INVERTING CIRCULANT MATRICES USING RECURRENCE EQUATIONS*

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ABSTRACT

The elements of the inverse of a circulant matrix having only 3 non-zero elements in each row (located in cyclically adjacent columns) are derived analytically from the solution of a recurrence equation. Expressing any circulant as a product containing these 3-element type circulants then provides an algorithm for inverting circulants in general. Extension to generalized inverses of circulants whose row sum is zero is also made.

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1. INTRODUCTION

A circulant is a square matrix of the form

$$C = \begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \cdots & c_{n-2} \\ c_{n-2} & c_{n-1} & c_0 & \cdots & c_{n-3} \\ \vdots & & & & \\ c_1 & c_2 & c_3 & \cdots & c_0 \end{bmatrix} . \tag{1}$$

It is determined by its first row and will be denoted by

$$C = (c_0, c_1, c_2, \cdots, c_{n-1}) . \tag{2}$$

Letting $P = (0, 1, 0, 0, \cdots, 0)$, then C can be expressed in terms of powers of P as

$$C = \sum_{i=0}^{n-1} c_i P^i . \tag{3}$$

We make considerable use of the following properties of circulants:

(i) $P^j = (0, 0, \cdots, 1, \cdots, 0)$ is a circulant with a one in the $j + 1$ position of the first row.

(ii) $P^n = P^j(P^{n-j}) = I = P^0$, and so $P^{n-j} = P^{-j} = (P^j)^{-1}$.

(iii) Multiplication of a circulant by P^j shifts all elements in each row j columns to the right in a cyclic manner, and multiplication by P^{-j} shifts them j columns to the left in a similar manner.

(iv) The inverse of a circulant is a circulant

$C^{-1} = A = (a_0, a_1, a_2, \cdots, a_{n-1})$.

(v) For $\underline{1}$ denoting a vector of ones, $C\underline{1} = s\underline{1}$ where s is the row sum of each row. When C^{-1} exists, its row sum is $1/s$ since $\underline{1} = C^{-1}C\underline{1} = sC^{-1}\underline{1}$.

(vi) The characteristic roots of C are

$$\lambda_i = \sum_{j=0}^{n-1} c_j r_i^j , \quad i=1,2,\cdots,n$$

where the r_i are the n roots of $r^n = 1$. The corresponding characteristic vector is $\underline{v}_i = (1\ r_i\ r_i^2 \cdots r_i^{n-1})'$.

(vii) Circulants commute in multiplication.

Circulant matrices and functions of circulant matrices arise in a variety of applications. Their uses in solid state physics are discussed in some detail by Löwdin, Pauncz, and de Heer [8] and by Gilbert [5]. The first of these considers 3 methods of obtaining the inverse of certain special circulant (overlap) matrices using Chebyshev polynomials of the first and second kind and giving some asymptotic results. The second presents a method of transforming to diagonal form, inverting, and then transforming back to original form, using the characteristic roots of a circulant C and the matrix which diagonalizes it, which are known. Abraham and Weiss [1] and Calaib and Appel [3] also utilize this fact, although their expressions for inverting overlap matrices are rather complex, involving Chebyshev polynomials and integrals that have to be evaluated by numerical integration or infinite sums of complex numbers.

In a more recent paper, Cline, Plemmons, and Worm [4] give an expression for the Moore-Penrose inverse of a circulant, using finite sums of terms involving the characteristic roots of the matrix and powers of a primitive root of unity. A related expression was obtained by Good [6] in a statistical application.

The purpose of this paper is to obtain a simple and readily computable inverse for a circulant with real elements. In section 2 we discuss a method of recurrence equations for obtaining inverses of certain patterned matrices. This method is applied in section 3 to the circulant $C = (a,b,0,0,\cdots,c)$ and simple closed form expressions for elements of C^{-1} are obtained. In section 4 we extend the procedure to general circulants. Finally, in section 5 we consider the inverse of $C + \lambda J = (c_0 + \lambda,\ c_1 + \lambda,\ c_2 + \lambda,\ \cdots,\ c_{n-1} + \lambda)$, and obtain the Moore-Penrose inverse for singular circulants C with $\sum_{i=0}^{n-1} c_i = 0$.

