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Investigating $f(R, \phi)$ Cosmology with Equation of State

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Abstract

The aim of this paper is to investigate the field equations of modified $f(R, \phi)$ theory of gravity, where R and ϕ represent the Ricci scalar and scalar potential respectively. We consider the Friedmann-Robertson-Walker space time for finding some exact solutions by using different values of equation of state parameter. In this regard, different possibilities of the exact solutions have been discussed for dust universe, radiation universe, ultra relativistic universe, sub-relativistic universe, stiff universe and dark energy universe. Mainly power law and exponential form of scale factor are choosen for the analysis.

Keywords: Modified gravity. Equation of state. Exact solutions.

1 Introduction

Recent explanations from astrophysical facts have unfolded an amazing picture of expanding universe. This cosmic expansion is well supported by cosmic microwave background anisotropy, galaxy clustering and high redshift supernovae experiments $[1]-[4]$. The phenomenon of dark energy and dark matter is another topic of discussion [5]. It has been shown that dark

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energy can be justified by using equation of state (EoS) parameter $\omega = p/\rho$, where ρ and p are the energy density and pressure of dark energy respectively [10] − [16]. The expansion of the universe is predicted to be accelerating when $\omega \approx -1$ [20]. Different ideas have been proposed for the explanation for dark energy. For example, one model affiliates dark energy with the vacant space and recommends the inclusion of cosmological constant in the field equations. In another opinion researchers have put emphases for the addition of scalar field in the action that may recognise the cosmic expansion. Quintessence [6], Chaplygin gas [7], phantom models [8] and k-essence [9] are different models consisting scalar field that have been investigated in recent past. Thus it is expected that alternative or modified models can give better explanation of the issues of dark energy.

Modern theories of gravity explain the concept of dark energy and dark matter which is responsible for current cosmic expansion. The previous decade is full of research work that dealt with $f(R)$ theory of gravity, where R denotes the Ricci scalar [17] − [19]. Some other well known modified theories include $f(G)$, $f(R,T)$, $f(G,T)$ and $f(R,G)$ theories of gravity, where G and T denotes the Gauss-Bonnet invariant and the trace of the energy momentum tensor respectively. These theories explain the weak field regimes but some modifications are still required to address the strong field for the expansion of universe.

Capozziello et. al. [21] introduced the scalar field in the action and discussed the $f(R,\phi)$ gravity. They concluded a new technique to address and solve the inconsistencies of General Relativity. Recently, density perturbations in the cosmic microwave background within general $f(R,\phi)$ models of gravity are investigated [24]. He concluded that density perturbation obtained from observations recovered naturally, with high precision. Thus it seems worthwhile to investigate further the $f(R,\phi)$ theories of gravity.

In this paper, we are Interested to explore the exact solution of Friedmann-Robertson-Walker (FRW) space-time in $f(R,\phi)$ gravity. The field equations are solved by assuming the conventional power law and exponential form of scale factor. The paper is planned as follows: In section **2**, field equations of $f(R,\phi)$ gravity are introduced. The solutions of filed equations along with their graphical behavior are discussed in section **3**. In last section, we summarize the results.

2 Modified Field Equations

An Interesting family of extended theories of gravity may be categorized as the scalar tensor theories [25]. Recently, Zubair et al. [26] used a generalized lagrangian in the presence of energy potential to investigate the field equations of modified $f(R,\phi)$ gravity. We consider a simplified action in the absence of energy potential which is given as follows

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{k^2}(f(R,\phi) + w(\phi)\phi_{;\alpha}\phi^{;\alpha})\right] + S_m, \quad (1)$$

where $f(R,\phi)$ is an analytic function of Ricci Scalar R and the scalar field ϕ. For our convince we use $f(R,\phi) \equiv f$. Here g is the determinant of metric tensor $g_{\mu\nu}$, S_m represents the action of matter. We use the convention $c = 1$ and $k^2 \equiv 8\pi G$. The field equations of $f(R,\phi)$ are obtained by varying the action in Eq. (1) with respect to the metric tensor

$$f_R R_{\mu\nu} - \frac{1}{2}[f + w(\phi)\phi_{;\alpha}\phi^{;\alpha}]g_{\mu\nu} + w(\phi)\phi_{;\mu}\phi_{;\nu} - f_{R;\mu\nu} + g_{\mu\nu}\Box f_R = k^2 T_{\mu\nu}, \quad (2)$$

where w is an arbitrary function of ϕ, ∇_μ represents the covariant derivative, $\Box \equiv \nabla_\mu\nabla^\nu$ and $f_R \equiv \frac{\partial f}{\partial R}$ etc. Eq. (2) is the fourth order partial differential equations in the metric tensor. If we replace $f = \phi R$ and $w(\phi) = \frac{\overline{w}}{\phi}$, where $\overline{w}$ is a dimensionless coupling constant, then $f(R,\phi)$ gravity becomes Brans-Dicke theory [29]. For the present work we choose the energy momentum tensor for perfect fluid as

