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Inverse Theory with Grossly Inadequate Data

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Summary

When only a few observations are available as data for an inverse problem, it is proposed that the best way to use them is to obtain bounds on various functionals of the structure. To do this, the model is found that has the smallest (or largest) value of the functional. In this way, for example, equations are derived for finding the model value that is exceeded somewhere by all structures satisfying the data, and thus this value must be exceeded in the Earth itself. The same techniques can be used to derive conditions for the existence of a solution, when a certain data set is given; this is an important problem in non-linear inverse theory.

Three examples are given, including the non-linear problem of electrical conductivity in the mantle. There, one- and two-data problems are solved and, by means of the existence theory, self-consistency criteria are defined for amplitude and phase measurements and for amplitude measurements at two different frequencies.

1. Introduction

In geophysics and planetary physics there is always a tendency to believe our knowledge is more precise than it really is. Recently, however, there has been a new emphasis on the proper assessment of uncertainties in inverse theory, which is the quantitative aspect of geophysical inference. A revolution in our understanding of inverse problems has come about, in my view, because of the work of Backus & Gilbert (1967, 1968, 1970; Backus 1970a, b, c). They have given a complete theory for linear inverse problems (where the observations are known to be linear functionals of the model) and, on this basis, built up an approximate treatment of non-linear problems. In this paper, a special part of inverse theory is examined: how to make definite inferences from data that the Backus–Gilbert theory demonstrates to be very incomplete. The reader will be expected to be acquainted with the essential concepts of Backus and Gilbert in what follows.

To make our objectives clearer, let us consider a simple example: suppose we wish to determine the density structure of the Earth using as data a finite number of observations of the Earth's normal mode frequencies. Even when the measurements are exact, there will be more than one model satisfying these data (in fact there are infinitely many), because a complete specification of the density requires an infinite number of measurements. Therefore we are uncertain of the true density; we merely know it is one of a class of models, but we cannot know which one. Backus & Gilbert (1968) show that this uncertainty can be expressed by saying that what we determine from our data is a smoothed version of the true density: there is a loss of detail analogous to the loss in resolution found with microscopes because of the finite wavelength of light. Details smaller than a certain size are invisible to an observer with only M data; conversely, if some of the observer's models exhibit such small details, they are superfluous. The 'resolving length' made available by the data can be computed with Backus–Gilbert theory and, as one would expect, the smaller M becomes, the poorer the resolution. Let us now suppose the number of measurements is so small that the resolution length exceeds the radius of the Earth. Is there anything we can say in such circumstances? I propose to show that there is.

We must abandon the goal of obtaining a density structure; instead, the data should be used to provide us with inequalities that must be satisfied by all models and therefore, presumably, by the Earth as well. The sort of conclusions one might reach could be, that the density in the Earth must exceed $10\,\mathrm{Mg\,m^{-3}}$, or that, if the density always increases with depth, the density at the centre of the Earth cannot be less than $13\,\mathrm{Mg\,m^{-3}}$. In other words, inadequate data cannot give us a detailed model, but they can be used to rule out certain classes of structures that otherwise might appear admissible.

The approach we adopt to achieve this goal is to choose some property of the Earth, and then find the model that makes it smallest: clearly, then, all models exceed or equal the value obtained. To be useful, the property chosen must be interesting and the problem that results should be mathematically tractable. In the next section we set out the general formulation for several sorts of property (e.g. the mean value of the model, the maximum value of the model). At the same time it appears that the apparatus developed is capable of answering the question of the self-consistency of the measurements, the problem of existence of solutions in the non-linear theory. It is remarkable to note that all the theory can be set up without introducing the linearization approximation commonly invoked to solve non-linear problems. After this we give some applications of the general theory to specific problems. The first two examples are really very simple; they are linear inverse problems and can be solved with relatively light algebra. The final, more serious example concerns a non-linear system in which I have been interested for some years: the inverse problem of electrical conductivity in the Earth. This example involves a moderate amount of numerical work and illustrates the difficulties of dealing with non-linear equations in a satisfactory way. Here we also consider how to include the effects of experimental errors.

2. Variational calculus on model-spaces

The preoccupation of this paper will be the various ways in which a variational formulation can be used in inverse theory. The principle is simple: a model is constructed that minimizes or maximizes some functional on the model space. One difficulty is choosing a suitable functional, one that yields geophysically interesting information. In this section, we examine several functionals and set up the conditions to be met at the extremal. Most of them will be exploited in the examples given later, where the problems of solving the resulting equations will naturally have to be considered. In the following development, we usually exemplify our equations with a scalar model that is spherically symmetric, i.e. $m(r)$, where r is the distance from the centre of the Earth; nonetheless, the results can be made valid with more general models and more general functions of position: a vector-valued model is considered in the first example.

The quantity that always comes to mind first in minimization studies is the squared value integral

$$I_2 := \int_0^a m^2(r)\,dr, \tag{2.1}$$

where a is the radius of the Earth. Following Backus & Gilbert (1967), we consider $m(r)$ to be an element, $\mathbf{m}$, in a Hilbert space of models, so that (2.1) can be written

$$I_2 = \|\mathbf{m}\|^2 = (\mathbf{m}, \mathbf{m}),$$

where $\|\mathbf{m}\|$ is the norm of the model (which must be finite for $\mathbf{m}$ to be in the space) and $(\mathbf{u}, \mathbf{v})$ is an inner product on the space defined here by

$$(\mathbf{u}, \mathbf{v}) = \int_0^a u(r)\, v(r)\, dr.$$

The measurements are taken to be the M real numbers $E_1, E_2, E_3, \ldots E_M$; they are related to the model $\mathbf{m}$ via the equations

$$E_i = F_i[\mathbf{m}],\ i = 1, 2, \ldots M. \tag{2.2}$$

Each F_i is a scalar-valued functional on the Hilbert space; it is the rule which allows us to predict the results of an experimental measurement if we actually know the structure. The simplest example of a functional is the linear* one

$$F[\mathbf{m}] = (\mathbf{m}, \mathbf{G}) = \int_0^a m(r)\, G(r)\, dr,$$

where $G(r)$ is a function independent of m (G is often called a Green's function for the problem).

