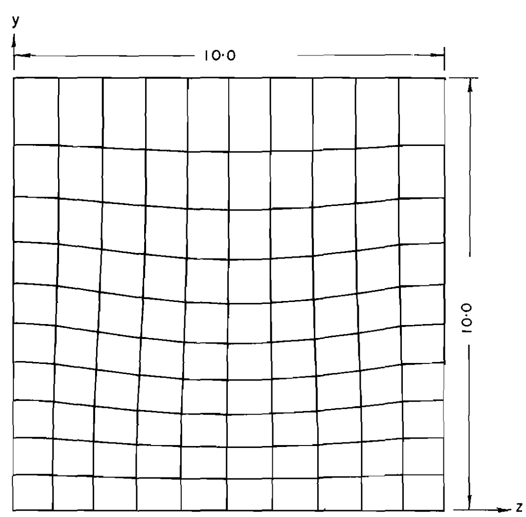

Fig. 61. Flownet from the $\psi\psi^*$ plane coincident with the face of the dam (i.e. i=1) obtained by plotting the magnitudes of z and y from the solution to Problem No. 4.

Conclusions

The methods defined in this report show promise for obtaining finite difference solutions to three-dimensional problems with free surfaces. By changing the conventional roles played by the variables of the problem, a free surface, with an unknown position in the physical space, can become a plane of known position in an inverse space, and consequently an inverse formulation such as that given in this report for potential flows has definite advantages. These advantages occur at the expense of more complex partial differential equations which must be solved.

The methods used in this report for solving the inverse partial differential equations from three-dimensional ideal fluid flows are practical with presently available high speed digital computers. The computer time required for a solution will depend upon a number of factors such as the number of finite difference grid points, the initialization used, the nature of the particular problem, and the accuracy required before terminating the iterative (i.e. cyclic) solution process. The solutions obtained in this report required approximately 15 to 20 minutes each of execution time in the UNIVAC 1108 system at the University of Utah. These problems used 2541 grid points (fewer were used for Problem Nos. 2 and 3) throughout the space of the problem. Since the three unknowns $x(\Phi, \psi, \psi^*)$, $y(\Phi, \psi, \psi^*)$, and $z(\Phi, \psi, \psi^*)$ must be solved simultaneously, three times this many finite difference grid points, or 7623, were actually used. Additional study should produce more efficient means for solving the resulting space boundary value problem than the method used in this report. At that time the merits of inverse formulations will be even greater.

The principal objectives of this study were to develop the inverse formulation and demonstrate its applicability in solving a three-dimensional problem. Consequently, major emphasis was not given to obtaining as accurate a solution as would be possible. For the problem investigated, greater accuracy could be achieved by giving special consideration to the singularities of the problem. Methods for improving the finite differences solution to two-dimensional problems in the vicinity of singularities can be modified to improve the finite difference solution to three-dimensional problems. The method of "patching in" an appropriate analytic solution should be quite easily adapted to three-dimensional flows. Despite the fact that no special consideration was given to the singularity, the methods used in this report yield what appear to be reasonably accurate solutions.

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APPENDIX A—DESCRIPTION OF BLOCK ITERATIVE METHODS INVESTIGATED FOR SIMULTANEOUS SOLUTIONS OF FIRST ORDER PARTIAL DIFFERENTIAL EQUATIONS

Since the basic inverse partial differential equations which describe ideal three-dimensional fluid flow are first order and three in number, and since these three equations cannot be combined by differentiation to give a single equation for each of the three dependent variables which is not extremely complex, an initial study investigated the feasibility of using block iterative methods to obtain finite difference solutions to simultaneous first order partial differential equations. This study utilized the first order inverse equations

$$\frac{\partial y}{\partial \psi} - \frac{\partial x}{\partial \Phi} = 0 \quad \ldots\ldots\ldots \quad \text{(A-1)}$$

and

$$\frac{\partial x}{\partial \psi} + \frac{\partial y}{\partial \Phi} = 0 \quad \ldots\ldots\ldots \quad \text{(A-2)}$$

which apply to ideal two-dimensional flows and which are derived by inverting the two equations which result from expressing the horizontal and vertical components of velocity in terms of the potential function and the two-dimensional stream function.

Actually none of the methods investigated that used second or higher order differences were convergent. They are described here to prevent other researchers from devoting effort investigating similar unsuccessful solution methods. The writer knows of no easier method for demonstrating convergence or nonconvergence of the block iterative methods used than to actually implement them in a computer program for solving a simple problem, and this has been done. This Appendix describes the methods used.

The first method solved the finite difference equations across the ψ equal constant grid lines simultaneously before proceeding to the next ψ equal constant line in the iterative process. The finite difference operator was based on a third degree polynomial passing through four consecutive grid points. For a grid network with $\Delta\Phi = \Delta\psi$, these operators for Eqs. A-1 and A-2 are:

$$-\frac{1}{3} y_{ij-1} - \frac{1}{2} y_{ij} + y_{ij+1} - \frac{1}{6} y_{ij+2} + \frac{1}{3} x_{i-1j} + \frac{1}{2} x_{ij} - x_{i+1j} + \frac{1}{6} x_{i+2j} = 0 \quad \ldots\ldots\ldots\ldots \quad \text{(A-3a)}$$

and

$$\frac{1}{3} x_{ij-1} - \frac{1}{2} x_{ij} + x_{ij+1} - \frac{1}{6} x_{ij+2} - \frac{1}{3} y_{i-1j} - \frac{1}{2} y_{ij} + y_{i+1j} - \frac{1}{6} y_{i+2j} = 0 \quad \ldots\ldots\ldots\ldots \quad \text{(A-4a)}$$

for i=2,3,... n-2

and for the final grid point adjacent to the final Φ equal constant boundary

$$-\frac{1}{3} y_{n-1j-1} - \frac{1}{2} y_{n-1j} + y_{n-1j+1} - \frac{1}{6} y_{n-1j+2} - \frac{1}{3} x_{nj} - \frac{1}{2} x_{n-1j} + x_{n-2j} + x_{n-2j} - \frac{1}{6} x_{n-3j} = 0 \quad \ldots\ldots\ldots \quad \text{(A-3b)}$$

and

$$-\frac{1}{3} x_{n-1j-1} - \frac{1}{2} x_{n-1j} + x_{n-1j+1} - \frac{1}{6} x_{n-1j+2} + \frac{1}{3} y_{nj} + \frac{1}{2} y_{n-1j} - y_{n-2j} + \frac{1}{6} y_{n-3j} = 0 \quad \ldots \quad \text{(A-4b)}$$

in which $i = 1 + \Phi/\Delta\Phi$ and $j = 1 + \psi/\Delta\psi$. For the last ψ equal constant grid line adjacent to the upper boundary of the problem, Eqs. A-3 and A-4 must be changed in a manner similar to the change between the (a) equation and the (b) equation such that the value of j does not exceed the number for the top ψ equal constant boundary.

