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# INVERSE-TYPE ESTIMATES ON $hp$-FINITE ELEMENT SPACES AND APPLICATIONS

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ABSTRACT. This work is concerned with the development of inverse-type inequalities for piecewise polynomial functions and, in particular, functions belonging to $hp$-finite element spaces. The cases of positive and negative Sobolev norms are considered for both continuous and discontinuous finite element functions.The inequalities are explicit both in the local polynomial degree and the local mesh size.The assumptions on the $hp$-finite element spaces are very weak, allowing anisotropic (shape-irregular) elements and varying polynomial degree across elements. Finally, the new inverse-type inequalities are used to derive bounds for the condition number of symmetric stiffness matrices of $hp$-boundary element method discretisations of integral equations, with element-wise discontinuous basis functions constructed via scaled tensor products of Legendre polynomials.

## 1. INTRODUCTION

Inverse-type estimates are widely used in the error analysis of many numerical methods for partial differential equations and integral equations. Classical inverse-type estimates are of the form

$$\|v\|_{H^s(\tau)} \leq C(s, r_\tau, m_\tau) h_\tau^{-s} \|v\|_{L^2(\tau)},$$

where $L^2(\tau)$ and $H^s(\tau)$ denote the standard (Hilbertian) Lebesgue and Sobolev spaces ($s \geq 0$), respectively, consisting of functions from a (usually, triangular or quadrilateral) set $\tau \subset \mathbb{R}^d$, $d = 2, 3$, $v$ being a polynomial function on $\tau$, $h_\tau := \operatorname{diam}(\tau)$, $m_\tau$ denotes the polynomial degree of $v$, and $r_\tau$ the shape-regularity constant. When the domain $\tau$ is anisotropic, i.e., its size varies substantially in different space directions, the diameter $h_\tau$ of $\tau$ does not provide the right scaling for the inverse-type estimate described above. Instead, a sharper estimate reads

$$\|v\|_{H^s(\tau)} \leq C(s, m_\tau) \rho_\tau^{-s} \|v\|_{L^2(\tau)},$$

where $\rho_\tau$ denotes the radius of the largest inscribed circle contained in $\tau$.

Explicit knowledge of the dependence of the constant $C(s, m_\tau)$ on the polynomial degree $m_\tau$ is available. In particular, it is known that

$$\|v\|_{H^s(\tau)} \leq C(s) m_\tau^{2s} \rho_\tau^{-s} \|v\|_{L^2(\tau)}$$

(see, e.g., [8]).

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Generalisations of the above inverse-type estimates for functions belonging to finite-element-type spaces have been presented in [5, 4], in particular, in [5], inverse-type inequalities of the form

$$C_1(s,m)\|\rho^s u\|_{H^s(\Omega)} \le \|u\|_{L^2(\Omega)} \le C_2(s,m)\|\rho^{-s}u\|_{H^{-s}(\Omega)},$$

with $H^{-s}(\Omega)$ and $\|\cdot\|_{H^{-s}(\Omega)}$ denoting the negative order Sobolev spaces and the corresponding norms defined by duality; $\rho$ is a mesh-dependent function representing the local $\rho_\tau$; $C_1(s,m)$ and $C_2(s,m)$ are positive constants depending on $s$ and on the (uniform) local polynomial degree $m$.

The results presented in this work extend the theory developed in [5] in three ways. First, the inverse estimates derived below are explicit in the polynomial degree of the finite element space; second, *hp*-finite element spaces are admissible, i.e., the local polynomial degree in each element can vary. Finally, hanging nodes in the mesh are now admissible in the analysis, allowing for meshes of greater generality (e.g., meshes emerging from adaptive algorithms). Throughout this work, the setting and much of the notation of [5] is followed. More specifically, inverse-type inequalities of the form

$$\|m^{-2s}\rho^s u\|_{H^s(\Omega)} \lesssim \|u\|_{L^2(\Omega)} \lesssim \|m^{2s}\rho^{-s}u\|_{H^{-s}(\Omega)}$$

are proven below, where $m$ is a *mesh-dependent function* representing the local polynomial degree $m_\tau$; here and in the following, $A \lesssim B$ if and only if there exists a constant $C$, independent of local mesh parameters and local polynomial degrees, such that $A \le CB$. We also define $A \sim B$ if and only if $A \lesssim B$ and $B \lesssim A$.

The paper is structured as follows. Section 2 introduces the admissible finite element spaces for our analysis. Section 3 contains the main results of this work, i.e., Theorems 3.3, 3.5 and 3.9. Finally, in Section 4, using the results of Section 3, we prove bounds for the condition number of stiffness matrices for the *hp*-version (conforming) boundary element method admitting discontinuous basis functions.

## 2. Finite element spaces

Let $\Omega \subset \mathbb{R}^d$ be a bounded $d$-dimensional subset of $\mathbb{R}^3$, for $d = 2$ or $d = 3$, i.e., when $d = 3$, $\Omega \subset \mathbb{R}^3$ is a bounded domain, and when $d = 2$, $\Omega$ is a piecewise smooth Lipschitz manifold in $\mathbb{R}^3$ (which may or may not have a boundary). (Of course, the case of $\Omega \subset \mathbb{R}^2$ is included through the trivial imbedding into $\mathbb{R}^3$.)

We assume the usual notion of a Hilbertian Sobolev space $H^s(\Omega)$ for a positive integer $s$ (see, e.g., [2]); Sobolev spaces with real positive index $s \ge 0$ are defined through the real or $K$-method of function space interpolation (see, e.g., [1]), and Sobolev spaces with $s < 0$ are defined through duality, in a standard fashion. Since we also consider Sobolev spaces on manifolds, we assume that the pull-backs are also sufficiently smooth.

We consider subdivisions $\mathcal{T}$ of the set $\Omega$, consisting of pair-wise disjoint elements $\tau \in \mathcal{T}$, with $\tau \subset \Omega$, constructed in a standard fashion, as mappings

$$\chi_\tau : \hat\tau \to \tau,$$

of a reference element $\hat\tau$ onto $\tau$. We consider two choices for $\hat\tau$: $\hat\tau := \hat\sigma^d$, with $\hat\sigma^d$ denoting the $d$-dimensional simplex, or $\hat\tau := \hat\kappa^d$, with $\hat\kappa^d = (-1,1)^d$. The above maps are assumed to be constructed so as to ensure that the union of the closures of the disjoint open elements $\tau \in \mathcal{T}$ forms a covering of the closure of $\Omega$, i.e., $\bar\Omega = \bigcup_{\tau\in\mathcal{T}} \bar\tau$.

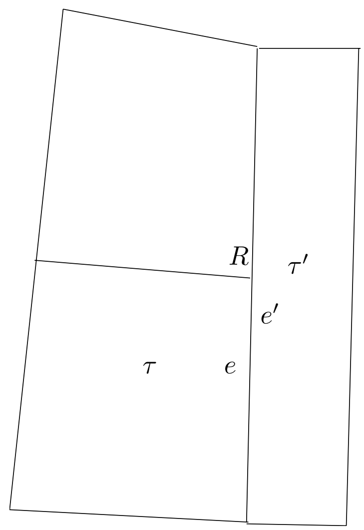


FIGURE 1. Regular hanging node

Hybrid meshes, i.e., meshes containing both (mapped) simplices and quadrilaterals/hexahedrals are allowed. Moreover, we allow non-conforming meshes containing *regular* hanging nodes, in the following sense.

**Definition 2.1.** Consider two neighbouring elements $\tau$ and $\tau'$, and let $e \subset \partial\tau$ and $e' \subset \partial\tau'$ denote the two element edges for which $e \cap e' \neq \emptyset$. If $e \neq e'$, then we say that there exists a hanging node in the subdivision, say $R$ (cf. Figure 1). We say that $R$ is a *regular hanging node* if

$$r^{-1} \leq \operatorname{meas}_1(e)/\operatorname{meas}_1(e') \leq r, \tag{2.1}$$

for $r > 0$ uniformly in $\mathcal{T}$; we call $r$ the *regularity constant of the subdivision* $\mathcal{T}$. We shall refer to a subdivision $\mathcal{T}$ as *regular* if it contains only one regular hanging node per edge.

We continue with some more notation.

**Definition 2.2.** For each $\tau \in \mathcal{T}$, we denote by $|\tau|$ its $d$-dimensional measure, by $h_\tau$ its diameter, by $\rho_\tau$ the diameter of the largest inscribed sphere contained in $\bar{\tau}$, and by $m_\tau := \min(1, \tilde{m}_\tau)$, where $\tilde{m}_\tau$ is the polynomial degree of the local finite element basis of $\tau$. Furthermore, if $t \subset \Omega$ is an arbitrary simplex or quadrilateral/hexahedron (not necessarily an element of $\mathcal{T}$), the local quantities $h_t$ and $\rho_t$ are defined completely analogously.

Also let $J_\tau$ denote the $3 \times d$ Jacobian of the map $\chi_\tau$, and define the Gram determinant of $\chi_\tau$ by

$$g_\tau(\hat{\mathbf{x}}) := \left(\det(J_\tau^T(\hat{\mathbf{x}})J_\tau(\hat{\mathbf{x}}))\right)^{1/2},$$

which appears in the integral change-of-variable formula:

$$\int_\tau f(\mathbf{x})\mathrm{d}\mathbf{x} = \int_{\hat{\tau}} f(\chi_\tau(\hat{\mathbf{x}}))g_\tau(\hat{\mathbf{x}})\mathrm{d}\hat{\mathbf{x}}.$$

Throughout this work, we make the following assumptions (cf. Assumptions 2.2 and 2.6 in [5]).

**Assumption 2.3** (Mapping Properties)**.** We have

$$\begin{aligned} D^{-1}|\tau| &\leq g_\tau(\hat{\mathbf{x}}) \leq D|\tau|, \\ E\rho_\tau^2 &\leq \lambda_{\min}(J_\tau^T(\hat{\mathbf{x}})J_\tau(\hat{\mathbf{x}})), \end{aligned}$$

uniformly in $\hat{x} \in \hat{\tau}$, with positive constants $D$ and $E$, independent of $\tau$, and $\lambda_{\min}(B)$, $\lambda_{\max}(B)$ denoting the minimum and the maximum eigenvalues of a square matrix $B$.