In order to minimize $\|\mathbf{m}\|^2$, we must assume that every E_i is Fréchet differentiable: this means that any small perturbation $\delta\mathbf{m}$ to the model $\mathbf{m}$ affects the functional F_i thus

$$F_i[\mathbf{m}+\delta\mathbf{m}] = F_i[\mathbf{m}] + (\delta\mathbf{m}, \mathbf{D}_i) + \mathrm{O}\|\delta\mathbf{m}\|^2, \tag{2.3}$$

where $\mathbf{D}_i$ is an element of the space and is called the Fréchet derivative of E_i with respect to $\mathbf{m}$. It is evident that a linear functional like $(\mathbf{m}, \mathbf{G})$ has a Fréchet derivative of $\mathbf{G}$, but in non-linear problems it is not always obvious what $\mathbf{D}_i$ is, or even if it exists (see Backus & Gilbert 1967, and Parker 1970 for some derivations in a geophysical context).

We are now ready to minimize $\|\mathbf{m}\|^2$. The condition must be included that the model fits the data; to do this, Lagrange multipliers, $\lambda_1, \lambda_2, \ldots \lambda_M$, must be introduced and an unrestrained functional, U, constructed:

$$U = \|\mathbf{m}\|^2 - \sum_{i=1}^{M} \lambda_i (F_i[\mathbf{m}] - E_i).$$

When a model $\mathbf{m}_0$ is found that causes U to be minimum, a small perturbation $\delta\mathbf{m}$ in $\mathbf{m}$ produces no change in U to first order. Using this condition and (2.3) we can easily obtain

$$2(\delta\mathbf{m}, \mathbf{m}_0) - \sum_{i=1}^{M} \lambda_i (\delta\mathbf{m}, \mathbf{D}_i) = 0$$

* A linear functional obeys

$$F[\alpha\mathbf{u} + \beta\mathbf{v}] = \alpha F[\mathbf{u}] + \beta F[\mathbf{v}]$$

for all $\mathbf{u}$ and $\mathbf{v}$ in the Hilbert space, where α and β are any real numbers.

or

$$\left(\delta\mathbf{m}, 2\mathbf{m}_0 - \sum_{i=1}^{M} \lambda_i \mathbf{D}_i\right) = 0,$$

for any small $\delta\mathbf{m}$. It follows from the completeness of the Hilbert space that

$$\mathbf{m}_0 = \tfrac{1}{2} \sum_{i=1}^{M} \lambda_i \mathbf{D}_i. \tag{2.4}$$

If the functionals F_i are all linear and linearly independent, (2.4) and (2.2) can be used together to obtain a unique model $\mathbf{m}_0$ that minimizes $\|\mathbf{m}\|^2$ (see Backus & Gilbert 1967). In non-linear problems, however, the Fréchet derivatives, $\mathbf{D}_i$, depend themselves on $\mathbf{m}$ (usually non-linearly) making (2.4) a much more difficult equation to solve. Indeed, solutions may not exist at all, because we do not know in a non-linear problem whether any model can give rise to the data $E_1, E_2, \ldots E_M$. We shall return to this problem in a moment.

Another quantity for which we may be able to find a stationary model is the integral of the model, given by

$$I_1 = \int_0^a m(r)\, dr.$$

Using the same procedure of Lagrange multipliers as before, we obtain the condition

$$\sum_{i=1}^{M} \mu_i D_i(r) = 1 \text{ for almost all } r \text{ in } (0, a). \tag{2.5}$$

The model $\mathbf{m}_0$ does not appear explicitly in (2.5) and when the data functionals F_i are linear (making every D_i independent of m) this condition appears to say nothing about $\mathbf{m}$ at all. In fact, Backus (1970a) shows that there is no upper or lower bound on I_1, or any other linear functional of the model, when the data functionals are themselves linear; the only exception occurs if the required functional can be constructed from linear combination of the data functionals. Non-linear functionals, F_i, may make (2.5) a useful condition in determining a model, but this question cannot be settled in general: every non-linear problem will be a special case.

Perhaps a more interesting functional on which to place a lower bound is the maximum value of the model itself. We need to find the model with the smallest maximum value, which also fits the data. To do this it is necessary to assume that $\mathbf{m}$ is an element of another space of functions, L^p (Riez & Nagy 1965). Here the norm is defined by

$$\|\mathbf{m}\|_p = \left(\int_0^a |m(r)|^p\, dr/a\right)^{1/p},$$

which must be finite for valid members of L^p. The reason for introducing this space is the intriguing property of the norm that, as p tends to infinity, $\|\mathbf{m}\|_p$ becomes the greatest maximum of $|m(r)|$, neglecting, of course, values of m defined on sets of zero measure (Hardy, Littlewood & Pólya 1959, p. 143). Fréchet differentiation can be extended to normed spaces without inner products (called Banach spaces), provided the inner product in (2.3) is replaced by a linear functional of $\delta\mathbf{m}$. Linear functionals in L^p can be written as integrals (Riez & Nagy 1965, p. 74) just as in Hilbert space, but the function D_i must belong to the space L^q, where $q = p/(p-1)$. Formally, nothing has changed when we employ L^p, except for the different restrictions on the model and its Fréchet derivatives.

Before proceeding to the limit of very large p, we carry out a minimization with Lagrange multipliers in the usual way: form the unrestricted functional U, with the Lagrange multipliers ν_i to account for the conditions in (2.2):

$$U = \int_0^a |m(r)|^p \, dr - \sum_{i=1}^{M} \nu_i (F_i[\mathbf{m}] - E_i);$$

it is convenient to work with the p-th power of the norm. Consider a small perturbation in U at its stationary value, due to the perturbation δm in m: there must be no change in U to first order in δm, so that to this order

$$0 = \int_0^a \left[p|m(r)|^p/m(r) - \sum_{i=1}^{M} \nu_i D_i(r) \right] \delta m \, dr,$$

where we have introduced the Fréchet derivatives D_i, which are assumed to belong to L^q. The first term under the integral belongs to L^q also and, since this space is complete, we have

$$p|m(r)|^p = m(r) \sum_{i=1}^{M} \nu_i D_i(r),$$

which must hold for almost all r in $(0, a)$. The consequences of this relationship as p becomes very large are

$$m(r) = \begin{cases} c, \text{ when } \sum \nu_i' D_i(r) > 0 \\ -c, \text{ when } \sum \nu_i' D_i(r) < 0, \end{cases} \tag{2.6}$$

where c is a constant and is the bound we are seeking. The value of c and the coefficients, ν_i', are chosen to make $m(r)$ satisfy the data, (2.2), as well as the condition (2.6).