Upon applying Eq. A-3 and then A-4 at each grid point across the ψ equal constant grid lines results in the following system of equations, written in matrix form.

```
      DO 21 I=2.LM
      IM=I-1
      XX=V1(IM.K)+DIF1*V1(IM.K)
      DFT=XX-X(I.M.K)
      X(I.M.K)=XX
      ABT=V3(IM.K)
   61 DBT=(ABT-DFT)/FM
      DO 22 J=2.MM
   22 X(I.J.K)=X(I.J.K)+ABT-DBT*FLOAT(J-1)
   21 CONTINUE
      DBT=(A(K)-XXM+X(L.M.K))/FM
      DO 35 J=2.MM
   35 X(L.J.K)=X(L.J.K)+A(K)-DBT*FLOAT(J-1)
      X(L.M.K)=XXM
      GO TO 20
   33 DO 25 I=2.LM
      IM=I-1
      XX=V2(IM.1)+DIF1*V2(IM.1)
      DFT=XX-X(I.M.K)
      X(I.M.K)=XX
      ABT=V2(IM.3)
   62 DBT=(ABT-DFT)/FM
      DO 23 J=2.MM
   23 X(I.J.K)=X(I.J.K)+ABT-DBT*FLOAT(J-1)
   25 CONTINUE
      DBT=(AA1-XXM+X(L.M.K))/FM
      DO 36 J=2.MM
   36 X(L.J.K)=X(L.J.K)+AA1-DBT*FLOAT(J-1)
      X(L.M.K)=XXM
      GO TO 20
   24 DO 26 I=2.LM
      IM=I-1
      XX=V2(IM.2)+DIF1*V2(IM.2)
      DFT=XX-X(I.M.K)
      X(I.M.K)=XX
   63 ABT=V2(IM.4)
      DBT=(ABT-DFT)/FM
      DO 27 J=2.MM
   27 X(I.J.K)=X(I.J.K)+ABT-DBT*FLOAT(J-1)
   26 CONTINUE
      DBT=(AA2-XXM+X(L.M.K))/FM
      DO 37 J=2.MM
   37 X(L.J.K)=X(L.J.K)+AA2-DBT*FLOAT(J-1)
      X(L.M.K)=XXM
   20 CONTINUE
      RETURN
   10 SUM=0.0
      DO 9 K=1.N
    9 SUM=SUM+XT(K)
      SUM=SUM/FLOAT(N)
   64 DO 11 K=1.N2
      IF(NIBD .EQ. 0) GO TO 65
      IF(NNOT .EQ. 0) GO TO 66
      SUM=XT(K)
       GO TO 65
   66 FK=K-KCET
      FK2=FK*FK
      SUM=B0+B1*FK+B2*FK2+B3*FK*FK2
   65 DIF=(SUM-XT(K))/XT(K)+1.0
      WRITE(6.109) K.DIF.SUM
      DO 12 I=2.LM
      IM=I-1
      IF(NIBT) 80.71.80
   71 XX=DIF*X(I.1.K)
      GO TO 72
   80 XX=DIF*V1(IM.K)
   72 V3(IM.K)=XX-X(I.1.K)
   12 X(I.1.K)=XX
      A(K)=SUM-X(L.1.K)
   11 X(L.1.K)=SUM
      IF(NIBD .EQ. 0) GO TO 67
      IF(NNOT .EQ. 0) GO TO 68
      SUM=XT(NM)
      GO TO 67
   68 FK=NM-KCET
      FK2=FK*FK
      SUM=B0+B1*FK+B2*FK2+B3*FK*FK2
   67 DIF=(SUM-XT(NM))/XT(NM)+1.0
      WRITE(6.109) NM.DIF.SUM
      DO 43 I=2.LM
      IM=I-1
      IF(NIBT) 73.74.73
   74 XX=DIF*X(I.1.NM)
      GO TO 75
   73 XX=DIF*V2(IM.1)
   75 V2(IM.3)=XX-X(I.1.NM)
   43 X(I.1.NM)=XX
      AA1=SUM-X(L.1.NM)
      X(L.1.NM)=SUM
      IF(NIBD .EQ. 0) GO TO 69
      IF(NNOT .EQ. 0) GO TO 70
      SUM=XT(N)
      GO TO 69
   70 FK=N-KCET
      FK2=FK*FK
      SUM=B0+B1*FK+B2*FK2+B3*FK*FK2
   69 DIF=(SUM-XT(N))/XT(N)+1.0
      WRITE(6.109) N .DIF.SUM
      DO 14 I=2.LM
      IM=I-1
      IF(NIBT) 76.77.76
   77 XX=DIF*X(I.1.N)
      GO TO 78
   76 XX=DIF*V2(IM.2)
   78 V2(IM.4)=XX-X(I.1.N)
   14 X(I.1.N)=XX
      AA2=SUM-X(L.1.N)
      X(L.1.N)=SUM
```

```
      RETURN
      END
@FOR.IS BZTOPB.BZTOPB
      SUBROUTINE ZTOPB(NBOT.IZFRST.ZFINL.ZEND1)
      COMMON X(21.11.11).Y(21.11.11).Z(21.11.11).C1(19.9).C2(19.9).C3(19
     $.9).C4(19.9).C5(19.9).A(19).B(19).G(19).H.FL.
     $FM.FN.DEL.ZF.Z1.ERR.L.LM.M.MM.N.NM.L2.M2.N2.MAX.MAXCT.NRT
     $.ZT(11)
C  COMPUTES Z-CORD. ON TOP OR BOTTOM.IF NBOT=0 THEN BOTTOM
C  IF IZFRST=0 DO NOT DETERMINE Z=C BY INTEGRATING NEAR FRONT
      W1=.4
      SUM1=Z(1.1.N)
      IF(NBOT .GT. 0) GO TO 1
      JJ=1
      J1=2
      GO TO 2
    1 JJ=M
      J1=MM
    2 IF(IZFRST .EQ. 0) GO TO 10
      ZT(1)=0.0
      SUM=0.0
      E1=.54166667
      E2=.04166667
      E3=.66666667
      E4=.08333333
      DO 3 J=3.M2
      J2=J-2
      JM=J-1
      JP=J+1
      J0=J+2
      DX1=E3*(X(4.J.1)-X(2.J.1))-E4*(X(5.J.1)-X(1.J.1))
      DX2=E3*(X(4.J.2)-X(2.J.2))-E4*(X(5.J.2)-X(1.J.2))
      DX3=E3*(X(4.J.3)-X(2.J.3))-E4*(X(5.J.3)-X(1.J.3))
      DY1=E3*(Y(3.JP.1)-Y(3.JM.1))-E4*(Y(3.J0.1)-Y(3.J2.1))
      DY2=E3*(Y(3.JP.2)-Y(3.JM.2))-E4*(Y(3.J0.2)-Y(3.J2.2))
      DY3=E3*(Y(3.JP.3)-Y(3.JM.3))-E4*(Y(3.J0.3)-Y(3.J2.3))
      D1=DX1/DY1
      D2=DX2/DY2
      D3=DX3/DY3
      ZT(2)=.5*(D1+D2)
      Z(1.J.2)=ZT(2)
      DO 4 K=3.NM
      KP=K+1
      DX4=E3*(X(4.J.KP)-X(2.J.KP))-E4*(X(5.J.KP)-X(1.J.KP))
      DY4=E3*(Y(3.JP.KP)-Y(3.JM.KP))-E4*(Y(3.J0.KP)-Y(3.J2.KP))
      D4=DX4/DY4
      ZT(K)=ZT(K-1)+E1*(D2+D3)-E2*(D1+D4)
      Z(1.J.K)=ZT(K)
      D1=D2
      D2=D3
    4 D3=D4
      ZT(N)=ZT(NM)+.5*(D3+D2)
      Z(1.J.N)=ZT(N)
      IF(NRT .GT. 100) GO TO 3
      WRITE(6.100) (ZT(K).K=1.N)
  100 FORMAT(1H .16F8.3)
    3 SUM=SUM+ZT(N)
      SUM=SUM/FLOAT(MM-3)
      WRITE(6.115) SUM.ZFINL
  115 FORMAT(' AVE Z='.F10.4.' ZFINAL='.F10.4)
      IF(IZFRST .LT. 0) SUM=ZFINL
      DO 6 J=3.M2
      DIF=(Z(1.J.N)-SUM)/Z(1.J.N)
      DO 21 K=2.NM
   21 Z(1.J.K)=Z(1.J.K)-DIF*Z(1.J.K)
    6 Z(1.J.N)=SUM
      IF(IZFRST .LT. -1) GO TO 24
      DO 5 I=1.LM
      DO 5 J=1.M
    5 Z(I.J.N)=SUM
      DO 7 J=1.M
      Z(L.J.1)=Z1
    7 Z(L.J.N)=ZEND1
   24 DO 8 K=2.N
      Z(1.2.K)=Z(1.3.K)
      Z(1.1.K)=Z(1.3.K)
      Z(1.MM.K)=Z(1.M2.K)
    8 Z(1.M.K)=Z(1.M2.K)
      IF(ABS(SUM-SUM1) .LT. .1) GO TO 10
      DIF=(SUM-SUM1)/FN
      DO 16 K=2.NM
      FK=DIF*FLOAT(K-1)
      DO 16 J=1.M
      DO 16 I=2.L
   16 Z(I.J.K)=Z(I.J.K)+FK
   10 NCT=0
      J=JJ
   20 SUM=0.0
      DO 11 K=2.NM
      KP=K+1
      KM=K-1
      DO 12 I=2.LM
      IP=I+1
      IM=I-1
      IF(NCT .GT. 0) GO TO 13
      CD=Y(I.J1.K)-Y(I.J.K)
      A1=CD*CD
      A2=.25*CD*(Y(I.J1.KP)-Y(I.J.KP)-Y(I.J1.KM)+Y(I.J.KM))
      A3=.25*(Y(IP.J1.K)-Y(IP.J.K)-Y(IM.J1.K)+Y(IM.J.K))/CD
      A31=1.0/(A3+1.)
      C1(IM.KM)=A31*(2.*A1+2.)
      C2(IM.KM)=A31*(A3-1.)