2. A GENERAL METHOD OF INVERSION

A method for inverting certain patterned matrices from recurrence equations is given in Kounias [7]. Suppose C is a patterned matrix of order n whose inverse is sought. Consider the equations

$$C\begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} = \begin{bmatrix} w \\ w^2 \\ \vdots \\ w^n \end{bmatrix} \tag{4}$$

written in an obvious way as $C\underline{z} = \underline{w}$. Their solution for $\underline{z}$ is $\underline{z} = C^{-1}\underline{w}$ and on writing

$$C^{-1} \equiv A \equiv \{a_{ij}\} \qquad \text{for } i,j = 1,2,\cdots,n \tag{5}$$

the i'th element of $\underline{z} = C^{-1}\underline{w}$ is $z_i = \sum_{j=1}^{n} a_{ij}w^j$ so that

$$a_{ij} = \text{coefficient of } w^j \text{ in } z_i \quad . \tag{6}$$

It is the pattern in C that gives rise to (5) being recurrent equations for the z_i. Solving them by traditional methods for solving such equations gives an expression for z_i to which (6) can be applied.

3. 3-ELEMENT CIRCULANTS

3.0 Summary

Consider first the particular circulant with only three non-zero elements in each row

$$C = (a, b, 0, \cdots, 0, c) \quad . \tag{7}$$

We find the inverse of this circulant for general n. It is apparent that any circulant with 3 consecutive non-zero elements in the first row and the remaining zero can be brought to the form of (7) by multiplication by P^j for appropriate j. Its inverse is then $C^{-1}P^{-j}$, representing just a cyclic shift of the elements of the rows of C^{-1}. Through inverting (7) we therefore obtain the inverse of any circulant that has only 3 non-zero elements in its first row, located in cyclically consecutive positions. Such a matrix we will call a 3-element circulant.

We denote the inverse of C by A, which is of course a circulant:

$$C^{-1} = A = (a_0 \; a_1 \; \cdots \; a_{n-1}) \quad .$$

For convenience, the expressions obtained in the remainder of this section for a_j, $j = 0, 1, \cdots, n-1$, the elements of A, are summarized in the following theorem. Effort has been made to express them in their most computable form.

Theorem 1 For y_1 and y_2 being roots of the quadratic equation $by^2 + ay + c = 0$, namely

$$y_1, y_2 = \left[- a \pm (a^2 - 4bc)^{\frac{1}{2}}\right] \Big/ 2b \ ,$$

elements of the inverse $C^{-1} = A = (a_0\ a_1\ \cdots\ a_{n-1})$ of the 3-element circulant $C = (a\ b\ 0\ \cdots\ 0\ c)$ of order n are as follows:

(a) C is singular when $a + b + c = 0$ or when n is even and $a = b + c$. These cases are therefore excluded in (b)—(e).

(b) When $a^2 - 4bc > 0$ and $b \neq 0$

$$a_j = \frac{1}{b(y_1 - y_2)} \left[\frac{y_1^{n-j}}{1 - y_1^n} - \frac{y_2^{n-j}}{1 - y_2^n} \right], \quad j=0,1,\cdots,n-1 \ .$$

(c) When $a^2 - 4bc < 0$

$$a_j = \frac{r^{n-j-1}\left[\sin(n - j)\theta + r^n \sin(j\theta)\right]}{b \sin\theta\left[1 - 2r^n \cos(n\theta) + r^{2n}\right]}, \quad j=0,1,\cdots,n-1 \ ,$$

where

$$r = (c/b)^{\frac{1}{2}} \quad \text{and} \quad \theta = \sin^{-1}(1 - a^2/4bc)^{\frac{1}{2}} \ .$$

(d) When $a^2 - 4bc = 0$, $(b \neq 0)$, then $y_1 = y_2 = y = - a/2b$ and

$$a_j = \frac{y^{n-1-j}}{b(1 - y^n)} \left[\frac{n}{1 - y^n} - j \right], \quad j=0,1,\cdots,n-1 \ .$$

(e) When $b = 0$ let $y = c/a$, $a \neq 0$, then

$$a_0 = \frac{1}{a(1 - y^n)}$$

and

$$a_j = a_0 y^{n-j}, \quad j=1,2,\cdots,n-1 \ .$$

3.1 Inversion by Recurrence Equations

The method of section 2 applied to C of (7) gives equations $C\underline{z} = \underline{w}$ of (4) whose typical form is

$$cz_{i-1} + az_i + bz_{i+1} = w^i \quad . \tag{8}$$

Kounias [7], for a matrix like (7) but lacking the off-diagonal corner elements, gives the general solution of (8) as