It is evident from (2.6) that models with negative values will sometimes occur as solutions to variational problems, not only when the smallest maximum is sought, but also in other cases as well. Often, the physical property represented by the model cannot assume a negative value (e.g. density or seismic velocity). Under these circumstances, any extremal solution with negative values is unrealistic and we can assume that the bound obtained on the required functional is too generous: it can be improved by restricting our attention to models that are everywhere positive (or at least non-negative). The device that allows us to include this restriction is quite simple: instead of minimizing the specified functional of m, minimize the functional of $f(m)$, where

$$f(m) = \begin{cases} m, & \text{if } m \geqslant 0 \\ |Sm|, & \text{if } m < 0, \end{cases}$$

and then let S tend to infinity. This has the effect of discriminating against negative values of m, because $f(m)$ then becomes very large. When $f(m)$ is used in the calculations concerning $\|\mathbf{m}\|^2$ and $\|\mathbf{m}\|_\infty$, we can easily obtain the following modifications to the equations. In place of (2.4) we have

$$m(r) = \begin{cases} \frac{1}{2} \sum_{i=1}^{M} \lambda_i D_i(r), & \text{where } \sum \lambda_i D_i(r) \geqslant 0 \\ 0, & \text{where } \sum \lambda_i D_i(r) < 0, \end{cases}$$

for the non-negative model with smallest $\|\mathbf{m}\|$. Equation (2.6) for the model with the smallest maximum value now becomes

$$m(r) = \begin{cases} c, & \text{where } \sum \nu_i' D_i(r) > 0 \\ 0, & \text{where } \sum \nu_i' D_i(r) < 0, \end{cases} \tag{2.7}$$

and c is, of course, positive.

The last type of functional to be considered is rather different: I wish to return now to the problem of the self-consistency of a set of data, E_i, when the associated functions are non-linear. The location of the boundary between admissible and inconsistent data in the Euclidian M-space of the data, $E_1, E_2 \ldots E_M$, can be made into a variational problem and examined with the calculus we have already developed. Suppose $M-1$ data are known to be self-consistent, i.e. there is at least one model, $\mathbf{m}$, satisfying (2.2); what is the largest (or smallest) value of E_M consistent with them? We set up the Lagrange multipliers χ_i and obtain the unrestrained functional U:

$$U = F_M[\mathbf{m}] - \sum_{i=1}^{M-1} \chi_i (F_i[\mathbf{m}] - E_i).$$

Performing the small perturbation in $\mathbf{m}$ and claiming that U is stationary, we obtain

$$\mathbf{D}_M = \sum_{i=1}^{M-1} \chi_i \mathbf{D}_i. \tag{2.8}$$

This is simply a statement that, on the boundary of the zone of permissible models, the Fréchet derivatives are linearly dependent. As with the other essentially non-linear equations we have obtained, there is no general algorithm for finding an $\mathbf{m}$ to satisfy (2.8). We can, however, give a solution in the case of the electrical conductivity problem.

It should be emphasized that the foregoing analysis is of a formal nature only, and that it is extremely difficult to give the conditions under which our results are valid. We have discussed a case in which no solution existed to the equations because there was no bound on the functional in question. Another difficulty is the possibility that the smallest value of the functional may be attained at a point in model space where the Fréchet differential does not exist; our results would not find this solution. Even when a solution has been found that is stationary, it must be determined whether it is a maximum or minimum: this can be done by examining the first neglected term in (2.3), which is normally a quadratic form. Obtaining this term is a very tedious operation algebraically, and most geophysicists will be content with a numerical verification (by small perturbations to the stationary solution) or a physical argument, which can often be found. There still remains the question of whether a particular minimum is the smallest value or not. This amounts to being able to prove, either that no other minima exist, or that the others are all ‘ worse ’. Such a task may be very difficult to achieve with full rigour (we do not achieve it in our non-linear example). It is sometimes possible to give a necessary condition for the validity of our equations by inspecting the various assumptions needed to arrive at the results. For example, we find that to bound the maximum value of the model, $\mathbf{m}$ must be in L^∞ (a rather obvious statement, since L^∞ is the space of bounded functions!) and every $\mathbf{D}_i$ must be in L^1. If the Fréchet derivatives do not meet this condition, then (2.6), although it may appear to make sense, may not be applicable; it is obviously wise to ensure the necessary conditions are met before using the equations (this is easy to do in the third example). We have not been rigorous in our definition of all the relevant spaces: there is a very difficult question regarding (2.8). The space of admissible models here may be very wide—in the third example we need a model that is a δ-function to satisfy (2.8). The subsequent results all seem very reasonable, so that in this case, at least, it appears the model space includes the space of distributions. Once more, we are faced with a problem in non-linear analysis to which no general solution has been found: these mathematical questions, however, will not interest many geophysicists. We proceed to the examples.