      C3(IM.KM)=A31*(A1-A2)
      C4(IM.KM)=A31*(A1+A2)
   13 A(IM)=C1(IM.KM)
      G(IM)=C2(IM.KM)
   12 B(IM)=C3(IM.KM)*Z(I.J.KM)+C4(IM.KM)*Z(I.J.KP)
      B(1)=B(1)+Z(1.J.K)
```

```
      B(L2)=B(L2)-G(L2)*Z(L,J,K)
      DO 14 I=2,L2
      IM=I-1
      A(I)=A(I)+G(IM)/A(IM)
   14 B(I)=B(I)+B(IM)/A(IM)
      I=L2
      TZ=B(I)/A(I)
      DIF=TZ-Z(LM,J,K)
      SUM=SUM+ABS(DIF)
      Z(LM,J,K)=TZ+W1*DIF
   15 IP=I
      I=I-1
      TZ=(B(I)-G(I)*TZ)/A(I)
      DIF=TZ-Z(IP,J,K)
      SUM=SUM+ABS(DIF)
      Z(IP,J,K)=TZ+W1*DIF
      IF(I .GT. 1) GO TO 15
   11 CONTINUE
      NCT=NCT+1
  112 FORMAT(' NCT=',I5,' SUM=',E12.6)
      IF(SUM .GT. 3000.) GO TO 64
      IF(SUM .GT. ERR .AND. NCT .LT. MAX) GO TO 20
   64 WRITE(6,204) NBOT,NCT,SUM
  204 FORMAT(' Z NBOT=',I3,' NCT=',I5,' SUM=',E12.6)
      IF(NRT .GT. 100) RETURN
      DO 31 K=1,N
   31 WRITE(6,100) (Z(I,J,K),I=1,L)
      RETURN
      END
@FOR,IS  XSIDE,XSIDE
      SUBROUTINE SIDEX(JBEG)
      INTEGER IEND(10)
      COMMON X(21,11,11),Y(21,11,11),Z(21,11,11),C1(19,9),C2(19,9),C3(19
     $,9),C4(19,9),C5(19,9),A(19),B(19),G(19),H,FL,
     $FM,FN,DEL,ZF,Z1,ERR,L,LDM,M,MM,N,NM,LD2,M2,N2,MAX,MAXCT,NRT
     $,ZT(11)
      W1=.4
      K=1
      KP=2
      F25=.25
      F5=.5
   10 NCT=0
      L2=LD2
      DO 30 J=JBEG,MM
   30 IEND(J)=LDM
   20 SUM=0.
      DO 1 J=JBEG,MM
      JM=J-1
      JP=J+1
      LM=LDM
      IF(IEND(J) .EQ. LDM) GO TO 31
      LM=IEND(J)
      L2=LM-1
   31 DO 2 I=2,LM
      IM=I-1
      IP=I+1
      IF(NCT .GT. 0) GO TO 3
      ZS=Z(I,J,KP)-Z(I,J,K)
      A3=.25*(Z(IP,J,KP)-Z(IP,J,K)-Z(IM,J,KP)+Z(IM,J,K))/ZS
      A31=1.0/(1.0+A3)
      IF(J .GT. 1) GO TO 12
      A1=2.*ZS*ZS
      C5(IM,1)=A31*(A1+2.)
      C5(IM,2)=A31*(A3-1.)
      C5(IM,3)=A31*A1
      GO TO 15
   12 A1=ZS*ZS
      A2=F25*ZS*(Z(I,JP,KP)-Z(I,JM,KP)-Z(I,JP,K)+Z(I,JM,K))
      C1(IM,JM)=A31*(2.+2.*A1)
      C2(IM,JM)=A31*(A3-1.)
      C3(IM,JM)=A31*(A1-A2)
      C4(IM,JM)=A31*(A1+A2)
    3 IF(J .GT. 1) GO TO 13
   15 A(IM)=C5(IM,1)
      G(IM)=C5(IM,2)
      B(IM)=C5(IM,3)*X(I,JP,K)
      GO TO 2
   13 A(IM)=C1(IM,JM)
      G(IM)=C2(IM,JM)
      B(IM)=C3(IM,JM)*X(I,JM,K)+C4(IM,JM)*X(I,JP,K)
    2 CONTINUE
      B(L2)=B(L2)-G(L2)*X(L,J,K)
      DO 4 I=2,L2
      I1=I-1
      A(I)=A(I)+G(I1)/A(I1)
    4 B(I)=B(I)+B(I1)/A(I1)
      I=L2
      XT=B(I)/A(I)
      DIF=XT-X(LM,J,K)
      SUM=SUM+ABS(DIF)
      X(LM,J,K)=XT+W1*DIF
    5 IP=I
      I=I-1
      XT=(B(I)-G(I)*XT)/A(I)
      DIF=XT-X(IP,J,K)
      SUM=SUM+ABS(DIF)
      X(IP,J,K)=XT+W1*DIF
      IF(I .GT. 1) GO TO 5
      LPLS=LM+1
      XT=X(LPLS,J,K)
      I=LM
   32 IF(X(I,J,K) .LT. XT) GO TO 1
      X(I,J,K)=XT-.015*FLOAT(LPLS-I)
      I=I-1
      IEND(J)=I
      GO TO 32
    1 CONTINUE
      NCT=NCT+1
      IF( SUM .GT. 3000.) GO TO 6
      IF(SUM .GT. ERR .AND. NCT .LT. MAX) GO TO 20
      WRITE(6,113) K,NCT,SUM
  113 FORMAT(' XSIDE=',I3,' NCT=',I5,' SUM=',E12.6)
      IF(NRT .GT. 100) GO TO 42
      DO 8 J=1,M
    8 WRITE(6,100) (X(I,J,K),I=1,L)
  100 FORMAT(1H ,16F6.3,5F7.3)
   42 IF(K .GT. 1) RETURN
      K=N
      KP=NM
      GO TO 10
    6 WRITE(6,114) K,NCT,SUM
  114 FORMAT(' XSIDE=',I3,' NCT=',I5,' SUM=',F12.5,' SOLUTION BLEW UP')
      STOP
      END
@FOR,IS SZINT,SZINT
      SUBROUTINE ZINTER
      COMMON X(21,11,11),Y(21,11,11),Z(21,11,11),C1(19,9),C2(19,9),C3(19
     $,9),C4(19,9),C5(19,9),A(19),B(19),C(19),H,FL,
     $FM,FN,DEL,ZF,Z1,ERR,L,LM,M,MM,N,NM,L2,M2,N2,MAX,MAXCT,NRT
      NCOUNT=1
   26 SUMM=0.0
      NCTMAX=0
      DO 25 J=2,MM
      JM=J-1
      JP=J+1
      NCT=0
   20 SUMZ=0.0
      DO 1 K=2,NM
      KM=K-1
      KP=K+1
      DO 2 I=2,LM
      IM=I-1
      IP=I+1
      IF(NCT .GT. 0) GO TO 3
      CD=.5*(Y(I,JP,K)-Y(I,JM,K))
      A1=CD*CD
      DDT=.125*(Y(I,JP,KP)-Y(I,JM,KP)-Y(I,JP,KM)+Y(I,JM,KM))
      A3=CD*DDT
      A2=.125*(Y(IP,JP,K)-Y(IP,JM,K)-Y(IM,JP,K)+Y(IM,JM,K))/CD
      CF=.5*(Y(I,J,KP)-Y(I,J,KM))
      CI=.5*(X(I,JP,K)-X(I,JM,K))
      DFT=Y(I,J,KP)+Y(I,J,KM)-2.*Y(I,J,K)
      DIP=.25*(X(IP,JP,K)-X(IP,JM,K)-X(IM,JP,K)+X(IM,JM,K))
      CG=.5*(Z(I,JP,K)-Z(I,JM,K))
      DFP=.25*(Y(IP,J,KP)-Y(IP,J,KM)-Y(IM,J,KP)+Y(IM,J,KM))
      DGT=.25*(Z(I,JP,KP)-Z(I,JM,KP)-Z(I,JP,KM)+Z(I,JM,KM))
      A5=2.*A2*CI*CF-CD*(CF*DGT+CG*DFT)-CI*DFP-CF*DIP
      DBIN=1.0/(1.0+A2)
      C1(IM,KM)=2.* (1.+A1)*DBIN
      C2(IM,KM)=DBIN*(A2-1.0)
      C3(IM,KM)=DBIN*(A1-A3)
      C4(IM,KM)=DBIN*(A1+A3)
      C5(IM,KM)=A5*DBIN
    3 A(IM)=C1(IM,KM)
      C(IM)=C2(IM,KM)
    2 B(IM)= C3(IM,KM)*Z(I,J,KM)+C4(IM,KM)*Z(I,J,KP)+C5(IM,KM)
      B(1)=B(1)+Z(1,J ,K)
      B(L2)=B(L2)-C(L2)*Z(L,J,K)
      DO 14 I=2,L2
      I1=I-1
      A(I)=A(I)+C(I1)/A(I1)
   14 B(I)=B(I)+B(I1)/A(I1)
      I=L2
      IP=LM
      ZT=B(I)/A(I)
      DIF=Z(IP,J,K)-ZT
      SUMZ=SUMZ+ABS(DIF)
      Z(IP,J,K)=ZT+.4*DIF
   15 IP=I
      I=I-1
      ZT=(B(I)-C(I)*ZT)/A(I)
      DIF=Z(IP,J,K)-ZT
      SUMZ=SUMZ+ABS(DIF)
      Z(IP,J,K)=ZT+.4*DIF
      IF(I .GT. 1) GO TO 15
    1 CONTINUE
      NCT=NCT+1
  100 FORMAT(1H ,16F8.3)
      IF(SUM .GT. 1.E5) GO TO 56
      IF(NCT .LT. MAX .AND. SUMZ.GT.ERR ) GO TO 20
   56 IF(NCT .GT. NCTMAX) NCTMAX=NCT
      IF(MOD(NCOUNT,NRT) .GT. 0) GO TO 86
      DO 61 K=1,M
   61 WRITE(6,100)   (Z(I,J,K),I=1,L)
   86 SUMM=SUMM+SUM
      WRITE(6,204) J,NCT,SUMZ
  204 FORMAT(' SOLUTION FOR Z IN PSI PSIS PLANE J=',I3,' NCT=',I5,' SUM=