$$z_i = \frac{w^{i+1}}{bw^2 + aw + c} + py_1^i + qy_2^i \tag{9}$$

where y_1 and y_2 are roots of

$$by^2 + ay + c = 0 \tag{10}$$

namely

$$y_1, y_2 = \frac{-a \pm \sqrt{a^2 - 4bc}}{2b} \quad . \tag{11}$$

In (9), the p and q are derived from boundary conditions on the z_i determined by the form of C in $C\underline{z} = \underline{w}$, these being, for (7)

$$z_0 = z_n \quad \text{and} \quad z_1 = z_{n+1} \quad . \tag{12}$$

This yields

$$z_i = \frac{w^{i+1}}{bw^2 + aw + c} + \frac{w(1 - w^n)}{(bw^2 + aw + c)(y_1 - y_2)} \left[\frac{(y_2 - w)y_1^i}{1 - y_1^n} - \frac{(y_1 - w)y_2^i}{1 - y_2^n} \right]$$

which, after writing $bw^2 + aw + c = b(w - y_1)(w - y_2)$, expressing its reciprocal in partial fractions and expanding those as infinite sums leads to

$$z_i = \frac{1}{b(y_1 - y_2)^2}\Big[(y_1 - y_2)\sum_{k=1}^{\infty}(y_1^k - y_2^k)w^{i-k}$$

$$+ \Big(\frac{y_2 y_1^i}{1 - y_1^n} - \frac{y_1 y_2^i}{1 - y_2^n}\Big)\sum_{k=1}^{\infty}(y_1^k - y_2^k)(w^{-k} - w^{n-k}) \tag{13}$$

$$- \Big(\frac{y_1^i}{1 - y_1^n} - \frac{y_2^i}{1 - y_2^n}\Big)\sum_{k=1}^{\infty}(y_1^k - y_2^k)(w^{1-k} - w^{n+1-k})\Big] .$$

Now in applying (4) to (13) to obtain elements of C^{-1} we use property (iv) of section 1, and seek only the elements of the first row of C^{-1}. Hence for using (4) we consider only z_1. Furthermore, since we then require only the coefficients of w^j in z_i for $j = 1,2,\cdots,n$, we can in z_1 ignore all powers of w that are outside the range 1 through n. Putting $i = 1$ in (13) and ignoring these powers gives

$$z_1' = \frac{1}{b(y_1 - y_2)^2}\Big[- y_1 y_2\Big(\frac{1}{1 - y_1^n} - \frac{1}{1 - y_2^n}\Big)\sum_{k=1}^{n-1}(y_1^k - y_2^k)w^{n-k}$$

$$+ \Big(\frac{y_1}{1 - y_1^n} - \frac{y_2}{1 - y_2^n}\Big)\sum_{k=1}^{n}(y_1^k - y_2^k)w^{n+1-k}\Big] .$$

The j'th element in the first row of $A = C^{-1}$ is then the coefficient of w^j in z_1' and this reduces to being

$$a_{1,j-1} = \frac{1}{b(y_1 - y_2)}\Big(\frac{y_1^{n+1-j}}{1 - y_1^n} - \frac{y_2^{n+1-j}}{1 - y_2^n}\Big), \qquad \text{for } j=1,2,\cdots,n . \tag{14}$$

It is easily verified, using (11), that $\sum_{j=1}^{n} a_{1,j-1} = 1/(a + b + c)$, as it should, in accord with property (v) of section 1. In terms of (2) and (3)

$$C_n = aI + bP_n + cP_n^{n-1} \tag{15}$$

and

$$C_n^{-1} = \sum_{j=0}^{n-1} a_j P_n^j \quad \text{for} \quad P^0 \equiv I \tag{16}$$

with a_j of (16) being $a_{1,j-1}$ from (14) so that

$$a_j = \frac{1}{b(y_1 - y_2)}\left(\frac{y_1^{n-j}}{1 - y_1^n} - \frac{y_2^{n-j}}{1 - y_2^n}\right) \quad \text{for } j=0,1,\cdots,n-1 \quad . \tag{17}$$

This is result (b) of Theorem 1.

3.2 Existence of Inverse

Before proceeding to calculate (17) for elements of the inverse we must notice 2 situations in which alternative forms are needed and 2 cases in which there is no inverse.