3. Currents in the core

The first and simplest application of the results in Section 2 is to the inverse problem of finding the current density distribution in the core from observations of the magnetic field. We take as our starting point a mechanism of the sort proposed by Bullard & Gellman (1954), but we are indifferent to the velocity field and the driving mechanism. The existence of toroidal magnetic fields denies us the possibility of ever finding the actual current distribution from magnetic observations at the surface. We seek the current distribution with the smallest r.m.s. value; the reason for this choice will be made clear in a moment. To simplify the problem even further, we assume only a knowledge of the dipole components of the field. Then the functional relating the currents to the dipole vector is (Panofsky & Phillips 1962, p. 130)

$$\mathbf{p} = \int_V \tfrac{1}{2}\mathbf{r} \times \mathbf{J}(\mathbf{r})\, dV, \tag{3.1}$$

where $\mathbf{p}$ is the magnetic moment of a current distribution $\mathbf{J}$ within the volume V, which for our purposes is a sphere, while r is the co-ordinate vector from the centre. To see more clearly that this is a linear functional, and to avoid confusion about bold-faced quantities, rewrite (3.1) in components with the Einstein summation convention

$$p_i = \int_V \tfrac{1}{2}\varepsilon_{ijk}\, r_j\, J_k\, dV, \qquad i = 1, 2, 3, \tag{3.2}$$

where ε_{ijk} is the alternating tensor. The generality of the abstract Hilbert space formulation allows us to deal with a vector-valued model J_i simply by redefining the inner product between two elements $\mathbf{u}$ and $\mathbf{v}$ to be

$$(\mathbf{u}, \mathbf{v}) = \int_V u_j\, v_j\, dV,$$

and carrying on as before. The Fréchet derivatives, because of linearity, are the kernels in (3.2) and are the vectors $\frac{1}{2}\varepsilon_{ijk}\, r_j$ with $i = 1, 2, 3$, one vector for each p_i. Applying (2.4) we obtain

$$J_i = \varepsilon_{ijk}\, r_j\, \lambda_k, \qquad i = 1, 2, 3$$

or

$$\mathbf{J} = \mathbf{r} \times \boldsymbol{\lambda}. \tag{3.3}$$

At this stage we should verify two essential properties of $\mathbf{J}$: first $\nabla\cdot\mathbf{J} = 0$, which must hold if the currents vary slowly in time; second, $\mathbf{J}\cdot\hat{\mathbf{r}}$ vanishes at the surface of the sphere, which implies no current flows in or out of the core. These properties confirm that $\mathbf{J}$ is a realistic model for this problem and that we need not apply additional constraints to improve our bound.

When (3.3) is substituted into (3.1), we find after some integrations that

$$\boldsymbol{\lambda} = 15\mathbf{p}/\pi b^5$$

where b is the radius of the core. It is then easy to deduce that

$$\int_V \mathbf{J}\cdot\mathbf{J}\,dV = 15p^2/2\pi b^5. \tag{3.4}$$

In itself, the smallest value of $\int \mathbf{J}\cdot\mathbf{J}\,dV$ is not very significant but, with it, we may place a lower bound on the electrical conductivity of the core.

The rate of heat generation, due to ohmic heating is given by

$$\dot{Q} = \frac{1}{\sigma}\int_V \mathbf{J}\cdot\mathbf{J}\,dV. \tag{3.5}$$

An upper bound on $\dot{Q}$ must be the total heat emerging from the Earth's interior (assuming steady-state conditions), but this is a crude over-estimate, because we know that much of this energy is generated in the crust by radioactivity. A more refined value for the heat escaping from great depths is calculated by Sclater & Franchetau (1970): they find $4\times10^{-7}\,\mathrm{cal\,cm^{-2}\,s^{-1}}$ (in SI units $17\,\mathrm{mW\,m^{-2}}$), although part of this heat flow must be due to mantle sources. If we integrate Sclater & Franchetau's figure over the Earth's surface and call it an upper bound on $\dot{Q}$ in the core, we can use (3.5) and (3.4) to set limits on σ. Taking $p = 8\times10^{22}\,\mathrm{A\,m^2}$ and $b = 3{\cdot}5\times10^6\,\mathrm{m}$, we conclude that $\sigma > 3{\cdot}3\,\Omega^{-1}\,\mathrm{m}^{-1}$. Current estimates put the conductivity three orders of magnitude higher, but they depend on assumptions about the chemical composition. Nonetheless, this rather low figure deters us from considering the effect of non-dipole fields and the problems of the steady-state assumption. In this case the data are so inadequate that they only put a very slight constraint on the class of acceptable models.

4. The densities of the planets

The following example was inspired by a lecture given by Cook (1971). He describes how astronomical techniques can be used to deduce the moment of inertia of some of the planets, and goes on to describe models of the planetary interiors based on this datum and various other assumptions. The question that Cook's talk suggested is what can be deduced from the mass and moment of inertia about the density of a planet? Equivalently, we are given $\bar{\rho}$, the mean density and C/Ma^2, where C is the moment of inertia, M is mass and a the radius of the planet.

With the results of Section 2, we can place a bound upon the greatest density. We need two assumptions: (a) the density, ρ, is always positive, and (b) ρ is a function of r alone, where r is the distance from the centre of mass. The functionals relating the data to the model are easily shown to be

$$\bar{\rho} = \int_0^a (3r^2/a^3)\,\rho(r)\,dr \tag{4.1}$$

$$C/Ma^2 = \int_0^a (2r^4/\bar{\rho}a^5)\,\rho(r)\,dr. \tag{4.2}$$

The Fréchet derivatives clearly are $3r^2/a^3$ for $\bar{\rho}$ and $2r^4/\bar{\rho}a^5$ for CM/a^2, because they are linear functionals. Equation (2.7) is the pertinent result from Section 2; it becomes

$$\rho(r) = \begin{cases} \rho_0, \text{ when } \alpha r^2 + \beta r^4 > 0 \\ 0, \text{ when } \alpha r^2 + \beta r^4 < 0, \end{cases}$$

where ρ_0 is the required lower bound on the maximum density. The discriminant function has only one root for $r > 0$ so that we must seek models of ρ that have a single discontinuity in ρ at r_1 and satisfy (4.1) and (4.2). Two types of models can occur: zero density inside or outside the critical radius. A little algebra will show for the first case that

$$r_1/a = (\tfrac{5}{2}C/Ma^2)^{\frac{1}{2}}$$

and

$$\rho_0/\bar{\rho} = (\tfrac{2}{5}Ma^2/C)^{\frac{3}{2}}$$

when $C/Ma^2 < 2/5$ and the zero density lies outside r_1. If $2/5 \leqslant C/Ma^2 < 2/3$, we find r_1/a is the root that lies in (0, 1) of the following equation:

$$x^4 + x^3 + (1 - \tfrac{5}{2}C/Ma^2)(x^2 + x + 1) = 0,$$

and

$$\rho_0/\bar{\rho} = 1/(1 - x^3),$$

with the zero density lying inside r_1. The former, simpler solution is the one of interest in astrophysics, because matter is usually more condensed towards the centre of a planet. When $C/Ma^2 > 2/3$ negative densities are required to fit the data.