Let $\{\mathbf{x}_p : p \in \mathcal{N}\}$ denote the set of nodes of $\mathcal{T}$ that are *not* hanging nodes, for some index set $\mathcal{N}$.

**Assumption 2.4** (Finite Element Space Properties)**.** For some $c_1, c_2, c_3 > 1$ and $M \in \mathbb{N}$, we assume that, for all $\tau, \tau' \in \mathcal{T}$ with $\bar{\tau} \cap \bar{\tau'} \neq \emptyset$, we have

$$h_\tau \leq c_1 h_{\tau'}, \quad \rho_\tau \leq c_2 \rho_{\tau'}, \quad m_\tau \leq c_3 m_{\tau'},$$

$$\max_{p \in \mathcal{N}} \#\{\tau \in \mathcal{T} : \mathbf{x}_p \in \bar{\tau}\} \leq M.$$

The above assumptions on the mesh are very weak, in the sense that they allow very general degenerate (anisotropic) meshes (in particular, power-graded and geometrically-graded meshes are allowed); we refer to Section 2 of [5] for a detailed discussion on the admissible meshes under the above assumptions.

**Definition 2.5** (Finite Element Spaces)**.** For $m$ and $\hat{\tau} \in \{\hat{\sigma}^d, \hat{\kappa}^d\}$, we define

$$\mathbb{P}^m(\hat{\tau}) = \begin{cases} \text{polynomials of total degree} \leq m \text{ on } \hat{\tau}, & \text{if } \hat{\tau} = \hat{\sigma}^d; \\ \text{polynomials of coordinate degree} \leq m \text{ on } \hat{\tau}, & \text{if } \hat{\tau} = \hat{\kappa}^d. \end{cases}$$

We also define the polynomial degree vector $\mathbf{m} = (m_\tau : \tau \in \mathcal{T})$, with $m_\tau$ as in Definition 2.2. Then, we set

$$\begin{aligned} \mathcal{S}_0^{\mathbf{m}}(\mathcal{T}) &= \{u \in L^\infty(\Omega) : u \circ \chi_\tau \in \mathbb{P}^{\tilde{m}_\tau}(\hat{\tau}), \tilde{m}_\tau \geq 0, \tau \in \mathcal{T}\}, \\ \mathcal{S}_1^{\mathbf{m}}(\mathcal{T}) &= \{u \in C^0(\Omega) : u \circ \chi_\tau \in \mathbb{P}^{\tilde{m}_\tau}(\hat{\tau}), \tilde{m}_\tau \geq 1, \tau \in \mathcal{T}\}. \end{aligned}$$

We present a generalisation of Proposition 2.9 (or Corollary 2.10) in [5].

**Proposition 2.6.** *Let $\tau \in \mathcal{T}$ and let $\hat{t}$ be any simplex which is contained in the associated unit element $\hat{\tau} \in \mathbb{R}^d$. Let $\hat{P} \in \mathbb{P}^{m_\tau}(\hat{t})$ denote any $d$-variate polynomial of degree $m_\tau$ on $\hat{t}$ and define $t = \chi_\tau(\hat{t})$, $P = \hat{P} \circ \chi_\tau^{-1}$, where $\chi_\tau$ is assumed to be affine, for simplicity. Then, for all $0 \leq s \leq k$, we have*

$$\|P\|_{H^s(t)} \lesssim m_\tau^{2s} \rho_{\hat{t}}^{-s} \rho_\tau^{-s} \|P\|_{L^2(t)}.$$

*Proof.* The proof is similar to the proof of Proposition 2.9 in [5]. Here we only note that the crucial difference is in the use of the well-known inverse estimate

$$\|v\|_{H^1(\hat{\tau})} \lesssim m_\tau^2 \|v\|_{L^2(\hat{\tau})},$$

for $v = |\hat{P} \circ \nu| \in \mathbb{P}^{m_\tau}(\hat{\tau})$ (see, e.g., (4.6.5) in [8]) with the notation of the proof of Proposition 2.9 in [5]. $\square$

## 3. Inverse estimates

We begin by introducing some notation.

**Definition 3.1** (Mesh Function)**.** For each $p \in \mathcal{N}$, set $\rho_p = \max\{\rho_\tau : \mathbf{x}_p \in \bar{\tau}\}$. The mesh function $\rho$ is the unique function in $\mathcal{S}_1^{\mathbf{1}}(\mathcal{T})$ such that $\rho(\mathbf{x}_p) = \rho_p$, for each $p \in \mathcal{N}$.

**Definition 3.2** (Polynomial Degree Function)**.** For each $p \in \mathcal{N}$, set $m_p = \max\{m_\tau : \mathbf{x}_p \in \bar{\tau}\}$. The polynomial degree function $m$ is the unique function in $\mathcal{S}_1^{\mathbf{1}}(\mathcal{T})$ such that $m(\mathbf{x}_p) = m_p$, for each $p \in \mathcal{N}$.

Clearly $\rho$ and $m$ are positive, continuous functions on $\Omega$, and by Assumption 2.4, it follows that if $\mathbf{x} \in \tau$, then $\rho(\mathbf{x}) \sim \rho_\tau$ and $m(\mathbf{x}) \sim m_\tau$, for all $\tau \in \mathcal{T}$.

The next result is a generalisation of Theorem 3.2 in [5].

**Theorem 3.3.** *Let $0 \le s \le 1$, $-\infty < \underline{\alpha} < \overline{\alpha} < \infty$, and $-\infty < \underline{\beta} < \overline{\beta} < \infty$. Then,*

$$\|\frac{\rho^\alpha}{m^\beta} u\|_{H^s(\Omega)} \lesssim \|\frac{\rho^{\alpha-s}}{m^{\beta-2s}} u\|_{L^2(\Omega)},$$

*uniformly in $\alpha \in [\underline{\alpha}, \overline{\alpha}]$ and $\beta \in [\underline{\beta}, \overline{\beta}]$, $u \in \mathcal{S}_1^{\mathbf{m}}(\mathcal{T})$.*

*Proof.* We have

$$\nabla\Big(\frac{\rho^\alpha}{m^\beta} u\Big) = \alpha u \frac{\rho^{\alpha-1}}{m^\beta} \nabla\rho - \beta u \frac{\rho^\alpha}{m^{\beta+1}} \nabla m + \frac{\rho^\alpha}{m^\beta} \nabla u.$$

For $v \in \mathbb{P}^m(\tau)$, we recall Markov's inequality

$$\|\nabla v\|_{L^\infty(\tau)} \le C m^2 \rho_\tau^{-1} \|v\|_{L^\infty(\tau)}, \tag{3.1}$$

where $C$ is a constant that depends only on the shape of $\tau$, but not its size (see, e.g., in Theorem 4.76 in [8] for $d = 2$; for $d = 3$ the proof is analogous). Using this together with Assumption 2.4 and Proposition 2.6, we have, respectively,

$$\begin{aligned}
\|\nabla\Big(\frac{\rho^\alpha}{m^\beta} u\Big)\|^2_{L^2(\tau)} \quad &\lesssim \quad \|\frac{\rho^{\alpha-1}}{m^\beta} u\|^2_{L^2(\tau)} \|\nabla\rho\|^2_{L^\infty(\tau)} + \|\frac{\rho^\alpha}{m^{\beta+1}} u\|^2_{L^2(\tau)} \|\nabla m\|^2_{L^\infty(\tau)} \\
&\quad + \|\frac{\rho^\alpha}{m^\beta}\|^2_{L^\infty(\tau)} \|\nabla u\|^2_{L^2(\tau)} \\
&\lesssim \quad \frac{\rho_\tau^{2\alpha-4}}{m_\tau^{2\beta}} \|u\|^2_{L^2(\tau)} \|\rho\|^2_{L^\infty(\tau)} + \frac{\rho_\tau^{2\alpha-2}}{m_\tau^{2\beta+2}} \|u\|^2_{L^2(\tau)} \|m\|^2_{L^\infty(\tau)} \\
&\quad + \frac{\rho_\tau^{2\alpha-2}}{m_\tau^{2\beta-4}} \|u\|^2_{L^2(\tau)} \\
&\lesssim \quad \frac{\rho_\tau^{2\alpha-2}}{m_\tau^{2\beta}} \|u\|^2_{L^2(\tau)} + \frac{\rho_\tau^{2\alpha-2}}{m_\tau^{2\beta}} \|u\|^2_{L^2(\tau)} + \frac{\rho_\tau^{2\alpha-2}}{m_\tau^{2\beta-4}} \|u\|^2_{L^2(\tau)} \\
&\lesssim \quad \frac{\rho_\tau^{2\alpha-2}}{m_\tau^{2\beta-4}} \|u\|^2_{L^2(\tau)} \lesssim \|\frac{\rho^{\alpha-1}}{m^{\beta-2}} u\|^2_{L^2(\tau)},
\end{aligned}$$

and the proof for $s = 1$ follows by summation over $\tau \in \mathcal{T}$. The proof for $s \in (0, 1)$ follows by interpolation. □

Before considering the case of $hp$-inverse estimates for functions belonging to $\mathcal{S}_0^{\mathbf{m}}(\mathcal{T})$, we present the following auxiliary result.

