Inversion and jump formulae for variation diminishing transforms.

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Summary. - *This is a study of the values of the determining function Φ as well as its derivatives and gaps in terms of the generating function f and its derivatives where f is the convolution transform of Φ with variation diminishing kernel.*

1. - Introduction.

In this paper we shall be interested in variation diminishing convolution transforms

$$(1.1) \qquad f(x) = \int_{-\infty}^{\infty} G(x-t)\varphi(t)dt$$

that is $G(t)$ is such that $f(x)$ never has more changes of sign than $\varphi(t)$ whenever $\varphi(t)$ is bounded and continuous in $(-\infty, \infty)$. It was shown by SCHÖNBERG [9] that for $G(t) \in L_1$ $G(t)$ is variation diminishing if and only if

$$(1.2) \qquad G(t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} E(s)^{-1} e^{st} ds,$$

$$(1.3) \qquad E(s) = e^{-cs^2+bs} \prod_{k=k}^{\infty} \left(1 - \frac{s}{a_k}\right) e^{s/a_k}$$

where $c \geq 0$, b and a_k for $k \geq 1$ are real $\sum_{k=1}^{\infty} a_k^{-2} < \infty$.

We shall be interested also in the related convolution STIELTJES transform

$$(1.4) \qquad f(x) = \int_{-\infty}^{\infty} G(x-t) e^{rt} d\alpha(t)$$

where r is a real number. In this paper we shall restrict ourselves to the case $c = 0$ in (1.3); this class contains as special cases the LAPLACE, STIELTJES, MEIJER and many other transforms (see [7; pp. 65-79]).

I. I. HIRSCHMAN and D. V. WIDDER in their book « The Convolution Tran-

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sform» [7 Ch. VI] and many papers (see [4], [5] [6] for instance) found an inversion theory for (1.1) and (1.4) based on the sequence $\{a_k\}$ and $P_m(D)$ defined by

$$P_m(D) = e^{b_m D} \prod_{k=1}^{m} \left(1 - \frac{D}{a_k}\right) e^{(1/a_k)D} \tag{1.5}$$

where $D = \frac{d}{dx}$, $e^{kD}f(x) = f(x+k)$ and $b_m = o(1)$ $(m \to \infty)$.

We shall find here a formula that gives $\varphi^{(n)}(t_0)$ in terms of $f(x)$ whenever $\varphi^{(n)}(t)$ satisfies a mild condition in a neighborhood of t_0 (the conditions vary for different classes of $\{a_k\}$ and $\{b_m\}$). The inversion problem is solved also when $\varphi^{(n)}(t_0)$ does not exist, but $\lim\limits_{h\to 0+} \varphi^{(n)}(t_0 \pm h) = \varphi^{(n)}(t_0 \pm)$ or $\varphi^{(n)}(t_0 \pm 0)$ exist where $g(t_0 \pm 0)$ are the numbers satisfying

$$\int_0^h [g(t_0 \pm y) - g(t_0 \pm 0)]dy = o(h) \qquad h \to 0+ \tag{1.6}$$

when such numbers exist. In this case we shall prove

$$\alpha\varphi^{(n)}(t_0+0) + (1-\alpha)\varphi^{(n)}(t_0-0) = \lim_{m\to\infty} D^n P_{m_r}(D)f(x) \tag{1.7}$$

where $\{m_r\}$ is a subsequence of m and $0 \le \alpha \le 1$, α depends on $\{b_m\}$, $\{a_k\}$ and $\{m_r\}$.

The above mentioned formula generalizes the HIRSCHMAN-WIDDER inversion formulae also in case $n = 0$.

Jump formulae which give $\varphi^{(n)}(t+) - \varphi^{(n)}(t-)$ or $\alpha(t+) - \alpha(t-)$ (and more general «left» and «right» values of $\varphi^{(n)}(t_0)$) in terms of $f(x)$ are also found. Jump formulae are known in three special cases of convolution transforms, the LAPLACE transform [10, p. 298] and [2], the STIELTJES transform [10, p. 351] and the second iterate of STIELTJES transform [1]. Our jump formulae will include these formulae (and give a somewhat better result even for these three transforms) and yield jump formulae for many known transforms (like MEIJER, Theta and iterates of LAPLACE transform).

We shall find that in applying the inversion operator $D^n P_m(D)$ on $f(x)$ the limit may vary for different orders of $\{a_k\}$ although these changes have no influence on $G(t)$.

Sometimes various subsequences of $D^n P_m(D) f(x)$ will tend to different limits and therefore we shall state and prove our theorems with conditions on $\{a_k\}$ and subsequences of it.

The paper is divided into two parts:

Part A: On properties of $\{a_k\}$ and $G_m^{(n)}(t)$.

Part B: Inversion and jump formulae.

Part A contains many estimations of $G_m^{(n)}(t)$.

$$G_m^{(n)}(t) = D^n P_m(D) G(t) \tag{1.8}$$

in sections 3 and 4 which shall be used further on for inversion and jump theorems but are interesting for themselves. In section 2 we introduce many conditions on sequences $\{a_k\}$ and subsequences and find some relations between them. The conditions mentioned above will be used in stating the theorems on inversion and jump formulae as well as the properties of $G_m^{(n)}(t)$.

Part B contains inversion and jump theorems as well as some theorems on $G_m^{(n)}(t)$ for special classes of $\{a_k\}$.

We shall generally follow the notation used by I. I. HIRSCHMAN and D. V. WIDDER in their book «The Convolution Transform». However, we shall introduce the definitions not mentioned in the introduction in the first time they are needed, even if they can be found in [7].

PART A

On properties of $\{a_k\}$ and $G_m^{(n)}(t)$.

2. - Conditions on $\{a_k\}$ and a new classification of $G(t)$.

In this section we shall be concerned with conditions on sequences related to $G(t)$, namely real sequences $\{a_k\}$ for which Σa_k^{-2} converge (to a certain $G(t)$ correspond a sequence $\{a_k\}$ and all its rearrangements). In fact the existence of various types of variation diminishing convolution transforms for which we shall later find inversion and jump formulae is shown in some of the examples of this section.

In [7; p. 140] the condition $C_m = o(S_m^{3/2})$ $(m \to \infty)$ where

$$C_m = \sum_{k=m+1}^{\infty} |a_k|^{-3} \quad \text{and} \quad S_m = \sum_{k=m+1}^{\infty} |a_k|^{-2} \tag{2.1}$$

was found useful for the inversion formula. The following lemma will establish equivalent conditions to $C_m = o(S_m^{3/2})$ $(m \to \infty)$.

LEMMA 2.1. - The assumptions

$$\sum_{k=m+1}^{\infty} |a_k|^{-2-\alpha} = o(S_m^{1+\frac{\alpha}{2}}) \quad (m \to \infty) \quad \text{for some fixed} \quad \alpha > 0 \tag{2.2}$$

and

$$\lim_{m\to\infty} (\max_{k>0} a_k^{-2}) S_m^{-1} = 0 \tag{2.3}$$

are equivalent.

COROLLARY 2.1. - Assumption (2.3) implies (2.2) for any $\alpha > 0$.

PROOF. - Let us assume (2.2) for some $\alpha > 0$. If (2.3) is not valid then a subsequence $\{m_r\}$ of $\{m\}$ exists such that $(\max_{k\geq m_r+1} a_k^{-2}) S_{m_r}^{-1} \geq \beta > 0$ for all $r \geq 1$. Therefore

$$\sum_{k=m_r+1}^{\infty} |a_k|^{-2-\alpha} \geq (\max_{k\geq m_r+1} a_k^{-2})^{1+\frac{\alpha}{2}} \geq \beta^{1+\frac{\alpha}{2}} S_{m_r}^{1+\frac{\alpha}{2}}$$

which contradicts (2.2).

Assuming (2.3) then

$$\sum_{k=m+1}^{\infty} |a_k|^{-2-\alpha} = \sum_{k=m+1}^{\infty} |a_k|^{-2} |a_k|^{-\alpha} \leq (\max_{k\geq m+1} |a_k|^{-\alpha}) S_m =$$

$$= ((\max_{k\geq m+1} a_k^{-2}) S_m^{-1})^{\alpha/2} S_m^{1+\frac{\alpha}{2}} = o(S_m^{1+\frac{\alpha}{2}}) \qquad (m \to \infty).$$

REMARK. - Assumption (2.3) is easier to verify (2.2), but assumption (2.2) with $\alpha = 1$, namely $C_m = o(S_m^{3/2})$ $(m \to \infty)$ is more convenient for proving some properties.

It is obvious that

$$0 \leq \liminf_{m\to\infty} (\max_{k>m} a_k^{-2}) S_m^{-1} \leq \limsup_{m\to\infty} (\max_{k>m} a_k^{-2}) S_m^{-1} \leq 1 \tag{2.4}$$

and therefore it is quite natural to consider the following assumption on $\{a_m\}$

$$\limsup_{m\to\infty} (\max_{k>m} a_k^{-2}) S_m^{-1} < 1 \tag{2.5}$$

and

$$\limsup_{m\to\infty} (\max_{k>m} a_k^{-2}) S_m^{-1} = 1. \tag{2.6}$$

One can easily see that (2.5) includes (2.3), but if we state instaed of (2.5)

$$0 < \limsup_{m\to\infty} (\max_{k>m} a_k^{-2}) S_m^{-1} < 1 \tag{2.7}$$

we obtain by (2.4) that (2.3), (2.6) and (2.7) form a full classification of the sequences $\{a_k\}$. However, since in all the above mentioned assumptions the order of the sequence $\{a_k\}$ is important, this is not a classification of $G(t)$.

Of all sequences $\{a_k\}$ that are related to $G(t)$ we can choose those which satisfy $|a_{k+1}| \geq |a_k|$ and call them absolutely monotonic ordered. Since assumptions (2.3), (2.6) and (2.7) depend only on absolute value of the $\{a_k\}$, we can classify $G(t)$ as follows: $G(t)$ satisfies (2.3), (2.6) or (2.7) if an absolutely monotonic ordered sequence $\{a_k\}$ related to $G(t)$ satisfies (2.3), (2.6) or (2.7) respectively. Obviously every $G(t)$ satisfies one and only one of the preceding assumptions.

In [7; p. 120] I. I. HIRSCHMAN and D. V. WIDDER formed the following well known classification of $G(t)$:

$G(t) \in$ class I if there are both positive and negative a_k,

$G(t) \in$ class II if $a_k > 0$ and $\Sigma\, a_k^{-1} = \infty$,

and $G(t) =$ class III if $a_k > 0$ and $\Sigma\, a_k^{-1} < \infty$.

LEMMA 2.2. - The intersection of any of the classes I, II or III with any of the classes: $G(t)$ satisfies (2.3), (2.7) or (2.6); is non-empty.

PROOF. - It is easy to verify that there are kernels $G(t)$ of classes I, II and III that satisfy (2.3) since all the examples given in [7; pp. 65-79] satisfy (2.3).

The following three examples are of kernels that belong to classes I, II and III respectively and satisfy (2.7) (The $G(t)$ are defined by $\{a_k\}$ and b, as in (1.2) and (1.3)).

EXAMPLE 1: Choose $b = 0$ and $a_k = (-2)^k$ $k \geq 1$.

EXAMPLE 2. - Choose $b = 0$, $a_1 = 1$ and $a_k = 2^{2^n}$ for k satisfying $\sum_{r=1}^{n} 2^{2^{r-1}} \leq k < \sum_{r=1}^{n+1} 2^{2^{r-1}}$ for $n \geq 1$.

It is easy to verify that $\Sigma\, a_k^{-2} < \infty$, $\Sigma\, a_k^{-1} = \infty$ and for $m = \sum_{r=1}^{n} 2^{2^{r-1}} - 2$, $n > 1$ we have $\max_{k>m} a_k^{-2} = a_{m+1}^{-2} = (2^{2^{n-1}})^{-2} = (2^{2^n})^{-1}$ and $S_m = (2^{2^{n-1}})^{-2} + \sum_{l=n}^{\infty} 2^{\,l} (2^{2^l})^{-2} = 2 \cdot (2^{2^n})^{-1} + \sum_{l=n+1}^{\infty} (2^{2^l})^{-1}$, therefore $2 \cdot (2^{2^n})^{-1} < S_m < 4 \cdot (2^{2^n})^{-1}$ and $G(t)$ satisfies (2.7).

EXAMPLE 3. - Choose $b = 0$ and $a_k = 2^k$ $(k \geq 1)$.

The following three examples are of kernels that belong to classes I, II and III respectively and satisfy (2.6).

EXAMPLE 4. - Chose $b = 0$ and $a_k = (-1)^k k!$ for $k \geq 1$.

EXAMPLE 5. - Choose $b = 0$, $i_1 = 0$, $i_n = 2i_{n-1} + n - 1$ and let $a_k = 2^{i_n}$ for $\sum_{r=1}^{n-1} 2^{i_r} < k \leq \sum_{r=1}^{n} 2^{i_r}$.

One can verify $\sum_{k=1}^{\infty} a_k^{-1} = \infty$ and $\sum_{k=1}^{\infty} a_k^{-2} < \infty$.

For $m = \sum_{k=1}^{n} 2^{i_r} - 1$ we have $S_m = 2^{-2i_n}(1 + o(1))$ $n \to \infty$ and $\max_{k>m} a_k^{-2} = a_{m+1}^{-2} = 2^{-2i_n}$ therefore $G(t)$ satisfies (2.6).

EXAMPLE 6. - Choose $b = 0$ and $a_k = k!$ for $k \geq 1$. Q.E.D.

Since the inversion and jump formulae depend on what assumption the sequence $\{a_k\}$ satisfies and not only on $G(t)$, we shall state our theorems with conditions on $\{a_k\}$ remembering that if $G(t)$ satisfies a certain assumption, an absolutely monotonic ordered related sequence satisfies the same assumption.

We shall find it useful to assume (2.5) instead of (2.7). In the following lemma we shall find an equivalent condition to (2.5) which is more convenient in applications.

LEMMA 2.3. - Assumption (2.5) which is $\limsup_{m\to\infty} (\max_{k>m} a_k^{-2}) S_m^{-1} < 1$ and

(2.8) $$S_m - \max_{k>m} a_k^{-2} \geq \alpha S_m \quad \text{for} \quad 0 < \alpha \quad \alpha \text{ fixed for all } m$$

are equivalent.

PROOF. - Formula (2.8) implies $1 - (\max_{k>m} a_k^{-2}) S_m^{-1} \geq \alpha > 0$, that is $(\max_{k>m} a_k^{-2}) S_m^{-1} \leq \leq 1 - \alpha$ which implies (2.5). To see that (2.5) implies (2.8) we should remember that for each finite m_0 we can find some $\alpha = \alpha(m_0)$, $0 < \alpha < 1$ such that $S_m - \max_{k>m} a_k^{-2} \geq \alpha(m_0) S_m$ for all $m \leq m_0$; the rest is similar to the proof in the other direction. Q.E.D.

Instead of $S_m - \max_{k>m} a_k^{-2}$ we can have natural and (as we see in Chapter 3) useful generalization

(2.9) $$S_m^{(r)} = S_m - \max_{m<k_1<\ldots<k_r} \sum_{p=1}^{r} a_{k_p}^{-2}.$$

We can write (2.8) in the form $S_m^{(1)} \geq \alpha S_m$ for $\alpha > 0$.

LEMMA 2.4. - If $S_m^{(1)} \geq \alpha S_m$ for $\alpha > 0$ α fixed for all m, then

(2.10) $$S_m^{(r)} \geq \alpha^r S_m.$$

PROOF. - We shall prove our lemma for any fixed m and r. If $a_{m+i}^{-2} = \max_{k\geq m+i} a_k^{-2}$ for $i = 1, 2, \ldots, r$ the proof of (2.10) would have been trivial.

For some fixed k_0 $a_{k_0} > a_{k_0+1}$ we can form the sequence $\{a_k^*\}$ where $a_{k_0}^* =$

$= a_{k_0+1}$, $a^*_{k_0+1} = a_{k_0}$ and for $k \neq k_0$, k_0+1 $a^*_k = a_k$. The sequence $\{a^*_k\}$ with S^*_m defined by $S^*_m = \sum_{k=m+1}^{\infty} (a^*_k)^{-2}$ satisfies $S^{*(1)}_m \geq \alpha S^*_m$ since for $m > k_0$ and $m < k_0$ $S_m = S^*_m$ and $S^{(1)}_m = S^{*(1)}_m$ and for $m = k_0$ we have

$$S^*_{k_0} - \max_{k>k_0} (a^*_k)^{-2} \geq S_{k_0} + (a^{-2}_{k_0+1} - a^{-2}_{k_0}) - \max_{k>k_0} a^{-2}_k > \alpha S_{k_0} + \alpha(a^{-2}_{k_0+1} - a^{-2}_{k_0}) = \alpha S^*_{k_0}.$$

With a finite number of changes of the above mentioned kind for $k > m$ we get a sequence $\{a^{**}_k\}$ which satisfies $S^{**}_m = S_m$, $S^{**(r)}_m = S^{(r)}_m$ and $S^{**(1)}_l \geq \alpha S^{**}_l$ for all $l > m$ and $(a^{**}_{m+i})^{-2} = \max_{k \geq m+i} (a^{**}_k)^{-2}$ for $i = 1, 2, \dots r$ and therefore the proof is completed. Q.E.D.

It is interesting to consider similar relations for subsequence of S_m. For instance, take a subsequence $\{m_l\}$ of the integers, the condition $S^{(r)}_{m_l} \geq \alpha S_{m_l}$ $\alpha > 0$ α fixed for all l. Of course, $S^{(r)}_{m_l} \geq \alpha S_{m_l}$ for a fixed $\alpha > 0$ implies for $p < r$ $S^{(p)}_{m_l} \geq \alpha S_{m_l}$.

Here $S^{(1)}_{m_l} \geq \alpha S_{m_l}$ $\alpha > 0$ for all l does not imply $S^{(r)}_{m_l} \geq \alpha_1 S_{m_l}$ for $\alpha_1 > 0$. This can be seen using Example 5 from the proof of Lemma 2.2 and using $m_l = \sum_{r=1}^{n} 2^{i_n} - 2$ (where i_n is defined there) we get

$$S^{(2)}_{m_l} = o(S^{(1)}_{m_l}) = o(S_{m_l}) \quad (l \to \infty) \quad \text{and} \quad S^{(1)}_{m_l} \geq \frac{1}{2} S_{m_l}.$$

However, the following lemma gives some information on $S^{(r)}_{m_l}$ in a special case.

LEMMA 2.5. - If $\lim_{l\to\infty} (\max_{k>m_l} a^{-2}_k) S^{-1}_{m_l} = 0$ then for $r = 1, 2 \dots$ there exists an α_r $\alpha_r > 0$ such that $S^{(r)}_{m_l} \geq \alpha_r S_{m_l}$ for all l.

PROOF. - $\lim_{l\to\infty} (\max_{k>m_l} a^{-2}_k) S^{-1}_{m_l} = 0$ implies for $l > l_0(n)$ $S^{(1)}_{m_l} \geq \left(1 - \frac{1}{n}\right) S_{m_l}$ this clearly yields $S^{(r)}_{m_l} \geq \left(1 - \frac{r}{n}\right) S_{m_l}$. Combining the above result with the fact that for each l_0 there exists a $\beta(l_0) \equiv \beta$ $\beta(l_0) > 0$ so that for $l \geq l_0$ $S^{(r)}_{m_l} \geq \beta S_{m_l}$, we complete the proof of this lemma. Q.E.D.

In Lemma 2.2 we showed that even for $G(t) \in \text{II}$ and $|a_{k+1}| \geq |a_k|$ $\limsup_{m\to\infty} a^{-2}_{m+1} S^{-1}_m$ need not be zero. The following lemma will solve the problem of $\liminf_{m\to\infty} a^{-2}_{m+1} S^{-1}_m$ in this case and some similar cases.

LEMMA 2.6. - Suppose $\sum_{k=1}^{\infty} |a_k|^{-\beta} = \infty$ for some $0 < \beta < 2$ and $|a_k| \leq |a_{k+1}|$ then $\liminf_{m\to\infty} a^{-2}_{m+1} S^{-1}_m = 0$.

PROOF. - Assume $\liminf\limits_{m\to\infty} a_{m+1}^{-1} S_m^{-2} = \alpha > 0$ which implies for $m > m_0$

$$a_{m+1}^{-2} S_m^{-1} > \frac{\alpha}{2} \quad \text{or} \quad S_m - S_{m+1} = a_{m+1}^{-2} > \frac{\alpha}{2} S_m \qquad S_m\left(1 - \frac{\alpha}{2}\right) > S_{m+1},$$

therefore $\sum\limits_{i=0}^{\infty} S_{m+i}^{l}$ converges for each $l > 0$. Since $a_{m+1}^{-2} < S_m$ $\sum\limits_{i=0}^{\infty} |a_{m+i}|^{-2l}$ converges for each $l > 0$, setting $2l = \beta$ we get a contradiction. Q.E.D.