We now apply these results to Jupiter: Cook deduces from the value of J_2 (the coefficient for the $l = 2$, $m = 0$ sperical harmonic in the gravitational potential) that $C/Ma^2 = 0{\cdot}25$, with the assumption of a hydrostatic model. The mean density of Jupiter, $\bar{\rho}$, is $1{\cdot}33 \text{ Mg m}^{-3}$ (Allen 1964); we conclude the density inside Jupiter must exceed $2{\cdot}74 \text{ Mg m}^{-3}$, a surprisingly high value. Similarly, inside Saturn the density must exceed $1{\cdot}69 \text{ Mg m}^{-3}$. These results could be improved upon: the value of C/Ma^2 is based on a hydrostatic model, with which the spherical symmetry of ρ is inconsistent. It is probable that a consistent treatment would yield even higher values for ρ_0.

5. Electrical conductivity in the mantle

5.1 *Introduction*

Measurements of geomagnetic fluctuations have long been used to infer the electrical conductivity deep within the Earth (Lahiri & Price 1939). The method relies upon the fact that a knowledge of the magnetic field components over the entire surface of the Earth can be used to separate the fields into two parts, one caused by sources outside the Earth, the other produced inside by eddy currents that flow in response to the external variations. The ratio of the internal to external fields is a response measure that depends on the conductivity structure and, provided the measure is known at all frequencies, it contains enough information to find the conductivity uniquely at all depths (Bailey 1970). All practical studies to date have employed data associated with frequencies of one cycle per year or higher, but there is some evidence (Chapman & Bartels 1962, p. 133) that there may be useful information at a period of 11·2 yr, the period of solar activity. If a response measurement can be made at this frequency, what can it tell us about the conductivity? The methods of this paper would seem the most appropriate with which to answer the question, because the other data are at frequencies so much higher than the 11-yearly line that it can be considered isolated from the rest. These remarks serve as a motivation for this investigation, but the problem is an excellent vehicle to illustrate the solution of a non-linear extremal problem. Furthermore, we shall derive new theorems about data compatibility that will be useful at any frequency and for any number of measurements.

5.2 *A one-datum problem*

First of all we shall define the problem more precisely. The data are considered to be functions of frequency, derived from their time-domain counterparts by Fourier transformation; they are therefore complex quantities, carrying amplitude and phase information. The response measure we use is the complex number E, given by

$$E = E_1 + iE_2 = -\ln W_l^m,$$

where W_l^m is the ratio of the vertical to horizontal magnetic field components (normalized by a geographic factor) at a single frequency and a particular spherical harmonic, degree l and order m. The complex quantity W_l^m can be estimated directly from the field records (Banks 1969); its amplitude is associated with E_1 (since $E_1 = -\ln|W_l^m|$) while its phase corresponds to E_2 (since $E_2 = -\arg W_l^m$). In nature, the low frequency disturbances are confined almost entirely to the $l = 1$, $m = 0$ harmonic, so that our numerical calculations will concentrate on this case. The simplest problem to consider is this: given a single observation of E, determine the model with the least maximum conductivity compatible with the measurement. The datum E is complex, but in Banks' work it was found that the amplitude of W_l^m could be estimated more precisely than its phase: therefore, a reasonable one-datum problem is to consider amplitude alone, or equivalently E_1, the real part of E. To set up the conditions that define the extremal solution, we shall need the Fréchet derivative of E with respect to σ, the conductivity. For this result and other derivations below we refer the reader to a previous paper of the author's (Parker 1970), in which the Backus–Gilbert technique was applied to the many-data inversion problem. The same notation will be used, with one exception: here Fréchet derivatives are denoted by D.

The Fréchet derivative of E with respect to conductivity at a radius, r, and radian frequency, ω, is given by

$$D(E; r, \omega) = D_1 + iD_2 = -\frac{i\omega\mu_0 r^2}{l(l+1)\,a}\left[\frac{R_l^m(r)}{R_l^m(a)}\right]^2 \exp(-E), \tag{5.1}$$

where D_1 and D_2 are the derivatives for E_1 and E_2; R_l^m is any non-singular solution of the ordinary differential equation

$$\frac{d}{dr} r^2 \frac{d}{dr} R_l^m + [i\omega\mu_0 \sigma r^2 - l(l+1)]\, R_l^m = 0. \tag{5.2}$$

It has been assumed that σ is a function only of r, the distance from the centre of the Earth. In terms of R_l^m, the datum E can be expressed thus

$$E = \ln\left\{\frac{1}{l(l+1)}\left[1 + \frac{a}{R_l^m}\frac{dR_l^m}{dr}\right]\right\}, \tag{5.3}$$

where R_l^m and its derivative are evaluated at the surface, $r = a$.

It was shown in Section 2 that models suitable as stationary solutions were made up of shells of constant, positive conductivity, σ_0, separated by regions of zero conductivity (we can obviously reject negative σ). For this type of behaviour in σ, (5.2) assumes a very simple solution

$$R_l^m(r) = \begin{cases} \alpha_1 r^l + \alpha_2 r^{-l-1}, & \text{where } \sigma = 0 \\ \beta_1 j_l(kr) + \beta_2 y_l(kr), & \text{where } \sigma = \sigma_0, \end{cases} \tag{5.4}$$