First, although $y_1 - y_2$ is part of the denominator of (17) it is also a factor of the numerator. Hence when $y_1 = y_2$, i.e. $a^2 = 4bc$, the factor $y_1 - y_2$ could be removed from both numerator and denominator of (17). However, equality of y_1 and y_2 means that the recurrence equation (8) will have a solution different from that given in (9) and correspondingly the elements of C^{-1} will differ from those in (17). They are derived in section 3.4, and summarized as result (d) of Theorem 1.

Second, (17) does not exist if y_1 or y_2 are unity. From (11) this means that $2b + a = \pm\sqrt{a^2 - 4bc}$, in turn implying $b(a + b + c) = 0$, i.e., $b = 0$ or $a + b + c = 0$. When $b = 0$, the form of the recurrence equation (8) is changed so that (9) is not its general solution and (17) does not give the corresponding elements of C^{-1}. They are derived in section 3.5 and stated as result (e) of Theorem 1.

We just observed that (17) does not exist when $a + b + c = 0$. This is so because C is then singular because, as is obvious from (7), $a + b + c$ is a factor of $|C|$. It is also clear that (17) does not exist for even values of n when either y_1 or y_2 are -1. From (11), this means (17) precludes an inverse when $a - 2b = \pm\sqrt{a^2 - 4bc}$, i.e., $b(a - b - c) = 0$. The case $b = 0$ has already been dealt with, but precluding $a = b + c$ requires substantiation It is readily available from property (vi) of section 1, which indicates that for a matrix like (7), of order n, the latent roots are $\lambda = a + br_i + cr_i^{n-1}$, where r_i satisfies $r^n = 1$. For n even, one such r_i is -1, which gives $\lambda = a - b - c$ which is zero for $a = b + c$ and there is then no inverse. Hence $a + b + c = 0$ and $a = b + c$ for even n constitute the exclusions of part (a) of Theorem 1.

3.3 Calculation of Inverse

Other than the exclusions just discussed, the calculation of (17) presents no problems when y_1 and y_2 are real. But when they are complex, calculation of (17) as it stands would involve manipulation of complex numbers even though the resulting value is always real. Avoidance of complex numbers can be achieved by using some elementary trigonometry.

Note that from (11) y_1 and y_2 will be complex only when $4bc > a^2$ and this can occur only when b and c have the same sign. Define

$$r = +\sqrt{\left(\frac{-a}{2b}\right)^2 + \left(\frac{\sqrt{4bc - a^2}}{2b}\right)^2} = +\sqrt{\frac{c}{b}} ,$$

which is always real, and

$$\theta = \sin^{-1}\left(\frac{\sqrt{4bc - a^2}}{2b} \Big/ r\right) = \sin^{-1}\sqrt{1 - \frac{a^2}{4bc}} \quad .$$

Then for $i = \sqrt{-1}$,

$$y_1 = r(\cos\theta + i\sin\theta) \quad \text{and} \quad y_2 = r(\cos\theta - i\sin\theta) \quad .$$

Substitution of these values into (17) along with utilizing de Moivre's theorem [that $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$] leads after a little reduction to (17) becoming

$$a_j = \frac{r^{n-j-1}[\sin(n-j)\theta + r^n \sin(j\theta)]}{b\sin\theta[1 - 2r^n\cos(n\theta) + r^{2n}]} \tag{18}$$

$$\text{for } j=0,1,\cdots,n-1 \quad .$$

This is result (b) of Theorem 1. It is real, and is quite computable. It is, of course, applicable only when $4bc > a^2$ and not when any of $a + b + c = 0$, $a - b - c = 0$ for even n, $b = 0$ or $a^2 = 4bc$ are true.

3.4 Inverse when $a^2 = 4bc$

Continuing to exclude $a + b + c = 0$, and $a - b - c$ for n even, we here derive C^{-1} for $a^2 = 4bc$. In this case $y_1 = y_2 = y = -a/2b$ of (11) and the recurrence equation (8) now has not (9) as its solution, but

$$z_i = \frac{w^{i+1}}{bw^2 + aw + c} + (p + qi)y^i$$

where p and q are again determined from the initial conditions (12). This leads after a little manipulation to

$$z_i = \frac{w(1 - w^n)}{by(y - w)(1 - y^n)} \left[\frac{ny^n}{1 - y^n} + \frac{y}{w - y} + i\right] y^i + \frac{w^{i+1}}{b(y - w)^2} \qquad (19)$$

for $i=1,2,\cdots,n$ and for $y= -a/2b$.