One can prove the following analogue to Lemma 2.1 for conditions on subsequences.

LEMMA 2.7. - Let $\{m_r\}$ be a subsequence of $\{m\}$ then the assumptions

$$\sum_{k=m_r+1}^{\infty} |a_k|^{-2-\alpha} = o(S_{m_r}^{1+\frac{\alpha}{2}}) \quad (r \to \infty) \quad \text{for some} \quad \alpha > 0 \tag{2.11}$$

and

$$\lim_{r\to\infty} (\max_{k>m_r} a_k^{-2}) S_{m_r}^{-1} = 0 \tag{2.12}$$

are equivalent.

PROOF. - The proof is similar to that of Lemma 2.1. Q.E.D.

COROLLARY 2.7a. - The statements

$$\liminf_{m\to\infty} (\max_{k>m} a_k^{-2}) S_m^{-1} = 0 \quad \text{and} \quad \liminf_{m\to\infty} S_m^{-1-\frac{\alpha}{2}} \left(\sum_{k=m+1}^{\infty} |a_k|^{-2-\alpha} \right) = 0$$

are equivalent.

However, $\Sigma |a_k|^{-\beta} = \infty$ for some β $(0 < \beta < \infty)$ does not imply $\liminf\limits_{m\to\infty} (\max\limits_{k>m} a_k^{-2}) S_m^{-1} =$ $= 0$ when we do not assume $|a_{k+1}| \geq |a_k|$ as the following example will show.

We shall take $\{a_k\} = \{k\}$ (with $\gamma = b$ this is related to the LAPLACE transform [7; p. 66]) and rearrange it in the following ways:

EXAMPLE 7. - Inserting 2^n after 5^n by the following method: 1, 3, 5, 2, 6, 7, 9, 10 ... 25, 4, 26, ... we get after some calculations $\lim\limits_{m\to\infty} (\max\limits_{k>m} a_k^{-2}) S_m^{-1} = \dfrac{1}{2}$.

EXAMPLE 8. - Similarly if we put $k!$ after $(3k)! + 1$ as in the following: 3, 4, 7, 1, 8, ... we get $\lim\limits_{m\to\infty} (\max\limits_{k>m} a_k^{-2}) S_m^{-1} = 1$.

The above examples indicate a general method of «spoiling» sequences, There is another series of conditions that are useful for inversion and jump formulae.

DEFINITION. - A sequence $\{a_k\}$ will be said to satisfy assumption $B(n, \{m_r\}, l)$ if

(2.13) $$\min_{k_1>m_r}(\max\{|a_{k_i}|\,|1 \leq i \leq n \quad k_1 < k_2 < \ldots < k_n\})(S_{m_r}^{(n-1)})^l < K$$

where K is independent of r.

REMARK. - It is easy to see that if $\{a_k\}$ satisfies $B(n, \{m_r\}, l)$ it satisfies also $B(p, \{m_r\}, l)$ with $p < n$.

DEFINITION. - A sequence $\{a_k\}$ is said to satisfy $B(n, \{m_r\})$ if it satisfies assumption $B(n, \{m_r\}, l)$ for some l.

DEFINITION. - A sequence $\{a_k\}$ is said to satisfy $B(n)$ if it satisfies $B(n, \{m\})$.

LEMMA 2.8. - If $\{a_k\}$ satisfies $|a_k| \leq |a_{k+1}|$ and $B(1)$ then $\{a_k\}$ satisfies $B(n)$ for all n.

PROOF. - Applying $|a_k| \leq |a_{k+1}|$ we can write instead of (2.13) $|a_{m+n}|(S_{m+n-1})^l < K$ and this is satisfied in case $\{a_k\}$ satisfies $B(1)$. Q.E.D.

LEMMA 2.9. - The inequality $\liminf_{r\to\infty}(\max_{k>m_r} a_k^{-2})S_{m_r}^{-1} > 0$ implies that $\{a_k\}$ satisfies $B(1, \{m_r\})$.

PROOF. - The inequality $\liminf_{r\to\infty}(\max_{k>m_r} a_k^{-2})S_{m_r}^{-1} \geq \alpha > 0$ implies

$$\limsup_{r\to\infty}(\min_{k>m_r}|a_k|)^2 S_{m_r} \leq \frac{1}{\alpha}$$

which yields our result immediately. Q.E.D.

However, it would be incorrect to assume that our assumption implies $\liminf_{r\to\infty}(\max a_k^{-2})S_{m_r}^{-1} > 0$. All examples given in [7; pp. 65-79] satisfy both $|a_{m+1}|\, S_m^l < K$ for some l and $\lim_{m\to\infty} a_{m+1}^{-2} S_m^{-1} = 0$. In fact, in all the examples we brought $\{a_k\}$ satisfies $B(n)$ for all n.

The following example will show that even an absolutely monotonic ordered sequence $\{a_k\}$ does not always satisfy $B(n, \{m\})$.

EXAMPLE 9. - $b = 0$, $a_k = \sqrt{k}\log k$, $\Sigma a_k^{-2} < \infty$ but $\left\{\sqrt{m+1}\log(m+1)\left(\sum_{k=m+1}^{\infty}\frac{1}{k(\log k)^2}\right)^l\right\}$ is not bounded by K for any l.

LEMMA 2.10. - If $\sum_{k=1}^{\infty} |a_k|^{-\beta} < \infty$ for some $\beta < 2$ then $\{a_k\}$ satisfies $B(n)$ for every n.

PROOF. - It is easy to see

$$S_m = \sum_{k=m+1}^{\infty} |a_k|^{-\beta} |a_k|^{-2-\beta} \leq (\min_{k>m} |a_k|)^{\beta-2} \sum_{k=m+1}^{\infty} |a_k|^{-\beta}.$$

Define

$$S_{m,\beta}^{(n)} = \sum_{k=m+1}^{\infty} |a_k|^{-\beta} - \max_{k_1<\ldots<k_n} \left(\sum_{l=1}^{n} |a_{k_l}|^{-\beta}\right)$$

$$S_m^{(n-1)} \leq \min_{k_i>m} (\max \{ |a_{k_i}| \,|\, 1 \leq i \leq n \;\; k_1 < \ldots < k_n \})^{\beta-2} S_{m,\beta}^{(n-1)}.$$

$\sum_{k=m+1}^{\infty} |a_k|^{-\beta}$ and $S_{m,\beta}^{(i)}$ are bounded by $\sum_{k=1}^{\infty} |a_k|^{-\beta}$ which is independent of m. We can find l so that $l(\beta-2) < -1$ and therefore using (2.13) we complete the proof. Q.E.D.

3. - Some properties of $G_m^{(n)}(t)$.

In this section some properties of $G_m^{(n)}(t)$, where

$$G_m^{(n)}(t) = D^n P_m(D) G(t) = D^n e^{b_m D} \prod_{k=1}^{m} \left(1 - \frac{D}{a_k}\right) e^{(1/a_k)D} G(t), \tag{3.1}$$

will be found; these properties shall depend on the sequences $\{b_m\}$ and $\{a_k\}$.

THEOREM 3.1. - For $n, m = 0, 1, 2, \ldots$ we have for $-\infty < t < \infty$

$$| G_m^{(n)}(t) | \leq \frac{A_n}{\left(\prod_{r=0}^{n} S_m^{(r)}\right)^{\frac{1}{2}}} \tag{3.2}$$

where A_n depends on n only, $S_m^{(o)} = S_m$ and $S_m^{(r)}$ is defined by (2.9).

PROOF. - The proof is by induction on n. For $n = 0$ our theorem is Lemma 7.1 of [7; p. 138]. Suppose that our theorem is valid for $n = 0, 1, \ldots, l-1$.

For a fixed m, $m \geq 0$ we have either

$$S_m \geq (2l+2) \max_{k \geq m+1} a_k^{-2} \tag{A}$$

or

(B)
$$S_m < (2l+2) \max_{k \geq m+1} a_k^{-2}.$$

In case (A) we have

$$\prod_{k=m+1}^{\infty}\left(1+\frac{\tau^2}{a_k^2}\right) = 1 + \tau^2 S_m + \sum_{p=2}^{\infty}\frac{\tau^{2p}}{p\,!}\sum_{\substack{k_1 \dots k_p \geq m+1\\ k_1 \neq k_r\ (q \neq r)}}^{\infty}(a_{k_1}\dots a_{k_p})^{-2},$$

$$\sum_{\substack{k_1,\dots,k_{2l+2}\geq m+1\\ k_q\neq k_r\ (q\neq r)}}^{\infty}(a_{k_1}\dots a_{k_{2l+2}})^{-2} = \sum_{\substack{k_1,\dots,k_{2l+1}\geq m+1\\ k_q\neq k_r\ (q\neq r)}}^{\infty}\left(S_m - \sum_{j=1}^{2l+1} a_{k_j}^{-2}\right)(a_{k_1}\dots a_{k_{2l+1}})^{-2}$$

(by assumption (A))

$$\geq \frac{S_m \cdot 1\,!}{2n+2}\cdot\sum_{\substack{k_1,\dots,k_{2l+1}\geq m+1\\ k_q\neq k_r\ (q\neq r)}}(a_{k_1}\dots a_{k_{2l+1}})^{-2} \geq \frac{S_m^{2l+2}(2l+2)\,!}{(2l+2)^{2l+2}}.$$

By the same method

$$\sum_{\substack{k_1,\dots,k_{l+1}\geq m+1\\ k_q\neq k_r\ (q\neq r)}}(a_{k_1}\dots a_{k_{l+1}})^{-2} \geq \frac{S_m^{l+1}(2l+1)\,!}{l\,!\,(2l+2)^{l+1}}$$

Now since all the coefficients of τ^{2r} in $\prod_{k=m+1}^{\infty}\left(1+\frac{\tau^2}{a_k^2}\right)$ are positive, we have

$$\prod_{k=m+1}^{\infty}\left(1+\frac{\tau^2}{a_k^2}\right) \geq 1 + \frac{\tau^{2l+2}}{(l+1)\,!}\sum_{\substack{k_1,\dots,k_{l+1}\geq m+1\\ k_q\neq k_r\ (q\neq r)}}(a_{k_1}\dots a_{k_{l+1}})^{-2} +$$

$$+\frac{\tau^{4l+4}}{(2l+2)\,!}\sum_{\substack{k_1,\dots,k_{2l+2}\geq m+1\\ k_q\neq k_r\ (q\neq r)}}(a_{k_1}\dots a_{k_{2l+2}})^{-2} \geq \left\{1+\frac{(S_m^{1/2}\tau)^{2l+2}}{(2l+2)^{l+2}}\right\}^2.$$

Now

$$|G_m^{(l)}(t)| \leq \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{|\tau|^l\,d\tau}{\left\{\prod_{k=m+1}^{\infty}\left(1+\frac{\tau^2}{a_k^2}\right)\right\}^{1/2}} \leq \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{|\tau|^l\,d\tau}{1+\frac{S_m^{l+1}\tau^{2l+2}}{(2l+2)^{l+1}}} =$$

$$= \frac{2(2l+2)^{\frac{l+1}{2}}}{(l+1)S_m^{\frac{l+1}{2}}\cdot 2\pi}\int_0^{\infty}\frac{dx}{1+x^2} = \frac{(2l+2)^{\frac{l-1}{2}}}{S_m^{\frac{l+1}{2}}} \leq (2l+2)^{\frac{l-1}{2}}\left(\prod_{r=0}^{l} S_m^{(r)}\right)^{-1/2}.$$

Case (B). By supposition (B) there is a $k_m \geq m+1$ such that $|a_{k_m}| < \frac{(2l+2)^{1/2}}{S_m^{1/2}}$. We define $g(a_{k_m}, u)$ by $g(a_{k_m}, u) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{e^{-s/a_{k_m}}}{1 - \frac{s}{a_{k_m}}} e^{su} ds$. Define $G_m^*(t)$ by

$$G_m^*(t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} (E_m^*(s))^{-1} e^{st} ds$$

where $E_m^*(s) = E_m(s)\left(1 - \frac{s}{a_{k_m}}\right)^{-1} e^{-s/a_{k_m}}$

$$|G_m^{(l)}(t)| = \left| \int_{-\infty}^{\infty} g(a_{k_m}, u) \frac{d^l}{dt^l} G_m^*(t-u) du \right| \leq$$

$$\leq \{\sup_u |g(a_{k_m}, u)|\} \int_{+\infty}^{\infty} \left| \frac{d^l}{dv^l} G_m^*(v) \right| dv$$

since $\frac{d^l}{dv^l} G_m^*(v)$ has at most l changes of sign (see [7; p. 92]) and as $\frac{d^l}{dv^l} G_m^*(v) = o(1)$ $(v \to \pm\infty)$ (see [7; pp. 108-110]) we have by induction

$$G_m^{(l)}(t) \leq |a_{k_m}| 2l \max_{-\infty<v<\infty} \left| \frac{d^{l-1}}{dv^{l-1}} G_m^*(v) \right|$$

$$\leq \frac{(2l+2)^{1/2}}{S_m^{1/2}} 2l \frac{A_{l-1}}{\left(\prod_{i=0}^{l-1} S_m^{*(i)}\right)^{1/2}} \leq \frac{(2l+2)^{1/2} 2l A_{l-1}}{\left(\prod_{i=0}^{l} S_m^{*(i)}\right)^{1/2}}$$

where $S_m^{*(r)}$ are the $S_m^{(r)}$ of the sequence $\{a_k^*\}: a_k^* = a_k\ k < k_m,\ a_k^* = a_{k-1}\ k \geq k_m$. Choose for completing the proof

$$A_n = \max\left((2n+2)^{\frac{(n-1)}{2}},\ (2n+2)^{1/2} \cdot 2n \cdot A_{n=1}\right). \qquad \text{Q.E.D.}$$

The following lemma will establish the behavior of the zeros of $G^{(n)}(t)$. We shall need the lemma in the remaining theorems of this chapter.

LEMMA 3.2. - There are exactly n points where changes of sign of $G^{(n)}(t)$ occur. At these points $G^{(n+1)}(t) \neq 0$. Any other point at which $G^{(n)}(\eta) = 0$ implies either $G(t) = 0$ for $\eta < t < \infty$ or $G(t) = 0$ for $-\infty < t \leq \eta$.

PROOF. - That $G^{(n)}(t)$ has exactly n changes of sign was proved in Theorem 10 of [5; p. 154]. The rest follows by the same arguments used in [7; p. 94].

We shall denote by $\zeta_{m,n,i}$ the $n+1$ distinct zeros of $G_m^{(n+1)}(t)$ at which $G_m^{(n+1)}(t)$ changes sign, where $\zeta_{m,n,i} < \zeta_{m,n,i+1}$.

THEOREM 3.3. - For $m, n = 0, 1, 2, \ldots$ and $B > 0$ we have

(a) $$G_m^{(n)}(\zeta_{m,n,i} - BS_m^{1/2}) \leq \frac{A_{n-1}}{BS_m^{1/2}\left(\prod_{i=0}^{n-1} S_m^{(i)}\right)^{1/2}}$$

(b) $$| G_m(\zeta_{m,n,n+1} + BS_m^{1/2}) | \leq \frac{A_{n-1}}{BS_m^{1/2}\left(\prod_{i=0}^{n-1} S_m^{(i)}\right)^{1/2}},$$

where A_n is independent of m.

PROOF. - $G_m^{(n)}(t)$ is monotonic increasing in the interval $(\zeta_{m,n,1} - BS_m^{1/2}, \zeta_{m,n,1})$ and $G_m^{(n-1)}(t)$ is monotonic increasing in $(-\infty, \zeta_{m,n,1})$ by the definitions of $\zeta_{m,n,1}$ and $\zeta_{m,n-1,1}$ respectively. Hence we have

$$BS_m^{1/2} G_m^{(n)}(\zeta_{m,n,1} - BS_m^{1/2}) \leq \int_{\zeta_{m,n,1}-BS_m^{1/2}}^{\zeta_{m,n,1}} G_m^{(n)}(t)dt \leq$$

$$\leq \begin{cases} 1 & \text{if} \quad n = 0 \\ G_m^{(n-1)}(\zeta_{m,n,1}) - G_m^{(n-1)}(\zeta_{m,n,1} - BS_m^{1/2}) \leq G_m^{(n-1)}(\zeta_{m,n,1}) & \text{if} \quad n \geq 1. \end{cases}$$

Using Theorem 3.1 the proof of case (a) follows. The proof of case (b) is similar. Q.E.D.

REMARK. - We can choose B of Theorem 3.3 to be a function of m as long as it is positive.

THEOREM 3.4. - For $m, n = 0, 1, 2 \ldots$ we have

(a) $$G_m^{(n)}(\zeta_{m,n,1}) \geq B_n S_m^{-n}\left(\prod_{i=0}^{n-2} S_m^{(i)}\right)^{1/2}$$

(b) $$| G_m^{(n)}(\zeta_{m,n,n+1}) | \geq B_n S_m^{-n}\left(\prod_{i=2}^{n-2} S_m^{(i)}\right)^{1/2},$$

where $\prod_0^{-1} = \prod_0^{-2} = 1$.

PROOF. - In case $n = 0$ our theorem was proved by I. I. HIRSCHMAN and D. V. WIDDER (see Lemma 4.1 [7; pp. 126-127]). We shall prove by induction case (a). Case (b) follows similarly. Assume (a) for $k = n - 1$. Since

$\zeta_{m,n,1}$ is the only maximum of $G_m^{(n)}(t)$ in $(-\infty, \zeta_{m,n-1,1})$ and $G_m^{(n)}(t)$ is positive there, we have for $B(m)>0$ (see Remark after Theorem 3.3)

$$B(m)S_m^{1/2}G_m^{(n)}(\zeta_{m,n,1}) \geq \int\limits_{\zeta_{m,n-1,1}-B(m)S_m^{1/2}}^{\zeta_{m,n-1,1}} G_m^{(n)}(t)dt =$$

$$= G_m^{(n-1)}(\zeta_{m,n-1,1}) - G_m^{(n-1)}(\zeta_{m,n-1,1} - B(m)S_m^{1/2}) \geq$$

$$\geq B_{n-1}S_m^{-n/2} - A_{n-2}\left[B(m)S_m^{1/2}\left(\prod_{i=0}^{n-2} S_m^{(i)}\right)^{1/2}\right]^{-1} > 0$$

$$G_m^{(n)}(\zeta_{m,n,1}) \geq B(m)^{-1}\left[B_{n-1} - A_{n-2}B(m)^{-1}\left(\prod_{i=0}^{n-2}(S_m/S_m^{(i)})\right)^{1/2}\right]S_m^{-\frac{n+1}{2}}.$$

We choose now $B(m) = B\left(\prod_{i=1}^{n-2}(S_m/S_m^{(i)})\right)^{1/2}$ where B is so large that $B_{n-1} - A_{n-2}\cdot B^{-1} > 0$.

For B_n we shall have now

$$B_n = B^{-1}(B_{n-1} - A_{n-1}B^{-1}) > 0$$

and therefore $G_m^{(n)}(\zeta_{m,n,1}) \geq B_{n-1}S_m^{-n}\left(\prod_{i=0}^{n-2} S_m^{(i)}\right)^{1/2}$. Q.E.D.

For convenience we shall define $S_m^{(i)}$ for negative i.

DEFINITION. - $S_m^{(n)} = S_m$ for $n = 0, -1, -2, -3, \ldots$.

COROLLARY 3.4. - Suppose for some n and $\{m_r\}$ $S_{m_r}^{(n-2)} \geq \alpha S_{m_r}$, $\alpha > 0$, independent of r, then

(a) $$G_{m_r}^{(n)}(\zeta_{m_r,n,1}) \geq B_n(\alpha)S_{m_r}^{-\frac{n+1}{2}}$$

(b) $$|G_{m_r}^{(n)}(\zeta_{m_r,n,n+1})| \geq B_n(\alpha)S_{m_r}^{-\frac{n+1}{2}},$$

where B_n are independent of r and depend on α for $n \geq 3$.

PROOF. - Since $S_{m_r}^{(n-2)} \geq \alpha S_{m_r}$ implies $S_{m_r}^{(i)} \geq \alpha S_{m_r}$ for $i \leq n-2$, we have by Theorem 3.4 a

$$G_{m_r}^{(n)}(\zeta_{m_r,n,1}) \geq \begin{cases} B_n\alpha^{n-2}S_{m_r}^{-\frac{n+1}{2}} & \text{for } n > 2 \\ B_nS_{m_r}^{-\frac{n+1}{2}} & \text{for } n \leq 2. \end{cases}$$

Similarly we show (b). Q.E.D.

REMARK. - Assuming $S_{m_r}^{(n-2)} \geq \alpha S_{m_r}$ we make no assumption for $n = 0, 1,$ or 2.