     $',E12.6)
   25 CONTINUE
   85 NCOUNT=NCOUNT+1
      WRITE(6,200) NCOUNT,SUMZ
  200 FORMAT(' NCOUNT=',I5,' SUM=',E12.6)
      IF(SUM .GT. 1.E5) STOP
      IF(NCTMAX .GT. 3 .AND. NCOUNT .LT. MAXCT) GO TO 26
      RETURN
      END
@FOR,IS SYINT,SYINT
      SUBROUTINE YINTER
      COMMON X(21,11,11),Y(21,11,11),Z(21,11,11),C1(19,9),C2(19,9),C3(19
     $,9),C4(19,9),C5(19,9),A(19),B(19),G(19),H,FL,
     $FM,FN,DEL,ZF,Z1,ERR,L,LM,M,MM,N,NM,L2,M2,N2,MAX,MAXCT,NRT
      W1=.4
      NCOUNT=1
   26 SUMM=0.
      NCTMAX=0
      DO 25 K=2,NM
```

```
      KM=K-1
      KP=K+1
      NCT=0
   20 SUM=0.0
      DO 1 J=2,MM
      JM=J-1
      JP=J+1
      DO 2 I=2,LM
      IM=I-1
      IP=I+1
       IF(NCT .GT. 0) GO TO 3
      CE=.5*(Z(I,J,KP)-Z(I,J,KM))
      DEP=.125*(Z(IP,J,KP)-Z(IP,J,KM)-Z(IM,J,KP)+Z(IM,J,KM))
      A1=CE*CE
      A2=.125*CE*(Z(I,JP,KP)-Z(I,JP,KM)-Z(I,JM,KP)+Z(I,JM,KM))
      A3=DEP/CE
      CH=.5*(X(I,J,KP)-X(I,J,KM))
      CG=Z(I,JP,K)-Z(I,JM,K)
      P1=A3*CH*CG
      P2=CE*(.5*(Y(I,J,KP)-Y(I,J,KM))*(Z(I,JP,K)+Z(I,JM,K)-2.*Z(I,J,K))+
     $.125*CG*(Y(I,JP,KP)-Y(I,JP,KM)-Y(I,JM,KP)+Y(I,JM,KM)))
      P3=.25*CH*(Z(IP,JP,K)-Z(IP,JM,K)-Z(IM,JP,K)+Z(IM,JM,K))+.125*CG*(
     $X(IP,J,KP)-X(IP,J,KM)-X(IM,J,KP)+X(IM,J,KM))
       B31=1.0/(1.0+A3)
      C1(IM,JM)=B31*(2.+2.*A1)
      C2(IM,JM)=B31*(A3-1.)
      C3(IM,JM)=B31*(A1-A2)
      C4(IM,JM)=B31*(A1+A2)
      C5(IM,JM)=B31*(P1-P2-P3)
    3 A(IM)=C1(IM,JM)
      G(IM)=C2(IM,JM)
    2 B(IM)=C3(IM,JM)*Y(I,JM,K)+C4(IM,JM)*Y(I,JP,K)+C5(IM,JM)
      B(1)=B(1)+Y(1,J,K)
      B(L2)=B(L2)-G(L2)*Y(L,J,K)
      DO 4 I=2,L2
      IM=I-1
      A(I)=A(I)+G(IM)/A(IM)
    4 B(I)=B(I)+B(IM)/A(IM)
      I=L2
      YT=B(I)/A(I)
      DIF=YT-Y(LM,J,K)
      SUM=SUM+ABS(DIF)
      Y(LM,J,K)=YT+W1*DIF
    5 IP=I
      I=I-1
      YT=(B(I)-G(I)*YT)/A(I)
      DIF=YT-Y(IP,J,K)
      SUM=SUM+ABS(DIF)
      Y(IP,J,K)=YT+W1*DIF
      IF(I .GT. 1) GO TO 5
      CE=.5*(Z(1,J,KP)-Z(1,J,KM))
      A1=CE*CE
      A2=.125*CE*(Z(1,JP,KP)-Z(1,JP,KM)-Z(1,JM,KP)+Z(1,JM,KM))
      A3=CE*(.5*(Y(1,J,KP)-Y(1,J,KM))*(Z(1,JP,K)+Z(1,JM,K)-2.*Z(1,J,K))+
     $.125*(Z(1,JP,K)-Z(1,JM,K))*(Y(1,JP,KP)-Y(1,JP,KM)-Y(1,JM,KP)+Y(1,
     $JM,KM)))
      Y(1,J,K)=(Y(2,J,K)+.5*((A1+A2)*Y(1,JP,K)+(A1-A2)*Y(1,JM,K)+A3))/
     $(1.+A1)
    1 CONTINUE
      NCT=NCT+1
  112 FORMAT(' NCT=',I5,' SUM=',E12.6)
      IF(SUM .GT. 3000.) GO TO 64
      IF(SUM .GT. ERR .AND. NCT .LT. MAX) GO TO 20
   64 IF(NCT .GT. NCTMAX) NCTMAX=NCT
      IF(MOD(NCOUNT,NRT) .GT. 0) GO TO 86
      DO 76 J=1,M
   76 WRITE(6,100)    (Y(I,J,K),I=1,L)
  100 FORMAT(1H ,16F8.3)
   86 SUMM=SUMM+SUM
      WRITE(6,204) K,NCT,SUM
  204 FORMAT(' SOLUTION FOR Y IN PHI-PSI PLANE K=',I5,' NCT=',I5,' SUM='
     $,E12.6)
   25 CONTINUE
      NCOUNT=NCOUNT+1
      WRITE(6,200) NCOUNT,SUMM,NCTMAX
  200 FORMAT(' NCOUNT=',I5,' SUMM=',E12.6,' NCTMAX=',I4)
      IF(SUMM .GT. 9000.) STOP
      IF(NCTMAX .GT.  3 .AND. NCOUNT .LT. MAXCT) GO TO 26
      RETURN
      END
@FOR,IS  XSPHI,XSPHI
      SUBROUTINE XINTER(JBEG)
      COMMON X(21,11,11),Y(21,11,11),Z(21,11,11),C1(19,9),C2(19,9),C3(19
     $,9),C4(19,9),C5(19,9),A(19),B(19),G(19),H,FL,
     $FM,FN,DEL,ZF,Z1,ERR,L,LDM,M,MM,N,NM,LD2,M2,N2,MAX,MAXCT,NRT
       REAL C6(19),C7(19),C8(19)
      INTEGER IEND(10)
      W1=.4
      NCOUNT=1
   26 SUMM=0.
      NCTMAX=0
      DO 25 K=2,NM
      KM=K-1
      KP=K+1
      L2=LD2
      DO 30 J=JBEG,MM
   30 IEND(J)=LDM
      NCT=0
   20  SUM=0.0
      DO 1 J=JBEG,MM
      JM=J-1
      JP=J+1
      LM=LDM
      IF(IEND(J) .EQ. LDM) GO TO 31
      LM=IEND(J)
      L2=LM-1
   31 DO 2 I=2,LM
      IM=I-1
      IP=I+1
      IF(NCT .GT. 0) GO TO 3
      CC=.5*(Z(I,J,KP)-Z(I,J,KM))
      CPHI=.25*(Z(IP,J,KP)-Z(IP,J,KM)-Z(IM,J,KP)+Z(IM,J,KM))
      B1=CC*CC
      B3=.5/CC*CPHI
      B31=1./(1.+B3)
      IF(J .GT. 1) GO TO 12
      B12=2.*B1
      C6(IM)=B31*(2.+B12)
      C7(IM)=B31*(B3-1.)
      C8(IM)=B31*B12
      GO TO 15
   12 B2=.125*CC*(Z(I,JP,KP)-Z(I,JP,KM)-Z(I,JM,KP)+Z(I,JM,KM))
      P1=.5*B3*(Y(I,J,KP)-Y(I,J,KM))*(Z(I,JP,K)-Z(I,JM,K))
      P2=CC*(.5*(X(I,J,KP)-X(I,J,KM))*(Z(I,JP,K)+Z(I,JM,K)-2.*Z(I,J,K))
     $+.125*(Z(I,JP,K)-Z(I,JM,K))*(X(I,JP,KP)-X(I,JP,KM)-X(I,JM,KP)+X(I,
     $JM,KM)))
      P3=.125*(Y(I,J,KP)-Y(I,J,KM))*(Z(IP,JP,K)-Z(IP,JM,K)-Z(IM,JP,K)+
     $Z(IM,JM,K))+.125*(Z(I,JP,K)-Z(I,JM,K))*(Y(IP,J,KP)-Y(IP,J,KM)-Y(IM
     $,J,KP)+Y(IM,J,KM))
       C5(IM,JM)=(P3-P1-P2)*B31
       C1(IM,JM)=(2.+2.*B1)*B31
       C2(IM,JM)=B31*(B3-1.)
       C3(IM,JM)=B31*(B1-B2)
       C4(IM,JM)=B31*(B1+B2)
    3 IF(J .GT. 1) GO TO 13
   15 A(IM)=C6(IM)
      B(IM)=C8(IM)*X(I,JP,K)
      G(IM)=C7(IM)
      GO TO 2
   13 A(IM)=C1(IM,JM)
      G(IM)=C2(IM,JM)
      B(IM)=C3(IM,JM)*X(I,JM,K)+C4(IM,JM)*X(I,JP,K)+C5(IM,JM)
    2 CONTINUE
      B(L2)=B(L2)-G(L2)*X(L,J,K)
      DO 4 I=2,L2
      I1=I-1
      A(I)=A(I)+G(I1)/A(I1)
    4 B(I)=B(I)+B(I1)/A(I1)
      I=L2
      XT=B(I)/A(I)
      DIF=XT-X(LM,J,K)
      SUM=SUM+ABS(DIF)
      X(LM,J,K)=XT+W1*DIF
    5 IP=I
      I=I-1
      XT=(B(I)-G(I)*XT)/A(I)
      DIF=XT-X(IP,J,K)
      SUM=SUM+ABS(DIF)
      X(IP,J,K)=XT+W1*DIF
      IF(I .GT. 1) GO TO 5
      LPLS=LM+1
      I=LM
      XT=X(LPLS,J,K)
   32 IF(X(I,J,K) .LT. XT) GO TO 1
      X(I,J,K)=XT-.015*FLOAT(LPLS-I)
      I=I-1
      IEND(J)=I
      GO TO 32
    1 CONTINUE