**Lemma 3.4.** *Let $\hat{\tau} \subset \mathbb{R}^d$, $d = 2, 3$ denote the reference element, and let $\hat{u} : \hat{\tau} \to \mathbb{R}$ be a polynomial of degree $m$. Also let the function $\hat{\eta}_\delta : \hat{\tau} \to \mathbb{R}$ be defined as*

$$\hat{\eta}_\delta(\hat{x}) = \begin{cases} 1, & \text{if } \operatorname{dist}(\hat{x}, \partial\hat{\tau}) \geq \delta; \\ \delta^{-1} \operatorname{dist}(\hat{x}, \partial\hat{\tau}), & \text{otherwise.} \end{cases}$$

*Then $\hat{\eta}_\delta \hat{u} \in H_0^1(\hat{\tau})$, and the following estimates hold:*

$$\|\hat{\eta}_\delta \hat{u}\|^2_{L^2(\hat{\tau})} \leq \|\hat{u}\|^2_{L^2(\hat{\tau})}, \tag{3.2}$$

$$\|(1-\hat{\eta}_\delta)\hat{u}\|^2_{L^2(\hat{\tau})} \lesssim \delta m^2 \|\hat{u}\|^2_{L^2(\hat{\tau})}, \tag{3.3}$$

$$\|\nabla(\hat{\eta}_\delta)\hat{u}\|^2_{L^2(\hat{\tau})} \lesssim \delta^{-1} m^2 \|\hat{u}\|^2_{L^2(\hat{\tau})}. \tag{3.4}$$

The proof Lemma 3.4 is given in the Appendix to enhance the clarity of the presentation.

The next result is a generalisation of Theorem 3.4 in [5].

**Theorem 3.5.** *Let $0 \leq s < \frac{1}{2}$, $-\infty < \underline{\alpha} < \overline{\alpha} < \infty$, and $-\infty < \underline{\beta} < \overline{\beta} < \infty$. Then,*

$$\|\frac{\rho^\alpha}{m^\beta} u\|_{H^s(\Omega)} \lesssim \|\frac{\rho^{\alpha-s}}{m^{\beta-2s}} u\|_{L^2(\Omega)},$$

*uniformly in $\alpha \in [\underline{\alpha}, \overline{\alpha}]$ and $\beta \in [\underline{\beta}, \overline{\beta}]$, $u \in \mathcal{S}_0^{\mathbf{m}}(\mathcal{T})$.*

*Proof.* We consider $s > 0$; for otherwise, the proof is trivial. For brevity, here we denote $\tilde{u} = \frac{\rho^\alpha}{m^\beta} u$.

We shall define the fractional order Sobolev space $H^s(\Omega)$ using the $K$-method of function space interpolation (see, e.g., [1]): $H^s(\Omega)$ is defined as the interpolation space between $L^2(\Omega)$ and $H^1(\Omega)$ with interpolation parameter $s$. The norm $\|\tilde{u}\|_{H^s(\Omega)}$ can be expressed through the, so-called, $K$-functional $K(t, \tilde{u})$, as

$$\|\tilde{u}\|_{H^s(\Omega)} := \left( \int_0^\infty t^{-2s} K^2(t, \tilde{u}) \frac{\mathrm{d}t}{t} \right)^{\frac{1}{2}},$$

where

$$K(t, \tilde{u}) := \left( \inf_{\substack{\tilde{u} = u_0 + u_1 \\ u_0 \in L^2(\Omega), u_1 \in H^1(\Omega)}} \left\{ \|u_0\|^2_{L^2(\Omega)} + t^2 \|u_1\|^2_{H^1(\Omega)} \right\} \right)^{\frac{1}{2}}.$$

We now construct a suitable splitting $u_0 + u_1 = \tilde{u}$. For $\tau \in \mathcal{T}$, we define $\eta_\delta^\tau$ by $\eta_\delta^\tau := \hat{\eta}_\delta \circ \chi_\tau^{-1}$, where $\hat{\eta}_\delta$ as in the statement of Lemma 3.4. We consider the splitting $\tilde{u} = u_0 + u_1$, defined element-wise by

$$u_0|_\tau := \begin{cases} (1 - \eta_\delta^\tau)\tilde{u}|_\tau, & 0 \leq t \leq \gamma; \\ \tilde{u}|_\tau, & t > \gamma \end{cases} \quad \text{and} \quad u_1|_\tau := \begin{cases} \eta_\delta^\tau \tilde{u}|_\tau, & 0 \leq t \leq \gamma; \\ 0, & t > \gamma \end{cases}$$

with $\gamma := \rho_\tau / m_\tau^2$. From Lemma 3.4, we know that $u_1 \in H^1(\Omega)$. Then, we have

$$K^2(t, \tilde{u}) \leq \|u_0\|^2_{L^2(\Omega)} + t^2 \|u_1\|^2_{H^1(\Omega)}$$

and, therefore,

$$\begin{aligned}\|\tilde{u}\|^2_{H^s(\Omega)} &\leq \int_0^\infty t^{-2s-1}\Big(\|u_0\|^2_{L^2(\Omega)}+t^2\|u_1\|^2_{H^1(\Omega)}\Big)\mathrm{d}t\\ &= \sum_{\tau\in\mathcal{T}}\int_0^\infty t^{-2s-1}\Big(\|u_0\|^2_{L^2(\tau)}+t^2\|u_1\|^2_{H^1(\tau)}\Big)\mathrm{d}t\\ &\leq \sum_{\tau\in\mathcal{T}}\int_0^\gamma t^{-2s-1}\Big(\|(1-\eta^\tau_\delta)\tilde{u}\|^2_{L^2(\tau)}+t^2\|\eta^\tau_\delta\tilde{u}\|^2_{H^1(\tau)}\Big)\mathrm{d}t\\ &\quad+\sum_{\tau\in\mathcal{T}}\int_\gamma^\infty t^{-2s-1}\|\tilde{u}\|^2_{L^2(\tau)}\mathrm{d}t.\end{aligned}\tag{3.5}$$

For a function $f:\tau\to\mathbb{R}$, we denote $\hat{f}:\hat{\tau}\to\mathbb{R}$, with $\hat{f}:=f\circ\chi_\tau^{-1}$. Then, the term $\|(1-\eta^\tau_\delta)\tilde{u}\|^2_{L^2(\tau)}$ can be bounded as follows:

$$\begin{aligned}\|(1-\eta^\tau_\delta)\tilde{u}\|^2_{L^2(\tau)} &\lesssim \frac{\rho_\tau^{2\alpha}}{m_\tau^{2\beta}}\|(1-\eta^\tau_\delta)u\|^2_{L^2(\tau)} \lesssim \frac{\rho_\tau^{2\alpha}}{m_\tau^{2\beta}}|\tau|\|(1-\hat{\eta}^\tau_\delta)\hat{u}\|^2_{L^2(\hat{\tau})}\\ &\lesssim \frac{\rho_\tau^{2\alpha}}{m_\tau^{2\beta-2}}\delta|\tau|\|\hat{u}\|^2_{L^2(\hat{\tau})} \lesssim \frac{\rho_\tau^{2\alpha}}{m_\tau^{2\beta-2}}\delta\|u\|^2_{L^2(\tau)} \lesssim m_\tau^2\delta\|\tilde{u}\|^2_{L^2(\tau)},\end{aligned}\tag{3.6}$$

where in the third inequality we have made use of (3.3), and in the second and fourth inequality, we made use of Assumption 2.3.

For the term $\|\eta^\tau_\delta\tilde{u}\|^2_{H^1(\tau)}$, we work as follows:

$$\begin{aligned}\|\eta^\tau_\delta\tilde{u}\|^2_{H^1(\tau)} &= \|\eta^\tau_\delta\tilde{u}\|^2_{L^2(\tau)}+\|\nabla(\eta^\tau_\delta\tilde{u})\|^2_{L^2(\tau)} \lesssim \|\tilde{u}\|^2_{L^2(\tau)}+|\tau|\rho_\tau^{-2}\|\hat{\nabla}(\hat{\eta}_\delta\hat{\tilde{u}})\|^2_{L^2(\hat{\tau})}\\ &\lesssim \|\tilde{u}\|^2_{L^2(\tau)}+|\tau|\rho_\tau^{-2}\big(\|\hat{\tilde{u}}\hat{\nabla}\hat{\eta}_\delta\|^2_{L^2(\hat{\tau})}+\|\hat{\eta}_\delta\hat{\nabla}\hat{\tilde{u}}\|^2_{L^2(\hat{\tau})}\big)\\ &\lesssim \|\tilde{u}\|^2_{L^2(\tau)}+|\tau|\frac{\rho_\tau^{2\alpha-2}}{m_\tau^{2\beta}}\|\hat{u}\hat{\nabla}\hat{\eta}_\delta\|^2_{L^2(\hat{\tau})}+|\tau|\rho_\tau^{-2}\|\hat{\nabla}\hat{\tilde{u}}\|^2_{L^2(\hat{\tau})}\\ &\lesssim \|\tilde{u}\|^2_{L^2(\tau)}+|\tau|\frac{\rho_\tau^{2\alpha-2}}{m_\tau^{2\beta-2}}\delta^{-1}\|\hat{u}\|^2_{L^2(\hat{\tau})}+|\tau|\rho_\tau^{-2}m_\tau^4\|\hat{\tilde{u}}\|^2_{L^2(\hat{\tau})}\\ &\lesssim \|\tilde{u}\|^2_{L^2(\tau)}+\frac{\rho_\tau^{2\alpha-2}}{m_\tau^{2\beta-2}}\delta^{-1}\|u\|^2_{L^2(\tau)}+\rho_\tau^{-2}m_\tau^4\|\tilde{u}\|^2_{L^2(\tau)}\\ &\lesssim \rho_\tau^{-2}(\delta^{-1}m_\tau^2+m_\tau^4)\|\tilde{u}\|^2_{L^2(\tau)};\end{aligned}\tag{3.7}$$

in the first inequality we made use of standard scaling (see, e.g., Theorem 3.1.2 [3]), and in the fourth inequality we applied (3.4) and Theorem 3.3. Using the estimates (3.6) and (3.7), (3.5) can be further bounded by

$$\|\tilde{u}\|^2_{H^s(\Omega)}\lesssim\sum_{\tau\in\mathcal{T}}\Big(\int_0^\gamma t^{-2s-1}\Big(m_\tau^2\delta+t^2\rho_\tau^{-2}(\delta^{-1}m_\tau^2+m_\tau^4)\Big)\mathrm{d}t+\int_\gamma^\infty t^{-2s-1}\mathrm{d}t\Big)\|\tilde{u}\|^2_{L^2(\tau)}.$$

Choosing $\delta:=(4\rho_\tau)^{-1}t$, for $0\leq t\leq\gamma$, and calculating the respective integrals, we deduce

$$\|\tilde{u}\|^2_{H^s(\Omega)}\lesssim\sum_{\tau\in\mathcal{T}}\Big(\frac{m_\tau^2}{\rho_\tau}\frac{\gamma^{-2s+1}}{-2s+1}+\frac{m_\tau^4}{\rho_\tau^2}\frac{\gamma^{-2s+2}}{-2s+2}+\frac{\gamma^{-2s}}{2s}\Big)\|\tilde{u}\|^2_{L^2(\tau)}.$$

Recalling that $\gamma=\rho_\tau/m_\tau^2$, the result follows. (We note that, with this choice of $\gamma$, we have $\delta\equiv(4\rho_\tau)^{-1}t<(2m_\tau)^{-2}$, for all $\tau\in\mathcal{T}$, yielding $\eta^\tau_\delta\in H^1_0(\tau)$.) $\square$

We continue with some intermediate results of a technical nature.