THEOREM 3.5. - Suppose for some $n \geq 0$ $S_{m_r}^{(n-2)} \geq \alpha S_{m_r}$ with $\alpha > 0$ fixed for all r, then for $1 \leq i \leq n+1$

$$|\zeta_{m_r, n, i} - b_{m_r}| \leq M_n S_{m_r}^{1/2},$$

where M_n are independent of r but for $n \geq 3$ depend on α.

PROOF. (A modification by I. I. HIRSCHMAN of the original proof). - It is enough to prove that both

(a) $$|\zeta_{m_r, n, 1} - b_{m_r}| \leq M_n S_{m_r}^{1/2}$$

and

(b) $$|\zeta_{m_r, n, n+1} - b_{m_r}| \leq M_n S_{m_r}^{1/2}$$

are satisfied. We shall prove (a) by induction on n. It is well known (see [5] and [7]) that (a) is valid for $n = 0, 1$. Assume (a) is valid for $n - 1$. The definition and simple properties of $\zeta_{m_r, n, i}$ imply

$$\frac{1}{2} G_{m_r}^{(k-1)}(\zeta_{m_r, k-1, 1})[\zeta_{m_r, k-1, 1} - \zeta_{m_r, k, 1}] \leq G_m^{(n-2)}(\zeta_{m_r, k-1, 1}).$$

Hence by Theorems 3.1 and 3.4 and using $S_{m_r}^{(i)} \geq \alpha S_{m_r}$ for $i \leq n-2$, we obtain

$$B_{n-1} S_{m_r}^{-n/2}[\zeta_{m_r, n-1, 1} - \zeta_{m_r, n, 1}] \leq A_{n-2} \alpha^{-n+2} S_{m_r}^{-\frac{n+1}{2}}$$

or $\zeta_{m_r, n-1, 1} - \zeta_{m_r, n, 1} \leq C_n S_{m_r}^{1/2}$. The induction hypothesis completes the proof of (a). The proof of (*b*) is similar. Q.E.D.

THEOREM 3.6. - Suppose for some $n \geq 0$ $S_{m_r}^{(n-2)} \geq \alpha S_{m_r}$ with $\alpha > 0$ fixed for all r, then

$$|G_{m_r}^{(n)}(\zeta_{m_r, n, i})| > C_n S_{m_r}^{-\frac{n+1}{2}}$$

where C_n are independent of r but for $n \geq 3$ depend on α.

PROOF. - For $i = 0$ and $i = n+1$ our theorem is reduced to Theorem 3.4, therefore it is proved in case $n = 0, 1$. The proof follows by induction.

Suppose our theorem is valid for $l \leq n-1$. The function $G_{m_r}^{(n)}(t)$ has no change of sign in $(\zeta_{m_r, n-1, i-1}, \zeta_{m_r, n-1, i})$. Hence

$$2C_{n-1}S_{m_r}^{-n/2} \leq |G_{m_r}^{(n-1)}(\zeta_{m_r, n-1, i}) - G_{m_r}^{(n-1)}(\zeta_{m_r, n-1, i-1})| =$$

$$= \int_{\zeta_{m_r, n-1, i-1}}^{\zeta_{m_r, n-1, i}} |G_{m_r}^{(n)}(x)|\, dx \leq (\zeta_{m_r, n-1, i} - \zeta_{m_r, n-1, i-1})\, |G_{m_r}^{(n)}(\zeta_{m_r, n, i})|\,.$$

Using Theorem 3.5 we get $\zeta_{m_r, n-1, i} - \zeta_{m_r, n-1, i-1} \leq 2M_{n-1}S_{m_r}^{1/2}$. Therefore $|G_{m_r}^{(n)}(\zeta_{m_r, n, i})| \geq C_{n-1}M_{n-1}^{-1}S_{m_r}^{-\frac{n+1}{2}}$. Q.E.D.

REMARK. - In Theorems 3.5 and 3.6 and Corollary 3.4 we do not require anything in case $n = 0, 1, 2$.

As was shown in Lemma 2.4 the assumption $S_m^{(1)} \geq \alpha S_m$ $\alpha > 0$ fixed for all m implies the assumptions of these theorems.

THEOREM 3.7. - Suppose for some n $S_{m_r}^{(n)} \geq \alpha S_{m_r}$ $\alpha > 0$ fixed for all r and $b_{m_r} = 0(S_{m_r}^{1/2})$ $(r \to \infty)$ then

$$S_{m_r}^{1/2} \int_{-\infty}^{\infty} |G_{m_r}^{(n+1)}(t)t^n|\, dt \leq M(n, \alpha)$$

where $M(n, \alpha)$ is independent of r.

PROOF. - Let $\{\xi_{m_r, n, i+1}\ 1 \leq i \leq n+4\} = \{\zeta_{m_r, n, i} 1 \leq i \leq n+1\} \cup \{0, -\infty, \infty\}$ where $\xi_{m_r, n, i} \leq \xi_{m_r, n, i+1}$ (the equality sign may occur only once if $0 = \xi_{m_r, n, i}$ for some i). In each interval $(\xi_{m_r, n, i}, \xi_{m_r, n, i+1})$ (possibly a point) $G^{(n+1)}(t)t^n$ has no changes of sign.

$$I_{m_r} = S_{m_r}^{1/2} \int_{-\infty}^{\infty} |G_{m_r}^{(n+1)}(t)t^n|\, dt = S_{m_r}^{1/2} \sum_{i=1}^{n+3} \left| \int_{\xi_{m_r, n, i}}^{\xi_{m_r, n, i+1}} G_m^{(n+1)}(t)t^n dt \right|.$$

Since $\xi_{m_r, n, 1} = -\infty$ and $\xi_{m_r, n, n+4} = \infty$, integrating by parts we obtain

$$I_{m_r} \leq 2S_{m_r}^{1/2} \sum_{i=2}^{n+3} \sum_{l=0}^{n} |\xi_{m_r, n, i}|^l\, |G_{m_r}^{(l)}(\xi_{m_r, n, i})|\, \frac{n!}{l!}.$$

Since $\{\xi_{m_r, n, i}\ 2 \leq i \leq n \leq +3\} = \{\zeta_{m_r, n, l}\ 1 \leq i \leq n+1\} \cup \{0\}$ we can complete the proof of this lemma using Theorem 3.1 and 3.5. Q.E.D.

THEOREM 3.8. - Suppose for some n $S^{(n-1)}_{m_r} \geq \alpha S_{m_r}$ $\alpha > 0$ fixed for all r and $b_{m_r} = 0(S^{1/2}_{m_r})$ $(s \to \infty)$, then

$$\int_{-\infty}^{\infty} | G^{(n)}_{m_r}(t) t^n | \, dt \leq M_1(n, \alpha)$$

where $M(n, \alpha)$ is independent of r.

PROOF. - Similar to that of Theorem 3.7. Q.E.D.

4. - Properties of $G^{(n)}_m(t)$ (continued).

The moments of $G_m(t)$ are $\mu_1(m) = b_m$ [7; pp. 55-56] and $\mu_n(m)$ are defined by

$$\mu_n(m) = \int_{-\infty}^{\infty} (t - b_m)^n G_m(t) dt. \tag{4.1}$$

THEOREM 4.1. - For $\delta > 0$ and $m = 1, 2, \ldots$ we have

(a) $$\int_{-\infty}^{b_m - \delta} G_m(t) dt \leq \frac{1}{\delta^{2n}} \mu_{2n}(m);$$

(b) $$\int_{b_m + \delta}^{\infty} G_m(t) dt \leq \frac{1}{\delta^{2n}} \mu_{2n}(m).$$

PROOF. - To prove (a) we write

$$\int_{-\infty}^{b_m - \delta} G_m(t) dt = \int_{-\infty}^{-\delta} G_m(t + b_m) dt \leq \int_{|t| \geq \delta} G_m(t + b_m) dt \leq \frac{1}{\delta^{2n}} \int_{-\infty}^{\infty} t^{2n} G_m(t + b_m) dt = \frac{1}{\delta^{2n}} \mu_{2n}(m).$$

The proof of (b) is similar. Q.E.D.

COROLLARY 4.1. - For each $\delta > 0$ we have

(a) $G_m(b_m - 2\delta) \leq \dfrac{4^{-n}}{\delta^{2n+1}} \mu_{2n}(m)$, (b) $G_m(b_m + 2\delta) \leq \dfrac{4^{-n}}{\delta^{2n+1}} \mu_{2n}(m)$.

Recall that $E_m(s)$ [7; p. 125] is

$$E_m(s) = e^{b_m s} \prod_{k=m+1}^{\infty} \left(1 - \frac{s}{a_k}\right) e^{s/a_k} \tag{4.2}$$

where $b_m = o(1)$ $(m \to \infty)$.

THEOREM 4.2. - Let $F_m(s) = E_m(s)^{-1} \cdot e^{b_m s}$ then

$$\mu_n(m) = (-1)^n F_m^{(n)}(0). \tag{4.3}$$

PROOF. - The proof follows immediately from

$$F_m(s) = \frac{e^{b_m s}}{E_m(s)} = \int_{-\infty}^{\infty} e^{-st} G_m(t + b_m) dt. \tag{4.4}$$

THEOREM 4.3. - For $n \geq 2$ and $m = 1, 2, \ldots$ there exists a constant $K(n)$ such that

$$|\mu_n(m)| \leq K(n) S_m^{n/2}. \tag{4.5}$$

PROOF. - By differentiation $(F_m'(s)/F_m(s)) = -\sum_{k=m+1}^{\infty} \left(\frac{1}{s - a_k} - \frac{1}{a_k}\right)$, therefore for $n > 1$ we obtain

$$\left| \left(\frac{F_m'(s)}{F_m(s)}\right)_{s=0}^{(n-1)} \right| = (n-1)! \left| \sum_{k=m+1}^{\infty} a_k^{-n} \right| \leq (n-1)!\, S^{n/2}. \tag{4.6}$$

We shall prove by induction that

$$|(F_m(s)^{-1})_{s=0}^{l}| \leq K_1(l) S_m^{l/2}. \tag{4.7}$$

For $l = 0$ and $l = 1$ (4.7) is trivial. Assume (4.7) is valid for $l \leq k$ then by LEIBNITZ formula

$$(F_m(s)^{-1})^{(k+1)} = \left(\frac{1}{F_m(s)} \cdot \frac{F_m'(s)}{F_m(s)}\right)^{k} = \sum_{j=0}^{k} \binom{k}{j} \left(\frac{1}{F_m(s)}\right)^{(j)} \left(\frac{F_m'(s)}{F_m(s)}\right)^{(k-1)}.$$

Using (4.6), $F_m'(0) = 0$ and the induction hypothesis, (4.7) is proved.
Now we prove (4.5) by induction. For $n = 2$ it is valid since $\mu_2(m) = S_m$. Assume (4.5) for all $n < k$. By LEIBNITZ formula

$$\mu_k(m) = (-1)^k F_m^{(k)}(0) = (-1)^k \left\{ \left(\frac{F_m'(s)}{F_m(s)}\right)_{s=0}^{(k-1)} - \sum_{r=1}^{k-1} \binom{k-1}{r} F_m^{(k-r)}(0) \left(\frac{1}{F_m(s)}\right)_{s=0}^{r} \right\}.$$

Taking absolute values and using (4.7) and the induction hypothesis we obtain (4.5) for $n = k$. Q.E.D.

THEOREM 4.4. - For $t \neq 0$ and for any k

$$G_m(t) = o(S_m^k) \qquad (m \to \infty). \tag{4.8}$$

PROOF. - Suppose $t > 0$, then since $\zeta_m = o(1)$ $m \to \infty$ for $m > m_0$ $t/2 > \zeta_m$. Due to the monotonic character of $G_m(t)$ for $t > \zeta_m$, Theorem 4.3 and Corollary 4.1, we complete the proof. The case $t > 0$ is similar. Q.E.D.

THEOREM 4.5. - If for some n $S_{m_r}^{(n-2)} \geq \alpha S_{m_r}$ $\alpha > 0$ α fixed for all r, then for $t \neq 0$ and any k

$$G_{m_r}^{(n)}(t) = o(S_{m_r}^k) \qquad r \to \infty. \tag{4.9}$$

PROOF. - The proof follows by induction. For $n = 0$ it is Theorem 4.4. Let us assume (4.9) is valid for $l = 0, 1, \ldots, n-1$. It is known by theorem 3.5 that $|\zeta_{m_r, n, i} - b_{m_r}| \leq M_n S_{m_r}^{1/2}$ and since $b_{m_r} = o(1)$ $r \to \infty$ and $S_{m_r} = o(1)$ $r \to \infty$ $G_{m_r}^{(n)}(t)$ is increasing or decreasing in $\left(\frac{t_0}{2}, \infty\right)$ when n is odd or even respectively and is increasing in $(-\infty, -t_0/2)$ for any positive t_0.

Therefore for $r > r_0$ we have

$$\frac{1}{2} t_0 G_{m_r}^{(n)}(-t_0) \leq \int_{-t_0}^{-t_0/2} G_{m_r}^{(n)}(t)dt \leq \int_{-\infty}^{-t_0/2} G_{m_r}^{(n)}(t)dt = G_{m_r}^{(n-1)}(-t_0/2)$$

and hence

$$G_{m_r}^{(n)}(-t_0) \leq \left(\frac{1}{2} t_0\right)^{-1} G_{m_r}^{(n-1)}(-t_0/2) = o(S_{m_r}^k) \qquad r \to \infty.$$

The case $t > 0$ is similar. Q.E.D.

REMARK. - One can easily see that in case $n = 0$, 1 or 2 we do not make any assumption on the special subsequence or on the order of the a_k's.

THEOREM 4.6. - Suppose for some n $\{a_k\}$ satisfies $B(n-2, \{m_r\})$ as defined in section 2. Then for $t \neq 0$ and $0 \leq p \leq n$ and for any $q > 0$

$$G_{m_r}^{(p)}(t) = o(S_{m_r}^q) \qquad r \to \infty. \tag{4.10}$$

PROOF. - Using the inequality (2.13) and a remark after it we see that whenever $\{a_k\}$ satisfies $B(n-2, \{m_r\})$ there exist $l > 0$ and $K > 0$ such that for $0 \leq p \leq n-2$ (no conclusion in case $n = 0, 1, 2$),

$$\{\min_{k_1 > m_r} (\max |a_{k_i}| \, | 1 \leq i \leq p \ k_1 < k_2 < \ldots < k_p)\} \cdot (S_{m_r}^{(p-1)})^l < K. \tag{4.11}$$

Let us define now $a_{m_r+r_j}$, $a_{m_r+r_j} \in \{a_k\}_{m_r+1}^{\infty}$,

$$|a_{m_r+r_j}| = \{\min |a_k| \, | \ k > m_r \ k \neq m_r + r_i \ 0 < i < j\} \tag{4.12}$$

and if there are some r_j with the above properties, we take the smallest one.

Let us also define $G_{m_r, j}(t)$ by

$$\prod_{i=0}^{j} \left(1 - \frac{D}{a_{m_r+r_i}}\right) G_{m_r}(t) \equiv G_{m_r, j}(t), \qquad G_{m_r}(t) \equiv G_{m_r, 0}(t). \tag{4.13}$$

$G_{m_r, j}(t)$ are subsequence of $G_m(t)$ with some rearrangement of the sequence $\{a_k\}$. $S_{m_r}^{(j)}$ is the second moment of $G_{m_r, j}(t)$. By Theorem 4.5 for $n = 0, 1, 2$ we have $G_{m_r, j}^{(n)}(t) = o(S_{m_r}^{(j)})^l$ $r \to \infty$ for all $l > 0$ $t \neq 0$ (see Remark after Theorem 4.5).

We shall prove our theorem by induction beginning with $n = 2$. For $n = 2$ $G_{m_r, j}^{(2)}(t) = o(S_{m_r}^{(j)})^q$ $r \to \infty$, for all $q > 0$ and $t \neq 0$.

Assume for some integer s $0 < s < n$

$$G_{m_r, j}^{(s)}(t) = o(S_{m_r}^{(j)})^q \ r \to \infty \ \text{for all } q > 0, \ t \neq 0 \ \text{and } 0 \leq j \leq n - s - 1. \tag{4.14}$$

By definition we have for $j > 0$

$$G_{m_r, j}(t) = (1 - (a_{m_r+r_j})^{-1} D) G_{m_r, j-1}(t)$$

and therefore

$$G_{m_r, j-1}^{(s+1)}(t) = a_{m_r+r_j} \{ G_{m_r, j-1}^{(s)}(t) - G_{m_r, j}^{(s)}(t) \}. \tag{4.15}$$

Since $S_{m_r}^{(j)} < S_{m_r}^{(j-1)}$ and since (4.10) implies $|a_{m_r+r_j}| (S_{m_r}^{(j-1)})^l < k$ we obtain

$$G_{m_r, j-1}^{(s+1)} = o(S_{m_r}^{(j-1)})^q \ r \to \infty \ \text{for all } q > 0, \ t \neq 0 \ \text{and } 0 \leq j \leq n - s - 1$$

which concludes our proof. Q.E.D.

REMARK. - Assumption (2.13) and its consequence (4.11) are quite complicated, but when $|a_k| \leq |a_{k+1}|$ for all k, it is enough to assume $(\min_{k > m} |a_k|)(S_m)^l < K$ for all m which is a simpler but stronger assumption (see Lemma 2.8). Still another assumption which implies (2.13) for all n even without ordering is $\Sigma |a_k|^{-\beta} < \infty$ for some $\beta < 2$ (see Lemma 2.10).

REMARK. - There is some overlapping between the Theorems 4.5 and 4.6.

THEOREM 4.7. - Suppose $G(t)$ to be defined by (1.2) and $E(s) = E_1(s) \cdot E_2(s)$ where $E(s)$, $E_1(s)$ and $E_2(s)$ satisfy (1.3). Let $H_i(t)$ be related to $E_i(s)$ by

$$H_i(t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} (E_i(s))^{-1} e^{st} ds \qquad i = 1,\ 2.$$

Assume $H_1(t) \in C^n(-\infty, \infty)$ and $H_2(t) \in C(-\infty, \infty)$. Then for each real a the function $G(t) - aH_1^{(n)}(t)$ has at most $2\left[\frac{n+2}{2}\right]$ changes of sign.

PROOF. - Define $\Psi_m(y)$ by $\Psi_m(y) = (1 - y^2)^m$ for $|y| < 1$ and $\Psi_m(y) = 0$ for $|y| \geq 1$. For $n < m$ $\Psi_m^{(n)}(y)$ is continuous and has exactly n changes of sign. For all $g(t) \in C^n(-\infty, \infty)$ $n < m$ we have

$$\lim_{h \to 0+} \int_{-\infty}^{\infty} g(x - t) \cdot \frac{1}{hQ_m} \frac{d^n}{dt^n}\left(\Psi_m\left(\frac{t}{h}\right)\right) dt = g^{(n)}(x) \tag{4.16}$$

where $Q_m = \int_{-1}^{1} (1 - y^2)^m dy$. By (4.16) we have

$$G(t) - aH_1^{(n)}(t) = \lim_{h \to 0+} \int_{-\infty}^{\infty} H_1(t - u) \left\{ H_2(u) - \frac{a}{hQ_m} \frac{d^n}{dt^n} \left[\Psi_m\left(\frac{u}{h}\right)\right] \right\} du.$$

Since $H_1(t)$ is variation diminishing, it is enough to show that $H_2(u) - \frac{a}{hQ_m} \frac{d^n}{dt^n} \Psi_m\left(\frac{u}{h}\right)$ has at most $2\left[\frac{n+2}{2}\right]$ changes of sign for $0 < h < h_0(a)$.

This follows from the continuity and boundedness of the derivative of $H_2(t)$ except for a possible discontinuity of $H_2'(t)$ at one point (but there the left and right derivatives are bounded) and simple properties of $\frac{1}{hQ_m} \frac{d^n}{dt^n} \Psi_m\left(\frac{t}{h}\right)$.

Q.E.D.

5. - On the convergence of $\int_{-\infty}^{\infty} G_m^{(n)}(u - t) e^{ct} d\alpha(t)$.

In this section we shall bring some theorems on convergence of $\int_{-\infty}^{\infty} G_m^{(n)}(u - t) e^{ct} d\alpha(t)$ related to same theorems for $n = 0$. In fact, one can use a direct method involving the asymptotic expansion of $G_m^{(n)}(t)$ but we shall prove them using the theorems for $n=0$ proved in [7; pp. 127-137].

LEMMA 5.1. - If

(1) $G(t) \in$ class I, $G(t)$ is non-finite,

(2) $\alpha(t) \in B.V.(a, b)$ in any finite interval $[a, b]$,

(3) $f(x) = \int_{-\infty}^{\infty} G(x-t)e^{ct}d\alpha(t)$ converges,

then

$$D^n P_m(D) f(x) = \int_{-\infty}^{\infty} G_m^{(n)}(x-t)e^{ct}d\alpha(t),$$

the integral converging uniformly for x in any finite interval.