      NCT=NCT+1
  112 FORMAT(' NCT=',I5,' SUM=',E12.6)
      IF(SUM .GT. 100.) STOP
      IF(SUM .GT. 3000.) GO TO 64
      IF(SUM .GT. ERR .AND. NCT .LT. MAX) GO TO 20
   64 IF(NCT .GT. NCTMAX) NCTMAX=NCT
      IF(MOD(NCOUNT,NRT) .GT. 0) GO TO 86
      DO 76 J=1,M
   76 WRITE(6,100)    (X(I,J,K),I=1,L)
  100 FORMAT(1H ,16F8.3)
   86 SUMM=SUMM+SUM
      WRITE(6,205) K,NCT,SUM
  205 FORMAT(' SOLUTION FOR X IN PHI-PSI PLANE K=',I3,' NCT=',I5,' SUM='
     $,E12.6)
   25 CONTINUE
      NCOUNT=NCOUNT+1
      WRITE(6,200) NCOUNT,SUMM,NCTMAX
  200 FORMAT(' NCOUNT=',I5,' SUMM=',E12.6,' NCTMAX=',I4)
      IF(SUMM .GT. 9000.) STOP
      IF(NCTMAX .GT.  3 .AND. NCOUNT .LT. MAXCT) GO TO 26
      RETURN
      END
@FOR,I  XDRAIN,XDRAIN
      SUBROUTINE DRAINX
      COMMON X(21,11,11),Y(21,11,11),Z(21,11,11),C1(19,9),C2(19,9),C3(19
     $,9),C4(19,9),C5(19,9),A(19),B(19),G(19),H,FL,
     $FM,FN,DEL,ZF,Z1,ERR,L,LM,M,MM,N,NM,L2,M2,N2,MAX,MAXCT,NRT
      L3=L-3
C  DERIVATIVES BASED ON FIRST AND SECOND ORDER DIFFERENCES
      D2=3.
      D3=1.5
      D4=.33333333
      D5=.66666667
C  FIRST ROW
      A(1)=X(L,1,1)
      DO 1 J=2,MM
      JP=J+1
      JM=J-1
      DD2=((X(L,J,2)-X(L,J,1))*(Z(L,JP,1)-Z(L,JM,1))*.5+D2*Y(LM,J,1)-D3*
     $Y(L2,J,1)+D4*Y(L3,J,1))/(Z(L,J,2)-Z(L,J,1))
      IF(J .EQ. 2) DD1=.5*DD2
      A(J)=A(JM)+.5*(DD1+DD2)
    1 DD1=DD2
      DD2=((X(L,M,2)-X(L,M,1))*(Z(L,M,1)-Z(L,MM,1))+D2*Y(LM,M,1)-D3*Y(L2
     $,M,1)+D4*Y(L3,M,1))/(Z(L,M,2)-Z(L,M,1))
      A(M)=A(MM)+.5*(DD1+DD2)
      IF(NRT .LT. 12)WRITE(6,100) (A(J),J=1,M)
  100 FORMAT(1H ,12F10.4)
      XB=X(L,1,1)
      DIF=(X(L,M,1)-A(M))/(A(M)-XB)
      DO 3 J=2,MM
    3 X(L,J,1)=A(J)+DIF*(A(J)-XB)
C  CENTER PORTION
      DO 2 K=2,NM
      A(1)=X(L,1,K)
```

```
      KM=K-1
      KP=K+1
      DO 4 J=2,MM
      JP=J+1
      JM=J-1
      DD2=(.5*(X(L,J,KP)-X(L,J,KM))*(Z(L,JP,K)-Z(L,JM,K))+6.*Y(LM,J,K)-
     $D2*Y(L2,J,K)+D5*Y(L3,J,K))/(Z(L,J,KP)-Z(L,J,KM))
      IF(J .EQ. 2) DD1=.5*DD2
      A(J)=A(JM)+.5*(DD1+DD2)
    4 DD1=DD2
      DD2=((X(L,M,KP)-X(L,M,KM))*(Z(L,M,K)-Z(L,MM,K))+6.*Y(LM,M,K)-D2*Y(
     $L2,M,K)+D5*Y(L3,M,K))/(Z(L,M,KP)-Z(L,M,KM))
      A(M)=A(MM)+.5*(DD1+DD2)
      IF(NRT .LT. 12)WRITE(6,100) (A(J),J=1,M)
      XB=X(L,1,K)
      DIF=(X(L,M,K)-A(M))/(A(M)-XB)
      DO 5 J=2,MM
    5 X(L,J,K)=A(J)+DIF*(A(J)-XB)
    2 CONTINUE
      A(1)=X(L,1,N)
      DO 6 J=2,MM
      JP=J+1
      JM=J-1
      DD2=((X(L,J,N)-X(L,J,NM))*(Z(L,JP,N)-Z(L,JM,N))*.5+D2*Y(LM,J,N)-D3
     $*Y(L2,J,N)+D4*Y(L3,J,N))/(Z(L,J,N)-Z(L,J,NM))
      IF(J .EQ. 2) DD1=.5*DD2
      A(J)=A(JM)+.5*(DD1+DD2)
    6 DD1=DD2
      DD2=((X(L,M,N)-X(L,M,NM))*(Z(L,M,N)-Z(L,MM,N))+D2*Y(LM,M,N)-D3*Y(
     $L2,M,N)+D4*Y(L3,M,N))/(Z(L,M,N)-Z(L,M,NM))
      A(M)=A(MM)+.5*(DD1+DD2)
      IF(NRT .LT. 12)WRITE(6,100) (A(J),J=1,M)
      XB=X(L,1,N)
      DIF=(X(L,M,N)-A(M))/(A(M)-XB)
      DO 7 J=2,MM
    7 X(L,J,N)=A(J)+DIF*(A(J)-XB)
      RETURN
      END
@FOR,I SUBIN,SUBIN
      SUBROUTINE INITAL(ZF,ZEND1)
      REAL KIZ
      COMMON X(21,11,11),Y(21,11,11),Z(21,11,11),C1(19,9),C2(19,9),C3(19
     $,9),C4(19,9),C5(19,9),A(19),B(19),G(19),H,FL,
     $FM,FN,DEL,FZ1,Z1,ERR,L,LM,M,MM,N,NM,L2,M2,N2,MAX,MAXCT,NRT
C  BOTTOM AND TOP(Y AND Z)
      CO=H/FL
      ZF=FN*DEL
      NC=N/2+1
      ZH3=(ZEND1-Z1)/FN
      FKZ2=0.
      FKZ1=0.
      DO 1 K=1,N
      DO 2 I=1,L
      Y(I,1,K)=0.0
      KIZ=(FLOAT(I-1)/FL)**3
      ZT=(1.-KIZ)*FKZ1+KIZ*FKZ2
      Z(I,1,K)=ZT
      Z(I,M,K)=ZT
      Y(I,M,K)=H-CO*FLOAT(I-1)
      DO 3 J=2,MM
    3 Z(I,J,K)=ZT
    2 CONTINUE
      FKZ1=FKZ1+DEL
      IF(K .EQ. 1) FKZ2=Z1
      FKZ2=FKZ2+ZH3
      IF(K .EQ. NM) FKZ2=ZF
    1 CONTINUE
      DO 4 I=1,L
      DELY=Y(I,M,1)/FM
      YT=DELY
      DO 4 J=2,MM
      Y(I,J,1)=YT
    4 YT=YT+DELY
      NCT=0
    5 SUM=0.
      DO 6 J=2,MM
      JP=J+1
      JM=J-1
      YT=.25*(Y(1,JP,1)+Y(1,JM,1)+2.*Y(2,J,1))
      DIF=YT-Y(1,J,1)
      SUM=SUM+ABS(DIF)
      Y(1,J,1)=YT+.6*DIF
      DO 6 I=2,LM
      YT=.25*(Y(I,JP,1)+Y(I,JM,1)+Y(I-1,J,1)+Y(I+1,J,1))
      DIF=YT-Y(I,J,1)
      SUM=SUM+ABS(DIF)
    6 Y(I,J,1)=YT+.6*DIF
      NCT=NCT+1
      IF(SUM .GT. .001 .AND. NCT .LT. 40) GO TO 5
      WRITE(6,100) NCT,SUM
  100 FORMAT(' INITIALIZING Y --  NCT=',I5,' SUM=',F12.6)
  101 FORMAT(1H0,11F10.3,20(/,1H ,11F10.3))
      DO 7 I=1,L
      DO 7 J=1,M
      YT=Y(I,J,1)
      DO 7 K=2,N
    7 Y(I,J,K)=YT
      DY1=Y(1,M,1)-Y(1,MM,1)
      X(1,M,1)=0.
      X(1,1,1)=0.
      DY3=Y(1,2,1)-Y(1,1,1)
      DO 8 I=2,L
      DY2=Y(I,M,1)-Y(I,MM,1)
      X(I,M,1)=X(I-1,M,1)+ .5*(DY1+DY2)
      DY1=DY2
      DY4=Y(I,2,1)-Y(I,1,1)
      X(I,1,1)=X(I-1,1,1)+ .5*(DY3+DY4)
    8 DY3=DY4
      DO 9 I=1,L
      DELX=(X(I,M,1)-X(I,1,1))/FM
      DO 9 J=2,MM
    9 X(I,J,1)=X(I,1,1)+DELX*FLOAT(J-1)
      NCT=0
   10 SUM=0.
      DO 11 J=2,MM
      JP=J+1
      JM=J-1
      DO 11 I=2,LM
      XT=.25*(X(I,JP,1)+X(I,JM,1)+X(I+1,J,1)+X(I-1,J,1))
      DIF=XT-X(I,J,1)
      SUM=SUM+ABS(DIF)
   11 X(I,J,1)=XT+.6*DIF
      NCT=NCT+1
      IF(SUM .GT. .001 .AND. NCT .LT. 40) GO TO 10
      WRITE(6,104) NCT,SUM
  104 FORMAT(' INITIALIZING X  --  NCT=',I5,' SUM=',E12.6)
      IF(NRT .LT. 50) WRITE(6,101) ((X(I,J,1),J=1,M),I=1,L)
      KPE=N+1
      DLX=.2*(X(L,M,1)-X(L,1,1))
      FNC=NC-1
      DO 12 K=2,NC
      KK=KPE-K
      FK1=(1.0-FLOAT(K-1)/FNC)**2