Again, in applying (6) to (19) we only need find elements of the first row of C^{-1} and so can consider (19) just for $i = 1$. This, after expanding the terms $(y - w)^{-1}$ and $(y - w)^{-2}$, becomes

$$z_1 = \frac{ny^{n-1}\, w(1 - w^n)}{b(1 - y^n)^2} \sum_{k=0}^{\infty} \left(\frac{w}{y}\right)^k - \frac{w^2}{by^2}\left(\frac{1 - w^n}{1 - y^n} - 1\right) \sum_{k=0}^{\infty} (k + 1)\left(\frac{w}{y}\right)^k . \qquad (20)$$

The coefficient of w^j for $j = 1,2,\cdots,n$ in this is

$$\frac{y^{n-j}}{b(1 - y^n)}\left(\frac{n}{1 - y^n} + 1 - j\right) \qquad (21)$$

and so for (16) the elements of the first row of C^{-1} are

$$a_j = \frac{y^{n-1-j}}{b(1 - y^n)}\left(\frac{n}{1 - y^n} - j\right) \quad \text{for } j=0,1,\cdots,n-1 . \qquad (22)$$

This is section (d) of Theorem 1. As before, it is easily verified that $\sum_{j=0}^{n-1} a_j = 1/(a + b + c)$.

3.5 2-Element Circulants

Section 3.2 indicates that when $b = 0$ in a 3-element circulant the recurrence equation is modified and requires a different solution. This is so because $b = 0$ reduces the matrix to being a 2-element circulant of the form

$$C = (a\ 0\ 0 \cdots 0\ c) \quad . \tag{23}$$

The recurrence equation (4) is now

$$cz_{i-1} + az_i = w^i$$

for which a general solution is

$$z_i = py^i + qw^i$$

with

$$y = - c/a \quad .$$

The coefficients p and q are derived from the boundary condition $z_0 = z_n$, so leading to

$$z_i = \frac{w}{c + aw}\left[w^i - \left(\frac{1 - w^n}{1 - y^n}\right)y^i\right] \quad .$$

Putting $i = 1$ and expanding $(c + aw)^{-1} = (1 - y/w)^{-1}/aw$ gives

$$z_1 = \frac{1}{a}\left[\sum_{k=0}^{\infty} y^k w^{1-k} - \frac{y(1 - w^n)}{1 - y^n}\sum_{k=0}^{\infty} y^k w^{-k}\right] \quad .$$

The coefficient of w in this is

$$\frac{1}{a} - \frac{1}{a}\left(\frac{- y}{1 - y^n}\right)y^{n-1} = \frac{1}{a(1 - y^n)}$$

and that of w^j for $j = 2,\cdots,n$ is

$$-\frac{1}{a}\left(\frac{-y}{1-y^n}\right)y^{n-j} = \frac{y^{n+1-j}}{a(1-y^n)} \quad .$$

Hence with C of (23) being of the form

$$C = aI + cP^{n-1}$$

its inverse is (16) with

$$a_0 = \frac{1}{a(1-y^n)} , \qquad \text{where } y = -c/a$$

and (24)

$$a_j = a_0 y^{n-j} \qquad \text{for } j = 1,2,\cdots,n-1 \quad .$$

This result is section (e) in Theorem 1. It is easily verified that $\sum_{j=0}^{n-1} a_j = 1/(a + c)$, as it should.

4. k-ELEMENT CIRCULANTS

The 3-element circulants are easy to invert in the manner described because they give rise to second-order recurrence equations whose solution depends on a quadratic equation, and this can always be solved. Inverting a k-element circulant, one having k consecutive cyclic elements with first and last being non-zero, will correspondingly come from solving a (k -1)-order recurrence equation, whose solution depends upon a polynomial equation of order k - 1 . Whenever this permits of simple factorization the recurrence equation can be solved and the elements of the inverse obtained by deriving the coefficient of w^j in z_i . Even with simple factorization of the polynomial, the algebra is likely to be tedious for k exceeding 3 or 4 . However, we now show how Theorem 1 can be used directly to obtain the inverse of a k-element circulant of order n .