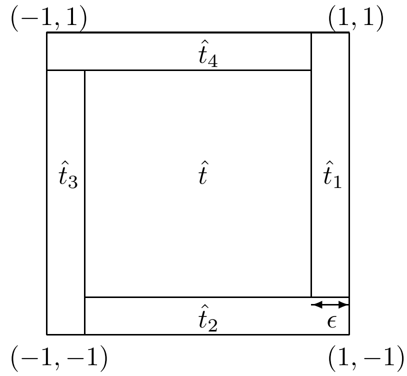


FIGURE 2. Splitting of $\hat{\tau} = \hat{\kappa}^2$

**Lemma 3.6.** *Let $\hat{\tau}$ be the reference element, and consider a set $\hat{t} \subset \hat{\tau}$, having the same shape as $\hat{\tau}$ and faces parallel to the faces of $\hat{\tau}$ such that* $\mathrm{dist}(\hat{t}, \partial\hat{\tau}) \geq \epsilon$, *for some $\epsilon > 0$ (i.e., $\hat{t}$ is $\epsilon$-away from the boundary of $\hat{\tau}$). Then, there exists a function $P_{\hat{t}} \in H^k(\mathbb{R}^d)$, such that*

$$P_{\hat{t}} \equiv 0 \quad on \quad \mathbb{R}^d \setminus \hat{t}, \quad 0 \leq P_{\hat{t}}(\mathbf{x}) \leq 1 \quad for\ all\, \mathbf{x} \in \hat{t}, \quad \frac{1}{2} \leq P_{\hat{t}}(\mathbf{x}) \leq 1 \quad for\ all\, \mathbf{x} \in \hat{t},$$

*and $\|D^i P_{\hat{t}}\|_{L^\infty(\hat{\tau})} \lesssim \epsilon^{-i}$, for $i = 1, 2, \ldots, k$.*

*Proof.* This is a standard mollifier argument. We can choose $P_{\hat{t}} = A_\epsilon^{\hat{t}}\, \mathrm{char}(\hat{t})$, where $A_\epsilon^{\hat{t}}$ is the standard mollification operator (see, e.g., [2]), and $\mathrm{char}(\omega)$ denotes the characteristic function of a set $\omega$. $\square$

We continue with the following technical result (cf. Lemma 3.5 in [5]).

**Lemma 3.7.** *Let $\hat{\tau}$ and $\mathbb{P}^m(\hat{\tau})$ be as in Definition 2.5. Then, for each $u \in \mathbb{P}^m(\hat{\tau})$, there exists a set $\hat{t} \subset \hat{\tau}$, having the same shape as $\hat{\tau}$ and faces parallel to the faces of $\hat{\tau}$, such that*

$$\rho_{\hat{t}} \sim 1 \quad and \quad \|u\|_{L^2(\hat{t})} \geq \frac{1}{2}\|u\|_{L^2(\hat{\tau})}.$$

*Proof.* Consider a set $\hat{t} \subset \hat{\tau}$ whose faces are parallel to the faces of $\hat{\tau}$ at a distance $\epsilon$ (see Figures 2 and 3 for a geometric representation of the setting for $\hat{\tau} = \hat{\kappa}^2$ and $\hat{\tau} = \hat{\sigma}^2$, respectively).

We first consider the case $\hat{\tau} = \hat{\kappa}^d$. To simplify the presentation consider the case $d = 2$; the case for $d \geq 3$ follows completely analogously, via a tensorization argument. To this end, we subdivide $\hat{\tau} \setminus \hat{t}$ into 4 subsets $\hat{t}_i$, $i = 1, 2, 3, 4$ as shown in Figure 2. Then for $\hat{t}_1$, we have

$$\begin{aligned}
\|u\|^2_{L^2(\hat{t}_1)} &= \int_{-1+\epsilon}^{1}\int_{1-\epsilon}^{1} u^2(x,y)\mathrm{d}x\mathrm{d}y = \int_{-1+\epsilon}^{1} \|u(\cdot,y)\|^2_{L^2(1-\epsilon,1)}\mathrm{d}y \\
&\leq \int_{-1+\epsilon}^{1} \epsilon\|u(\cdot,y)\|^2_{L^\infty(1-\epsilon,1)}\mathrm{d}y \leq \int_{-1}^{1} \epsilon\|u(\cdot,y)\|^2_{L^\infty(-1,1)}\mathrm{d}y \\
&\leq \int_{-1}^{1} 2\epsilon C m^2\|u(\cdot,y)\|^2_{L^2(-1,1)}\mathrm{d}y = 2\epsilon C m^2\|u\|^2_{L^2(\hat{\tau})},
\end{aligned}$$

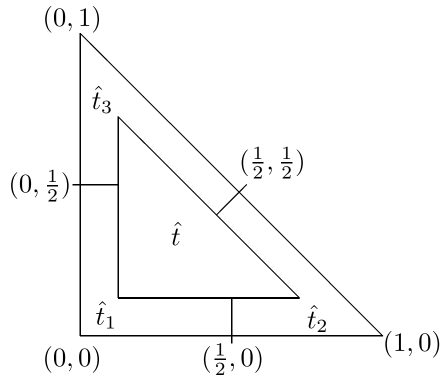


FIGURE 3. Splitting of $\hat{\tau} = \hat{\sigma}^2$

where in the third inequality we have used Bernstein's inequality

$$\|v\|_{L^\infty(\tau)} \lesssim m^d |\tau|^{-\frac{1}{2}} \|v\|_{L^2(\tau)} \tag{3.8}$$

with $d = 1$ (see, e.g., Theorem 3.92 in [8] for $d = 1$, Theorem 4.76 in [8] for $d = 2$; for $d = 3$ the proof is analogous). Quite similarly, we can deduce analogous bounds for $\hat{t}_i$, $i = 2, 3, 4$, or when $\hat{\tau}$ is a simplex, to obtain

$$\|u\|^2_{L^2(\hat{\tau} \setminus \hat{t})} = \sum_{i=1}^{4} \|u\|^2_{L^2(\hat{t}_i)} \leq 8\epsilon C m^2 \|u\|^2_{L^2(\hat{\tau})}.$$

Selecting $\epsilon = (\frac{32}{3} C m^2)^{-1}$, we deduce $\|u\|^2_{L^2(\hat{\tau} \setminus \hat{t})} \leq \frac{3}{4} \|u\|^2_{L^2(\hat{\tau})}$. Using this, we have, respectively,

$$\|u\|^2_{L^2(\hat{t})} = \|u\|^2_{L^2(\hat{\tau})} - \|u\|^2_{L^2(\hat{\tau} \setminus \hat{t})} \geq \|u\|^2_{L^2(\hat{\tau})} - \frac{3}{4} \|u\|^2_{L^2(\hat{\tau})} = \frac{1}{4} \|u\|^2_{L^2(\hat{\tau})}.$$

Thus, $\hat{t} \subset \hat{\tau}$ has the same shape as $\hat{\tau}$ (and faces parallel to those of $\hat{\tau}$, too) with $\operatorname{dist}(\hat{t}, \partial\hat{\tau}) \gtrsim \epsilon \sim m^{-2}$. Hence, $1 \geq \rho_{\hat{t}} \gtrsim 1 - \epsilon \sim 1$, and the proof is complete for $\hat{\tau} = \hat{\kappa}^2$.

Now, we consider the case $\hat{\tau} = \hat{\sigma}^d$. We first consider the case $d = 2$ in detail. We subdivide $\hat{\tau} \setminus \hat{t}$ into 3 subsets $\hat{t}_i$, $i = 1, 2, 3$ as shown in Figure 3. Then for $\hat{t}_1$, we have

$$\begin{aligned}
\|u\|^2_{L^2(\hat{t}_1)} &= \int_0^\epsilon \int_0^{\frac{1}{2}} u^2(x, y) \mathrm{d}x \mathrm{d}y + \int_\epsilon^{\frac{1}{2}} \int_0^\epsilon u^2(x, y) \mathrm{d}x \mathrm{d}y \\
&\leq \int_0^{\frac{1}{2}} \epsilon \|u(x, \cdot)\|^2_{L^\infty(0,\epsilon)} \mathrm{d}x + \int_\epsilon^{\frac{1}{2}} \epsilon \|u(\cdot, y)\|^2_{L^\infty(0,\epsilon)} \mathrm{d}y \\
&\leq \int_0^{\frac{1}{2}} \epsilon \|u(x, \cdot)\|^2_{L^\infty(0,\frac{1}{2})} \mathrm{d}x + \int_0^{\frac{1}{2}} \epsilon \|u(\cdot, y)\|^2_{L^\infty(0,\frac{1}{2})} \mathrm{d}y \\
&\leq \epsilon C m^2 \|u\|^2_{L^2(A_1)}, \qquad (3.9)
\end{aligned}$$

where $A_1 = (0, \frac{1}{2})^2$.