where the complex wave number k is defined by

$$k = (\mu_0 \omega \sigma_0)^{\frac{1}{2}} \exp(i\pi/4),$$

and the functions j_l and y_l are spherical Bessel functions (see Abramowitz & Stegun 1965). Because of the continuity of the electric and magnetic fields, R_l^m and dR_l^m/dr are everywhere continuous, and therefore the constants $\alpha_1, \alpha_2, \beta_1, \beta_2$ are adjusted to arrange this condition at every interface; α_2 or β_2 must, of course, vanish for the interval that includes $r = 0$. The condition required for an extremal solution is that a linear combination of the Fréchet derivatives must vanish at the radii, $r_1, r_2, \ldots r_n$, where the jumps in σ occur. In the one-datum case this merely implies that the Fréchet derivative itself should vanish there. To illustrate the practical procedure for finding solutions we discuss the one-discontinuity system with $\sigma = \sigma_0$ when $r > r_1$ and $\sigma = 0$ when $r < r_1$. The explicit dependence of the problem on ω is eliminated by introducing a dimensionless conductivity $\tilde{\sigma}$, defined by $\mu_0 \omega \sigma_0 a^2$. A value of $\tilde{\sigma}$ is chosen and models with r_1 in the range* $[0, a)$ are investigated; in particular $D_1(r_1)$ is evaluated. If this quantity vanishes, then the model is a possible candidate as a stationary solution, because the Fréchet derivative does indeed vanish at the jump in σ, in accordance with (2.7). When such a zero in $D_1(r_1)$ is located, the function must be evaluated for other values of r to ensure that the zero-crossing at r_1 is the only one. For every value of $\tilde{\sigma}$ above 13·070 there is a single value of r_1 satisfying all the conditions. When $\tilde{\sigma}$ is less than this, D_1 has no zeros at all in the range $[0, a)$, no matter what r_1 is chosen to be; this implies that a uniform conductivity from centre to surface is the appropriate model in this range of conductivities.

The root search described above is easily programmed for a computer; the value of $\tilde{\sigma}$ is varied and when a root, r_1, has been found we then calculate E_1 from (5.3). In this way we find a class of extremal solutions, but we do not yet know if these are the best mini-max models: for example, are there any solutions with $\sigma = \sigma_0$ in $(0, r_1)$ and $\sigma = 0$ in (r_1, a)? There are, but they produce larger values of $\tilde{\sigma}$ for a given E_1. Presumably, these belong to a saddle-point set. On physical grounds, we can support our conviction that the first class of solutions is the one with minimum σ_0 by noting that the conducting material is near the surface, where it can have the greatest 'effect'. More complicated models (with several zero-conductivity shells) could not be found, although the author was unable to prove that such solutions do not exist. A fairly extensive search failed to find any for two-discontinuity systems and, in view of that fact that more and more conditions must be satisfied as the number of shells increases, it is plausible to assume that the one-discontinuity system is the only type that yields solutions in the one-datum problem.

Figs 1 and 2 show the results of the calculations with $l = 1$. The dimensionless conductivity, $\tilde{\sigma}$, simplifies the display; the datum E_1 is already dimensionless. As might be expected, $\tilde{\sigma}$ increases with increasing E_1. When E_1 is below 0·56149, the minimum σ model is the one that is constant throughout the whole sphere, and then the radius of the discontinuity is shown as zero in Fig. 2. Larger values of E_1 give a shell of decreasing thickness and increasing conductivity right at the surface. In fact, as E_1 becomes very large,

$$1 - r_1/a \to C/\tilde{\sigma}^{\frac{1}{2}} + O(\tilde{\sigma}^{-1}),$$

$$E_1 \to \tfrac{1}{2}\ln(\tilde{\sigma}/4) + O(\tilde{\sigma}^{-\frac{1}{2}}),$$

* The value $r = 0$ is excluded, because the Fréchet derivative always vanishes there and discontinuity in σ at this point is immaterial.

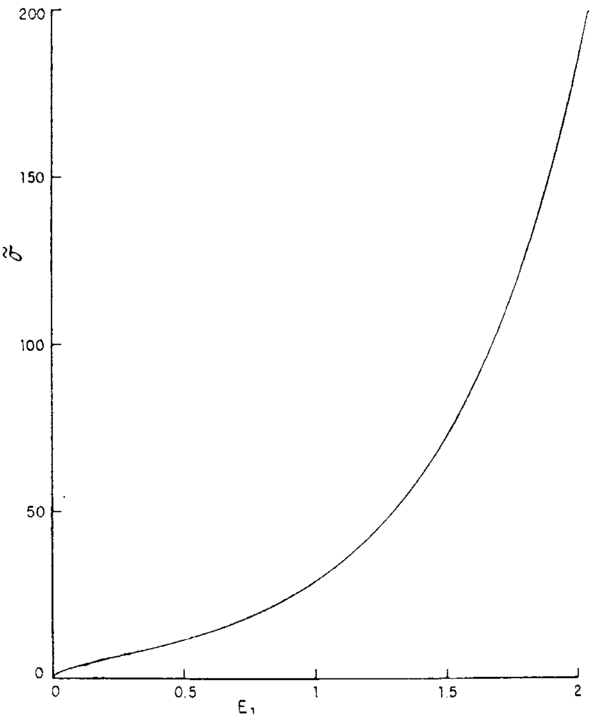

FIG. 1. The lower bound on the dimensionless maximum conductivity, $\tilde{\sigma} = \mu_0\, \omega\mu_0\, a^2$, consistent with an observed value of $E_1 = -\ln|W_1{}^\circ|$. The curve has been computed for $l = 1$ in a spherically symmetric conductor.

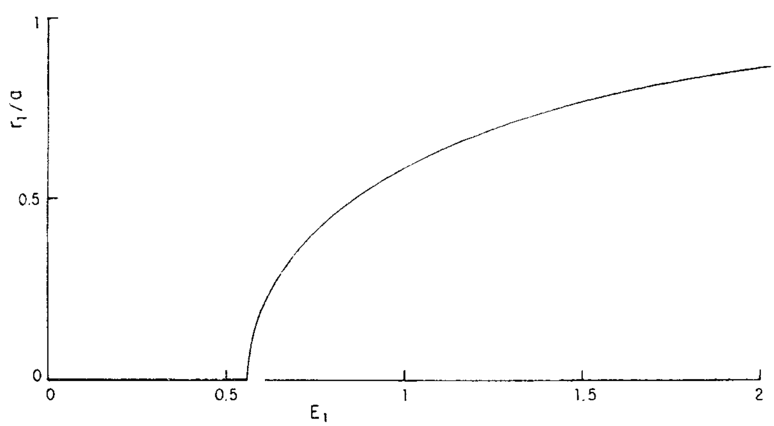

FIG. 2. The models that attain the lower bounds shown in Fig. 1 consist of a constant conductivity shell enclosing a zero-conductivity core. The radius of the non-conducting core is r_1, which is shown as a function of E_1. When $E_1 < 0{\cdot}56149$, the appropriate model is conducting throughout and then r_1 is shown to be zero.

where $C \sim 1{\cdot}74588$ and is the smallest root of the equation

$$\operatorname{Im} \cos (x \exp i\pi/4) = -3\pi/8.$$

These results are derived from asymptotic expansions of the spherical Bessel functions. This completes the discussion of the one-datum case.