PROOF. - The case $c=n=0$ is Theorem 5.1.a [7; p. 127]. The case $n=0$ $c \neq 0$ is not different and we shall refer to it as proved in the above mentioned theorem. There are constants $\alpha_0^*, \ldots, \alpha_k^*$ such that

$$D^n P_m(D) = \sum_{j=0}^{n} \alpha_j^* P_{m+j}(D) \exp\left(\left(b_{m+j} - b_m - \sum_{r=m+1}^{m+j} a_r^{-1}\right) D\right). \tag{5.1}$$

Since our lemma is valid for $n=0$ for all m we obtain

$$D^n P_m(D) f(x) = \sum_{j=0}^{n} \alpha_j^* P_{m+j}(D) \exp\left(\left(b_{m+j} - b_m - \sum_{r=m+1}^{n+j} a_r^{-1}\right) D\right) f(x) =$$

$$= \sum_{j=0}^{n} \alpha_j^* \int_{-\infty}^{\infty} G_{m+j}\left(x - t + b_{m+j} - b_m - \sum_{r=m+1}^{n+j} a_r^{-1}\right) e^{ct} d\alpha(t).$$

By Theorem 5.1.a of [7; p. 127] these integrals are uniformly convergent in any finite interval and the sum is equal to $\int_{-\infty}^{\infty} G_m^{(n)}(x-t)e^{ct}d\alpha(t)$. Q.E.D.

LEMMA 5.2. - If

(1) $G(t) \in$ class II,

(2) $\alpha(t) \in B.V.(a, b)$ $-\infty < a < b < \infty$,

(3) $f(x) = \int_{-\infty}^{\infty} G(x-t)e^{ct}d\alpha(t)$ converges for $x > \gamma_c$, then

$$D^n P_m(D) f(x) = \int_{-\infty}^{\infty} G_m^{(n)}(x-t)e^{ct}d\alpha(t),$$

the integral converges uniformly in any finite interval

$$\gamma_c - b + b_m - \sum_{j=1}^{m} a_k^{-1} < x_1 \leq x \leq x_2 < \infty.$$

PROOF. - Similar to the proof of Lemma 5.1, but using Theorem 5.2.a [7; p. 129] instead of Theorem 5.1.a of [7; p. 127], we get Lemma 5.2. Q.E.D.

LEMMA 5.3. - If

(1) $G(t) \in$ class III and $G(t)$ is non-finite,

(2) $\alpha(t) \in B.V.(a, b)$ $T < a < b < \infty$,

(3) $f(x) = \int_{-\infty}^{\infty} G(x-t)e^{ct}d\alpha(t)$ converges for $x > T + b + \sum_{r=1}^{\infty} a_r^{-1}$, then

$$D^n P_m(D)f(x) = \int_{-\infty}^{\infty} G_m^{(n)}(x-t)e^{ct}d\alpha(t)$$

and the integral converges uniformly for $x \in [x_1, x_2]$ $T + b_m + \sum_{r=m+1}^{\infty} a_r^{-1} < x_1 \leq$ $\leq x \leq x_2 < \infty$.

PROOF. - Similar to the proof of Lemma 5.1, but using Theorem 5.3.a of [7; p. 132] instead of Theorem 5.1.a of [7; p. 127], we complete the proof of Lemma 5.3. Q.E.D.

LEMMA 5.4. - Suppose

(1) $G(t) \in$ class I and $G(t)$ is non-finite,

(2) $\alpha(t) \in B.V.(a, b)$ $-\infty < a < b < \infty$,

(3) $f(x) = \int_{-\infty}^{\infty} G(x-t)e^{ct}d\alpha(t)$ converges,

then for m sufficiently large:

A) $\alpha_1 < c < \alpha_2$ $(\alpha_1 = \max_{a_k<0}[-\infty, a_k]$ $\alpha_2 = \min_{a_k<0}[\infty, a_k])$ implies that

$$\int_{x_2}^{x_2} e^{-cx} D^n P_m(D)f(x)dx = \int_{-\infty}^{\infty} G_m^{(n)}(x_2-t)e^{-c(x_2-t)}\alpha(t)dt - \int_{-\infty}^{\infty} G_m^{(n)}(x_1-t)e^{-c(x_1-t)}\alpha(t)dt;$$

B) $c \geq a_2$ implies that $\alpha(+\infty)$ exists and that

$$\int_{x_1}^{\infty} e^{-cx} D^n P_m(D)f(x)dx = \int_{-\infty}^{\infty} G_m^{(n)}(x_1-t)e^{-c(x_1-t)}[\alpha(+\infty)-\alpha(t)]dt;$$

C) $c \leq \alpha_1$ implies that $\alpha(-\infty)$ exists and that

$$\int_{-\infty}^{x_2} e^{-cx} D^n P_m(D) f(x) dx = \int_{-\infty}^{\infty} G_m^{(n)}(x_2 - t) e^{-c(x_2-t)} [\alpha(t) - \alpha(-\infty)] dt;$$

D. we have for $-\infty < c < \infty$

$$e^{-cx} D^n P_m(D) f(x) = \int_{-\infty}^{\infty} \{ G_m^{(n+1)}(x-t) - c G_m^{(n)}(x-t) \} e^{-c(x-t)} \alpha(t) dt.$$

PROOF. - Using equation (5.1) for $D^n P_m(D)$ and the α_j defined there

$$(5.2)\quad e^{-cx} D^n P_m(D) f(x) = \left[\sum_{j=0}^{n} \alpha_j^* G_{m+j}\left(x - t + b_{m+j} - b_m - \sum_{r=m+1}^{m+j} a_r^{-1}\right) e^{-c(x-t)} \alpha(t) \right]_{-\infty}^{\infty} -$$

$$- \sum_{j=0}^{n} \alpha_j^* \int_{-\infty}^{\infty} \frac{\partial}{\partial t} \left\{ G_{m+j}\left(x - t + b_{m+j} - b_m - \sum_{r=m+1}^{m+j} a_r^{-1}\right) e^{-c(x-t)} \right\} \alpha(t) dt =$$

$$= \int_{-\infty}^{\infty} G_m^{(n+1)}(x-t) e^{-c(x-t)} \alpha(t) dt - c \int_{-\infty}^{\infty} G_m^{(n)}(x-t) e^{-c(x-t)} \alpha(t) dt,$$

the integrals converging uniformly in any finite interval. This concludes the proof of D.

By (5.2) we have

$$(5.3)\quad e^{-cx} D^n P_m(D) f(x) = -$$

$$- \sum_{j=0}^{n} \alpha_j^* \int_{-\infty}^{\infty} \frac{\partial}{\partial t} \left\{ G_{m+j}\left(x - t + b_{m+j} - b_m - \sum_{r=m+1}^{m+j} a_r^{-1}\right) e^{-c(x-t)} \right\} \alpha(t) dt.$$

Treating each integral in the sum on the right hand by the method of Theorem 6.1.a [7; pp. pp. 132-134] and applying (5.1) again to $G(t)$ we obtain, since an integral and finite sum are interchangeable, results A, B and C of this lemma. Q.E.D.

LEMMA 5.5. - Suppose

(1) $G(t) \in$ class II,

(2) $\alpha(t) \in B.V.(a, b)$ $-\infty < a < b < \infty$,

(3) $f(x) = \int_{-\infty}^{\infty} G(x - t)e^{ct}d\alpha(t)$ converges for $x > \gamma_c$,

then conclusions A, B and D of Lemma 5.4 are valid for m sufficiently large.

PROOF. - Similar to that of Lemma 5.4 but using Lemma 5.2 and Theorem 6.2.a of [7; p. 136] instead of Lemma 5.1 and Theorem 6.1.a of [7; p. 132].

LEMMA 5.6. - Suppose

(1) $G(t) \in$ class III, $G(t)$ is non-finite,

(2) $\alpha(t) \in B.V.(a, b)$ $T < a < b < \infty$,

(3) $f(x) = \int_{-\infty}^{\infty} G(x - t)e^{ct}d\alpha(t)$ converges for $x > T + b + \sum_{k=1}^{\infty} a_k^{-1}$.

Then for x, x_1 and $x_2 > T$ and m sufficiently large conclusion A, B and D of Lemma 5.4 are valid.

PROOF. - Similar to that of Lemma 5.4 but using Theorem 6.3.a of [7; p. 137] instead of Theorem 6.1.a of [7; p. 132]. Q.E.D.

PART B

Inversion and jump formulae

6. - General inversion formulae.

In this section we shall obtain inversion formulae for the cases where $\{a_k\}$ satisfies either $B(n-2, \{m_r\})$ or $S_{m_r}^{(n-2)} \geq \alpha S_{m_r}$ $\alpha > 0$ independent of r; both are satisfied by any subsequence m_r automatically for $n = 0$, 1 and 2.

Whenever $\sum_{k=1}^{\infty} |a_k|^{-\beta} < \infty$ for some β $0 < \beta < 2$ $\{a_k\}$ satisfies $B(n)$ (that is $B(n, \{m_r\})$ for every $\{m_r\}$) for all n by Lemma 2.10.

In case $\sum_{k=1}^{\infty} |a_k|^{-\beta} = \infty$ for some β, $0 < \beta < 2$ and $|a_k| \leq |a_{k+1}|$, by Lemmas 2.5 and 2.6, there exists a subsequence $\{m_r\}$ of $\{m\}$ such that $S_{m_r}^{(n)} \geq \alpha_n S_{m_r}$ for all n. Therefore the above mentioned conditions do not restrict the kernels.

We shall bring the inversion theorems for (1.1) for three classes I, II, III and prove them together.

THEOREM 6.1. - Suppose

(1) $G(t) \in$ class I and $G(t)$ is non-finite,

(2) $\varphi(t) \in L_1(a, b) \quad -\infty < a < b < \infty$,

(3) $f(x) = \int_{-\infty}^{\infty} G(x-t)\varphi(t)dt$ converges,

(4) for some real x and some $n \geq 0$ there exists a $\delta > 0$ such that $\varphi^{(n)}(t)$ exist a.e. in $(x-\delta, x+\delta)$ and for $n > 0$ we assume that $\varphi^{(n-1)}(t)$ is absolutely continuous;

(5) either $\{a_k\}$ satisfy $B(n-2, \{m_r\})$ or $S_{m_r}^{(n-2)} \geq \alpha S_{m_r}$ $\alpha > 0$ for n mentioned in (4) (no assumption in case $n = 0$, 1 or 2).

Then

(a) if $\varphi^{(n)}(t)$ is continuous at $t = x$, then

$$\lim_{r\to\infty} D^n P_{m_r}(D)f(x) = \varphi^{(n)}(x);$$

(b) if both $\varphi^{(n)}(x \pm 0)$ exist (defined by (1.6)) $\varphi^{(n)}(x+0) = \varphi^{(n)}(x-0)$ and $b_{m_r} = 0(S_{m_r}^{1/2})$, then

$$\lim_{r\to\infty} D^n P_{m_r}(D)f(x) = \varphi^{(n)}(x+0) \qquad (= \varphi^{(n)}(x-0));$$

(c) if $S_{m_r}^{1/2} = o(b_{m_r})$ $(r \to \infty)$ $b_{m_r} > 0$ for $r \geq r_0$ $\varphi^{(n)}(x-)$ exists and one of the following two assumptions is satisfied:

(I) $\varphi^{(n)}(t)$ is bounded in $(x, x+\delta_1)$ for some δ_1, $0 < \delta_1 < \delta$,

(II) $S_{m_r}^{\frac{1}{2}-\gamma} = o(b_{m_r})$ $(r \to \infty)$ for some γ, $0 < \gamma < \frac{1}{2}$,

then

$$\lim_{r\to\infty} D^n P_{m_r}(D)f(x) = \varphi^{(n)}(x\ -);$$

(d) if $S_{m_r}^{1/2} = o(|b_{m_r}|)$ $(r \to \infty)$ $b_{m_r} < 0$ for $r \geq r_0$ $\varphi^{(n)}(x+)$ exists and one of the following two assumptions is satisfied:

(I) $\varphi^{(n)}(t)$ is bounded in $(x-\delta_1, x)$ for some δ_1, $0 < \delta_1 < \delta$,

(II) $S_{m_r}^{\frac{1}{2}-\gamma} = o(b_{m_r})$ $(r \to \infty)$ for some γ, $0 < \gamma < \frac{1}{2}$,

then

$$\lim_{r\to\infty} D^n P_{m_r}(D)f(x) = \varphi^{(n)}(x+).$$

REMARK. - Conclusions (a) and (b) for $n=0$ are the well known results of I. I. HIRSCHMAN and D. V. WIDDER (see Theorem 5.1.b [7; p. 128] and Theorem 7.1.a [7; p. 139]).

THEOREM 6.2 - Suppose

(1) $G(t) \in$ class II,

(2) assumptions (2), (4), and (5) of Theorem 6.1 are satisfied,

(3) $f(x) = \int_{-\infty}^{\infty} G(x-t)\varphi(t)dt$ converges for $x > \gamma_c$.

Then conclusions (a), (b), (c) and (d) of Theorem 2.1 are valid.

THEOREM 6.3. - Suppose

(1) $G(t) \in$ class III and $G(t)$ is non-finite,

(2) $\varphi(t) \in L_1(a, b)$ $T < a < b < \infty$,

(3) $f(x) = \int_{-\infty}^{\infty} G(x-t)\varphi(t)dt$ converges for $x > T + b + \sum_{k=1}^{\infty} a_k^{-1}$,

(4) assumption (4) of Theorem 6.1 is satisfied for some real $x > T$.

Then conclusions (a), (b), (c) and (d) of Theorem 6.1 are valid (for x satisfying (4)).

REMARK. - Conclusions (a) of Theorems 6.2 and 6.3 for the case $n=0$ are Theorem 5.2.b [7; p. 131] and Theorem 5 3.b [7; p. 132] Conclusions (b) of Theorems 6.2 and 6.3 for the case $n=0$ are in Theorem 7.1.a [7; p. 139].

We shall obtain now some lemmas that will be used in the proof of Theorems 6.1, 6.2 and 6.3 as well as in the theorems of the following sections.

LEMMA 6.4 - Suppose $\{a_k\}$ satisfies $S_{m_r}^{(n-2)} \geq \alpha S_{m_r}$ $\alpha > 0$ independent of r and $G(t)$ belongs either to class I and is non-finite or to class II, then to every real t_0 and each $\delta > 0$ corresponds a r_0 $r_0 = r_0(\delta, t_0) > 0$ such that for each $r > r_0$ $G(t+t_0) - aG_{m_r}^{(n)}(t)$ has at most one change of sign in $(-\infty, -\delta)$ and at most one change of sign in (δ, ∞).

PROOF. - Let $\delta > 0$ be given; by Theorem 3.5 there exists a r_0 such that

$$\zeta_{m_r, n, 1} \geq -\delta \quad \text{for} \quad r \geq r_0. \tag{6.1}$$

Since $S_m = o(1)$ $m \to \infty$ we have for C_n of Theorem 3.6

$$C_n S_{m_r}^{-\frac{n+1}{2}} \leq \max_t |G_{m_r}^{(n)}(t)| \quad \text{for} \quad r \geq r_0. \tag{6.2}$$

Suppose first that $a \geq 1$. For all $G(t) \in$ class I and $m \geq m_1$ we have (see [7; pp. 108-110])

$$\lim_{t \to \pm\infty} \frac{G_m^{(n)}(t)}{G(t+t_0)} = 0.$$

By Theorem 4.7 the functions $G(t+t_0) - aG_{m_r}^{(n)}(t)$ have at most $2\left[\frac{n+2}{2}\right]$ changes of sign. Theorem 3.6 and inequality (6.2) imply that for $r \geq r_1$ $G(t+t_0) - aG_{m_r}^{(n)}(t)$ has at least, and therefore exactly, $2\left[\frac{n+2}{2}\right]$ changes of sign and has only one on the left of $\zeta_{m_r,n,1}$. This implies that (in case $a \geq 1$ and $G(t) \in$ class I) for $r \geq r_1$ $G(t+t_0) - aG_{m_r}^{(n)}(t)$ has at most one change of sign in $(-\infty, -\delta)$. If $G(t) \in$ class II, then by [7; pp. 108-110] we obtain for $m \geq m_2$ $\lim_{t \to -\infty} \frac{G_m^{(n)}(t)}{G(t+t_0)} = 0$, therefore $G(t+t_0) - aG_m^{(n)}(t)$ has at least $2\left[\frac{n+2}{2}\right] - 1$ changes of sign for $m \geq m_2$. By Theorem 4.7 and by the continuity of the function $G(t+t_0) - aG_m^{(n)}(t)$ there is for $r \geq r_2$ where $r_2 \geq r_1$ only one change of sign to the left of $\zeta_{m_r,n,1}$; that is: there is at most one change of sign in $(-\infty, -\delta)$. This proves our Lemma for $a \geq 1$ and $(-\infty, -\delta)$. By Theorems 4.6 and 6.1 $G_{m_r}^{(n)}(t)$ is monotonic in $(-\infty, -\delta)$ and there is a constant $r_3 > 0$ such that for $r \geq r_3$ $G_{m_r}^{(n)}(-\delta) < G(-\delta + t_0)$. Since $G(t+t_0) - G_{m_r}^{(n)}(t)$ has no changes of sign in $(-\infty, -\delta)$ for $r \geq r_3$, $G(t+t_0) - aG_{m_r}^{(n)}(t)$ for $a \leq 1$ will also have none. Choosing $r_0 = \max(r_1, r_2, r_3)$ completes our proof for $(-\infty, -\delta)$. The case of (δ, ∞) is similar. Q.E.D.

LEMMA 6.5. - Suppose $G(t) \in$ class I or class II and $\{a_k\}$ satisfies $S_{m_r}^{(n-2)} \geq \alpha S_{m_r}$ $\alpha > 0$, then for each $\delta > 0$ and every real t_0 corresponds a $r_0 = r_0(\delta, t_0)$ such that for $r \geq r_0$ $G_{m_r}^{(n)}(t)/G(t+t_0)$ is monotonic in $(-\infty, -\delta)$ and in (δ, ∞).

PROOF. - Suppose $h(t) = G_{m_r}^{(n)}(t)/G(t+t_0)$ is not monotonic in $(-\infty, -\delta)$ (for example). Then there are three points $t_1 < t_2 < t_3 < -\delta$ such that, $h(t_1), h(t_3) < h(t_2)$ (or $h(t_1), h(t_3) > h(t_2)$). Choosing $h(t_1), h(t_3) < \frac{1}{a} < h(t_3)$ we see that $G(t+t_0) - aG_{m_r}^{(n)}(t)$ has at least two changes of sign.

This, for $r \geq r_0$ contradicts Lemma 6.4.

LEMMA 6.6. - Suppose:

(1) $G(t) \in$ class I and is non finite or $G(t) \in$ class II.

(2) $\varphi(t) \in L(a, b)$ for $\infty < a < b < \infty$.

(3) $f(x) = \int_{-\infty}^{\infty} G(x-t)\varphi(t)dt$ converges for some x.

(4) $S_{m_r}^{(n-2)} \geq \alpha S_{m_r}$ $\alpha > 0$ independent of r.

Then for all x $-\infty < x < \infty$, $\delta > 0$ and $l > 0$.

$$\int_{x+\delta}^{\infty} G_{m_r}^{(n)}(x-t)\varphi(t)dt = o(S_{m_r}^l) \qquad r \to \infty \tag{6.3}$$

and

$$\int_{-\infty}^{x-\delta} G_{m_r}^{(n)}(x-t)\varphi(t)dt = o(S_{m_r}^l) \qquad r \to \infty. \tag{6.4}$$

PROOF. - By Lemma 6.5 and BONNET'S theorem for integrals

$$I_{m_r} = \int_{-\infty}^{x-\delta} G_{m_r}^{(n)}(x-t)\varphi(t)dt = (G_{m_r}^{(n)}(\delta)/G(x_1 - x + \delta)) \int_{\zeta}^{x-\delta} G(x_1 - t)\varphi(t)dt.$$

Since for $G(t) \in$ class I or II we have $G(t) > 0$ and taking x_1 for which $\int_{-\infty}^{\infty} G(x_1 - t)\varphi(t)dt$ converges we obtain by Theorem 4.5 $I_{m_r} = o(S_{m_r}^l)$ for any l. The proof of (6.3) is similar. Q.E.D.

LEMMA 6.7. - Let assumptions (1), (2) and (3) of Lemma 6.6 be satisfied and for some n $\{a_k\}$ satisfy $B(n-2, \{a_k\})$ then for this n (6.3) and (6.4) are valid for all $l > 0$, $\delta > 0$ and $-\infty < x < \infty$.