Suppose

$$C = (c_0, c_1, \cdots, c_{k-1}, 0, 0, \cdots, 0) = \sum_{i=0}^{k-1} c_i P^i \tag{25}$$

is a k-element circulant of order n . We note that multiplication by appropriate P^j puts any k-element circulant into this form; and any circulant of order n may be regarded as an n-element circulant. Suppose the polynomial corresponding to (25)

$$c_0 + c_1 x + c_2 x^2 + \cdots + c_{k-1} x^{k-1} = 0 \tag{26}$$

has roots $\delta_1, \delta_2, \cdots, \delta_{k-1}$. Then it follows that

$$C = c_{k-1}(P - \delta_1 I)\,(P - \delta_2 I) \cdots (P - \delta_{k-1} I) \tag{27}$$

is a product of k - 1 two-element circulants. If k - 1 is even, (k - 1)/2 pairs of roots may be formed as $(\delta_{i1}, \delta_{i2})$ $i = 1,2,\cdots,(k-1)/2$, where complex conjugate roots will occur in pairs, to yield

$$C = \prod_{i=1}^{(k-1)/2} (P - \delta_{i1} I)(P - \delta_{i2} I) = \prod_{i=1}^{(k-1)/2} (P^2 + \alpha_i P + \gamma_i) \tag{28}$$

where

$$\alpha_i = -(\delta_{i1} + \delta_{i2}) \quad \text{and} \quad \gamma_i = \delta_{i1}\delta_{i2} \; .$$

Thus C is the product of (k - 1)/2 three-element circulants, and

$$C^{-1} = \prod_{i=1}^{(k-1)/2} (P^2 + \alpha_i P + \gamma_i)^{-1} \tag{29}$$

each of which can be obtained explicitly from Theorem 1. If $(k - 1)$ is odd, C is the product of one 2-element and $(k - 2)/2$ three-element circulants and an expression similar to (29) is obtained.

Thus the inversion of a k-element circulant reduces to finding the roots of a polynomial of order $k - 1$ and the multiplication of $(k - 1)/2$, or $1 + (k - 2)/2$, circulants.

5. THE INVERSE OF $C + \lambda J$

Let J be a square matrix with every element unity, that is $J = \sum_{i=0}^{n-1} P^i$. In this section we consider the inverse of $C + \lambda J$ both when C is non-singular and when the row sums of C are zero. This latter case arises frequently in statistical applications, as for example in Anderson [2].

5.1 C Non-Singular

Let s denote the row sum of C so that $CJ = sJ$. Solving

$$(C + \lambda J)(C^{-1} + \phi J) = I$$

for ϕ we obtain

$$(C + \lambda J)^{-1} = C^{-1} - \frac{\lambda}{s(s + n\lambda)} J \quad . \tag{30}$$

Thus when k elements in the row of an n-element circulant are all the same, the inverse of that n-element circulant is easily derived from (30) using the inverse of an $(n - k)$-element circulant. It also follows from (30) that

$$(C + \lambda J)^{-1} - (C + \theta J)^{-1} = \frac{-(\lambda - \theta)}{(s + n\lambda)(s + n\theta)} J \quad . \tag{31}$$

5.2 C Singular—Generalized Inverse

When the row sum of C is zero the characteristic root $\sum_{i=0}^{n-1} c_i(1)^i$ corresponding to the vector $\underline{1}$ is zero. This is the only zero root when C has rank n - 1 and $(C + \lambda J)$ then has one root $n\lambda$ and the remaining are the non-zero roots of C . This is an important case since this is precisely the situation for the information matrix obtained from an experimental design known as a connected block design developed cyclically from an initial set of blocks [Anderson (2)]. We show that when the row sum of C is zero the circulant

$$Q = (C + \lambda J)^{-1} \tag{32}$$

is a generalized inverse of C (meaning only that $CQC = C$) . First, if s_Q is the row sum of Q, property (v) of section 1 gives

$$s_Q = \frac{1}{s_{C+\lambda J}} = \frac{1}{s_C + n\lambda}$$

and

$$QJ = s_Q J \quad .$$

Also by definition, $Q(C + \lambda J) = I$ so that

$$QC = I - \lambda QJ = I - \lambda s_Q J \quad .$$

When C has a zero row sum

$$s_C = 0 \quad \text{and} \quad JC = 0 \tag{33}$$

and

$$s_Q = 1/n\lambda, \; QJ = (1/n\lambda)J \quad \text{and} \quad QC = I - n^{-1}J \quad . \tag{34}$$

Then

$$CQC = C - n^{-1}CJ = C \quad . \tag{35}$$

Hence for C having zero row sum, $Q = (C + \lambda J)^{-1}$ is a generalized inverse of C . Similarly

$$C(Q + \tau J)C = C + \tau s_C JC = C \, , \tag{36}$$

showing for any τ that $Q + \tau J = (C + \lambda J)^{-1} + \tau J$ is also a generalized inverse of C . From this, we establish the following theorem.