5.3 *Two-data problems*

Logically, the next subject to be investigated is the behaviour of σ_0 when a knowledge of the phase datum, E_2, is included. Now the function that must vanish at the σ discontinuities is some linear combination of D_1 and D_2. The search procedure employed here is very similar to that of the one-datum case: we fix σ and try to find r_1 and r_2, computing E_1 and E_2 when a successful model has been found. To simplify the search, the linear combination

$$L(r) = D_1(r)\, D_2(r_1) - D_2(r)\, D_1(r)$$

is formed, since it always vanishes at $r = r_1$: we need now only find the zeros of $L(r_2)$. In fact, the search procedure has not always required, because models could be found in which $L(r)$ had only the one zero at $r = r_1$, and thus these models satisfy (2.7) already. We distinguish three types of model: type I in which $\sigma > 0$ with $r > r_1$, type II where $\sigma > 0$ with $r < r_1$ and type III where $\sigma > 0$ with $0 < r_1 < r < r_2 < a$. For a given value of $\tilde{\sigma}$ all three types might be required, but a continuous curve in the $E_1 - E_2$ plane could always be constructed. It will be important later to recall that type I models always exhibit the largest magnitudes of E_2. Again, models with more than two discontinuities could not be found to satisfy the conditions, and it will be assumed that they do not exist.

The results for $l = 1$ are illustrated in Fig. 3. It will be apparent that there is a zone in which no solutions fall; this could be attributed to an inadequate search procedure, but it was felt more likely that we were mapping the true zone of existence of solutions in the $E_1 - E_2$ plane. The precise location of the boundary clearly calls for the use of equation (2.8), which in our case becomes

$$D_1(r) = \chi D_2(r) \quad \text{for almost all } r \text{ in } (0, a). \tag{5.5}$$

At first sight (5.5) is not particularly attractive equation to solve for σ. Fortunately, however, we can guess a solution by looking at Fig. 3 and the attendant calculations: models with the largest phases are always type I with very great conductivity. This suggests that we may find a solution to (5.5) with an infinitesimally thin conducting shell at the surface, enclosing a non-conductor. Equations (5.1) and (5.4) show that, in the interior, D_1 and D_2 both behave like r^{2l+2} as functions of r, and thus (5.5) is satisfied. We need only note that D_1 and D_2 remain continuous in the limit of large σ to be assured that (5.5) is obeyed in the shell also; a more detailed behaviour of R_l^m is required to calculate E_1 and E_2. The response of a thin shell may be deduced from our equations (5.3) and (5.4) or, more immediately, from the work of Price (1949) who studied such shells as models of the ocean. For the complex response E, we obtain

$$E = \ln [(l+1-i\gamma)/(l+1)\, l],$$

where γ is a dimensionless variable given by $\gamma = \tilde{\sigma}(1 - r_1/a)$, and while $r_1 \to a$ in such a way as to keep γ finite. Separating E into real and imaginary parts gives the equation for the boundary in the $E_1 - E_2$ plane between allowed and forbidden data pairs. Thus we arrive at the inequality

$$\exp(-E_1) \leqslant l \cos E_2, \tag{5.6}$$

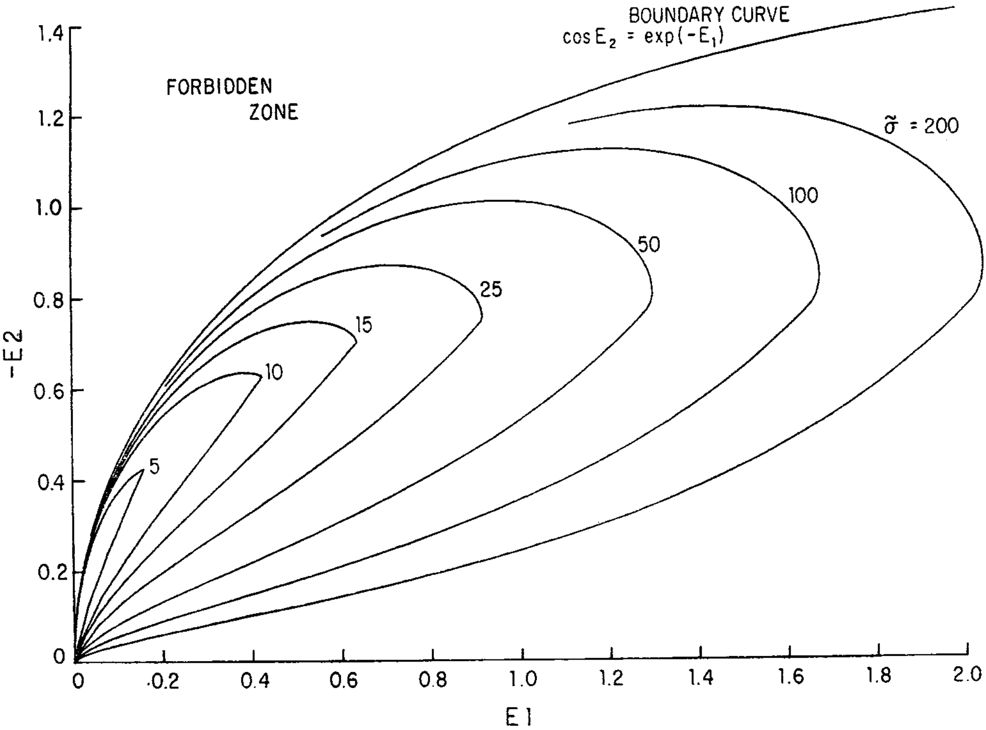

FIG. 3. Contours of the smallest maximum conductivity, $\tilde{\sigma}$, against E_1, the amplitude datum, and E_2 the phase datum, when $l = 1$. Note the region where there is no solution compatible with a given data pair.

for any two permissible data E_1 and E_2. The boundary when $l = 1$ is shown in Fig. 3. It is important to observe that another solution to (5.5) can be obtained by choosing a model with infinite conductivity in $(0, r_1)$ and zero conductivity in (r_1, a). In this case we find the lower boundary in Fig. 3, which is, of course, the E_1 axis, $E_2 = 0$.