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu - p g_{\mu\nu}, \quad (3)$$

where $u_\mu = g_{00}(1,0,0,0)$ is the velocity four vector. We consider the FRW universe model

$$ds^2 = dt^2 - a^2[dx^2 + dy^2 + dz^2], \quad (4)$$

where the scale factor a is the function of cosmic time t . The corresponding Ricci scalar is

$$R = -6\left[\frac{\dot{a}^2 + a\ddot{a}}{a^2}\right], \quad (5)$$

where dot represents the time derivative. Using Eqs. (3) and (4) in Eq. (2), the field equations becomes

$$-3\frac{\ddot{a}}{a}f_R - \frac{1}{2}[f - w(\phi)\dot{\phi}^2] + 3\frac{\dot{a}}{a}f_{R,0} = k^2\rho, \quad (6)$$

$$[\frac{\ddot{a}}{a}+2(\frac{\dot{a}}{a})^2]f_R+\frac{1}{2}[f+w(\phi)\dot{\phi}^2]-2\frac{\dot{a}}{a}f_{R,0}-f_{R,00}=k^2p. \quad (7)$$

These are complicated and highly non-linear differential equations and it is difficult to solve them without imposing some physical assumptions.

3 Solutions of Field equations

We use two types namely exponential law [22] and power law technique [23] for investigating the exact solutions. For current analysis, we propose $f(R,\phi)=f_1(R)+f_2(\phi)$. We further consider $f_1(R)=f_0R^{z+1}$ [27], where f_0 and z are arbitrary constants. $f_2(\phi)$ is also given by [28]

$$f_2(\phi)=\frac{w_0\gamma^2n^2a_0^{2/n}(sn\gamma+2n\gamma+6n-2)}{sn\gamma+2n\gamma-2}\phi^{s+2-\frac{2}{n\gamma}}, \quad (8)$$

where w_0, m, n and γ are arbitrary constants. We also choose $w(\phi)=w_0\phi^m$ and $\phi\equiv\phi(t)=a^\beta$, where β is an arbitrary constant [30, 31]. We further consider different forms of EoS parameters and field equations for dark energy universe (ω=-1), ultra relativistic universe (ω=$\frac{1}{2}$), radiation universe (ω=$\frac{1}{3}$), Sub-relativistic universe (ω=$\frac{1}{4}$), stiff universe (ω=1) and dust Case (ω=0) for FRW model.

3.1 Dark energy universe

For investigating the solutions of dark energy universe, we use ω=-1. The field equations (6) and (7) becomes

$$[-2\frac{\ddot{a}}{a}+2(\frac{\dot{a}}{a})^2]f_R+w(\phi)\dot{\phi}^2+\frac{\dot{a}}{a}f_{R,0}-f_{R,00}=0, \quad (9)$$

which is a very complicated differential equation. We can observe that f_R is involved in Eq. (9). Therefore $f_1(R)$ and $w(\phi)$ are only used for finding the solution of Dark energy universe. Substituting the values in Eq. (9), we get

$$\frac{1}{a^2(\dot{a}^2+a\ddot{a})^2}f_0\left(-\frac{6(\dot{a}^2+a\ddot{a})}{a^2}\right)^z\Big[2\dot{a}^6+4z^3a\dot{a}^4\ddot{a}-z^3a^2\dot{a}^2\ddot{a}^2+z^2a\dot{a}^4\ddot{a}-z^2a^4\ddddot{a}\ddot{a}$$
$$+4z^3a\dot{a}^4\ddot{a}-za^3\ddddot{a}\dot{a}^2-za^4\ddddot{a}\ddot{a}-z^2a^3\ddot{a}^3-z^3a^4\dddot{a}^2+3za^3\dot{a}\ddot{a}\dddot{a}-3za^2\dot{a}^3\dddot{a}$$
$$+2a\dot{a}^4\ddot{a}-2a^2\dot{a}^2\ddot{a}^2+za^4\dddot{a}^2+z^2a^3\dot{a}\ddot{a}\dddot{a}+7\ddot{a}^2a^2\dot{a}^2-3\ddot{a}^3a^3z+8z^2\dot{a}^2a^2\ddot{a}^2-2z\dot{a}^6$$
$$+z^2a^2\dot{a}^3\dddot{a}-az\dot{a}^4\ddot{a}-z^2a^3\dot{a}^2\ddddot{a}-2z^3a^3\dot{a}\ddot{a}\dddot{a}-8z^2\dot{a}^6-2a^3\ddot{a}^3\Big]+w_0a^{m\beta+2\beta}\beta^2\dot{a}^2=0 \tag{10}$$