PROOF. - By lemma 6.6 our Lemma is valid for $n = 0, 1, 2$. We recall the notation of $a_{m_r+r_j}$ and $G_{m_r,j}(t)$ defined in (4.11) and (4.12) in the proof of Theorem 4.6. Using Lemma 6.6 and since $G_{m_r,j}(t)$ are a subsequence of $G_m(t)$ with rearranged $\{a_k\}$ we have for all $\delta > 0$, $l > 0$, $-\infty < x < \infty$, $j \geq 0$ and $n = 0$, 1 or 2

$$\int_{x+\delta}^{\infty} G_{m_r,j}^{(n)}(x-t)\varphi(t)dt = o(S_{m_r}^{(j)})^l \qquad r \to \infty.$$

Assume for some integer s, $0 < s < n$, $\delta > 0$, $l > 0$, $-\infty < x < \infty$ and $0 \leq j \leq n - s - 1$

$$\int_{x+\delta}^{\infty} G_{m_r,j}^{(s)}(x-t)\varphi(t)dt = o(S_{m_r}^{(j)})^l \qquad (r \to \infty) \tag{6.5}$$

now we use (4.14) $G_{m_r,j-1}^{(s+1)}(t) = a_{m_r+r_j}\{G_{m_r,j-1}^{(r)}(t) - G_{m_r,j}^{(s)}(t)\}$ and in a method similar to that of Theorem 4.6 complete the proof of (6.3). The proof of (6.4) is similar. Q.E.D.

LEMMA 6.8. - Suppose:

(1) $G(t) \in$ class III and $G(t)$ is non finite.

(2) $\varphi(t) \in L_1(a, b)$ $T < a < b < \infty$.

(3) $f(x) = \int_{-\infty}^{\infty} G(x-t)\varphi(t)dt$ converges for $x > T + b + \sum_{k=1}^{\infty} a_k^{-1}$.

Then for $x > T$ for all $l > 0$

$$\int_{x+\delta}^{\infty} G_m(x-t)\varphi(t)dt = 0 \quad \text{for} \quad m > m_0 \tag{6.6}$$

and

$$\int_{-\infty}^{x-\delta} G_m^{(n)}(x-t)\varphi(t)dt = o(S_m^l) \qquad m \to \infty. \tag{6.7}$$

PROOF. - Equation (6.6) is obvious since $G_m(t) = 0$ for $-\infty < t < b_m + \sum_{k=m+1}^{\infty} a_k^{-1}$ [7; p. 114] and $b_m + \sum_{k=m+1}^{\infty} a_k^{-1} = o(1)$ $m \to \infty$.

By Lemma 2.10 our $\{a_k\}$ which satisfy $\Sigma a_k^{-1} < \infty$ satisfy $B(n)$ for all n. The proof of (6.7) follows by induction. The case $n = 0$ is for all $l > 0$, $\delta > 0$ and $x > T$

$$\int_{-\infty}^{x-\delta} G_{m,j}(x-t)\varphi(t)dt = o(S_m^{(j)})^l \qquad m \to \infty$$

can be proved by the method of Lemma 6.6, since the monotonic character of $G_{m,j}(x-t)/G(x_1-t)$ in $(-\infty, x-\delta)$ for $m > m_0$ is a simple consequence of Theorem 6.1.b. of [7 p. 95].

Assume for $n = n_0 - 1 \geq 0$

$$\int_{-\infty}^{x-\delta} G_{m,j}^{(n_0-1)}(x-t)\varphi(t)dt = o(S_m^{(j)})^l \qquad m \to \infty$$

we complete the proof using equation (4.14) $S_m^{(j)} < S_m^{(j-1)}$ and (2.13). Q.E.D.

PROOF OF THEOREMS 6.1, 6.2 AND 6.3. - By Lemmas 5.1, 5.2 and 5.3 we have for Theorems 6.1, 6.2 and 6.3 respectively

$$D^n P_{m_r}(D)f(x) = \left\{ \int_{-\infty}^{x-\eta} + \int_{x-\eta}^{x+\eta} + \int_{x+\eta}^{\infty} \right\} G_{m_r}^{(n)}(x-t)\varphi(t)dt \equiv I_{m_r,1} + I_{m_r,2} + I_{m_r,3}. \tag{6.8}$$

(For the proof of Theorem 6.3 $\{m_r\} = \{m\}$ and $x > T$).

We obtain, by Lemmas 6.6 and 6.7 for all $\eta > 0$, $-\infty < x < \infty$ and $l > 0$ (in case $G(t)$ belongs to class I or II)

$$(6.9) \qquad I_{m_r,1} = o(S^l_{m_r}) \quad r \longrightarrow \infty \quad I_{m_r,3} = (S^l_{m_r}) \quad r \longrightarrow \infty.$$

Using Lemma 6.8 for $G(t) \in$ class III (for Theorem 6.3) we obtain for $x > T$, $\eta > 0$, $l > 0$

$$(6.10) \qquad I_{m_r,1} = o(S^l_{m_r}) \quad r \longrightarrow \infty \quad I_{m_r,3} = 0 \quad \text{for} \quad m_r > m_0.$$

Integrating by parts, assuming $\eta < \delta$ (δ defined in the Theorems) we have

$$I_{m_r,2} = \int_{x-\eta}^{x+\eta} G_{m_r}(x-t)\varphi^{(n)}(t)dt +$$
$$+ \sum_{k=0}^{n-1} \{ G_{m_r}^{(n-k-1)}(\eta)\varphi^{(k)}(x-\eta) - G_{m_r}^{(n-k-1)}(-\eta)\varphi^{(k)}(x+\eta) \}.$$

Using Theorems 4.5 and 4.6 we obtain, since $\varphi^{(k)}(x \pm \eta)$ for $k < n$ are bounded,

$$I_{m_r,2} = \int_{x-\eta}^{x+\eta} G_{m_r}(x-t)\varphi^{(n)}(t)dt + o(S^l_{m_r}) \equiv I_{m_r} + o(S^l_{m_r}) \quad r \longrightarrow \infty.$$

It is enough now for the calculation of $D^n P_{m_r}(D)f(x)$ to evaluate I_{m_r} in cases (a), (b), (c) and (d). The estimations are the same for classes I, II and III.

CASE (a). - Following the method by HIRACHMAN and WIDDER in proving Theorem 5.1.b, 5.2.b and 5 3.b [7; Ch. VI] we obtain

$$|I_{m_r} - \varphi^{(n)}(x)| \leq \varepsilon + o(1) \quad r \longrightarrow \infty \quad \text{where} \quad \varepsilon = \varepsilon(\eta) = o(1) \quad \eta \longrightarrow o+.$$

CASE (b). - By a method similar to that used in the proof of Theorem 7.1.a of [7; p. 139] we get $|I_{m_r} - \varphi^{(n)}(x-o)| \leq o(1) + \varepsilon M$ $r \longrightarrow \infty$ where M is a constant and $\varepsilon = \varepsilon(\eta) = o(1)$ $\eta \longrightarrow 0+$.

CASE (c). - Since for $r > r_0$ $b_{m_r} > 0$ we have for $\eta > 0$ and $r > r_0$

$$0 \leq \int_{-\eta}^{0} G_{m_r}(z)dz \leq \int_{-\infty}^{0} G_{m_r}(z)dz = \int_{-\infty}^{-b_{m_r}} G_{m_r}(t + b_{m_r})dt \leq$$

$$\leq \int\limits_{|t|\geq b_{m_r}} G_{m_r}(t + b_{m_r})dt \leq \frac{1}{b_{m_r}^2} \int\limits_{-\infty} t^2 G_{m_r}(t + b_{m_r})dt =$$

$$= S_{m_r}/b_{m_r}^2 = o(1) \qquad r \to \infty.$$

This implies also $\lim\limits_{r\to\infty} \int\limits_0^\infty G_{m_r}(z)dz = 1$. Combining it with Lemmas 6.6, 6.7 and 6.8 with $\varphi(t) \equiv 1$ we get for $\eta > 0$ $\lim\limits_{r\to\infty} \int\limits_0^\eta G_{m_r}(z)dz = 1$.

We divide now $I_{m_r 2}$ into two parts

$$I_{m_r 2} = \left\{ \int\limits_{x-\eta}^{x} + \int\limits_{x}^{x+\eta} \right\} G_{m_r}(x - t)\varphi^{(n)}(t)dt \equiv I_{m_r}(1) + I_{m_r}(2).$$

In order to calculate $I_{m_r}(1)$ we choose $\eta > 0$ so that $|\varphi^{(n)}(x - z) - \varphi^{(n)}(x -)| < \varepsilon$ for $0 < z < \eta$ and obtain

$$|I_{m_r}(1) - \varphi^{(n)}(x -)| = \left| \int\limits_0^\eta G_{m_r}(z)[\varphi^{(n)}(x - z) - \varphi^{(n)}(x -)]dz \right| + o(1) \leq$$

$$\leq \varepsilon \int\limits_0^\eta G_{m_r}(z)dz + o(1) \leq \varepsilon + o(1) \qquad r \to \infty.$$

The estimation of $I_{m_r}(2)$ will be different for cases (*c*, I) and (*c*; II).

In case (*c*, I) we have $|\varphi^{(n)}(x - z)| < M$ where $-\eta \leq z \leq 0$

$$|I_{m_r}(2)| \leq \left| \int\limits_{-\eta}^0 G_{m_r}(z)\varphi^{(n)}(x - z)dz \right| \leq O(1) \int\limits_{-\eta}^0 G_{m_r}(z)dz = o(1) \qquad r \to \infty.$$

To estimate $I_{m_r}(2)$ in case (c, II) we recall that $G_{m_r}(t)$ is monotonic increasing in $(-\infty, \zeta_{m_r})$ where ζ_{m_r} is the only maximum of $G_{m_r}(t)$.

Since $S_{m_r}^{1/2} = o(b_{m_r})$ $r \to \infty$, $b_{m_r} > 0$ for $r > r_0$ and $|\zeta_{m_r} - b_{m_r}| < 8S_{m_r}^{1/2}$ (Lemma 4.1 of [7; p. 126]) $G_{m_r}(t)$ is monotonic increasing for $r > r_1$ in $\left(-\infty, \frac{1}{2}b_{m_r}\right)$.

In order to show that $I_{m_r}(2) = o(1)$ $r \to \infty$ it is enough to show that $\lim\limits_{r\to\infty} G_{m_r}(0) = 0$ because

$$|I_{m_r}(2)| = \left| \int\limits_{-\eta}^0 G_{m_r}(z)\varphi^{(n)}(x - z)dz \right| \leq O(1) \int\limits_{-\eta}^0 G_{m_r}(z)dz \leq O(1)\eta\, G_{m_r}(0).$$

$$\frac{1}{2}b_{m_r}G_{m_r}(0) \le \int_0^{\frac{1}{2}b_{m_r}} G_{m_r}(z)dz \le \int_{-\infty}^{-\frac{1}{2}b_{m_r}} G_{m_r}(z+b_{m_r})dz \le$$

$$\le \int_{|z|\ge\frac{1}{2}b_{m_r}} G_{m_r}(z-b_{m_r})dz \le \frac{2^{2k}}{b_{m_r}^{2k}}\int_{-\infty}^{\infty} z^{2k}G_{m_r}(z+b_{m_r})dz \le$$

$$\le \frac{2^{2k}}{b_{m_r}^{2k}}\mu_{2k}(m_r) \le \frac{2^{2k}}{b_{m_r}^{2k}} S_{m_r}^k \cdot K(2k).$$

Hence for every $k \ge 1$

$$G_{m_r}(0) \le 2^{2k+1}K(2k)\cdot(b_{m_r}^{2k+1})^{-1}S_{m_r}^k.$$

Choosing k large enough we obtain $G_{m_r}(0) = o(1)$ $r \to \infty$. The proof of case (d) is similar to that of case (c). Q.E.D.

Theorems 6.1, 6.2 and 6.4 are included in analogous inversion formulae for

$$f(x) = \int_{-\infty}^{\infty} G(x-t)e^{ct}d\alpha(t). \tag{6.11}$$

The proofs of the inversion formulae for (6.11) use the limit

$$\lim_{r\to\infty}\int_{-\infty}^{\infty} G_{m_r}^{(n)}(x-t)\varphi(t)dt$$

calculated already while proving Theorems 6.1, 6.2 and 6.3.

THEOREM 6.9. - Suppose

(1) $G(t) \in$ class I and is non-finite,

(2) $\alpha(t) \in B.V.(a, b)$ for all a, b $-\infty < a < b < \infty$,

(3) $f(x) = \int_{-\infty}^{\infty} G(x-t)e^{ct}d\alpha(t)$ converges,

(4) $\alpha^{(n)}(t)$ exists for some $-\infty < x < \infty$, $n \ge 0$ and $\delta > 0$ a.e in $(x-\delta, x+\delta)$ and in case $n > 1$ we add the condition that $\alpha^{(n-1)}(t)$ is absolutely continuous there;

(5) for $n>2$ either $S_{m_r}^{(n-2)} \geq \alpha S_{m_r}$ $\alpha > 0$ independent of r or $\{a_k\}$ satisfy $B(n-2, \{m_r\})$;

(6) $\alpha^{(n)}(t)$ is continuous at $t = x$.

Then

A) $\alpha_1 < c < \alpha_2$ and $\alpha(t)$ satisfying (4) and (6) for both x and x_1 imply

$$\lim_{r\to\infty} \int_{x_1}^{x} e^{-cx} D^n P_{m_r}(D) f(x) dx = \sum_{j=0}^{n} c^j \binom{n}{j} (\alpha^{(n-j)}(x) - \alpha^{(n-j)}(x_1)).$$

B) $c \geq \alpha_2$ implies that $\alpha(\infty)$ exists and

$$\lim_{r\to\infty} - \int_{x}^{\infty} e^{-cx} D^n P_{m_r}(D) f(x) dx = \sum_{j=0}^{n} c^j \binom{n}{j} \alpha^{(n-j)}(x) - c^n \alpha(\infty).$$

C. $c \leq \alpha_1$ implies that $\alpha(-\infty)$ exists and

$$\lim_{r\to\infty} \int_{-\infty}^{x} e^{-cx} D^n P_{m_r}(D) f(x) dx = \sum_{j=0}^{n} c^j \binom{n}{j} \alpha^{(n-j)}(x) - c^n \alpha(-\infty).$$

D. For $n \geq 1$ we have

$$\lim_{r\to\infty} c^{-cx} D^{n-1} P_{m_r}(D) f(x) = \sum_{j=1}^{n} c^j \left\{ \binom{n}{j} - \binom{n-1}{j-1} \right\} \alpha^{(n-j)}(x) + \alpha^{(n)}(x).$$

REMARK. - The cases $n = 0$ of A, of B and of C were proved [7, p. 135]. In case $n = 1$ $c = 0$ D yields a generalization of Theorem 5.1.b of [7, p. 128]. D with any $n \geq 1$ and $c = 0$ is a generalization of Theorem 6.1, part (a), since here $\varphi(t)$ does not necessarily exist in $-\infty < t < \infty$ but only in a neighborhood of x.

PROOF. - Using Lemma 5.4A we obtain

$$\int_{x_1}^{x_2} e^{-cx} D^n P_{m_r}(D) f(x) dx = e^{-cx_2} \int_{-\infty}^{\infty} G_{m_r}^{(n)}(x_2 - t) e^{ct} \alpha(t) dt - e^{-cx_1} \int_{-\infty}^{\infty} G_{m_r}^{(n)}(x_1 - t) e^{ct} \alpha(t) dt.$$

To complete the proof of A we use Theorem 6.1(a) with $\varphi(t)$ equal to $e^{ct}\alpha(t)$. The assumptions of Theorem 6.1(a) are satisfied since whenever $\alpha(t)$ has k derivatives in some interval so has $e^{ct}\alpha(t)$ and the same is true about continuity of the k-*th* derivative at a point.

Cases B and C are similar but using B and C of Lemma 5.4 and taking $e^{ct}(\alpha(\infty) - \alpha(t))$ and $e^{ct}(\alpha(t) - \alpha(-\infty))$ for $\varphi(t)$ respectively.

PROOF OF D. - Using Lemma 5.4D we obtain

$$e^{-cx}D^{n-1}P_{m_r}(D)f(x) = e^{-cx}\int_{-\infty}^{\infty} G_{m_r}^{(n)}(x - t)e^{ct}\alpha(t)dt +$$

$$- ce^{cx}\int_{-\infty}^{\infty} G_{m_r}^{(n-1)}(x - t)e^{ct}\alpha(t)dt.$$

Theorem 6.1(a) for n and $n - 1$ with $\varphi(t)$ equal to $e^{ct}\alpha(t)$ implies D by some simple calculations. Q.E.D.

Like Theorem 6.9 one can state and prove eleven more theorems which are the analogues to Theorems 6.1, 6.2 and 6.3 in each of the four cases (a), (b), (c) and (d) (the analogue of 6.1(a) is 6.9). We shall not write these theorems since both their statements and proofs are easily derived froms Theorems 6.1, 6.2, 6.3, 6.9 and Lemmas 5.4, 5.5 and 5.6. We want to mention that the analogues of (c) and (d) are new also in case $n = 0$. In this case the assumption $S_{m_r}^{1/2-\gamma} = o(b_{m_r})$ $r \to \infty$ for some γ $0 < \gamma < \frac{1}{2}$ is not necessary since $\alpha(t)$ is bounded in every finite interval (see (c, I) of Theorem 6.1).

The combination of (c) and (d) for each of Theorems 6.1, 6.2 and 6.3 whenever both $\varphi^{(n)}(x \pm)$ exist (in this case also $S_{m_r}^{1/2-\gamma} = o(b_{m_r})$ $r \to \infty$ is unnecessary) form a *jump formula*.

By the same method we can obtain a jump formula for the convolution STIELTJES transform.

7. - The inversion formula in the case $\lim\limits_{m\to\infty} \inf\limits_{k>0} (\max a_k^{-2})S_m^{-1} = 0$.

The class of convolution transforms whose related sequence $\{a_k\}$ satisfies for some order the assumption

$$\liminf_{m\to\infty} (\max_{k>0} a_k^{-2})S_m^{-1} = 0 \tag{7.1}$$

is large, since we know by Lemma 2.5 that whenever $\Sigma |a_k|^{-\beta} = \infty$ for some β $0 < \beta < 2$ and $|a_{k+1}| \leq |a_k|$ (7.1) is satisfied. All the well-known examples of convolution transforms ((1.1), (1.2) and (1.3) with $c = 0$) that where given by HIRSCHMAN and WIDDER in their book «The convolution transform» [7, pp. 65-69] satisfy (7.1) whenever $|a_{k+1}| \geq |a_k|$.

We shall need some lemmas on properties of $G_m(t)$ when $\{a_k\}$ satisfy (7.1) that will be used only in this section and in the following one.

Whenever (7.1) is satisfied, there exists a subsequence of integers $\{m_r\}$ such that $\lim_{r\to\infty}(\max_{k>m_r} a_k^{-2})S_{m_r}^{-1}=0$. In the following lemma we shall use the special subsequence $\{m_r\}$.

LEMMA 7.1. - If for some subsequence of integers $\{m_r\}$ $\lim_{r\to\infty}(\max_{k>m_r} a_k^{-2})S_{m_r}^{-1}=0$ and for some real λ, $b_{m_r}-\lambda S_{m_r}^{1/2}=o(S_{m_r}^{1/2})$ $(r\to\infty)$, then we have uniformly in t $(-\infty<t<\infty)$

$$\lim_{r\to\infty} S_{m_r}^{1/2}\frac{d^n}{dt^n}G_{m_r}(S_{m_r}^{3/2}(t+\varepsilon_{m_r}))=\frac{1}{\sqrt{2\pi}}\frac{d^n}{dt^n}\{e^{-(t-\lambda)^2/2}\} \tag{7.2}$$

where $n=0,1,2\ldots$ and $\varepsilon_{m_r}=o(1)$ $(r\to\infty)$ (ε_{m_r} independent of t).

PROOF. - Using Lemma 2.7 we obtain

$$\sum_{k=m_r+1}^{\infty}|a_k|^{-3}=o(S_{m_r}^{3/2}) \qquad r\to\infty. \tag{7.3}$$

The proof is now similar to that of Theorem 7.2a [7, p. 140]. We define

$$F_m(z)=\exp(b_mS_m^{-1/2}z)\prod_{k=m+1}^{\infty}\left(1-\frac{z}{a_kS_m^{1/2}}\right)\exp(z/a_kS_m^{1/2}).$$

In order to complete the proof we have to note that

(a) the assumption $b_{m_r}-\lambda S_{m_r}^{1/2}=o(S_{m_r}^{1/2})$ $r\to\infty$ (which we use now in stead of $b_m=o(S_m^{1/2})$ $m\to\infty$) implies $\lim_{r\to\infty}F_{m_r}(z)=e^{\lambda z}e^{-z^2/2}$ uniformly in $|z|\le R$ for any $R>0$ (instead of $\lim_{m\to\infty}F_m(z)=e^{-z^2/2}$);

(b) using the method of proving Theorem 3.1, case (A), and by Lemma 2.5 we have for each l and $m>m_0(l)$ $|F_m(iy)|\ge(1+K(l)y^{2l+2})^2$ where $K(l)$ is independent of m. Q.E D.