Another two-data problem of some interest is the one where amplitude data are known at two frequencies, ω_1 and ω_2. Here we only consider the question of self-consistency of data pairs: it is enough to observe that the thin, surface-shell model gives rise to Fréchet derivatives satisfying (2.8) in this case also. Hence, after a little algebra, we arrive at the relation

$$E_2' - E_1' \leqslant \tfrac{1}{2} \ln \left[\rho^2 + (1-\rho^2) \exp(-2E_1')/l^2\right], \tag{5.7}$$

where $\rho = \omega_2/\omega_1 > 1$ and $E_1' = \mathrm{Re}E(\omega_1) = E_1(\omega_1)$ and $E_2' = \mathrm{Re}E(\omega_2) = E_1(\omega_2)$. The perfectly conducting core model is similarly a viable solution to (2.8); it yields

$$E_2' - E_1' \geqslant 0 \quad \text{with} \quad \omega_2 > \omega_1. \tag{5.8}$$

Equations (5.8) and (5.7) can be used to put bounds on $dE_1/d\omega$ by letting ω_2 approach ω_1:

$$0 \leqslant \frac{dE_1}{d\omega} \leqslant \frac{1}{\omega}\left(1 - \frac{\exp(-2E_1)}{l^2}\right).$$

These conditions can naturally be used when more than two data are available, simply by taking pairs of measurements from the set; then they are only necessary conditions for the existence of a solution.

5.4 *Applications*

The response of the Earth at a period of 11·2 yr has not yet been measured, but we shall apply the results of the previous section to the higher frequency data. For the numerical values we shall draw upon the analysis of Banks (1969). The lowest frequency signal Banks considers is the line at one cycle per year; however, it is found to be predominantly $l = 2$ in spherical harmonic composition, which is unfortunate for us, because our analyses have assumed $l = 1$. We pass on to the measurement at a period of 100 days: scrutinizing Figs 10 and 11 of Banks, we find $|W_1{}^0| = 0{\cdot}49 \pm 0{\cdot}030$ and a phase of 131°. Converted to our data conventions, this means $E_1 = 0{\cdot}722$ and* $E_2 = -0{\cdot}506$ (we ignore the error estimate for the moment). A one-datum analysis on E_1 alone (interpolating from tables prepared to draw Fig. 1) gives $\tilde{\sigma} = \mu_0 \omega \sigma_0 a^2 = 17{\cdot}6$. The implication, after the insertion of the various numerical values, is that $\sigma_0 = 0{\cdot}47 \Omega^{-1}\,\mathrm{m}^{-1}$, or that the conductivity in the mantle must exceed $0{\cdot}47\,\Omega^{-1}\,\mathrm{m}^{-1}$, from this single datum (we exclude the possibility that such high frequency variations can tell us anything about the core). The addition of the phase information, E_2, raises the value of $\tilde{\sigma}$ to 31·5 (see Fig. 3), and thus we conclude that all models fitting these two data exceed $0{\cdot}84\,\Omega^{-1}\,\mathrm{m}^{-1}$ somewhere. Banks' Fig. 17 shows some models with a conductivity below this value everywhere but, for the comparison to be fair, we must include the effect of the experimental error.

It is possible to account for error statistics in our treatment without approximation: we calculate a probability figure for the bound. This is done simply by integrating the joint probability distribution of the data over the region in the data space (e.g. the $E_1 - E_2$ plane in Fig. 3) implied by the inequality. For example, to find the probability that $\tilde{\sigma} \geqslant 25$ in Fig. 3, the probability distribution for the data pair must be integrated over the whole zone outside the $\tilde{\sigma} = 25$ contour. Unfortunately, Banks has made no estimates of the errors involved in the phase determinations. The error procedure outlined above can, of course, be used in the one-datum case rather simply: assuming a gaussian error law for $|W_1{}^0|$ with standard deviation 0·030, we infer that $|W_1{}^0| \leqslant 0{\cdot}524$ with a probability of 0·90, or that $E_1 \geqslant 0{\cdot}646$ with the same probability. This implies that $\tilde{\sigma} \geqslant 15{\cdot}6$, so that the conductivity in the mantle exceeds $0{\cdot}42\,\Omega^{-1}\,\mathrm{m}^{-1}$ with 90 per cent confidence, based only on the amplitude of the 0·01 cycles per day response and its estimated error.

A different application is the use of the consistency conditions (5.7) and (5.8). Fig. 2 of the author's paper (Parker 1970), which is based on Banks' work, shows clearly that (5.8) is violated by the data values, since $E_2' < E_1'$ when $\omega_2 > \omega_1$ for many pairs of points; equation (5.7) is also disobeyed by a number of the observations. Both these remarks must be understood to be made ignoring the estimated uncertainties. Parker reported that he could not find any conductivity model to fit Banks' data exactly, and conjectured that none exists: the observation that many of the data values are inconsistent proves the conjecture to be true.

6. Conclusions

We have seen how definite conclusions can be reached from very incomplete data. Such data can arise, not only when the number of observations is small, as it was in our examples, but also if the inverse problem is intrinsically non-unique: for example, no matter how many gravity observations there are outside a body, they can never determine uniquely the density inside. Therefore, the extremal method described here may be more widely applicable than is at first apparent. There is no reason in principle why the method should not be used when an adequate data set is available

* There is a difference of 180° between Banks' phases and ours because of different Fourier transform conventions.

but, if a detailed model can be constructed, it will be more satisfying to most geophysicists than a mere inequality.

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