Theorem 2 When C is a circulant of order n and rank n - 1 with row sum $s_C = 0$, then

$$Q^* = (C + \lambda J)^{-1} - (1/\lambda n^2)J = Q - (1/\lambda n^2)J \tag{37}$$

is the unique Moore-Penrose inverse of C for any non-zero λ .

Proof Because $Q^* = Q + \tau J$ for $\tau = -1/\lambda n^2$, from (35) we see that $CQ^*C = C$. Also, from (33), (34) and (37) we find that $Q^*C = I - (1/n)J$ which is symmetric; and by property (vii) of section 1 $Q^*C = CQ^*$, and so CQ^* is also symmetric. Finally

$$Q^*CQ^* = (I - n^{-1}J)Q^* = Q^* - (s_{Q^*}/n)J$$

where from (37) and (34)

$$s_{Q^*} = s_Q - n/\lambda n^2 = 1/n\lambda - 1/n\lambda = 0$$

and so $Q^*CQ^* = Q^*$, and the proof is complete.

It is noteworthy from (32) and (35) that $Q = (C + \lambda I)^{-1}$ is a generalized inverse of C for any non-zero λ . Although for the case of non-singular C an expression for the difference

$$D = (C + \lambda J)^{-1} - (C + \theta J)^{-1}$$

was obtained in (31) on the basis of C^{-1} in (30) existing, the analogous result for C^{-1} not existing can also be derived. This is so because by direct multiplication

$$D(C + \theta J) = (C + \lambda J)^{-1}(C + \theta J) - I$$
$$= QC + \theta QJ - I$$
$$= \frac{\theta - \lambda}{n\lambda} J \qquad \text{from (34).}$$

Hence on using $J(C + \theta J)^{-1} = (1/n\theta)J$ from (34),

$$D = \frac{(\theta - \lambda)}{n\lambda} J(C + \theta J)^{-1} = \frac{\theta - \lambda}{n^2\lambda\theta} J \, , \tag{38}$$

which is exactly (31) with $s = 0$.

5.3 Calculations for 3-Element C

When C is non-singular, $(C + \lambda J)^{-1}$ is obtained from (30) using section (b), (c) or (d) of Theorem 1 for C^{-1} .

When C is singular with $a + b + c = 0$, the inverse of $C + \lambda J$ must be derived by adaptation of the general procedure used in section 3. The results are summarized in the following theorem.

Theorem 3 When C is a 3-element circulant of order n and rank $n - 1$, with non-zero elements a, b and c with $a + b + c = 0$, then $(C + \lambda J)^{-1} = A$ $= (a_0, a_1, \cdots, a_{n-1})$ where:

(a) For $y = - c/(a + c) \neq 1$

$$a_j = \frac{1}{\lambda n^2} - \frac{1}{a + 2c}\left[\frac{-(n + 1)}{2n} + \frac{j}{n} + \frac{1}{n(1 - y)} - \frac{y^{n-j}}{1 - y^n}\right], \quad j=0,1,2,\cdots,n-1$$

and

(b) for $y = - c/(a + c) = 1$, i.e., $b = c = -\frac{1}{2}a$,

$$a_j = \frac{1}{\lambda n^2} - \frac{n^2 - 1}{12nc} + \frac{j(n - j)}{2nc}, \qquad j=0,1,\cdots,n-1 \quad .$$

Proof Equation (4) using $C + \lambda J$ in place of C gives rise to the recurrence equation

$$cz_{i-1} + az_i + bz_{i+1} + \lambda \sum_1^n z_i = w^i \tag{39}$$

with the same boundary conditions as in (12), namely

$$z_0 = z_n \quad \text{and} \quad z_1 = z_{n-1} \quad . \tag{40}$$

Summing (39) and using (40) and $a + b + c = 0$ gives

$$\sum_1^n z_i = \frac{1}{\lambda n} \sum_1^n w^i = \frac{w(1 - w^n)}{\lambda n(1 - w)} \quad . \tag{41}$$

Then, subtracting from (39) the equivalent equation having right-hand side w^{i+1} we get on again using $a + b + c = 0$

$$cz_{i-1} + (a - c)z_i - (2a + c)z_{i+1} + (a + c)z_{i+2} = w^i(1 - w) \quad . \tag{42}$$