It may be observed that power law form of scale factor

$$a(t)=\alpha t^k, \tag{11}$$

where k is an arbitrary real number, is a solution of Eq. (10) with the constraint equation

$$w_0\beta^2k^2+2kf_0=0. \tag{12}$$

We may choose different values of k for finding the solutions of energy density and pressure. For example, $k=0$ gives trivial solution. For non-trivial solution and for the sake of simplicity we choose $k=1$. We also choose $w_0=-2$ and $f_0=-\beta^2$, so that the constraint equation is satisfied. With the help of above constraint equation, we obtain the value of ρ and p.

$$\frac{1}{4t^2}[2w_0\beta^2k^2+12k^2f_0+w_0n^2\beta^2a_0^{\frac{2}{n}}\alpha^{\frac{-2}{n}}t^{\frac{-2k}{n}+2}]=\rho. \tag{13}$$

$$\frac{-1}{4t^4}[192f_0k^3-384f_0k^2+12f_0k^2t^2-2w_0\beta^2k^2t^2-8f_0kt^2+144f_0k+w_0n^2\beta^2a_0^{\frac{2}{n}}t^4\alpha^{\frac{-2}{n}}t^{\frac{-2k}{n}}]=p. \tag{14}$$

The behavior of energy density and pressure for power law solutions is decreasing, which can be shown in figure 1. The graphical behavior shows that there is singularity, which could be eliminated by splitting the domain into two parts i.e. set of positive Real numbers and set of negative Real numbers.

Now, we find the exact solution of eq. (10) using exponential law, that is

$$a(t)=e^{lt}, \tag{15}$$

where l is an arbitrary real number. Using Eq. (15) in Eq. (10), we get the constraint equation

$$w_0\beta^2l^2=0. \tag{16}$$

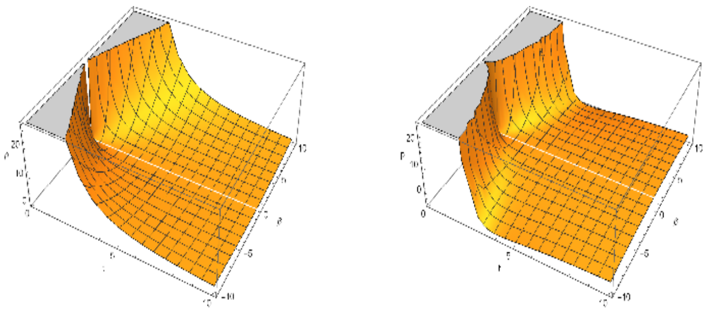


Figure 1: Energy density and pressure for power law solution

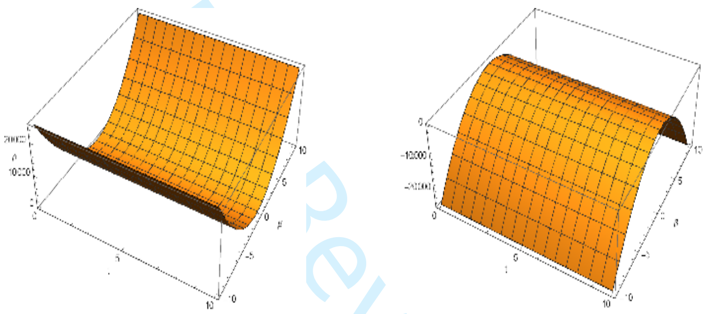


Figure 2: Dark energy universe for exponential law

From Eq. (16), it can be observed that either $w_0 = 0$ or $l = 0$ or $\beta = 0$. If we choose $w_0 = 0$ then field equations of $f(R, \phi)$ will be disturbed. Similarly, the value of β is also non-zero for non-trivial solution. So, there is only possibility to consider $l = 0$ for trivial solution. Thus it is concluded that $f(R, \phi) = f_1(R) + f_2(\phi)$ does not give a non trivial solution of field equations in $f(R, \phi)$ gravity for exponential form of scale factor a. With the help of this constraint equation, we get the value of ρ and p.

$$3l^2 f_0(-12l^2)^z + \frac{1}{4}a_0^{2/n} w_0 \beta^2 n^2 (e^{lt})^{-2/n} + \frac{1}{2} w_0 \beta^2 l^2 - 3l^2 f_0(-12l^2)^z z = \rho. \quad (17)$$

$$-3l^2 f_0(-12l^2)^z - \frac{1}{4}a_0^{2/n} w_0 \beta^2 n^2 (e^{lt})^{-2/n} + \frac{1}{2} w_0 \beta^2 l^2 + 3l^2 f_0(-12l^2)^z z = p. \quad (18)$$

The graphical behavior of ρ and p for exponential law is shown in figure 2. The energy density is positive and much higher than the density of power law solution but pressure is negative, which is the validation of our result that universe is expanding.