      XF=DLX*(1.-FK1)
      DO 13 J=1,M
      X(1,J,KK)=X(1,J,1)
   13 X(1,J,K)=X(1,J,1)
      DO 14 I=2,L
      FI=XF*(FLOAT(I-1)/FL)**1.5
      X(I,M,K)=X(I,M,1)-FI
      X(I,M,KK)=X(I,M,K)
      DO 14 J=1,MM
      FJ=FLOAT(J-1)/FM
      XX=X(I,J,1)-FJ*FI
      X(I,J,KK)=XX
   14 X(I,J,K)=XX
   12 CONTINUE
      DO 15 I=1,L
      DO 15 J=1,M
   15 X(I,J,N)=X(I,J,1)
      RETURN
      END
@FOR,I ZDRAIN,ZDRAIN
      SUBROUTINE ZDRANF(ZEND1)
      COMMON X(21,11,11),Y(21,11,11),Z(21,11,11),C1(19,9),C2(19,9),C3(19
     $,9),C4(19,9),C5(19,9),A(19),B(19),G(19),H,FL,
     $FM,FN,DEL,ZF,Z1,ERR,L,LM,M,MM,N,NM,L2,M2,N2,MAX,MAXCT,NRT
     $,ZT(11)
      DY(Y1,Y2,Y3,D2,D3,D4)=D2*Y1-D3*Y2+D4*Y3
      DX2(X1,X2,X3,X4,D4,D5)=X1-D4*X2-.5*X3-D5*X4
      DX(X1,X2,X3,X4,E3,E4)  = E3*(X1-X2)-E4*(X3-X4)
      DMM(X1,X2,X3,X4,D4,D5)=D4*X1+.5*X2-X3+D5*X4
      DXM(X1,X2,X3,X4,D1,D2,D3,D4)=D1*X1-D2*X2+D3*X3-D4*X4
      N3=N-3
      L3=L-3
      D1=1.83333333
      D2=3.0
      D3=1.5
      D4=.33333333
      D5=.16666667
      E1=.541666667
      E2=.041666667
      E3=.666666667
      E4=.083333333
      DO 2 J=2,M
      Z(L,J,1)=Z1
      JP=J+1
      JM=J-1
      J0=J+2
      J2=J-2
      DY1=DY(Y(LM,J,1),Y(L2,J,1),Y(L3,J,1),D2,D3,D4)
      DY2=DY(Y(LM,J,2),Y(L2,J,2),Y(L3,J,2),D2,D3,D4)
      DY3=DY(Y(LM,J,3),Y(L2,J,3),Y(L3,J,3),D2,D3,D4)
      SX1=D2*X(L,J,2)-D1*X(L,J,1)-D3*X(L,J,3)+D4*X(L,J,4)
      SX2=X(L,J,3)-.5*X(L,J,2)-D4*X(L,J,1)-D5*X(L,J,4)
      SX3=E3*(X(L,J,4)-X(L,J,2))-E4*(X(L,J,5)-X(L,J,1))
      IF(J .GT. 2) GO TO 3
      DD1=(DY1+SX1*DX2(Z(L,3,1),Z(L,1,1),Z(L,2,1),Z(L,4,1),D4,D5))/
     $DX2(X(L,3,1),X(L,1,1),X(L,2,1),X(L,4,1),D4,D5)
      DD2=(DY2+SX2*DX2(Z(L,3,2),Z(L,1,2),Z(L,2,2),Z(L,4,2),D4,D5))/
     $DX2(X(L,3,2),X(L,1,2),X(L,2,2),X(L,4,2),D4,D5)
      DD3=(DY3+SX3*DX2(Z(L,3,3),Z(L,1,3),Z(L,2,3),Z(L,4,3),D4,D5))/
     $DX2(X(L,3,3),X(L,1,3),X(L,2,3),X(L,4,3),D4,D5)
      GO TO 4
    3 IF(J .EQ. MM) GO TO 5
      IF(J .EQ. M) GO TO 6
      DD1=(DY1+SX1*DX(Z(L,JP,1),Z(L,JM,1),Z(L,J0,1),Z(L,J2,1),E3,E4))/
     $ DX(X(L,JP,1),X(L,JM,1),X(L,J0,1),X(L,J2,1),E3,E4)
      DD2=(DY2+SX2*DX(Z(L,JP,2),Z(L,JM,2),Z(L,J0,2),Z(L,J2,2),E3,E4))/
     $ DX(X(L,JP,2),X(L,JM,2),X(L,J0,2),X(L,J2,2),E3,E4)
      DD3=(DY3+SX3*DX(Z(L,JP,3),Z(L,JM,3),Z(L,J0,3),Z(L,J2,3),E3,E4))/
     $ DX(X(L,JP,3),X(L,JM,3),X(L,J0,3),X(L,J2,3),E3,E4)
      GO TO 4
    5 DD1=(DY1+SX1*DMM(Z(L,JP,1),Z(L,J,1),Z(L,JM,1),Z(L,J2,1),D4,D5))/
     $ DMM(X(L,JP,1),X(L,J,1),X(L,JM,1),X(L,J2,1),D4,D5)
      DD2=(DY2+SX2*DMM(Z(L,JP,2),Z(L,J,2),Z(L,JM,2),Z(L,J2,2),D4,D5))/
     $ DMM(X(L,JP,2),X(L,J,2),X(L,JM,2),X(L,J2,2),D4,D5)
      DD3=(DY3+SX3*DMM(Z(L,JP,3),Z(L,J,3),Z(L,JM,3),Z(L,J2,3),D4,D5))/
     $ DMM(X(L,JP,3),X(L,J,3),X(L,JM,3),X(L,J2,3),D4,D5)
      GO TO 4
    6 J3=J-3
      DD1=(DY1+SX1*DXM(Z(L,J,1),Z(L,JM,1),Z(L,J2,1),Z(L,J3,1),D1,D2,D3,
     $D4))/DXM(X(L,J,1),X(L,JM,1),X(L,J2,1),X(L,J3,1),D1,D2,D3,D4)
      DD2=(DY2+SX2*DXM(Z(L,J,2),Z(L,JM,2),Z(L,J2,2),Z(L,J3,2),D1,D2,D3,
     $D4))/DXM(X(L,J,2),X(L,JM,2),X(L,J2,2),X(L,J3,2),D1,D2,D3,D4)
      DD3=(DY3+SX3*DXM(Z(L,J,3),Z(L,JM,3),Z(L,J2,3),Z(L,J3,3),D1,D2,D3,
     $D4))/DXM(X(L,J,3),X(L,JM,3),X(L,J2,3),X(L,J3,3),D1,D2,D3,D4)
    4 ZT(2)=Z1+.5*(DD1+DD2)
```

```
      DO 7 K=3,NM
      I=K+1
      KM=K-1
      KP=K+2
      K2=K-2
      K3=K+3
      DY4=DY(Y(LM,J,I),Y(L2,J,I),Y(L3,J,I),D2,D3,D4)
      IF(K .EQ. NM) GO TO 13
      IF(K .EQ. N2) GO TO 15
      SX4=E3*(X(L,J,KP)-X(L,J,K))-E4*(X(L,J,K3)-X(L,J,KM))
      GO TO 9
   13 SX4=D1*X(L,J,N)-D2*X(L,J,NM)+D3*X(L,J,N2)-D4*X(L,J,N-3)
      GO TO 9
   15 SX4=D4*X(L,J,N)+.5*X(L,J,NM)-X(L,J,N2)+D5*X(L,J,N-3)
    9 IF(J .GT. 2) GO TO 8
      DD4=(DY4+SX4*DX2(Z(L,3,I),Z(L,1,I),Z(L,2,I),Z(L,4,I),D4,D5))/
     $DX2(X(L,3,I),X(L,1,I),X(L,2,I),X(L,4,I),D4,D5)
      GO TO 14
    8 IF(J .EQ. MM) GO TO 10
      IF(J .EQ. M) GO TO 11
      DD4=(DY4+SX4*DX(Z(L,JP,I),Z(L,JM,I),Z(L,JO,I),Z(L,J2,I),E3,E4))/
     $ DX(X(L,JP,I),X(L,JM,I),X(L,JO,I),X(L,J2,I),E3,E4)
      GO TO 14
   10 DD4=(DY4+SX4*DMM(Z(L,JP,I),Z(L,J,I),Z(L,JM,I),Z(L,J2,I),D4,D5))/
     $ DMM(X(L,JP,I),X(L,J,I),X(L,JM,I),X(L,J2,I),D4,D5)
      GO TO 14
   11 DD4=(DY4+SX4*DXM(Z(L,J,I),Z(L,JM,I),Z(L,J2,I),Z(L,J3,I),D1,D2,D3,
     $D4))/DXM(X(L,J,I),X(L,JM,I),X(L,J2,I),X(L,J3,I),D1,D2,D3,D4)
   14 ZT(K)=ZT(KM)+E1*(DD2+DD3)-E2*(DD1+DD4)
      DD1=DD2
      DD2=DD3
    7 DD3=DD4
      ZT(N)=ZT(NM)+.5*(DD2+DD3)
      DIF=(ZEND1-ZT(N))/(ZT(N)-Z1)
      DO 17 K=2,NM
   17 Z(L,J,K)=ZT(K)+DIF*(ZT(K)-Z1)
      WRITE(6,258) J,Z1,(ZT(K),K=2,N)
  258 FORMAT(1H ,I2,11F11.4)
    2 Z(L,J,N)=ZEND1
      DO 12 K=1,N
   12 Z(L,1,K)=Z(L,2,K)
      NMID=N/2-1
      NMID2=NMID+2
      DO 22 K=1,NMID
      KP=K+1
      DO 22 J=2,MM
      JP=J+1
      SLOP1=(X(L,J,K)-X(L,J,KP))/(Z(L,J,KP)-Z(L,J,K))
      XDIF=X(L,JP,KP)-X(L,J,KP)
      SLOP2=(Z(L,JP,KP)-Z(L,J,KP))/XDIF
      IF(SLOP1 +.11 .GT. SLOP2 .AND. SLOP1-.11 .LT. SLOP2) GO TO 22
      Z(L,JP,KP)=XDIF*SLOP1+Z(L,J,KP)
   22 CONTINUE
      K=N
   24 KM=K-1
      DO 23 J=2,MM
      JP=J+1
      SLOP1=(X(L,J,K)-X(L,J,KM))/(Z(L,J,K)-Z(L,J,KM))
      XDIF=X(L,JP,KM)-X(L,J,KM)
      SLOP2=(Z(L,JP,KM)-Z(L,J ,KM))/XDIF
      IF(SLOP1 +.11 .GT. SLOP2 .AND. SLOP1-.11 .LT. SLOP2) GO TO 23
      Z(L,JP,KM)=Z(L,J,KM)-XDIF*SLOP1
   23 CONTINUE
      K=K-1
      IF(K .GT. NMID2) GO TO 24
      RETURN
      END
```