LEMMA 7.2. - If $\lim_{r\to\infty}(\max_{k>m_r} a_k^{-2})S_{n_r}^{-1}=0$ and for some real λ $b_{m_r}-\lambda S_{m_r}^{1/2}=$ $=o(S_{m_r}^{1/2})$ $r\to\infty$, then

$$\lim_{r\to\infty}\int_0^{\infty}G_{m_r}(t)dt=N(\lambda) \tag{7.4}$$

and

$$\lim_{r\to\infty}\int_{-\infty}^{0} G_{m_r}(t)dt = 1 - N(\lambda), \tag{7.5}$$

where

$$N(\lambda) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\lambda} e^{-u^2/2}du. \tag{7.6}$$

PROOF. - We have by FATOU'S Lemma and Lemma 7.1 (for $n = 0$)

$$\liminf_{r\to\infty}\int_{0}^{\infty} G_{m_r}(t)dt \geq \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} e^{-(t-\lambda)^2/2}dt = N(\lambda)$$

and

$$\liminf_{r\to\infty}\int_{-\infty}^{0} G_{m_r}(t)dt \geq \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} e^{-(t-\lambda)^2/2}dt = 1 - N(\lambda).$$

The equality $\int_{-\infty}^{\infty} G_m(t)dt = 1$ for all m completes the proof. Q.E.D.

THEOREM 7.3. - Suppose

(1) $G(t) \in$ class I, $G(t)$ is non-finite,

(2) $\varphi(t) \in L_1(a, b)$ for all a, b $-\infty < a < b < \infty$,

(3) $\lim_{r\to\infty}(\max_{k>m_r} a_k^{-2})S_{m_r}^{-1} = 0$,

(4) $b_{m_r} - \lambda S_{m_r}^{1/2} = o(S_{m_r}^{1/2})$ $r \to \infty$ (λ is real),

(5) $f(x) = \int_{-\infty}^{\infty} G(x-t)\varphi(t)dt$ converges,

(6) both $\varphi(x \pm 0)$ exist or for $n > 0$

$$\varphi(t) = \begin{cases} \sum_{l=0}^{n} \frac{1}{l!} b_l(t-x)^l + o(t-x)^n & t \to x^+ \\ \sum_{l=2}^{n} \frac{1}{l!} c_l(t-x)^l + o(t-x)^n & t \to x^- \end{cases}$$

and for $l < n$ $c_l = b_l$.

Then

$$\lim_{r\to\infty} D^n P_{m_r}(D)f(x) = \begin{cases} N(\lambda)\varphi(x-0) + (1-N(\lambda))\varphi(x+0) & \text{for } n = 0 \\ N(\lambda)c_n + (1-N(\lambda))b_n & \text{for } n > 0 \end{cases} \tag{7.7}$$

REMARK - If instead of assumption (6) we assumed, as in the former section, that $\varphi^{(n)}(t)$ exists, a.e. in some neighborhood of x, $\varphi^{(n-1)}(t)$ is absolutely continuous there and in addition $\varphi^{(n)}(x \pm 0)$ exists, we would obtain in (7.7) $\varphi^{(n)}(x-0)$ and $\varphi^{(n)}(x+0)$ instead of c_n and b_n respectively as a special case of our theorem.

We state the inversion formula for $G(t)$ of classes II and III and then prove them together with that for $G(t)$ of class I.

THEOREM 7.4. - If $G(t) \in$ class II and assumptions (2), (3), (4), (5), and (6) of Theorem 7.3 are satisfied, then (7.7) is valid.

THEOREM 7.5. - Suppose

(1) $G(t) \in$ class III and is non-finite,

(2) $\varphi(t) \in L_1(a, b)$ for all a, b $T < a < b < \infty$,

(3) $f(x) = \int_{-\infty}^{\infty} G(x-t)\varphi(t)dt$ converges for $x > T + b + \sum_{k=1}^{\infty} a_k^{-1}$,

(4) assumptions (3), (4) and (6) (for $x > T$) of Theorem 7.3 are satisfied, then (7.7) is valid for $x > T$.

PROOF OF THEOREMS 7.3, 7.4 AND 7.5. - By Lemmas 6.6, 6.7 and 6.8 the proof of Theorems 6.1-6.3 and Lemma 2.5

$$D^n P_{m_r}(D) f(x) = \int_{x-\eta}^{x+\eta} G_{m_r}^{(n)}(x-t)\varphi(t)dt + o(1) \qquad r \to \infty$$

for all $\eta > 0$. Assume η so small that in the case where $n > 0$

$$\left| \varphi(t) - \sum_{l=0}^{n} \frac{1}{l\,!} b_l (t-x)^l \right| \leq \varepsilon \, |t-x|^n \qquad x < t \leq x + \eta \tag{7.8}$$

and

$$\left| \varphi(t) - \sum_{l=0}^{n} \frac{1}{l\,!} c_l (t-x)^l \right| \leq \varepsilon \, |t-x|^n \qquad x - \eta \leq t < x. \tag{7.9}$$

Estimates (7.8) and (7.9) and the fact that for $l < n$ $b_l = c_l$ imply

$$\left| \int_{x-\eta}^{x+\eta} G_{m_r}^{(n)}(x-t)\varphi(t)dt - N(\lambda)c_n - (1 - N(\lambda))b_n \right| =$$

$$\leq \left| \int_{x-\eta}^{x+\eta} G_{m_r}^{(n)}(x-t) \sum_{l=0}^{n-1} \frac{1}{l\,!} b_l(t-x)^l dt \right| + \left| \int_{x-\eta}^{x} G_{m_r}^{(n)}(x-t) \frac{c_n}{n\,!}(t-x)^n dt - N(\lambda)c_n \right|$$

$$+ \left| \int_{x}^{x+\eta} G_{m_r}^{(n)}(x-t) \frac{1}{n\,!} b_n(t-x)^n dt - (1-N(\lambda))b_n \right| +$$

$$+ \varepsilon \int_{x-\eta}^{x+\eta} | G_{m_r}^{(n)}(x-t) | \, | t-x |^n \, dt \equiv I(r,\ 1) + I(r,\ 2) + I(r,\ 3) + I(r,\ 4).$$

We shall prove $I(r, j) = o(1)$ $r \to \infty$ for $j = 1, 2, 3$ and $|I(r, 4)| < \varepsilon M$.

The condition $\lim_{r\to\infty} (\max_{k>m_r} a_k^{-2}) S_{m_r}^{-1} = 0$ implies, by Lemma 2.4, the existence of $\alpha_p > 0$ independent of r such that $S_{m_r}^{(p)} \geq \alpha_p S_{m_r}$ for all p; therefore, by Theorem 4.5 we obtain for $\eta > 0$ and any integers n and l

$$G_{m_r}^{(n)}(\eta) = o(S_{m_r}^l), \qquad G_{m_r}^{(n)}(-\eta) = o(S_{m_r}^l) \qquad r \to \infty. \tag{7.10}$$

Using integration by parts and (7.10) we get

$$I(r,\ 1) = o(S_{m_r}^l) \qquad r \to \infty \tag{7.11}$$

and for $I(r, 2)$

$$I(r,\ 2) = \left| \int_{x-\eta}^{x} G_{m_r}(x-t)c_n dt - N(\lambda)c_n \right| + o(S_{m_r}^l) \qquad r \to \infty.$$

Combination of Lemmas 6.6 and 6.7 with $\varphi(t) \equiv 1$, and Lemma 7.2 yield

$$\left| \int_0^{\eta} G_{m_r}(t)dt - N(\lambda) \right| = o(1) \qquad r \to \infty \tag{7.12}$$

from which it is easily seen that $I(r, 2) = o(1)$ $r \to \infty$. By the same method $I(r, 3) = o(1)$ $r \to \infty$. We have

$$I(r,\ 4) = \varepsilon \int_{x-\eta}^{x+\eta} | G_{m_r}^{(n)}(x-t) | \, | t-x |^n \, dt \leq \varepsilon \int_{-\infty}^{\infty} | G_{m_r}^{(n)}(t) | \, | t |^n \, dt.$$

Theorem 3.8, the assumption of which is satisfied here, implies $|I(r, 4)| < \varepsilon M(n, \alpha)$. This concludes the proof in case $n > 0$.

In the case $n = 0$ we have

$$\left\{ \int_{x-\eta}^{x} + \int_{x}^{x+\eta} \right\} G_{m_r}(x-t)\varphi(t)dt \equiv J(r,\ 1) + J(r,\ 2).$$

It is enough to show

$$| J(r,\ 1) - N(\lambda)\varphi(x-0) | < \varepsilon M \quad \text{and} \quad | J(r,\ 2) - (1 - N(\lambda))\varphi(x+0) | < \varepsilon M$$

where M does not depend on r.

Let us define $\alpha(u) = \int_0^u [\varphi(x-t) - \varphi(x-0)]dt$ and chose η so small that $|\alpha(u)| < \varepsilon u$ (possible since by definition $\alpha(u) = o(u)$ $u \to 0+$). By estimation (7.12) and Theorem 3.8 for $n = 1$ we have

$$| J(r,\ 1) - N(\lambda)\varphi(x-0) | \leq o(1) + \left| \int_x^{x+\eta} G_{m_r}(t)[\varphi(x-t) - \varphi(x-0)]dt \right| \leq$$

$$\leq o(1) + o(S^l_{m_r}) + \left| \int_x^{x+\eta} G'_{m_r}(t)\alpha(t)dt \right| \leq o(1) + \varepsilon \int_x^{x+\eta} | G'_m(t)t | \, dt$$

$$\leq \varepsilon M + o(1) \qquad r \to \infty.$$

Similarly we get $|J(r,\ 2) - (1 - N(\lambda))\varphi(x+0)| < \varepsilon M$ which concludes the proof of these theorems. Q.E.D.

We shall show now by applying Theorem 7.3 in case $n = 0$ to the STIELTJES transform that by ordering the zeros of $E(s)$ in different ways and by a suitable choice of the sequence $\{b_m\}$ we get real inversion formulae with quite different properties.

The STIELTJES transform $F(x)$ of $\varphi(t)$ defined by $F(x) = \int_0^\infty \frac{\varphi(t)}{x+t} dt$ where $\varphi(t) \in L(\varepsilon,\ R)$ for any $0 < \varepsilon < R < \infty$. By suitable change of variables (see 7, p. 68]) the STIELTJES transform becomes a convolution transform with $E(s) = \frac{\sin \pi s}{\pi s}$.

Suppose both $\varphi(x_0 \pm 0)$ exist for some $x_0 > 0$. Let λ be a real number r, l integers and $\varepsilon_n (n \geq 1)$ a sequence satisfying $\varepsilon_n = o(n^{-1/2})$ $(n \to \infty)$. By applying Theorem 7.3 with $n = 0$ to $E(s) = \frac{\sin \pi s}{\pi s}$ and by a suitable ordering of the zeros of $E(s)$ we obtain the operator

$$P_m^*(D) \overset{\text{Def}}{=\!=} P_{2m+r+l}(D) = e^{\left(\lambda\sqrt{\frac{2}{m}}+\varepsilon_m\right)D} \prod_{k=1}^{m+l}\left(1-\frac{D}{k}\right)\prod_{k=1}^{m+r}\left(1-\frac{D}{k}\right)$$

where $b=0$, $b_m = -\sqrt{\frac{2}{m}} - \varepsilon_m + 0\left(\frac{1}{m}\right)$ $(m \to \infty)$.

Now as is easily seen (see also [7, p. 68]) we have

$$\lim_{m\to\infty} \frac{(-1)^{m+r+1}}{(m+r)!\,(m+l)!}\{x^{2m+l+r+1}F^{(m+r+1)}(x)\}^{(m+l)}_{x=x_0\exp(-b_m)} =$$

$$= N(\lambda)\varphi(x_0+0) + (1-N(\lambda))\varphi(x_0-0).$$

The case $\lambda = r = 0$, $l = -2$ and $\varepsilon_m = 0$ $m \geq 1$ is given in [10, p. 350].

Suppose both $\varphi(x_0 \pm 0)$ exist for some $x_0 > 0$. Let μ, r be real numbers. By applying Theorem 7.3 as in the first example to $E(s) = \frac{\sin \pi s}{\pi s}$ for another suitable ordering of zeros of $E(s)$ we get the operator

$$P_m^{**}(D) \overset{\text{Def}}{=\!=} P_{[m+\lambda\sqrt{m}]+[m+\mu\sqrt{m}]}(D) = \prod_{k=1}^{[m+\mu\sqrt{m}]}\left(1+\frac{D}{k}\right)\prod_{k=1}^{[m+\lambda\sqrt{m}]}\left(1-\frac{D}{k}\right),$$

where $[\alpha] = \max\{n \mid n \leq \alpha,\ n \text{ integer}\}$. Now it is easily seen that

$$\lim_{m\to\infty} K_{m,\lambda,\mu}\left(\frac{d}{dx}\right)^{[m+\mu\sqrt{m}]}\{x^{[m+\lambda\sqrt{m}]+[m+\mu\sqrt{m}]+1}F^{([m+\lambda\sqrt{m}])}(s)\}_{x=x_0} =$$

$$= N\left(\frac{\mu-\lambda}{\sqrt{2}}\right)\varphi(x_0+0) + \left(1-N\left(\frac{\mu-\lambda}{\sqrt{2}}\right)\right)\varphi(x_0-0)$$

where $k_{m,\lambda,\mu} = \dfrac{(-1)^{[m+\mu\sqrt{m}]+1}}{[m+\lambda\sqrt{m}]!\,[m+\mu\sqrt{m}]!}$.

Suppose both $\varphi(x \pm 0)$ exist; we can apply the same theorem to the same kernel by a different ordering of zeros and get the operator

$$P_m^{***}(D) \overset{\text{Def}}{=\!=} \prod_{k=1}^{\left[\frac{mx}{x_0}\right]}\left(1+\frac{D}{k}\right)\prod_{k=1}^{m}\left(1-\frac{D}{k}\right).$$

Now it is easily seen that

$$\lim_{m\to\infty} \frac{(-1)^{m+1}}{m!\left[m\frac{x}{x_0}\right]!}\left(\frac{d}{dx}\right)^{\left[\frac{mx}{x_0}\right]}\{x^{m+\left[\frac{x}{x_0}m\right]+1}F^{(m+1)}(x)\}_{x=x} = \frac{1}{2}\{\varphi(x+0)+\varphi(x-0)\}.$$

It should be noted here that in the last example we get the values of $\varphi(x)$ at almost all the points $x > 0$ by means of the values of $F^{(k)}(x_0)$ $(k \geq 0)$. We can obtain similar results for the generalized STIELTJES transform and iterates of the STIELTJES transform.

The inversion for the convolution STIELTJES transform, which generalize Theorems 7.3, 7.4 and 7.5 and use them as lemmas, are outlined in the next section.

8. - Jump formulae for the case $\liminf_{m\to\infty} (\max_{k>m} a_k^{-2}) S_m^{-1} = 0$.

In this section we shall find a jump formula for a rather general class of convolution transforms that will contain as special cases the known jump formulae for the LAPLACE transform [10, p. 298] and [2], for the STIELTJES transform [10, p. 351] and the iterated STIELTJES transform [1].

We shall derive the formula first by heuristic considerations and then prove it directly.

Let us recall Theorem 7.3 where for b_m satisfying $b_m - \lambda S_m^{1/2} = o(S_m^{1/2})$ (in case $\lim_{m\to\infty} (\max_{k>m} a_k^{-2}) S_m^{-1} = 0$) and some other assumptions we have

$$\lim_{m\to\infty} P_m(D) f(x) = N(\lambda)\varphi(x-0) + (1 - N(\lambda))\varphi(x+0). \tag{8.1}$$

For $\{b_m^*\}$ eefined by $b_m^* = b_m + (\lambda_1 - \lambda) S_m^{1/2}$, which implies $b_m^* - \lambda_1 S_m^{1/2} = o(S_m^{1/2})$ $m \to \infty$, using Theorem 7.3 again, we have

$$\lim_{m\to\infty} e^{-(\lambda_1-\lambda)S_m^{1/2} D} P_m(D) = N(\lambda_1)\varphi(x-0) + (1 - N(\lambda_1))\varphi(x+0). \tag{8.2}$$

Combining (8 1) and (8.2) we obtain

$$\lim_{m\to\infty} \frac{1}{N(\lambda) - N(\lambda_1)} \{ e^{-(\lambda_1-\lambda)\, S_m^{1/2} D} - 1 \} P_m(D) f(x) = \varphi(x+0) - \varphi(x-0) \tag{8.3}$$

which is a jump formula; (this is a simple corollary of Theorem 7.3).

Since (8.3) is valid for all λ_1 we have

$$\varphi(x+0) - \varphi(x-0) = \lim_{\lambda_1\to\lambda} \lim_{m\to\infty} \frac{1}{N(\lambda_1) - N(\lambda)} \{ e^{-(\lambda_1-\lambda)S_m^{1/2} D} - 1 \} P_m(D) f(x)$$

(changing formally the order of limits)

$$= \lim_{m\to\infty} \lim_{\lambda_1\to\lambda} \frac{\lambda - \lambda_1}{N(\lambda) - N(\lambda_1)} \left(\frac{1}{\lambda - \lambda_1} (e^{-(\lambda_1-\lambda)S_m^{1/2} D} - 1) \right) P_m(D) f(x)$$

$$= \sqrt{2\pi}\, e^{\lambda^2/2} \lim_{m\to\infty} S_m^{1/2} D P_m(D) f(x),$$

Thus formally we obtain

$$\varphi(x+0)-\varphi(x-0)=\sqrt{2\pi}\, e^{\lambda^2/2}\lim_{m\to\infty} S_m^{1/2}DP_m(D)f(x). \tag{8.4}$$

The following three theorems state generalizations of (8.4) for kernels of classes I, II and III and will be proved together.

THEOREM 8.1. - Suppose

(1) $G(t)\in$ class I and $G(t)$ is non-finite,

(2) $\varphi(t)\in L_1(a, b)$ for all a, b $-\infty<a<b<\infty$,

(3) $b_{m_r}-\lambda S_{m_r}^{1/2}=o(S_{m_r}^{1/2})$ $\quad(r\to\infty)\quad$ (λ is real),

(4) $\lim\limits_{r\to\infty}(\max\limits_{k>m_r} a_k^{-2})S_{m_r}^{-1}=0$,

(5) $f(x)=\int_{-\infty}^{\infty} G(x-t)\varphi(t)dt$ converges,

(6) both $\varphi(x\pm 0)$ exist, or for $n>0$

$$\varphi(t)=\begin{cases}\sum\limits_{l=0}^{n}\dfrac{1}{l!}b_l(t-x)^l+o(t-x)^n & t\to x+\\ \sum\limits_{l=0}^{n}\dfrac{1}{l!}c_l(t-x)^l+o(t-x)^n & t\to x-\end{cases}$$

Then

$$\lim_{r\to\infty}\sqrt{2\pi}\, e^{\lambda^2/2}S_{m_r}^{1/2}\left(D^{n+1}P_{m_r}(D)f(x)-\sum_{l=0}^{n-1}G_{m_r}^{(n-l)}(0)(b_l-c_l)\right)= \tag{8.5}$$
$$=\begin{cases}\varphi(x+0)-\varphi(x-0) & \text{for } n=0\\ b_n-c_n & \text{for } n>0.\end{cases}$$

THEOREM 8.2. - If $G(t)\in$ class II and assumptions (2), (3), (4), (5) and (6) of Theorem 8 1 are satisfied, then conclusion (8.5) is valid.

THEOREM 8.3. - Suppose

(1) $G(t)\in$ class III, $G(t)$ is non-finite,

(2) $\varphi(t)\in L_1(a, b)$ for all a, b satisfying $T<a<b<\infty$,

(3) assumptions (3), (4) and (6), for $x>T$, of Theorem 8.1 are satisfied,

(4) $f(x)=\int_{-\infty}^{\infty} G(x-t)\varphi(t)dt$ converges for $x>T+b+\sum\limits_{k=1}^{\infty}a_k^{-1}$,

then conclusion (8.5) is valid for $x>T$.

REMARK. - We cannot use our estimates for the limit of $G_{m_r}(0)$ in (8.5) as given in Lemma 7.1, since (7.2) implies only

$$\left| S_{m_r}^{\frac{n+1}{2}} G_{m_r}^{(n)}(o) - \frac{1}{\sqrt{2\pi}} \frac{d^n}{dt^n} (e^{-(t-\lambda)^2/2})_{t=0} \right| < \varepsilon(r_0) \quad \text{for} \quad r > r_0,$$

or $\left| G_{m_r}^{(n)}(0) - K \cdot S_{m_r}^{-\frac{n+1}{2}} \right| < \varepsilon(r_0) S_{m_r}^{-\frac{(n+1)}{2}}$ where $K = \frac{1}{\sqrt{2\pi}} \frac{d^n}{dt^n} (e^{-(t-\lambda)^2/2})_{t=0}$.