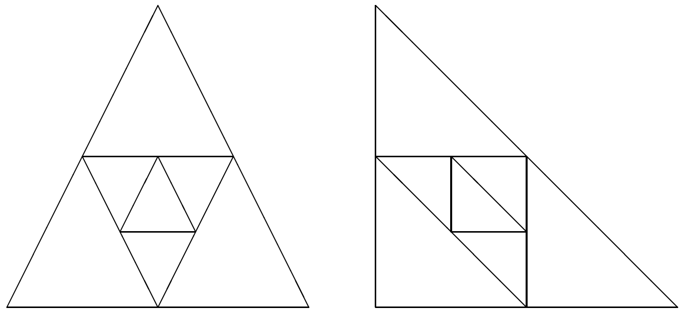


FIGURE 4. Splitting of $\hat{\tau} = \hat{\sigma}^3$ (not drawn to scale). (a) The splitting of the face of the canonical tetrahedron that is an equilateral triangle. (b) One of the 3 faces of the canonical tetrahedron that are right triangles.

For $\hat{t}_2$, we make the (linear) change of variables $(x, y) \to (\tilde{x}, \tilde{y})$, where $\tilde{x} = x + y$ and $\tilde{y} = y$. Then, we have

$$\begin{aligned}
\|u\|^2_{L^2(\hat{t}_2)} &\leq \int_0^{\epsilon} \int_{\frac{1}{2}}^{1} u^2(\tilde{x} - \tilde{y}, \tilde{y}) \mathrm{d}\tilde{x} \mathrm{d}\tilde{y} + \int_{\epsilon}^{\frac{1}{2}} \int_{1-\epsilon}^{1} u^2(\tilde{x} - \tilde{y}, \tilde{y}) \mathrm{d}\tilde{x} \mathrm{d}\tilde{y} \\
&\leq \int_{\frac{1}{2}}^{1} \epsilon \|u(\tilde{x} - \cdot, \cdot)\|^2_{L^\infty(0,\epsilon)} \mathrm{d}\tilde{x} + \int_{\epsilon}^{\frac{1}{2}} \epsilon \|u(\cdot - \tilde{y}, \tilde{y})\|^2_{L^\infty(1-\epsilon,1)} \mathrm{d}\tilde{y} \\
&\leq \int_{\frac{1}{2}}^{1} \epsilon \|u(\tilde{x} - \cdot, \cdot)\|^2_{L^\infty(0,\frac{1}{2})} \mathrm{d}\tilde{x} + \int_{0}^{\frac{1}{2}} \epsilon \|u(\cdot - \tilde{y}, \tilde{y})\|^2_{L^\infty(\frac{1}{2},1)} \mathrm{d}\tilde{y} \\
&\leq \epsilon C m^2 \|u\|^2_{L^2(A_2)}, \qquad (3.10)
\end{aligned}$$

where $A_2$ denotes the parallelogram with vertices $(\frac{1}{2}, 0), (1, 0), (\frac{1}{2}, \frac{1}{2}), (0, \frac{1}{2})$.

For $\hat{t}_3$, we make the (linear) change of variables $(x, y) \to (\tilde{x}, \tilde{y})$, where $\tilde{x} = x$ and $\tilde{y} = x + y$. Then, completely analogously to the case of $t_2$, we obtain

$$\|u\|^2_{L^2(\hat{t}_3)} \leq \epsilon C m^2 \|u\|^2_{L^2(A_3)}, \qquad (3.11)$$

where $A_3$ denotes the parallelogram with vertices $(\frac{1}{2}, 0), (\frac{1}{2}, \frac{1}{2}), (0, 1), (0, \frac{1}{2})$.

Combining (3.9), (3.10), and (3.11), we deduce

$$\|u\|^2_{L^2(\hat{\tau} \setminus \hat{t})} \leq 3\epsilon C m^2 \|u\|^2_{L^2(\hat{\tau})},$$

and selecting $\epsilon = (4Cm^2)^{-1}$, the result follows.

The case $d = 3$ follows by considering the splitting described in Figure 4, whereby the internal lines are understood to be the traces of the intersection of the domain $\hat{t}$ with planes perpendicular to the faces. Considering now the corresponding prisms that contain each of the parts of $\hat{t}$ and are in turn contained in the tetrahedron, the result follows by a suitable change of variables, in an analogous fashion to the two-dimensional case. $\square$

Combining the two lemmata above, we have the following result.

**Proposition 3.8.** *Let $u \in \mathbb{P}^m(\hat\tau)$, $\hat t \subset \hat\tau$ as in Lemma 3.7 with $\operatorname{dist}(\hat t, \partial\hat\tau) \gtrsim m^{-2}$, and $P_{\hat t}$ as in Lemma 3.6. Then, we have*

$$\|uP_{\hat t}\|_{H^k(\hat\tau)} \lesssim m^{2k}\|u\|_{L^2(\hat\tau)}. \tag{3.12}$$

*Proof.* We prove the result for $k = 1$; then, the result for general $k \in \mathbb{N}$ follows by induction.

We have, respectively,

$$\begin{aligned}\|\nabla(uP_{\hat t})\|_{L^2(\hat\tau)} &\leq \|(\nabla u)P_{\hat t}\|_{L^2(\hat\tau)} + \|u(\nabla P_{\hat t})\|_{L^2(\hat\tau)}\\ &\leq \|P_{\hat t}\|_{L^\infty(\hat\tau)}\|\nabla u\|_{L^2(\hat\tau)} + \|\nabla P_{\hat t}\|_{L^\infty(\hat\tau)}\|u\|_{L^2(\hat\tau)}\\ &\lesssim m^2\|u\|_{L^2(\hat\tau)} + m^2\|u\|_{L^2(\hat\tau)} \sim m^2\|u\|_{L^2(\hat\tau)},\end{aligned}$$

where in the last inequality we have made use of Proposition 2.6 and the $L^\infty$-bounds of $P_{\hat t}$ and its first derivative, from Lemma 3.6. □

The next result is a generalisation of Theorem 3.6 in [5].

**Theorem 3.9.** *Let $i \in \{0, 1\}$, $0 \leq s \leq k$, $-\infty < \underline{\alpha} < \overline{\alpha} < \infty$, $-\infty < \underline{\beta} < \overline{\beta} < \infty$, and assume that $\tilde m_\tau \geq i$, for all $\tau \in \mathcal{T}$. Then,*

$$\|\frac{\rho^{s+\alpha}}{m^{2s+\beta}}u\|_{L^2(\Omega)} \lesssim \|\frac{\rho^\alpha}{m^\beta}u\|_{H^{-s}(\Omega)},$$

*uniformly in $\alpha \in [\underline{\alpha}, \overline{\alpha}]$ and $\beta \in [\underline{\beta}, \overline{\beta}]$, $u \in \mathcal{S}_i^{\mathbf{m}}(\mathcal{T})$.*

*Proof.* The result is clear for $s = 0$. We prove it for $s = k \in \mathbb{N}$; the result then follows by interpolation. Without loss of generality we assume $u \neq 0$, for otherwise the result is trivial. We want to construct $w \in H^k(\Omega)$ such that

$$|(\frac{\rho^\alpha}{m^\beta}u, w)| \gtrsim \|\frac{\rho^{k+\alpha}}{m^{2k+\beta}}u\|^2_{L^2(\Omega)}$$

and

$$\|w\|_{H^k(\Omega)} \lesssim \|\frac{\rho^{k+\alpha}}{m^{2k+\beta}}u\|_{L^2(\Omega)};$$

then the result follows immediately from the definition of the dual norm.

To construct $w$, we work as follows. For any $\tau \in \mathcal{T}$, we have $\hat u := u \circ \chi_\tau^{-1} \in \mathbb{P}^{m_\tau}(\hat\tau)$, and by Lemma 3.7, there exists $\hat t(\tau) \subset \hat\tau$ such that

$$\rho_{\hat t(\tau)} \sim 1 \quad \text{and} \quad \|\hat u\|_{L^2(\hat t)} \geq \frac{1}{2}\|\hat u\|_{L^2(\hat\tau)}.$$

Scaling yields

$$\rho_{t(\tau)} \sim \rho_\tau \quad \text{and} \quad \|u\|_{L^2(t(\tau))} \geq \frac{1}{2}\|u\|_{L^2(\tau)}, \tag{3.13}$$

for $t(\tau) := \hat t(\tau) \circ \chi_\tau^{-1}$. Let $\tau \subset \Omega$ be an element, and consider $t(\tau) \subset \tau$ as above. Then, from the proof of Lemma 3.7, we have $\operatorname{dist}(t(\tau), \partial\tau) \gtrsim m_\tau^{-2}$. Now, using Lemma 3.6 and Proposition 3.8, there exists a function $P_{t(\tau)} \in H^k(\mathbb{R}^d)$ such that

$$P_{t(\tau)} \equiv 0 \quad \text{on} \quad \mathbb{R}^d \setminus \tau, \quad 0 \leq P_{t(\tau)}(\mathbf{x}) \leq 1 \quad \text{for all } \mathbf{x} \in \tau,$$
$$\frac{1}{2} \leq P_{t(\tau)}(\mathbf{x}) \leq 1 \quad \text{for all } \mathbf{x} \in t,$$

and

$$\|uP_{t(\tau)}\|_{H^k(\tau)} \lesssim m_\tau^{2k}\rho_\tau^{-k}\|u\|_{L^2(\tau)}, \tag{3.14}$$

with the last inequality resulting from (3.12) through a standard scaling argument.