APPENDIX C—EXAMPLE OF SOLUTION GIVING

$x(\Phi, \psi, \psi^{*}),\ y(\Phi, \psi, \psi^{*}),\ z(\Phi, \psi, \psi^{*})$

X-COORDINATES FOR K= 1

.000	.894	1.760	2.596	3.398	4.164	4.889	5.570	6.206	6.795	7.336	7.829	8.272	8.662	8.992	9.243	9.321	9.336	9.351	9.366	9.381
.000	.900	1.770	2.609	3.415	4.183	4.910	5.594	6.232	6.823	7.367	7.863	8.310	8.705	9.042	9.308	9.428	9.443	9.458	9.473	9.488
.000	.913	1.797	2.648	3.464	4.241	4.976	5.666	6.311	6.909	7.461	7.965	8.421	8.827	9.176	9.453	9.627	9.652	9.667	9.682	9.697
.000	.936	1.842	2.713	3.546	4.337	5.084	5.786	6.441	7.050	7.612	8.128	8.598	9.019	9.385	9.686	9.893	10.001	10.015	10.030	10.036
.000	.971	1.909	2.807	3.663	4.472	5.234	5.949	6.617	7.238	7.814	8.345	8.830	9.269	9.655	9.982	10.217	10.371	10.433	10.450	10.472
.000	1.019	2.000	2.934	3.816	4.646	5.425	6.154	6.835	7.469	8.057	8.602	9.103	9.558	9.963	10.314	10.572	10.756	10.838	10.883	10.951
.000	1.087	2.124	3.098	4.010	4.862	5.656	6.398	7.089	7.733	8.332	8.888	9.400	9.868	10.288	10.657	10.936	11.145	11.240	11.323	11.424
.000	1.186	2.292	3.312	4.252	5.122	5.928	6.678	7.375	8.025	8.629	9.189	9.708	10.182	10.611	10.992	11.291	11.522	11.636	11.730	11.865
.000	1.341	2.529	3.590	4.551	5.432	6.243	6.995	7.692	8.340	8.941	9.500	10.017	10.492	10.924	11.309	11.631	11.890	12.068	12.252	12.267
.000	1.621	2.873	3.955	4.922	5.803	6.611	7.357	8.046	8.685	9.276	9.824	10.331	10.798	11.227	11.614	11.953	12.236	12.450	12.585	12.630

Y-COORDINATES FOR K= 1

.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000
.866	.861	.849	.829	.804	.774	.739	.702	.662	.621	.577	.531	.482	.431	.377	.319	.258	.195	.130	.065	.000
1.739	1.731	1.706	1.666	1.614	1.552	1.482	1.407	1.326	1.242	1.153	1.061	.963	.860	.750	.634	.512	.386	.257	.128	.000
2.625	2.611	2.570	2.506	2.424	2.328	2.221	2.105	1.983	1.855	1.720	1.580	1.433	1.277	1.113	.939	.757	.570	.380	.190	.000
3.523	3.502	3.442	3.350	3.234	3.099	2.950	2.792	2.625	2.451	2.270	2.081	1.884	1.676	1.457	1.227	.988	.744	.496	.248	.000
4.442	4.411	4.325	4.197	4.039	3.860	3.665	3.460	3.246	3.024	2.794	2.555	2.306	2.046	1.773	1.490	1.199	.901	.601	.300	.000
5.398	5.350	5.226	5.049	4.839	4.607	4.360	4.103	3.837	3.563	3.282	2.991	2.691	2.379	2.056	1.724	1.384	1.040	.693	.347	.000
6.419	6.336	6.150	5.908	5.632	5.338	5.030	4.715	4.392	4.063	3.727	3.384	3.032	2.672	2.303	1.926	1.544	1.159	.773	.387	.000
7.568	7.395	7.108	6.773	6.417	6.049	5.673	5.293	4.909	4.521	4.130	3.733	3.332	2.926	2.515	2.100	1.682	1.262	.842	.421	.000
9.000	8.550	8.100	7.650	7.200	6.750	6.300	5.850	5.400	4.950	4.500	4.050	3.600	3.150	2.700	2.250	1.800	1.350	.900	.450	.000

Z-COORDINATES FOR K= 1

.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	1.000
.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	1.000
.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	1.000
.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	1.000
.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	1.000
.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	1.000
.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	1.000
.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	1.000
.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	1.000
.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	1.000

X-COORDINATES FOR K= 2

.000	.884	1.742	2.573	3.371	4.132	4.853	5.530	6.161	6.746	7.282	7.768	8.202	8.581	8.899	9.149	9.317	9.336	9.351	9.366	9.381
.000	.889	1.752	2.587	3.388	4.152	4.875	5.554	6.187	6.773	7.311	7.799	8.235	8.616	8.937	9.191	9.368	9.419	9.434	9.449	9.464
.000	.903	1.780	2.626	3.437	4.209	4.939	5.624	6.262	6.853	7.395	7.887	8.328	8.715	9.043	9.306	9.494	9.596	9.600	9.615	9.630
.000	.927	1.827	2.693	3.520	4.306	5.046	5.740	6.386	6.984	7.532	8.032	8.481	8.876	9.215	9.491	9.698	9.831	9.890	9.892	9.894
.000	.964	1.896	2.790	3.640	4.443	5.197	5.903	6.558	7.164	7.720	8.227	8.685	9.090	9.440	9.730	9.954	10.110	10.197	10.219	10.226
.000	1.014	1.991	2.920	3.798	4.622	5.393	6.110	6.775	7.389	7.953	8.468	8.932	9.345	9.704	10.006	10.245	10.419	10.521	10.535	10.578
.000	1.085	2.120	3.092	4.001	4.847	5.633	6.362	7.036	7.657	8.227	8.746	9.214	9.632	9.997	10.306	10.556	10.743	10.863	10.907	10.922
.000	1.188	2.296	3.316	4.255	5.121	5.920	6.658	7.339	7.964	8.536	9.056	9.526	9.944	10.310	10.622	10.878	11.074	11.204	11.243	11.258
.000	1.349	2.542	3.607	4.571	5.450	6.258	7.000	7.683	8.309	8.880	9.398	9.864	10.278	10.640	10.950	11.207	11.408	11.549	11.599	11.614
.000	1.638	2.899	3.988	4.961	5.845	6.653	7.395	8.075	8.698	9.265	9.779	10.239	10.646	11.000	11.300	11.547	11.742	11.888	12.004	12.055

Y-COORDINATES FOR K= 2

.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000
.849	.845	.832	.811	.785	.753	.717	.678	.636	.592	.547	.500	.450	.398	.344	.287	.228	.169	.108	.048	.000
1.707	1.698	1.672	1.630	1.575	1.510	1.437	1.357	1.274	1.186	1.094	.999	.899	.795	.687	.573	.456	.337	.216	.096	.000
2.578	2.564	2.521	2.455	2.369	2.268	2.156	2.035	1.907	1.774	1.636	1.493	1.344	1.189	1.027	.858	.683	.505	.324	.145	.000
3.468	3.446	3.383	3.288	3.167	3.026	2.871	2.706	2.533	2.353	2.168	1.976	1.778	1.572	1.357	1.134	.904	.668	.430	.194	.000
4.384	4.352	4.263	4.131	3.968	3.782	3.580	3.367	3.146	2.917	2.683	2.442	2.193	1.937	1.672	1.397	1.115	.826	.532	.244	.000
5.344	5.294	5.168	4.988	4.773	4.534	4.280	4.014	3.741	3.461	3.175	2.883	2.584	2.279	1.965	1.642	1.311	.973	.630	.294	.000
6.376	6.292	6.105	5.860	5.580	5.280	4.966	4.643	4.314	3.979	3.640	3.296	2.946	2.592	2.231	1.864	1.490	1.110	.724	.344	.000
7.543	7.371	7.083	6.748	6.389	6.017	5.638	5.253	4.864	4.473	4.078	3.682	3.282	2.880	2.474	2.066	1.653	1.237	.819	.396	.000
9.000	8.550	8.100	7.650	7.200	6.750	6.300	5.850	5.400	4.950	4.500	4.050	3.600	3.150	2.700	2.250	1.800	1.350	.900	.450	.000