PROOF OF THEOREMS 8.1, 8.2 AND 8.3. - By Lemmas 6.6, 6.7 and 6.8 and the proof of Theorems 6.1, 6.2 and 6.3 we have

$$\text{(8.6)} \qquad \sqrt{2\pi}\, e^{\lambda^2/2} S_{m_r}^{1/2} D^{n+1} P_{m_r}(D) f(x) = \sqrt{2\pi}\, e^{\lambda^2/2} S_{m_r}^{1/2} \int_{-\infty}^{\infty} G_{m_r}^{(n+1)}(x-t)\varphi(t)dt =$$

$$= \sqrt{2\pi}\, e^{\lambda^2/2} S_{m_r}^{1/2} \left\{ \int_{-\infty}^{x-\delta} + \int_{x-\delta}^{x+\delta} + \int_{x+\delta}^{\infty} \right\} G_{m_r}^{(n+1)}(x-t)\varphi(t)dt \equiv I(m_r,\ 1) + I(m_r,\ 2) + I(m_r,\ 3)$$

where $I(m_r,\ 1) = o(S_{m_r}^l)$ and $I(m_r,\ 3) = o(S_{m_r}^l)$ $r \to \infty$ for any integer l and any $\delta > 0$.

It is sufficient to estimate

$$I_r = \left| I(m_r,\ 2) - \sqrt{2\pi}\, e^{\lambda^2/2} S_{m_r}^{1/2} \sum_{l=0}^{n-1} G_{m_r}^{(n-l)}(0)(b_l - c\,) - b_n + c_n \right|.$$

We choose δ so that

$$\text{(8.7)} \qquad \left| \varphi(t) - \sum_{l=0}^{n} \frac{1}{l\,!} b_l (t-x)^l \right| \leq \varepsilon \left| t - x \right|^n \quad \text{for} \quad x < t \leq x + \delta$$

and

$$\text{(8.8)} \qquad \left| \varphi(t) - \sum_{l=0}^{n} \frac{1}{l\,!} c_l (t-x)^l \right| \leq \varepsilon \left| t - x \right|^n \quad \text{for} \quad x - \delta \leq t < x.$$

By (8.7), (8.8) and the triangle inequality we obtain

$$I_r = \sqrt{2\pi}\, e^{\lambda^2/2} S_{m_r}^{1/2} \sum_{l=0}^{n-1} \left| \int_{x-\delta}^{x} \frac{b_l}{l\,!} (t-x)^l G_{m_r}^{(n+l)}(x-t)dt - G_{m_r}^{(n-1)}(0) b_l \right| +$$

$$+ \sqrt{2\pi}\, e^{\lambda^2/2} S_{m_r}^{1/2} \sum_{l=0}^{n-1} \left| \int_{x}^{x+\delta} \frac{c_l}{l\,!} (t-x)^l G_{m_r}^{(n+1)}(x-t)dt + G_{m_r}^{(n-l)}(0) c_l \right| +$$

$$+\left|\sqrt{2\pi}\, e^{\lambda^2/2} S_{m_r}^{1/2} \int_{x-\delta}^{x} \frac{b_n}{n!}(t-x)^n G_{m_r}^{(n+1)}(x-t)dt - b_n\right| +$$

$$+\left|\sqrt{2\pi}\, e^{\lambda^2/2} S_{m_r}^{1/2} \int_{x}^{x+\delta} \frac{c_n}{n!}(t-x)^n G_{m_r}^{(n+1)}(x-t)dt + c_n\right| +$$

$$+\, \varepsilon \sqrt{2\pi}\, e^{\lambda^2/2} S_{m_r}^{1/2} \int_{x-\delta}^{x+\delta} |G_{m_r}^{(n+1)}(x-t)|\,|x-t|^n dt \equiv I_r(1) + I_r(2) + I_r(3) + I_r(4) +$$

$$+\, I_r(5).$$

$$I_r(5) \leq \varepsilon \sqrt{2\pi}\, e^{\lambda^2/2} S_{m_r}^{1/2} \int_{-\infty}^{\infty} |G_{m_r}^{(n+1)}(t)|\,|t^n|\, dt.$$

By Theorem 3.7 we obtain

$$I_r(5) \leq \varepsilon \sqrt{2\pi}\, e^{\lambda^2/2} M(n, \alpha) \tag{8.9}$$

where the existence of α and the assumptions of Theorem 3.7 are implied by Lemma 2.4. Integrating by parts and using Theorem 4.5 for $t = \pm\delta \mp 0$ we obtain for all $l > 0$

$$I_r(1) = o(S_{m_r}^l), \qquad I_r(2) = o(S_{m_r}^l) \qquad r \to \infty. \tag{8.10}$$

From Lemma 7.1 we derive now

$$\lim_{r\to\infty} \sqrt{2\pi}\, e^{\lambda^2/2} S_{m_r}^{1/2} \int_{x-\delta}^{x} G'_{m_r}(x-t)dt = -\lim_{r\to\infty} \sqrt{2\pi}\, e^{\lambda^2/2} S_{m_r}^{1/2}(G_{m_r}(0) - G_{m_r}(\delta)) = -1 \tag{8.11}$$

and similarly we obtain

$$\lim_{r\to\infty} \sqrt{2\pi}\, e^{\lambda^2/2} S_{m_r}^{1/2} \int_{x}^{x+\delta} G'_{m_r}(x-t)dt = 1. \tag{8.12}$$

Integrating by parts using Theorem 4.5 and limits (8.11) and (8.12) we obtain for any $l > 0$

$$I_r(3) = o(S_{m_r}^l) \quad \text{and} \quad I_r(4) = o(S_{m_r}^l) \qquad r \to \infty. \tag{8.13}$$

Combining (8.9), (8.10) and (8.13) we complete the proof for $n > 0$. The case $n = 0$ is similarly proved using the function $\alpha(u) = \int_0^u [\varphi(x-t) - \varphi(x-0)]dt$ and the method used in the proof of Theorems 7.3, 7.4 and 7.5. Q.E.D.

Application to the Stieltjes transform. - Using substitutions of [7, p. 66-68] and section 7 for STIELTJES transform

$$F(x) = \int_0^\infty \frac{\varphi(t)dt}{x+t}, \tag{8.14}$$

namely taking $E(s) = \frac{\sin \pi s}{\pi s}$ and $a_{2k} = k$ $a_{2k-1} = -k$ we obtain by some calculation in the case where both $\varphi(x_0 \pm 0)$ exist

$$\lim_{r\to\infty} \sqrt{2\pi}\,\frac{(-1)^{m+1}}{m\,!\,m\,!}\sqrt{\frac{2}{m}}\left(x\frac{d}{dx}\right)\left\{x^{2m+1}F^{(m+1)}(x)\right\}^{(m)}_{x=x_0} = \varphi(x_0+0) - \varphi(x_0-0) \tag{8.15}$$

which is a completely new formula. Similar formulae which give $\varphi(x+0) - \varphi(x-0)$ in terms of $F^{(k)}(x_0)$ where x_0 is a fixed point can be achieved following the method of section 7.

The inversion and jump formulae for the convolution transforms achieved in the theorems of section 7 and this one can be generalized for the case convolution STIELTJES transform.

THEOREM 8.4. - Suppose

(1) $G(t) \in$ class I and is non-finite,

(2) $\alpha(t) \in B.V.(a, b)$ for all $-\infty < a < b < \infty$,

(3) $f(x) = \int_{-\infty}^{\infty} G(x-t)e^{ct}d\alpha(t)$ converges,

(4) $\lim_{r\to\infty} (\max_{k>m_r} a_k^{-1}) S_{m_r}^{-1} = 0$,

(5) $b_{m_r} - \lambda S_{m_r}^{1/2} = o(S_{m_r}^{1/2}) \qquad r \to \infty$,

(6) $\alpha(t)$ satisfies for some x and n

$$\alpha(t) = \begin{cases} \sum_{l=0}^{n} \frac{b_l(x)}{l\,!}(t-x)^l + o(t-x)^n & t \to x+ \\ \sum_{l=0}^{n} \frac{c_l(x)}{l\,!}(t-x)^l + o(t-x)^n & t \to x- \end{cases}$$

(for $n = 0$ this assumption is obviously satisfied).

Then for $B_l(x)$ and $c_l(x)$ defined by

$$B_l(x) = \sum_{j=0}^{l}\binom{l}{j}c^j b_{l-j}(x), \quad C_l(x) = \sum_{j=0}^{l}\binom{l}{j}c^j c_{l-j}(x) \quad \text{and} \quad \sum_{l=0}^{-2}\ldots = \sum_{l=0}^{-1}\ldots = 0$$

we have:

A) $\alpha_1 < c < \alpha_2$ and (6) existing for the same n also for x_1 implies

I.
$$\lim_{r\to\infty} \sqrt{2\pi}\, e^{\lambda^2/2} S_{m_r}^{1/2} \Big\{ \int_{x_1}^{x} e^{-cx} D^{n+1} P_{m_r}(D) f(x) + - \sum_{l=0}^{n-1} G_{m_r}^{(n-l)}(0)[B_l(x) - C_l(x) - B_l(x_1) + C_l(x_1)] \Big\} =$$
$$= B_n(x) - C_n(x) - (B_n(x_1) - C_1(x_1)).$$

II.
$$\lim_{r\to\infty} \Big\{ \int_{x_1}^{x} e^{-cx} D^n P_{m_r}(D) f(x) dx - \sum_{l=0}^{n-1} G_{m_r}^{(n-l-1)}(0)[B_l(x) - C_l(x) -$$
$$- B_l(x_1) - C_l(x_1)] \Big\} = N(\lambda)(C_n(x) - C_n(x_1)) + (1 - N(\lambda))(B_n(x) - B_n(x_1)).$$

B) $c \geq \alpha_2$ implies

I.
$$\lim_{r\to\infty} \sqrt{2\pi}\, e^{\lambda^2/2} S_{m_r}^{1/2} \Big\{ - \int_{x}^{\infty} e^{-cx} D^{n+1} P_{m_r}(D) f(x) dx -$$
$$- \sum_{l=0}^{n-1} G_{m_r}^{(n-l)}(0)[B_l(x) - C_l(x)] \Big\} = B_n(x) - C_n(x).$$

II.
$$\lim_{r\to\infty} \Big\{ - \int_{x}^{\infty} e^{-cx} D^n P_{m_r}(D) f(x) dx - \sum_{l=0}^{n-1} G_{m}^{(n-l-1)}(0)[B_l(x) - C_l(x)] \Big\}$$
$$= N(\lambda) C_n(x) + (1 - N(\lambda)) B_n(x) - c^n \alpha(\infty).$$

C) $c \leq \alpha_1$ implies

I.
$$\lim_{r\to\infty} \sqrt{2\pi}\, e^{\lambda^2/2} S_{m_r}^{1/2} \Big\{ \int_{-\infty}^{x} e^{-cx} D^{n+1} P_{m_r}(D) f(x) dx +$$
$$- \sum_{l=0}^{n-1} G_{m_r}^{(n-l)}(0)[B_l(x) - C_l(x)] \Big\} = B_n(x) - C_n(x).$$

II.
$$\lim_{r\to\infty} \Big\{ \int_{-\infty}^{x} e^{-cx} D^n P_{m_r}(D) f(x) dx - \sum_{l=0}^{n-1} G_{m_r}^{(n-l-1)}(0)[B_l(x) - C_l(x)] \Big\} =$$
$$= N(\lambda) C_n(x) + (1 - N(\lambda)) B_n(x) - c^n \alpha(-\infty).$$

D) For every real c

$$\text{I.}\qquad \lim_{r\to\infty} \sqrt{2\pi}\, e^{\lambda^2/2} S_{m_r}^{1/2} \Big\{ e^{-cx} D^n P_{m_r}(D) f(x) - \sum_{l=0}^{n-1} G_{m_r}^{(n-l)}(0)[B_l(x) - C_l(x)] + c \sum_{l=0}^{n-1} G_{m_r}^{(n-l-1)}(0)[B_l(x) - C_l(x)] \Big\} = B_n(x) - C_n(x),$$

and in case $n \geq 1$

$$\text{II.}\qquad \lim_{r\to\infty} \Big\{ e^{-cx} D^{n-1} P_{m_r}(D) f(x) - \sum_{l=0}^{n-1} G_{m_r}^{(n-l-1)}(0)[B_l(x) - C_l(x)] + c\Big(\sum_{r=0}^{n-2} G_{m_r}^{(n-r-2)}(0)[B_r(x) - C_r(x)]\Big) \Big\} = N(\lambda)(C_n(x) - C_{n-1}(x)) + (1 - N(\lambda))(B_n(x) - B_{n-1}(x)).$$

THEOREM 8.5. - Suppose assumptions (2), (3), (4), (5) and (6) of Theorem 8.4 are satisfied and $G(t) \in$ class II, then conclusions A, B and D of Theorem 8.4 are valid.

THEOREM 8.6. - Suppose

(1) $G(t) \in$ class III and is non-finite,

(2) $\alpha(t) \in B(a, b)$ for all $T < a < b < \infty$,

(3) $f(x) = \int_{-\infty}^{\infty} G(x - t) e^{ct} d\alpha(t)$ converges for $x > \gamma_c$,

(4) assumptions (4), (5) and (6) for x greater than T) of Theorem 8.4 are satisfied,

then conclusions, A, B and D of Theorem 8.4 are valid for $x > T$.

PROOF OF THEOREMS 8.4, 8.5 AND 8.6. - Using Lemma 5.4, 5.5, 5.6, the fact that under assumption (6) of Theorem 8.4 we have

$$e^{-c(x-t)}\alpha(t) = \begin{cases} \sum_{l=0}^{n} B_l(x) \dfrac{(t-x)^l}{l\,!} + o(t-x)^n & t \to x+ \\ \sum_{l=0}^{n} C_l(x) \dfrac{(t-x)^l}{l\,!} + o(t-x)^n & t \to x- \end{cases}$$

and some computations similar to those of Theorem 6.9, we derive our theorems from Theorems 7.3, 7.4, 7.5, 8.1, 8.2 and 8.3. Q.E.D.

A special case of Theorem 8.4 which is rather interesting (8.4. D.I. in case $n = 0$, $\lambda = 0$ and $m_r = m$) is:

COROLLARY 8.4.a. - Suppose assumptions (1), (2) and (3) of Theorem 8.4 are satisfied, $\lim_{m\to\infty}(\max_{k>m} a_k^{-2})S_k^{-1} = 0$ and $b_m = o(S_m^{1/2})$ $m \to \infty$, then

$$(8.16) \qquad \lim_{m\to\infty} \sqrt{2\pi}\, S_m^{1/2} e^{-cx} P_m(D) f(x) = \alpha(x+) - \alpha(x-).$$

Interpreting (8.16) for the STIELTJES transform gives its known jump formula [10, p. 351] and for the second iterate of STIELTJES transform it gives the known jump formula (see [1]). Of course, (8.16) yields a simple jump formula for the convolution STIELTJES transform, since similar corollaries can be stated for $G(t) \in$ class II or class III. For the LAPLACE transform (8.16), when $G(t) \in$ class II, yields its known jump formula [10, p. 298].

9. - Inversion and jump theorems for the case $\lim_{r\to\infty}(\max_{k>m_r} a_k^{-2})S_{m_r}^{-1} = 1$.

Whenever the assumption $S_m - \max_{k>m} a_k^{-2} = S_m^{(1)} \geq \alpha S_m$ for some α independent of m $0 < \alpha < 1$, $m = 1, 2, \ldots$, fails, there exists a subsequence of $\{m\}$, $\{m_r\}$ satisfying

$$\lim_{r\to\infty}(\max_{k>m_r} a_k^{-2})S_{m_r}^{-1} = 1 \quad \text{or} \quad S_{m_r}^{-1} = o(\max_{k>m} a_k^{-1}) \qquad r \to \infty.$$

Define $(a_{m_r}^*)^{-1} = \max_{k>m} a_k^{-1}$ (this is unique choice for $r \geq r_0$). We can choose a subsequence m_r such that $\operatorname{sgn} a_{m_r}^* =$ constant.

The following two theorems are needed for the proof of the inversion and jump formulae.

THEOREM 9.1. - If for an infinite subsequence $\{m_r\}$ of integers $S_{m_r}^{(1)} = o(\max_{k>m_r} a_k^{-2})$ $r \to \infty$, where all $a_{m_r}^*$ are of equal sign ($a_{m_r}^*$ are those a_k satisfying $(a_{m_r}^*)^{-2} = \max_{k>m_r} a_k^{-2}$), and for some real λ $b_{m_r} - \lambda S_{m_r}^{1/2} = o(S_{m_r}^{1/2})$ $r \to \infty$, then

$$(9.1) \qquad \lim_{r\to\infty} \int_{-\infty}^{0} G_{m_r}(t) = \begin{cases} e^{-\lambda-1} & \lambda > -1 \\ 1 & \lambda \leq -1 \end{cases} \text{ if } a_{m_r}^* > 0, \qquad \begin{cases} 0 & \lambda \geq 1 \\ 1 - e^{\lambda-1} & \lambda < 1 \end{cases} \text{ if } a_{m_r}^* < 0$$

PROOF. - Suppose first $a_{m_r}^* > 0$. We define $G_{m_r+1}^*(t)$ by

$$(9.2) \qquad G_{m_r}(t) = \int_{-\infty}^{\infty} g_{m_r}^*(u) G_{m_r}^*(t-u)\, du$$

where

$$g^*_{m_r}(u) = \begin{cases} e^{-1} a^*_{m_r} \exp\left(a^*_{m_r}(u - b_{m_r})\right) & u < (a^*_{m_r})^{-1} + b_{m_r} \\ 0 & u > (a^*_{m_r})^{-1} + b_{m_r}. \end{cases}$$

We have (see [7, pp. 55-56])

$$\int_{-\infty}^{\infty} t G^*_{m_r+1}(t)dt = 0$$

and

$$\int_{-\delta}^{\delta} G^*_{m_r+1}(t)dt \geq 1 - \frac{1}{\delta^2} S^{(1)}_{m_r}. \tag{9.3}$$

Denote for $n,\ r \geq 1$ $\delta \equiv \delta_{n,r} \equiv (n a^*_{m_r})^{-1}$. Given $\varepsilon > 0$ and n, then for $r \geq \geq r_0(n,\ \varepsilon)$

$$1 \geq \int_{-\delta}^{\delta} G^*_{m_r+1}(t)dt \geq 1 - \frac{1}{\delta^2} S^{(1)}_{m_r} \geq 1 - \varepsilon.$$

For $t < t_0 \equiv (a^*_{m_r})^{-1} + b_{m_r} - \delta = \left(1 - \frac{1}{n}\right)(a^*_{m_r})^{-1} + b_{m_r}$

$$G_{m_r}(t) = \int_{-\infty}^{\infty} g^*_{m_r}(u) G^*_{m_r+1}(t-u)du \geq \tag{9.4}$$

$$\geq \int_{t-\delta}^{t+\delta} G^*_{m_r+1}(t-u) g^*_{m_r}(u)du \geq \left(\min_{|t-u|<\delta} g^*_{m_r}(u)\right) \int_{t-\delta}^{t+\delta} G^*_{m_r+1}(t-u)du \geq$$

$$\geq e^{-1}(a^*_{m_r})^{-1} \ \exp\ \left(a^*_{m_r}(-b_{m_r} + t - \delta)\right)(1 - \varepsilon).$$

We shall estimate $\int_0^\infty G_{m_r}(t)dt$ as $r \to \infty$. Suppose $\lambda > -1$. For $n > (\lambda + 1)^{-1}$ and $r > r_1(n,\ \lambda,\ \varepsilon) > r_0$ $t_0 > 0$ and therefore

$$\int_0^\infty G_{m_r}(t)dt \geq e^{-1} a^*_{m_r}(1 - \varepsilon) \int_0^{t_0} \exp\left(a^*_{m_r}(t - b_{m_r} - \delta)\right)dt =$$

$$= e^{-1}(a^*_{m_r})(1 - \varepsilon) \exp\left(-a^*_{m_r}\delta\right) \int_0^{t_0} \exp\left(a^*_{m_r}(t - b_{m_r})\right)dt =$$

$$= e^{-\left(1+\frac{1}{n}\right)}(1-\varepsilon)\int_{-\lambda+o(1)}^{1-\frac{1}{n}} e^{z}dz = (1-\varepsilon)e^{-\frac{1}{n}}(e^{-\frac{1}{n}} - e^{-\lambda-1+o(1)}).$$

Since n and ε are arbitrary

$$\liminf_{r\to\infty}\int_0^\infty G_{m_r}(t)dt \geq 1 - e^{-(\lambda+1)}.$$

By the same method

$$\liminf_{r\to\infty}\int_{-\infty}^0 G_{m_r}(t)dt \geq e^{-(\lambda+1)}.$$

Since $\int_{-\infty}^{\infty} G_{m_r}(t)dt = 1$ we conclude the proof in the case $\lambda > -1$. Suppose $\lambda \leq -1$. We have the trivial estimation $\int_0^\infty G_{m_r}(t)dt \geq 0$. Also by the above argument (using (9.4)) we obtain $\lim_{r\to\infty}\int_{-\infty}^0 G_{m_r}(t)dt \geq 1$ and this concludes the proof in the case $a^*_{m_r} > 0$.