Defining $u_i = z_i/w^i(1 - w)$ the recurrence equation then depends on the cubic

$$(a + c)w^3x^3 - (2a + c)w^2x^2 - (a - c)wx + c = 0 \tag{43}$$

which has roots $x = 1/w$, $1/w$ and $- c/w(a + c)$. Hence, writing

$$y \equiv - c/(a + c) \tag{44}$$

the solution for z_i is

$$z_i = (\alpha + \beta i + \gamma y^i)(1 - w) + \delta w^i(1 - w) \tag{45}$$

where α, β, γ and δ are constants to be determined from the boundary conditions (40), the sum (41), and the general recurrence relationship (42). After considerable detail z_i is found to be

$$z_i = \frac{w(1-w^n)}{n(1-w)}\left(\frac{1}{\lambda n} - \frac{1}{a+2c}\left\{\frac{n+1}{2} - i + \frac{1-wy}{(1-w)(1-y)}\right.\right. \qquad (46)$$

$$\left.\left. - \frac{n}{w-y}\left[\frac{(1-y)w^i}{1-w^n} - \frac{(1-w)y^i}{1-y^n}\right]\right\}\right) .$$

On putting $i = 1$, ignoring terms in w^n and expanding $(1-w)^{-1}$ and $(1-w)^{-2}$, we find that the coefficient of w^{j+1} in z_1 for $j = 0,1,\cdots,n-1$, which we call a_j reduces to

$$a_j = \frac{1}{\lambda n^2} - \frac{1}{a+2c}\left[\frac{-(n+1)}{2n} + \frac{j}{n} + \frac{1}{n(1-y)} - \frac{y^{n-j}}{1-y^n}\right] \qquad (47)$$

$$\text{for } j=0,1,2,\cdots,n-1 .$$

Thus is established part (a) of Theorem 3.

It is clear that (47) does not hold for $a + 2c = 0$ or $y = 1$. From (44), these conditions are equivalent and together with $a + b + c = 0$ they imply $a = -2c$ and $b = c$. With these values the roots of (43) are all $1/w$ and so the solution to (42) is, in place of (45)

$$z_i = (\alpha + \beta i + \gamma i^2)(1-w) + \delta w^i(1-w) \qquad (48)$$

After further tedious manipulation to determine α, β, γ and δ of (48), using (40), (41) and (42), all with $a = -2c$ and $b = c$, the expression for z_i becomes

$$z_i = \frac{w(1 - w^n)}{\lambda n^2(1 - w)} + \frac{w(1 - w^n)}{2nc(1 - w)^2} \left\{ [2 + n(1 - w)] \left(i - \frac{n + 1}{2}\right) \right.$$

$$\left. - \left[i^2 - \frac{(n + 1)(2n + 1)}{6}\right] (1 - w) - 2w \left(\frac{1}{1 - w} - \frac{nw^{i-1}}{1 - w^n}\right) \right\} .$$

Putting $i = 1$, ignoring terms in w^n and expanding negative powers of $(1 - w)$ we reduce z_1 to

$$z_1' = \left[\frac{1}{\lambda n^2} - \frac{(n - 1)(n - 5)}{12nc}\right] \sum_{k=1}^{\infty} w^k - \frac{n - 1}{2nc} \sum_{k=1}^{\infty} kw^k$$

$$+ \frac{1}{c} \sum_{k=2}^{\infty} (k - 1)w^k - \frac{1}{nc} \sum_{k=2}^{\infty} \tfrac{1}{2}k(k - 1)w^k .$$

The coefficient of w in this is

$$a_{11} = \frac{1}{\lambda n^2} - \frac{(n - 1)(n - 5)}{12nc} - \frac{n - 1}{2nc} = \frac{1}{\lambda n^2} - \frac{n^2 - 1}{12nc}$$

and the coefficient of w^j for $j > 1$ is

$$a_{1j} = \frac{1}{\lambda n^2} - \frac{(n - 1)(n - 5)}{12nc} - \frac{(n - 1)j}{2nc} + \frac{j - 1}{c} - \frac{j(j - 1)}{2nc}$$

which simplifies to

$$a_{1j} = a_{11} + \frac{(j - 1)(n + 1 - j)}{2nc} \quad \text{for } j=1,2,\cdots,n .$$

Hence the elements of the first row of $(C + \lambda J)^{-1}$ are

$$a_j = \frac{1}{\lambda n^2} - \frac{n^2 - 1}{12nc} + \frac{j(n - j)}{2nc} \quad \text{for } j=0,1,\cdots,n-1 \tag{49}$$

as stated in Theorem 3(b).

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