We are now in a position to define

$$w = \sum_{\tau \in \mathcal{T}} m_\tau^{-4k-\beta} \rho_\tau^{2k+\alpha} u P_{t(\tau)}.$$

Then,

$$\begin{aligned} |(\frac{\rho^\alpha}{m^\beta} u, w)| &\gtrsim \sum_{\tau \in \mathcal{T}} m_\tau^{-4k-2\beta} \rho_\tau^{2k+2\alpha} \int_\tau u^2 P_{t(\tau)} \geq \sum_{\tau \in \mathcal{T}} m_\tau^{-4k-2\beta} \rho_\tau^{2k+2\alpha} \int_{t(\tau)} u^2 P_{t(\tau)} \\ &\geq \sum_{\tau \in \mathcal{T}} m_\tau^{-4k-2\beta} \rho_\tau^{2k+2\alpha} \frac{1}{2} \int_{t(\tau)} u^2 \geq \sum_{\tau \in \mathcal{T}} m_\tau^{-4k-2\beta} \rho_\tau^{2k+2\alpha} \frac{1}{8} \int_\tau u^2 \\ &\gtrsim \| \frac{\rho^{k+\alpha}}{m^{2k+\beta}} u \|_{L^2(\Omega)}^2, \end{aligned}$$

where in the third step we used the fact that $P_{t(\tau)} \geq \frac{1}{2}$ on $t(\tau)$, and in the fourth step we made use of (3.13).

Also, we have

$$\|w\|_{H^k(\Omega)}^2 = \sum_{\tau \in \mathcal{T}} \|w\|_{H^k(\tau)}^2 \leq \sum_{\tau \in \mathcal{T}} m_\tau^{-8k-2\beta} \rho_\tau^{4k+2\alpha} \|u P_{t(\tau)}\|_{H^k(\tau)}^2,$$

and, using (3.14), we obtain

$$\begin{aligned} \|w\|_{H^k(\Omega)}^2 &\lesssim \sum_{\tau \in \mathcal{T}} m_\tau^{-8k-2\beta} \rho_\tau^{4k+2\alpha} m_\tau^{4k} \rho_\tau^{-2k} \|u\|_{L^2(\tau)}^2 \\ &\lesssim \sum_{\tau \in \mathcal{T}} m_\tau^{-4k-2\beta} \rho_\tau^{2k+2\alpha} \|u\|_{L^2(\tau)}^2 \lesssim \| \frac{\rho^{k+\alpha}}{m^{2k+\beta}} u \|_{L^2(\Omega)}^2, \end{aligned}$$

and the result now follows. □

*Remark* 3.10. In [5], where the local polynomial degree is assumed to be uniform and constant, the relation

$$c(m_\tau) \|u\|_{L^\infty(\tau)} |\tau|^{\frac{1}{2}} \leq \|u\|_{L^2(\tau)} \leq \|u\|_{L^\infty(\tau)} |\tau|^{\frac{1}{2}} \tag{3.15}$$

was used extensively in the proofs presented in that work, with $c(m) > 0$ being included in the generic constants. When the explicit dependence on the polynomial degree of $u$ is required, as is the case in the present work, (3.15) becomes

$$\|u\|_{L^\infty(\tau)} |\tau|^{\frac{1}{2}} m_\tau^{-d} \lesssim \|u\|_{L^2(\tau)} \lesssim \|u\|_{L^\infty(\tau)} |\tau|^{\frac{1}{2}},$$

using Bernstein's inequality (3.8). Note that the lower bound involves the polynomial degree raised to the negative power equal to the dimension of the computational domain. However, the dimension of the computational domain is not present in conjunction with the polynomial degree in the classical $hp$-inverse estimates (for polynomials) of the form

$$\frac{h^s}{m^{2s}} \|u\|_{H^s(\tau)} \lesssim \|u\|_{L^2(\tau)} \lesssim \frac{m^{2s}}{h^s} \|u\|_{H^{-s}(\tau)},$$

and, thus, the same is expected for the inverse estimates on $hp$-finite element spaces. Hence, we systematically avoided using $\|u\|_{L^\infty(\tau)}$ in the proofs of Theorems 3.3, 3.5 and 3.9 above, making use instead of tensor-product-type constructions.

Next, we present an application of the above developments, regarding the estimation of the condition number of stiffness matrices arising in $hp$-boundary element methods.

## 4. The conditioning of $hp$-boundary element matrices emerging from discontinuous subspaces

Most boundary integral equations arising from elliptic partial differential equations have a variational formulation in a low order Sobolev space. A typical example is the classical single layer potential for the Laplacian, whose solutions belong to $H^{-1/2}(\Gamma)$, where $\Gamma$ is either a domain in $\mathbb{R}^d$ or a $d$-dimensional manifold in $\mathbb{R}^{d+1}$, satisfying the same assumptions as $\Omega$ in Section 2. We are interested in approximating the solutions to such integral equations using the $hp$-version boundary element method and in studying the conditioning of the resulting stiffness matrices.

We consider the general linear integral equation

$$(\lambda I + \mathcal{K})u_{\mathrm{an}}(x) := \lambda u_{\mathrm{an}}(x) + \int_\Gamma k(x,y)u_{\mathrm{an}}(y)\mathrm{d}y = f(x), \quad x \in \Gamma,$$

for some given scalar $\lambda \in \mathbb{R}$, kernel function $k$, analytic solution $u_{\mathrm{an}}$ and sufficiently smooth right-hand side $f$. The corresponding weak form is

$$\text{Find } \ u_{\mathrm{an}} \in H^s(\Gamma) \text{ such that } a(u_{\mathrm{an}}, v) := ((\lambda I + \mathcal{K})u_{\mathrm{an}}, v) = (f, v) \quad \forall v \in H^s(\Gamma),$$

where we assume $-1 \leq s < 1/2$, and $s$ is such that

$$a(u,u) \sim \|u\|^2_{H^s(\Gamma)}. \tag{4.1}$$

*Remark* 4.1. The equivalence (4.1) is assumed in the subsequent discussion. However, for particular cases of integral operators, it is possible to prove that relations of type (4.1) hold, for corresponding values of $s$; we refer, e.g., to [7] for details.

*Remark* 4.2. The spaces $\tilde{H}^s(\Gamma)$, defined by

$$\tilde{H}^s(\Gamma) := \{u \in H^s(\mathbb{R}) : \mathrm{supp} \subset \bar{\Gamma}\}, \quad \text{with norm} \quad \|u\|_{\tilde{H}^s(\Gamma)} := \|u\|_{H^s(\mathbb{R})},$$

are often used in the analysis of integral equations. Since for $-1 \leq s < 1/2$, we have $\tilde{H}^s(\Gamma) \equiv H^s(\Gamma)$, we shall be working with classical Sobolev spaces instead.

Let $\mathcal{T}$ be a subdivision of $\Gamma$, consisting of quadrilateral elements, satisfying the assumptions in Section 2. Since $s < 1/2$, we can choose an element-wise discontinuous finite-dimensional subspace $S^{\mathbf{m}}(\mathcal{T})$, constructed from tensor-product Legendre-polynomial local basis functions.

The corresponding discrete problem reads

$$\text{Find } \ u \in S^{\mathbf{m}}(\mathcal{T}) \text{ such that } a(u,v) := ((\lambda I + \mathcal{K})u, v) = (f, v) \quad \forall v \in S^{\mathbf{m}}(\mathcal{T}), \tag{4.2}$$

where $u$ is the $hp$-boundary element approximation to $u_{\mathrm{an}}$.

Let $L_i$ denote the Legendre polynomial of degree $i$ defined on $(-1,1)$. It is known that the Legendre polynomials form an orthogonal basis of $L^2(-1,1)$. Hence, for any $\hat{u} \in \mathbb{P}^m(-1,1) \subset L^2(-1,1)$, there exist $\hat{U}_i \in \mathbb{R}$, $i \in \mathbb{N}_0$, such that

$$\hat{u} = \sum_{i=0}^{m} \hat{U}_i L_i;$$

the corresponding Parseval's identity reads

$$\|\hat{u}\|^2_{L^2(-1,1)} = \sum_{i=0}^{m} \hat{U}_i^2 \frac{2}{2i+1}. \tag{4.3}$$

On the reference element $\hat{\tau} \equiv \bar{\kappa}^d := (-1,1)^d$, $d = 2,3$, we consider the tensor-product polynomial basis

$$\operatorname{span}\{L_{i_1} \dots L_{i_d} : 0 \le i_j \le m,\ i_j \in \mathbb{N}_0, j = 1, \dots, d\}.$$

Thus, every polynomial $\hat{u} : \hat{\tau} \to \mathbb{R}$ of degree at most $m$ in each variable can be expressed in terms of the above basis, i.e., there exist $U^{\hat{\tau}}_{i_1,\dots,i_d} \in \mathbb{R}$ such that

$$\hat{u}(\hat{x}_1, \dots, \hat{x}_d) = \sum_{\substack{0 \le i_j \le m \\ j=1,\dots,d}} U^{\hat{\tau}}_{i_1,\dots,i_d} \prod_{j=1}^{d} L_{i_j}(\hat{x}_j).$$

Using (4.3), along with Fubini's Theorem, we deduce

$$\|\hat{u}\|^2_{L^2(\hat{\tau})} = \sum_{\substack{0 \le i_j \le m \\ j=1,\dots,d}} \big(\hat{U}^{\hat{\tau}}_{i_1,\dots,i_d}\big)^2 \prod_{j=1}^{d} \frac{2}{2i_j + 1}. \tag{4.4}$$

For a typical $u \in S^{\mathbf{m}}(\mathcal{T})$, we have $u|_\tau := \hat{u}_\tau \circ \chi_\tau^{-1}$, where

$$\hat{u}_\tau(\hat{x}_1, \dots, \hat{x}_d) = \sum_{\substack{0 \le i_j \le m_\tau \\ j=1,\dots,d}} U^{\tau}_{i_1,\dots,i_d} \prod_{j=1}^{d} L^{\tau}_{i_j}(\hat{x}_j),$$

for some real numbers $U^{\tau}_{i_1,\dots,i_d}$ with $0 \le i_j \le m_\tau$, $j = 1, \dots, d$, $\tau \in \mathcal{T}$.