The proof of (9.1) in the case $a^*_{m_r} < 0$ is similar. Q.E.D.

THEOREM 9.2. - **If the assumptions of Theorem 9.1 are satisfied for $\lambda = 0$, then**

$$\lim_{r\to\infty} S^{1/2}_{m_r} G_{m_r}(\zeta_{m_r}) = 1 \tag{9.5}$$

where ζ_{m_r} is the only maximum point of $G_{m_r}(t)$ and for a fixed real ν

$$\lim_{r\to\infty} S^{1/2}_{m_r} G_{m_r}(\nu S^{1/2}_{m_r} + o(S^{1/2}_{m_r})) = \begin{cases} e^{\nu-1} & \text{if } \nu < 1 \text{ and } a^*_{m_r} > 0 \\ e^{-\nu-1} & \text{if } \nu > -1 \text{ and } a^*_{m_r} < 0. \end{cases} \tag{9.6}$$

PROOF. - From the fact that $S_{m_r} \sim (a^*_{m_r})^{-2}$ $r \to \infty$ and the proof of case B of Lemma 7.1 of [7, pp. 138-139], but taking $1+\varepsilon$ in place of 2, we get

$$S^{1/2}_{m_r} G_{m_r}(\zeta_{m_r}) \leq (1+\varepsilon)^{1/2}$$

and since ε is arbitrary

$$\limsup_{r\to\infty} S^{1/2}_{m_r} G_{m_r}(\zeta_{m_r}) \leq 1. \tag{9.7}$$

Since ζ_{m_r} are the maxima points of $G_{m_r}(t)$, we get from (9.4)

$$S_{m_r}^{1/2} G_{m_r}(\zeta_{m_r}) \geq S_{m_r}^{1/2} \, | a_{m_r}^* | \, e^{-\frac{2}{n}}(1-\varepsilon) = e^{-\frac{2}{n}}(1-\varepsilon)(1+o(1)) \qquad r \to \infty.$$

Letting n tend to infinity and ε to zero, we obtain

$$\liminf_{r\to\infty} S_{m_r}^{1/2} G_{m_r}(\zeta_{m_r}) \geq 1. \tag{9.8}$$

Combining (9.7) with (9.8) we obtain (9.5).
For $\nu < 1$ we have, using (9.4),

$$S_{m_r}^{1/2} G_{m_r}(S_{m_r}^{1/2}(\nu + o(1))) \geq S_{m_r}^{1/2} a_{m_r}^* e^{-1} \exp\left(a_{m_r}^*((\nu + o(1))S_{m_r}^{1/2} - \frac{1}{n}(a_{m_r}^*)^{-1})\right)(1-\varepsilon)$$
$$\geq e^{\nu - 1 - \frac{1}{n}}(1-\varepsilon)(1+o(1)) \qquad r \to \infty.$$

Hence for $\nu < 1$ and $a_{m_r}^* > 0$ we obtain

$$\liminf_{r\to\infty} S_{m_r}^{1/2} G_{m_r}(\nu S_{m_r}^{1/2} + o(S_{m_r}^{1/2})) \geq e^{\nu - 1}. \tag{9.9}$$

By (9.2) we have for $t < b_{m_r} + \left(1 - \frac{2}{n}\right)(a_{m_r}^*)^{-1}$

$$S_{m_r}^{1/2} G_{m_r}(t) = S_{m_r}^{1/2} \left\{ \int_{|t-u|>\delta} + \int_{|t-u|\leq\delta} \right\} g_{m_r}^*(u) G_{m_r+1}^*(t-u) du.$$

For $t = \nu S_{m_r}^{1/2} + o(S_{m_r}^{1/2})$, $r \to \infty$ we obtain

$$S_{m_r}^{1/2} G_{m_r}((\nu + o(1))S_{m_r}^{1/2}) \leq \varepsilon S_{m_r}^{1/2} \max_u g_{m_r}^*(u) + (1-\varepsilon) S_{m_r}^{1/2} \max_{|t-u|\leq\delta} g_{m_r}^*(u) \leq$$
$$\leq \varepsilon S_{m_r}^{1/2} a_{m_r}^* + (1-\varepsilon) S_{m_r}^{1/2} e^{-1} a_{m_r}^* \exp\left(a_{m_r}^*(\nu S_{m_r}^{1/2} + \delta - b_{m_r} + o(S_{m_r}^{1/2}))\right) \leq$$
$$\leq \varepsilon(1 + o(1)) + (1-\varepsilon) e^{\nu - 1 + \frac{1}{n}}(1 + o(1)) \qquad r \to \infty.$$

Hence

$$\limsup_{r\to\infty} S_{m_r}^{1/2} G_{m_r}(\nu S_{m_r}^{1/2} + o(S_{m_r}^{1/2})) \leq e^{\nu - 1}. \tag{9.10}$$

Combining (9.9) with (9.10) we derive (9.6) for $\nu < 1$ and $a_{m_r}^* > 0$. The case $\nu > -1$ and $a_{m_r}^* < 0$ is similar. Q.E.D.

The inversion formula will be described in the following theorem.

THEOREM 9.3. - Suppose

(1) $G(t) \in$ class I, $G(t)$ is non-finite,

(2) $\varphi(t)$ is integrable on every finite interval,

(3) $f(x) = \int\limits_{-\infty}^{\infty} G(x-t)\varphi(t)dt$ converges,

(4) the infinite subsequence of integers $\{m_r\}$ satisfies for some real λ, $b_{m_r} - \lambda S_{m_r}^{1/2} = o(S_{m_r}^{1/2})$, $S_{m_r}^{(1)} = o(\max\limits_{k>m_r} a_k^{-2})$ $r \to \infty$ and $a_{m_r}^*$ is of constant sign (where $(a_{m_r}^*)^{-2} = \max\limits_{k-m_r} a^{-2})$;

(5) for a real x and $\delta > 0$ $\varphi^{(n)}(t)$ exists for some $n \geq 0$ a.e. in $(x-\delta, x+\delta)$ and in case $n \geq 1$ we make the additional assumption that $\varphi^{(n-1)}(t)$ is absolutely continuous in $(x-\delta, x+\delta)$;

(6) $\{a_k\}$ satisfy $B(n-2, \{m_r\})$ for the same n of assumption (5) (the void assumption in case $n = 0$, 1 or 2);

(7) both $\varphi^{(n)}(x \pm 0)$ exist.

Then

$$\lim_{r\to\infty} D^n P_{m_r}(D) f(x) = \begin{cases} (1-e^{-\lambda-1})\varphi^{(n)}(x-0) + e^{-\lambda-1}\varphi^{(n)}(x+0) & \text{if } \lambda > -1 \\ \varphi^{(n)}(x+0) & \text{if } \lambda \leq -1 \end{cases} \; a_{m_r}^* > 0$$
$$\qquad \begin{cases} (1-e^{\lambda-1})\varphi^{(n)}(x+0) + e^{\lambda-1}\varphi^{(n)}(x-0) & \text{if } \lambda < 1 \\ \varphi^{(n)}(x-0) & \text{if } \lambda \geq 1 \end{cases} \; a_{m_r}^* < 0 \tag{9.11}$$

PROOF. - By Lemma 5.1 we have

$$D^n P_{m_r}(D) f(x) = \left\{ \int\limits_{-\infty}^{x-\delta} + \int\limits_{x-\delta}^{x-\delta} + \int\limits_{x+\delta}^{\infty} \right\} G_{m_r}^{(n)}(x-t)\varphi(t)dt \equiv I(r, 1) + I(r, 2) + I(r, 3).$$

By Lemmas 6.7 and 6.8 we have for every $\delta > 0$ and $l > 0$

$$I(r, 1) = o(S_{m_r}^l) \quad \text{and} \quad I(r, 3) = o(S_{m_r}^l) \qquad r \to \infty.$$

Integrating by parts n times and using Theorem 4.6 we obtain

$$I(r, 2) = \int\limits_{x-\delta}^{x+\delta} G_{m_r}(x-t)\varphi^{(n)}(t)dt + o(S_{m_r}^l) = \left\{ \int\limits_{x-\delta}^{x} + \int\limits_{x}^{x+\delta} \right\} G_{m_r}(x-t)\varphi^{(n)}(t)dt +$$
$$+ o(S_{m_r}^l) \equiv I(r, 2, 1) + I(r, 2, 2) + o(S_{m_r}^l) \qquad r \to \infty.$$

Suppose $a^*_{m_r} > 0$ and $\lambda > -1$. Using the estimation of $I(r, 1)$ for $\varphi(t) \equiv 1$ and Theorem 9.1 we obtain

$$|I(r, 2, 1) - (1 - e^{-\lambda - 1})\varphi^{(n)}(x - 0)| = \left| \int_0^\delta G_{m_r}(z)[\varphi^{(n)}(x - z) - \varphi^{(n)}(x - 0)]dz \right| + o(1) \qquad r \to \infty.$$

Define

$$\Psi(t) = \int_0^t [\varphi^{(n)}(x - z) - \varphi^{(n)}(x - 0)]dz$$

$$|I(r, 2, 1) - (1 - e^{-\lambda - 1})\varphi^{(n)}(x - 0)| = \left| \int_0^\delta G'_{m_r}(z)\Psi(z)dz \right| + o(1)$$

(since $\Psi(t) = o(t)$ $t \to 0+$ and using Theorem 3.8)

$$= o\left(\int_0^\delta | G'_{m_r}(z)z | \, dz \right) + o(1) = o(1) \qquad r \to \infty.$$

Similarly we can derive

$$\lim_{r \to \infty} I(r, 2, 2) = e^{-\lambda - 1}\varphi(x + 0).$$

Therefore

$$\lim_{r \to \infty} D^n P_{m_r}(D)f(x) = (1 - e^{-\lambda - 1})\varphi^{(n)}(x - 0) + e^{-\lambda - 1}\varphi^{(n)}(x + 0).$$

The proof of the other three cases is similar. Q.E.D.

Analogous results for classes II and III, with the obvious necessary changes (like dropping assumption (6) in case of $G(t) \in$ class III) and (9.11) being of interest only $a^*_{m_r} > 0$ (since $G(t) \in$ class II or III imply $a_m > 0$), can be achieved (see also Theorems 6.1, 6.2 and 6.3), but we shall not write them here.

The jump formula for the convolution transform satisfying $S^{(1)}_{m_r} = o(\max_{k > m_r} a_k^{-2})$ $r \to \infty$ is stated in the following theorem.

THEOREM 9.4. - Suppose

(1) assumptions (1), (2), (3), (4) and (5) of Theorem 9.3 are satisfied,

(2) $\{a_k\}$ satisfy $B(n - 1, \{m_r\})$ (the void assumption in the case n $= 0$ or $n = 1$),

(3) both $\varphi^{(n)}(x\pm)$ exist (x defined in (5) of Theorem 9 3), then for $\lambda > -1$ and $a^*_{m_r} > 0$ or $\lambda < 1$ and $a^*_{m_r} < 0$

$$\lim_{r\to\infty} \exp(1+\lambda \operatorname{sgn} a^*_{m_r}) S^{1/2}_{m_r} D^{n+1} P_{m_r}(D) f(x) = \varphi^{(n)}(x+) - \varphi^{(n)}(x-). \tag{9.12}$$

PROOF. - By the method used in proving Theorem 9.3 we have for $a^*_{m_r} > 0$ and $\lambda > -1$

$$e^{\lambda+1} S^{1/2}_{m_r} D^{n+1} P_{m_r}(D) f(x) = e^{\lambda+1} S^{1/2}_{m_r} \left\{ \int_{-\infty}^{x-\delta} + \int_{x-\delta}^{x+\delta} + \int_{x+\delta}^{\infty} \right\} G^{(n+1)}_{m_r}(x-t)\varphi(t)dt \equiv$$

$$\equiv I(r,\ 1) + I(r,\ 2) + I(r,\ 3).$$

As in Theorem 9.3 we can show

$$I(r,\ 1) = o(S^l_{m_r}) \quad \text{and} \quad I(r,\ 3) = o(S^l_{m_r}) \qquad r\to\infty$$

$$I(r,\ 2) = \left\{ \int_{x-\delta}^{x} + \int_{x}^{x+\delta} \right\} G'_{m_r}(x-t)\varphi^{(n)}(t)dt + o(S^l_{m_r}) =$$

$$= I(r,\ 2,\ 1) + I(r,\ 2,\ 2) + o(S^l_{m_r}) \qquad r\to\infty.$$

In order to calculate $I(r,\ 2,\ 1)$ we derive, using Theorem 9.2,

$$S^{1/2}_{m_r} \int_0^{\delta} G'_{m_r}(z)dz = S^{1/2}_{m_r} G_{m_r}(\delta) - S^{1/2}_{m_r} G_{m_r}(0) = -e^{-\lambda-1} + o(1) \qquad r\to\infty. \tag{9.13}$$

Given $\varepsilon < 0$ there is a $\delta > 0$ such that $|\varphi^{(n)}(x-z) - \varphi^{(n)}(x-)| > \varepsilon$ for $0 < z \leq \delta$ and by (9.13) we have

$$|I(r,\ 2,\ 1) + \varphi^{(n)}(x-)| \leq e^{\lambda+1} S^{1/2}_{m_r} \int_0^{\delta} |G'_{m_r}(z)|\ |\varphi^{(n)}(x-z) - \varphi^{(n)}(x-)|\, dz + o(1)$$

$$\leq \varepsilon e^{\lambda+1} S^{1/2}_{m_r} (2G_{m_r}(\zeta_{m_r}) + G_{m_r}(0) + G_{m_r}(\delta)) + o(1) \leq$$

$$\leq \varepsilon 4 e^{\lambda+1}(1 + o(1) + o(1) \qquad r\to\infty.$$

This completes the proof in the case $\lambda > -1$ and $a^*_{m_r} > 0$. The proof of our theorem in the case where $\lambda < 1$ and $a^*_{m_r} < 0$ is similar. Q.E.D.

Similar results for $G(t)$ belonging to classes II and III can be achieved, (see for partial results the first author's thesis [3], Chapter 7) but we shall not introduce them here.

Results analogous to Theorems 6 9, 8.5, 8.6, and 8.7 for inversion and jump formulae of the convolution STIELTJES transform where a subsequence $\{m_r\}$ which satisfies $S^{(1)}_{m_r} = o(\max_{k>m_r} a_k^{-2})$ $r \to \infty$ exists, can also be achieved. (For partial results see [3], Champter 7.) We shall intrnduce one of the above mentioned results, others will be omitted being similar.

THEOREM 9.5. - Suppose

(1) $G(t) \in$ class II,

(2) $\alpha(t)$, is of bounded variation in any finite interval,

(3) $f(x) = \int_{-\infty}^{\infty} G(x-t)e^{ct}d\alpha(t)$ converge for $x > \gamma_c$,

(4) assumptions (4) of Theorem 9.3 and (2) of Theorem 9.4 are satisfied,

(5) for a real x and $\delta > 0$, $\alpha^{(n)}(t)$ exists for some $n \geq 0$ a.e. in $(x-\delta, x+\delta)$ and in case $n \geq 1$ we make the additional assumption that $\alpha^{(n-1)}(t)$ is absolutely continuous;

(6) both $\alpha^{(n)}(x-)$ and $\alpha^{(n)}(x+)$ exist.

Then for $\lambda > -1$

$$\lim_{r\to\infty} e^{\lambda+1} S^{1/2}_{m_r} e^{-cx} D^n P_{m_r}(D) f(x) = \alpha^{(n)}(x+) - \alpha^{(n)}(x-). \tag{9.14}$$

REMARK. - In case $n = 0$ only assumptions (1), (2), (3) and (4) are needed.

PROOF. - Similarly to the derivation of Theorem 6.9 from Theorem 6.1 we can obtain our theorem from the analogous Theorem for $G(t) \in$ class II to Theorem 9.4. Q.E.D.

10. - Some remarks on inversion and jump formula for the case.

$$0 < \liminf_{m\to\infty} (\max_{k>m} a_k^{-2}) S_m^{-1} \leq \limsup_{m\to\infty} (\max_{k>m} a_k^{-2}) S_m^{-1} < 1$$

Inversion and jump formulae were found for the general case in section 6 and in the two extreme cases $\liminf_{m\to\infty} (\max_{k>m} a_k^{-2}) S_m^{-1} = 0$ (in sections 7 and 8) and $\limsup_{m\to 0} (\max_{k>m} a_k^{-2}) S_m^{-1} = 1$ (in section 9). The results that were achieved suggest that we can assume less on $\varphi(t)$ and its derivatives in a neighborhood of x if we hnow more about the sequence $\{a_k\}$.

In the case $\limsup_{m\to\infty} (\max_{k>m} a_k^{-2}) S_m^{-1} < 1$ we have $S_m - \max_{k>m} a_k^{-2} \geq \alpha S_m$ $\alpha > 0$ independent of m (see section 2) and therefore $S^{(i)} \geq \alpha^i S_m$ (see Lemma 2.4). This allows us to use all the theorems of section 3, and therefore by methods similar to those of sections 7 and 8 we obtain an inversion formula for the case $b_{m_r} = o(S_{m_r}^{1/2})$ $r \to \infty$ and

$$\lim_{r\to\infty} \int_{-\infty}^{0} G_{m_r}(t)dt = A; \tag{10.1}$$

and a jump formula for the case $b_{m_r} = o(S_{m_r}^{1/2})$ $r \to \infty$ and

$$\lim_{r\to\infty} S_{m_r}^{1/2} G_{m_r}(0) = B > 0. \tag{10.2}$$

A subsequence $\{m_r\}$, for which the limits (10.1) and (10.2) exist, can always be found since $0 \leq \int_{-\infty}^{0} G_m(t)dt \leq 1$, $0 \leq S_m^{1/2} G_m \leq \sqrt{2}$ (see [7, p. 138], and we can choose b_{m_r} so that $\frac{3}{16} \leq S_{m_r}^{1/2} G_{m_r}(0) \leq \sqrt{2}$ (see [7, p. 126]) (for $B > 0$).

The inversion results will be those of § 7 taking A instead of $N(\lambda)$ and the jump formulae will be those of section 8 taking B^{-1} instead of $\sqrt{2\pi} e^{\lambda^2}$.

EXAMPLE: Choose $a_k = q^{k-1} > 1$ and $b_m = 0$.

$$\int_{-\infty}^{0} G_m(t)dt = \int_{-\infty}^{0} G(q^m t) q^m dt = \int_{-\infty}^{0} G(t)dt = A$$

$$S_m^{1/2} G_m(0) = \frac{1}{\sqrt{1-q^{-2}}} q^{-m} \cdot G(0) q^m = \frac{1}{\sqrt{1-q^{-2}}} (G(0) = B.$$

We also want to mention that even in the case $\limsup_{m\to\infty} (\max_{k>m} a_k^{-2}) S_m^{-1} = 1$ we can have inversion and jump formulae if (10.1) and (10.2) are satisfied for a sequence that does not satisfy $S_{m_r}^{(1)} = o(\max_{k>m_r} a_k^{-2})$ $r \to \infty$; in this case we shall need restrictions stronger than those used in section 7 on the determining function $\varphi(t)$ in the neighborhood of x.

Assuming $\limsup_{m\to\infty} (\max_{k>m} a_k^{-2}) S_m^{-1} < 1$ we have $S_m^{(1)} \geq \alpha S_m$ $0 < \alpha$ and therefore (Lemma 2.4) $S_m^{(i)} \geq \alpha^i S_m$. In Theorem 6.1(*b*) we can take $\limsup_{m\to\infty} (\max_{k>m} a_k^{-2}) S_m^{-1} < 1$ instead of assumption (5) and

$$\varphi(t) = \sum_{l=0}^{n} \frac{b_l}{l!}(t-x)^l + o(t-x)^n \qquad t \to x$$

instead of assumption (4); dropping the assumption on the existence of $\varphi^{(n)}(x \pm 0)$ we obtain

$$\lim_{n\leftarrow\infty} D^n P_m(D) f(x) = b_n .$$

Analogous results can be achieved for kernels of classes II and III.

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