Z-COORDINATES FOR K= 2

1.019	.993	.975	.962	.951	.942	.933	.925	.920	.917	.919	.930	.952	.990	1.048	1.126	1.222	1.323	1.413	1.473	1.492

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 9.000 8.550 8.100 7.650 7.200 6.750 6.300 5.850 5.400 4.950 4.500 4.050 3.600 3.150 2.700 2.250  1.800  1.350   .900   .450   .000
Z-COORDINATES FOR   K=   9
 8.042 8.032 8.025 8.019 8.015 8.011 8.007 8.002 7.997 7.990 7.981 7.971 7.958 7.944 7.928 7.912  7.895  7.880  7.867  7.859  7.856
 8.042 8.033 8.026 8.021 8.017 8.013 8.009 8.004 7.999 7.992 7.983 7.972 7.959 7.945 7.929 7.912  7.895  7.880  7.867  7.859  7.856
 8.042 8.032 8.025 8.020 8.015 8.011 8.007 8.002 7.997 7.989 7.980 7.970 7.957 7.943 7.927 7.911  7.894  7.879  7.867  7.859  7.856
 8.042 8.031 8.023 8.017 8.012 8.008 8.003 7.998 7.992 7.984 7.975 7.964 7.951 7.937 7.921 7.904  7.888  7.873  7.861  7.853  7.850
 8.042 8.029 8.021 8.014 8.009 8.004 7.999 7.993 7.987 7.979 7.970 7.959 7.947 7.933 7.917 7.901  7.886  7.871  7.860  7.852  7.849
 8.041 8.027 8.017 8.010 8.004 7.998 7.993 7.987 7.980 7.972 7.963 7.953 7.941 7.927 7.913 7.897  7.883  7.869  7.858  7.851  7.848
 8.040 8.024 8.013 8.005 7.998 7.992 7.986 7.980 7.973 7.965 7.956 7.946 7.934 7.921 7.907 7.893  7.879  7.867  7.856  7.850  7.847
 8.039 8.022 8.010 8.001 7.994 7.987 7.980 7.973 7.965 7.957 7.947 7.937 7.926 7.913 7.900 7.887  7.874  7.863  7.854  7.848  7.846
 8.039 8.021 8.010 8.001 7.994 7.986 7.979 7.971 7.961 7.951 7.940 7.928 7.915 7.901 7.886 7.871  7.857  7.844  7.834  7.828  7.825
 8.039 8.025 8.018 8.012 8.006 8.000 7.994 7.986 7.977 7.967 7.956 7.943 7.930 7.917 7.904 7.891  7.880  7.871  7.864  7.860  7.859
X-COORDINATES FOR   K=  10
  .000  .751 1.502 2.242 2.964 3.661 4.328 4.961 5.558 6.115 6.633 7.109 7.544 7.935 8.283 8.586  8.842  9.052  9.213  9.324  9.381
  .000  .755 1.510 2.254 2.978 3.678 4.347 4.981 5.579 6.137 6.655 7.131 7.566 7.957 8.304 8.607  8.863  9.072  9.233  9.344  9.408
  .000  .768 1.534 2.288 3.022 3.729 4.404 5.042 5.642 6.203 6.721 7.198 7.632 8.023 8.369 8.670  8.925  9.133  9.292  9.402  9.463
  .000  .789 1.576 2.348 3.097 3.816 4.499 5.145 5.749 6.312 6.833 7.310 7.743 8.133 8.477 8.776  9.029  9.234  9.390  9.497  9.555
  .000  .821 1.638 2.436 3.206 3.941 4.637 5.290 5.901 6.467 6.989 7.466 7.[illegible] 8.286 8.628 8.924  9.173  9.374  9.526  9.629  9.683
  .000  .867 1.725 2.558 3.355 4.109 4.819 5.482 6.098 6.667 7.191 7.667 8.098 8.483 8.822 9.113  9.358  9.554  9.701  9.799  9.848
  .000  .932 1.847 2.722 3.550 4.325 5.049 5.721 6.342 6.915 7.438 7.914 8.342 8.723 9.057 9.344  9.582  9.773  9.915 10.007 10.049
  .000 1.028 2.016 2.941 3.800 4.596 5.331 6.011 6.636 7.209 7.732 8.205 8.630 9.006 9.334 9.615  9.847 10.031 10.166 10.252 10.288
  .000 1.182 2.260 3.233 4.118 4.929 5.672 6.355 6.981 7.553 8.073 8.541 8.960 9.330 9.651 9.924 10.149 10.325 10.453 10.532 10.561
  .000 1.463 2.620 3.672 4.519 5.334 6.079 6.760 7.384 7.951 8.465 8.927 9.338 9.69910.01110.274 10.490 10.658 10.777 10.849 10.873
Y-COORDINATES FOR   K=  10
  .000  .000  .000  .000  .000  .000  .000  .000  .000  .000  .000  .000  .000  .000  .000  .000   .000   .000   .000   .000   .000
  .808  .804  .792  .772  .746  .714  .677  .638  .595  .550  .504  .456  .408  .358  .309  .258   .207   .155   .104   .052   .000
 1.624 1.616 1.592 1.551 1.497 1.432 1.358 1.278 1.191 1.101 1.008  .913  .815  .716  .616  .515   .413   .310   .207   .104   .000
 2.458 2.445 2.406 2.343 2.258 2.158 2.044 1.920 1.789 1.653 1.512 1.368 1.221 1.073  .922  .771   .618   .464   .310   .155   .000
 3.320 3.301 3.244 3.152 3.033 2.893 2.736 2.567 2.389 2.204 2.015 1.822 1.625 1.427 1.227 1.025   .821   .617   .412   .206   .000
 4.224 4.196 4.114 3.987 3.826 3.640 3.435 3.218 2.990 2.756 2.517 2.274 2.028 1.779 1.528 1.276  1.023   .768   .513   .256   .000
 5.187 5.144 5.027 4.853 4.640 4.400 4.142 3.872 3.593 3.307 3.017 2.723 2.427 2.128 1.827 1.525  1.222   .917   .612   .306   .000
 6.241 6.168 5.992 5.752 5.475 5.173 4.856 4.529 4.195 3.857 3.514 3.169 2.822 2.473 2.123 1.771  1.418  1.065   .710   .355   .000
 7.452 7.296 7.018 6.687 6.329 5.956 5.574 5.187 4.796 4.403 4.008 3.611 3.212 2.813 2.413 2.012  1.611  1.208   .806   .403   .000
 9.000 8.550 8.100 7.650 7.200 6.750 6.300 5.850 5.400 4.950 4.500 4.050 3.600 3.150 2.700 2.250  1.800  1.350   .900   .450   .000
Z-COORDINATES FOR   K=  10
 9.000 9.000 9.000 9.000 9.000 9.000 9.000 9.000 9.000 9.000 9.000 9.000 9.000 9.000 9.000 9.000  9.000  9.000  9.000  9.000  9.000
 9.000 9.000 9.000 9.000 9.000 9.000 9.000 9.000 9.000 9.000 9.000 9.000 9.000 9.000 9.000 9.000  9.000  9.000  9.000  9.000  9.000
 9.000 9.000 9.000 9.000 9.000 9.000 9.000 9.000 9.000 9.000 9.000 9.000 9.000 9.000 9.000 9.000  9.000  9.000  9.000  9.000  9.000
 9.000 9.000 9.000 9.000 9.000 9.000 9.000 9.000 9.000 9.000 9.000 9.000 9.000 9.000 9.000 9.000  9.000  9.000  9.000  9.000  9.000
 9.000 9.000 9.000 9.000 9.000 9.000 9.000 9.000 9.000 9.000 9.000 9.000 9.000 9.000 9.000 9.000  9.000  9.000  9.000  9.000  9.000
 9.000 9.000 9.000 9.000 9.000 9.000 9.000 9.000 9.000 9.000 9.000 9.000 9.000 9.000 9.000 9.000  9.000  9.000  9.000  9.000  9.000
 9.000 9.000 9.000 9.000 9.000 9.000 9.000 9.000 9.000 9.000 9.000 9.000 9.000 9.000 9.000 9.000  9.000  9.000  9.000  9.000  9.000
 9.000 9.000 9.000 9.000 9.000 9.000 9.000 9.000 9.000 9.000 9.000 9.000 9.000 9.000 9.000 9.000  9.000  9.000  9.000  9.000  9.000
 9.000 9.000 9.000 9.000 9.000 9.000 9.000 9.000 9.000 9.000 9.000 9.000 9.000 9.000 9.000 9.000  9.000  9.000  9.000  9.000  9.000
 9.000 9.000 9.000 9.000 9.000 9.000 9.000 9.000 9.000 9.000 9.000 9.000 9.000 9.000 9.000 9.000  9.000  9.000  9.000  9.000  9.000
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