Let $A$ denote the symmetric stiffness matrix of the discrete problem (4.2). Then, for $\mathbf{U} := (U^{\tau}_{i_1,\dots,i_d} : 0 \le i_j \le m_\tau,\ j = 1, \dots, d,\ \tau \in \mathcal{T})$, we have

$$\mathbf{U}^T A \mathbf{U} = a(u,u) \sim \|u\|^2_{H^s(\Gamma)} \quad \text{and} \quad \mathbf{U}^T \mathbf{U} = \sum_{\tau \in \mathcal{T}} \sum_{\substack{0 \le i_j \le m \\ j=1,\dots,d}}^{m_\tau} \big(U^{\tau}_{i_1,\dots,i_d}\big)^2.$$

Thus, if we show the bounds

$$\Lambda_{\min} \sum_{\tau \in \mathcal{T}} \sum_{\substack{0 \le i_j \le m \\ j=1,\dots,d}}^{m_\tau} \big(U^{\tau}_{i_1,\dots,i_d}\big)^2 \lesssim \|u\|^2_{H^s(\Gamma)} \lesssim \Lambda_{\max} \sum_{\tau \in \mathcal{T}} \sum_{\substack{0 \le i_j \le m \\ j=1,\dots,d}}^{m_\tau} \big(U^{\tau}_{i_1,\dots,i_d}\big)^2,$$

we shall immediately have $\lambda_{\max}(A) \lesssim \Lambda_{\max}$ and $\lambda_{\min}(A) \gtrsim \Lambda_{\min}$.

**Lemma 4.3.** *If $0 \le s < 1/2$, we have*

$$\lambda_{\min}(A) \gtrsim \min_{\tau \in \mathcal{T}} \{m_\tau^{-d} |\tau|\} \quad \text{and} \quad \lambda_{\max}(A) \lesssim \max_{\tau \in \mathcal{T}} \{m_\tau^{4s} |\tau| \rho_\tau^{-2s}\}.$$

*Proof.* Let $d = 2$. We have

$$\|u\|^2_{L^2(\Gamma)} \le \|u\|^2_{H^s(\Gamma)} \lesssim \|\frac{\rho^{-s}}{m^{-2s}} u\|^2_{L^2(\Gamma)}, \tag{4.5}$$

where the first inequality is trivial, and the second inequality follows from Theorem 3.5. Parseval's identity (4.4), along with scaling, yields

$$\|u\|^2_{L^2(\Gamma)} = \sum_{\tau \in \mathcal{T}} \|u\|^2_{L^2(\tau)} \sim \sum_{\tau \in \mathcal{T}} |\tau| \|\hat{u}\|^2_{L^2(\hat{\tau})} = \sum_{\tau \in \mathcal{T}} |\tau| \sum_{i,j=0}^{m_\tau} \big(U^{\tau}_{i,j}\big)^2 \frac{2}{2i+1} \frac{2}{2j+1}.$$

Hence, in view of (4.5), we have

$$\|u\|^2_{H^s(\Gamma)} \geq \|u\|^2_{L^2(\Gamma)} \gtrsim \sum_{\tau\in\mathcal{T}} |\tau| \sum_{i,j=0}^{m_\tau} \left(U^\tau_{i,j}\right)^2 m_\tau^{-2} \gtrsim \min_{\tau\in\mathcal{T}}\{m_\tau^{-2}|\tau|\} \sum_{\tau\in\mathcal{T}} \sum_{i,j=0}^{m_\tau} \left(U^\tau_{i,j}\right)^2,$$

and, thus, the lower bound on $\lambda_{\min}(A)$ follows.

On the other hand, Parseval's identity gives

$$\begin{aligned}
\|\frac{\rho^{-s}}{m^{-2s}}u\|^2_{L^2(\Gamma)} &= \sum_{\tau\in\mathcal{T}} \|\frac{\rho^{-s}}{m^{-2s}}u\|^2_{L^2(\tau)} \sim \sum_{\tau\in\mathcal{T}} m_\tau^{4s}\rho_\tau^{-2s}\|u\|^2_{L^2(\tau)} \\
&\sim \sum_{\tau\in\mathcal{T}} m_\tau^{4s}\rho_\tau^{-2s}|\tau|\|\hat{u}\|^2_{L^2(\hat{\tau})} \\
&= \sum_{\tau\in\mathcal{T}} m_\tau^{4s}\rho_\tau^{-2s}|\tau| \sum_{i,j=0}^{m_\tau} \left(U^\tau_{i,j}\right)^2 \frac{2}{2i+1}\frac{2}{2j+1} \\
&\lesssim \sum_{\tau\in\mathcal{T}} m_\tau^{4s}\rho_\tau^{-2s}|\tau| \sum_{i,j=0}^{m_\tau} \left(U^\tau_{i,j}\right)^2 \\
&\leq \max_{\tau\in\mathcal{T}}\{m_\tau^{4s}\rho_\tau^{-2s}|\tau|\} \sum_{\tau\in\mathcal{T}} \sum_{i,j=0}^{m_\tau} \left(U^\tau_{i,j}\right)^2.
\end{aligned}$$

Hence, in view of (4.5), the upper bound on $\lambda_{\max}(A)$ is shown.

The corresponding bounds for $d = 3$ follow in a completely analogous fashion. □

**Lemma 4.4.** *If $-1 \leq s \leq 0$, we have*

$$\lambda_{\min}(A) \gtrsim \min_{\tau\in\mathcal{T}}\{m_\tau^{4s-d}|\tau|\rho_\tau^{-2s}\} \quad \textit{and} \quad \lambda_{\max}(A) \lesssim \max_{\tau\in\mathcal{T}}|\tau|.$$

*Proof.* Let $d = 2$. We have

$$\|\frac{\rho^{-s}}{m^{-2s}}u\|^2_{L^2(\Gamma)} \lesssim \|u\|^2_{H^s(\Gamma)} \leq \|u\|^2_{L^2(\Gamma)}, \tag{4.6}$$

where the first inequality follows from Theorem 3.9, and the second inequality follows from the dual imbedding. Parseval's identity (4.4), along with scaling, yields

$$\begin{aligned}
\|\frac{\rho^{-s}}{m^{-2s}}u\|^2_{L^2(\Gamma)} &= \sum_{\tau\in\mathcal{T}} \|\frac{\rho^{-s}}{m^{-2s}}u\|^2_{L^2(\tau)} \\
&\sim \sum_{\tau\in\mathcal{T}} m_\tau^{4s}\rho_\tau^{-2s}\|u\|^2_{L^2(\tau)} \sim \sum_{\tau\in\mathcal{T}} m_\tau^{4s}|\tau|\rho_\tau^{-2s}\|\hat{u}\|^2_{L^2(\hat{\tau})} \\
&= \sum_{\tau\in\mathcal{T}} m_\tau^{4s}|\tau|\rho_\tau^{-2s} \sum_{i,j=0}^{m_\tau} \left(U^\tau_{i,j}\right)^2 \frac{2}{2i+1}\frac{2}{2j+1} \\
&\gtrsim \sum_{\tau\in\mathcal{T}} m_\tau^{4s}|\tau|\rho_\tau^{-2s} \sum_{i,j=0}^{m_\tau} \left(U^\tau_{i,j}\right)^2 m_\tau^{-2}.
\end{aligned}$$

Hence, in view of (4.6), we have

$$\|u\|^2_{H^s(\Gamma)} \geq \|\frac{\rho^{-s}}{m^{-2s}}u\|^2_{L^2(\Gamma)} \gtrsim \min_{\tau\in\mathcal{T}}\{m_\tau^{4s-2}|\tau|\rho_\tau^{-2s}\} \sum_{\tau\in\mathcal{T}} \sum_{i,j=0}^{m_\tau} \left(U^\tau_{i,j}\right)^2,$$

and, thus, the lower bound on $\lambda_{\min}(A)$ follows.

On the other hand, Parseval's identity gives

$$\begin{aligned}\|u\|^2_{L^2(\Gamma)} &= \sum_{\tau\in\mathcal{T}}\|u\|^2_{L^2(\tau)} \sim \sum_{\tau\in\mathcal{T}}|\tau|\|\hat{u}\|^2_{L^2(\hat{\tau})} = \sum_{\tau\in\mathcal{T}}|\tau|\sum_{i,j=0}^{m_\tau}\left(U^\tau_{i,j}\right)^2\frac{2}{2i+1}\frac{2}{2j+1}\\ &\lesssim \sum_{\tau\in\mathcal{T}}|\tau|\sum_{i,j=0}^{m_\tau}\left(U^\tau_{i,j}\right)^2 \leq \max_{\tau\in\mathcal{T}}\{|\tau|\}\sum_{\tau\in\mathcal{T}}\sum_{i,j=0}^{m_\tau}\left(U^\tau_{i,j}\right)^2.\end{aligned}$$

Hence, in view of (4.6), the upper bound on $\lambda_{\max}(A)$ is shown.

The corresponding bounds for $d=3$ follow in a completely analogous fashion. □

*Remark* 4.5. A different upper bound for $\lambda_{\max}$ is presented in Lemma 4.4 of [6], for nodal finite element bases using arguments involving dual Sobolev embedding. It would be interesting to investigate further the sharpness of each bound with respect to different meshes.

Finally, recalling that the condition number cond$(A)$ of a symmetric positive definite matrix $A$ is given by

$$\mathrm{cond}(A) = \frac{\lambda_{\max}(A)}{\lambda_{\min}(A)},$$

an upper bound for the condition number of $A$ is now immediate.

**Theorem 4.6.** *The condition number of the stiffness matrix $A$, arising in the boundary element method* (4.2) *with tensor-product Legendre local polynomial basis, can be bounded as follows: if $0\leq s<1/2$, we have*

$$\mathrm{cond}(A) \lesssim \frac{\max_{\tau\in\mathcal{T}}\{m_\tau^{4s}|\tau|\rho_\tau^{-2s}\}}{\min_{\tau\in\mathcal{T}}\{m_\tau^{-d}|\tau|\}},$$

*and if $-1\leq s<0$, we have*

$$\mathrm{cond}(A) \lesssim \frac{\max_{\tau\in\mathcal{T}}|\tau|}{\min_{\tau\in\mathcal{T}}\{m_\tau^{4s-d}|\tau|\rho_\tau^{-2s}\}}.$$

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## Appendix: Proof of Lemma 3.4

*Proof.* Since $\hat{\eta}_\delta \in H^1_0(\hat{\tau})$ and $\hat{u}\in C^\infty(\hat{\tau})$, we have $\hat{\eta}_\delta\hat{u}\in H^1_0(\hat{\tau})$. The proof of the estimate (3.2) is trivial.

Next, we prove the bounds (3.3) and (3.4) in two dimensions via a tensor-product construction. The proof for three dimensions is analogous. Let $\hat{\tau}=(-1,1)^2$, and consider the splitting of $\hat{\tau}$ into 5 subregions as drawn in Figure 5. Consider $\hat{\eta}_\delta$ on $\hat{\tau}_1\subset\hat{\tau}$. It is immediate that

$$\hat{\eta}_\delta(x,y) = \delta^{-1}(x+1), \quad \text{for } (x,y)\in\hat{\tau}_1.$$

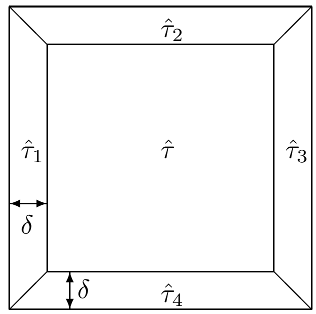


FIGURE 5. Domain of definition for $\hat{\eta}_\delta$

Thus

$$\begin{aligned}
\|(1-\hat{\eta}_\delta)\hat{u}\|^2_{L^2(\hat{\tau}_1)} &\leq \int_{-1}^{1}\int_{-1}^{-1+\delta}(1-\delta^{-1}(x+1))^2\hat{u}^2\mathrm{d}x\mathrm{d}y \\
&\leq \int_{-1}^{1}\|\hat{u}\|^2_{L^\infty((-1,-1+\delta)\times\{y\})}\int_{-1}^{-1+\delta}(1-\delta^{-1}(x+1))^2\mathrm{d}x\mathrm{d}y \\
&\lesssim \int_{-1}^{1}\delta\|\hat{u}\|^2_{L^\infty((-1,-1+\delta)\times\{y\})}\mathrm{d}y \leq \int_{-1}^{1}\delta\|\hat{u}\|^2_{L^\infty((-1,1)\times\{y\})}\mathrm{d}y \\
&\lesssim \int_{-1}^{1}\delta m^2\|\hat{u}\|^2_{L^2((-1,1)\times\{y\})}\mathrm{d}y = \delta m^2\|\hat{u}\|^2_{L^2(\hat{\tau})},
\end{aligned}$$

where in the fifth inequality we made use of Bernstein's inequality (3.8) in one dimension. Also, we have

$$\begin{aligned}
\|\nabla(\hat{\eta}_\delta)\hat{u}\|^2_{L^2(\hat{\tau}_1)} &= \|\frac{\partial\hat{\eta}_\delta}{\partial x}\hat{u}\|^2_{L^2(\hat{\tau}_1)} = \|\delta^{-1}\hat{u}\|^2_{L^2(\hat{\tau}_1)} \leq \int_{-1}^{1}\int_{-1}^{-1+\delta}\delta^{-2}\hat{u}^2\mathrm{d}x\mathrm{d}y \\
&\leq \int_{-1}^{1}\delta^{-1}\|\hat{u}\|^2_{L^\infty((-1,-1+\delta)\times\{y\})}\mathrm{d}y \\
&\leq \int_{-1}^{1}\delta^{-1}\|\hat{u}\|^2_{L^\infty((-1,1)\times\{y\})}\mathrm{d}y \\
&\lesssim \int_{-1}^{1}\delta^{-1}m^2\|\hat{u}\|^2_{L^2((-1,1)\times\{y\})}\mathrm{d}y = \delta^{-1}m^2\|\hat{u}\|^2_{L^2(\hat{\tau})}.
\end{aligned}$$

The proof for $\hat{\tau}_i$, $i=2,3,4$ is completely analogous.

Now, we consider the case $\hat{\tau}=\hat{\sigma}^d$. First, we consider the case $d=2$. We subdivide $\hat{\tau}\setminus\hat{t}$ into 3 subsets $\hat{t}_i$, $i=1,2,3$ as shown in Figure 3, with $\epsilon=\delta$. We observe that for the quadrilateral with vertices $(0,0),(\delta,\delta),(\delta,\frac{1}{2}),(0,\frac{1}{2})$, we have $\hat{\eta}_\delta=\delta^{-1}x$, and for the quadrilateral with vertices $(0,0),(\delta,\delta),(\frac{1}{2},\delta),(\frac{1}{2},0)$, we have

$\hat{\eta}_\delta = \delta^{-1}y$. Thereby, for $\hat{t}_1$, we have

$$
\begin{aligned}
\|(1-\hat{\eta}_\delta)\hat{u}\|^2_{L^2(\hat{t}_1)} &\leq \int_0^\delta \int_0^{\frac12} (1-\frac{y}{\delta})^2 u^2(x,y)\mathrm{d}x\mathrm{d}y \\
&\quad + \int_\delta^{\frac12} \int_0^\delta (1-\frac{x}{\delta})^2 u^2(x,y)\mathrm{d}x\mathrm{d}y \\
&\lesssim \int_0^{\frac12} \delta \|u(x,\cdot)\|^2_{L^\infty(0,\delta)}\mathrm{d}x + \int_\delta^{\frac12} \delta \|u(\cdot,y)\|^2_{L^\infty(0,\delta)}\mathrm{d}y \\
&\leq \int_0^{\frac12} \delta \|u(x,\cdot)\|^2_{L^\infty(0,\frac12)}\mathrm{d}x + \int_0^{\frac12} \delta \|u(\cdot,y)\|^2_{L^\infty(0,\frac12)}\mathrm{d}y \\
&\lesssim \delta m^2 \|\hat{u}\|^2_{L^2(A_1)}, \qquad (4.7)
\end{aligned}
$$

where $A_1 = (0,\frac12)^2$. Also, we have

$$
\begin{aligned}
\|\nabla(\hat{\eta}_\delta)\hat{u}\|^2_{L^2(\hat{t}_1)} &\leq \int_0^\delta \int_0^{\frac12} \big(\frac{\partial}{\partial y}(1-\frac{y}{\delta})\big)^2 u^2(x,y)\mathrm{d}x\mathrm{d}y \\
&\quad + \int_\delta^{\frac12} \int_0^\delta \big(\frac{\partial}{\partial x}(1-\frac{x}{\delta})\big)^2 u^2(x,y)\mathrm{d}x\mathrm{d}y \\
&\lesssim \delta^{-1} m^2 \|\hat{u}\|^2_{L^2(A_1)}, \qquad (4.8)
\end{aligned}
$$

similarly as in the case of $\hat{\tau} = \hat{\kappa}^2$. For $\hat{t}_2$, we make the (linear) change of variables $(x,y) \to (\tilde{x},\tilde{y})$, where $\tilde{x} = x+y$ and $\tilde{y} = y$. Observing also that for the quadrilateral with vertices $(\frac12,0), (\frac12,\delta), (1-\sqrt{2}\delta,\delta), (1,0)$, we have $\hat{\eta}_\delta = \delta^{-1}(x+y-1)$, we deduce

$$
\begin{aligned}
\|(1-\hat{\eta}_\delta)\hat{u}\|^2_{L^2(\hat{t}_2)} &\leq \int_0^\delta \int_{\frac12}^1 (1-\frac{\tilde{y}}{\delta})^2 u^2(\tilde{x}-\tilde{y},\tilde{y})\mathrm{d}\tilde{x}\mathrm{d}\tilde{y} \\
&\quad + \int_\delta^{\frac12} \int_{1-\delta}^1 (1-\frac{\tilde{x}-1}{\delta})^2 u^2(\tilde{x}-\tilde{y},\tilde{y})\mathrm{d}\tilde{x}\mathrm{d}\tilde{y} \\
&\lesssim \delta m^2 \|\hat{u}\|^2_{L^2(A_2)}, \qquad (4.9)
\end{aligned}
$$

where $A_2$ denotes the parallelogram with vertices $(\frac12,0), (1,0), (\frac12,\frac12), (0,\frac12)$. Completely analogously, we have

$$\|\nabla(\hat{\eta}_\delta)\hat{u}\|^2_{L^2(\hat{t}_2)} \lesssim \delta^{-1} m^2 \|\hat{u}\|^2_{L^2(A_2)}. \qquad (4.10)$$

For $\hat{t}_3$, we make the (linear) change of variables $(x,y) \to (\tilde{x},\tilde{y})$, where $\tilde{x} = x$ and $\tilde{y} = x+y$. Then, analogously to the case of $t_2$, we obtain

$$\|(1-\hat{\eta}_\delta)\hat{u}\|^2_{L^2(\hat{t}_3)} \lesssim \delta m^2 \|\hat{u}\|^2_{L^2(A_3)}, \qquad (4.11)$$

where $A_3$ denotes the parallelogram with vertices $(\frac12,0), (\frac12,\frac12), (0,1)$, and $(0,\frac12)$. Straightforward calculation also yields

$$\|\nabla(\hat{\eta}_\delta)\hat{u}\|^2_{L^2(\hat{t}_3)} \lesssim \delta^{-1} m^2 \|\hat{u}\|^2_{L^2(A_3)}. \qquad (4.12)$$

Combining (4.7), (4.9), and (4.11), the bound (3.3) follows, and combining (4.8), (4.10), and (4.12), the bound (3.4) follows.

The case $d = 3$ follows by considering the splitting described in Figure 4, whereby the internal lines are understood to be the traces of the intersection of the domain

$\hat{t}$ with planes perpendicular to the faces, completely analogously to the proof of Lemma 3.7. □

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