3.2 Ultra Relativistic Fluid

For this case, we use $\omega = \frac{1}{2}$ and then subtracting the field equations (6) and (7) gives

$$[5\frac{\ddot{a}}{a} + 4(\frac{\dot{a}}{a})^2]f_R + \frac{1}{2}w(\phi)\dot{\phi}^2 + \frac{3}{2}f - 7\frac{\dot{a}}{a}f_{R,0} - 2f_{R,00} = 0. \quad (19)$$

Substituting all the values in eq (19), we get

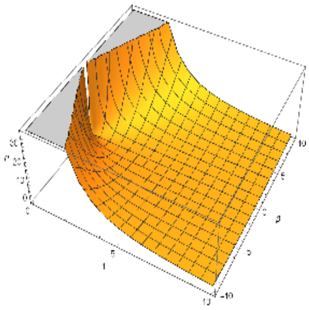
$$f_0(z+1)\left(-\frac{6(\dot{a}^2 + a\ddot{a})}{a^2}\right)^z\Big[5\frac{\ddot{a}}{a} + 4\frac{\dot{a}^2}{a^2} + \frac{11z}{(a^2\dot{a}^2 + a^3\ddot{a})}(2\dot{a}^2 - \frac{17}{6}a\dot{a}\ddot{a} - a^2\dddot{a}) - \frac{z^2}{18(\dot{a}^2 + a\ddot{a})^2}(12a\dot{a}^2 - 17a^2\dot{a}\ddot{a} - 6a^3\ddddot{a}) + \frac{z}{\dot{a}^2 + a\ddot{a}}(-2\ddot{a}^2 + 16\frac{\dot{a}^2\ddot{a}}{a} - 8a\ddddot{a} + 2\ddddot{a} + 8\dot{a}\dddot{a} - 12\frac{\dot{a}^4}{a^2}) - \frac{z}{(\dot{a}^2 + a\ddot{a})^2}(12\frac{\dot{a}^3\ddot{a}}{a} - 17\dot{a}^2\ddot{a}^2 - 6a\dot{a}\ddot{a}\dddot{a} + a\ddddot{a}) + \frac{1}{2}w_0a^{\beta m+2\beta-2}\beta^2\dot{a}^2 + 8\dot{a}\ddddot{a} - 12\frac{\dot{a}^4}{a^2}) - \frac{z}{(\dot{a}^2 + a\ddot{a})^2}(12\frac{\dot{a}^3\ddot{a}}{a} - 17\dot{a}^2\ddot{a}^2 - 6a\dot{a}\ddot{a}\dddot{a} + a\ddddot{a}) + \frac{1}{2}w_0a^{\beta m+2\beta-2}\beta^2\dot{a}^2\Big] + \frac{3}{2}f_0[-\frac{6(\dot{a}^2 + a\ddot{a})}{a^2}]^{z+1} + \frac{3}{2(sn\gamma + 2n\gamma - 2)}a_0^{2/n}w_0n^2\gamma^2a^{s\beta+2\beta-\frac{2\beta}{n\gamma}} = 0 \quad (20)$$

Now, Firstly we find the power law solution (i.e $a(t) = \alpha t^k$). For obtaining the constraint equation, if we use $s = -2 + \frac{2}{n\gamma}$ then we obtain the undefined form. Due to this singularity we are unable to find the solution of above mentioned differential equation. Actually this undefined form is due to the value of s, thats why we can not use this value. To eliminate this singularity form, we fix more parameters . Now we use $z = 0$, $m = -2$, $s = -2$, $n = \beta$, $k = 1$, $\gamma = 1$ and obtain the constraint equation

$$20f_0 - 2w_0\beta^2 + 3a_0^{2/\beta}w_0\beta^2\alpha^{-2} = 0. \quad (21)$$

There exists many solutions by choosing parametric values. But for our choice, we take $f_0 = -\beta^2$, $w_0 = 20/3$, $\alpha = 1$ and $a_0 = 1$. By using constraint equation (21), the solution of ρ and p is

$$\frac{1}{4t^2}[2w_0\beta^2 + 12f_0 + w_0\beta^2a_0^{2/\beta}\alpha^{-2}] = \rho. \quad (22)$$



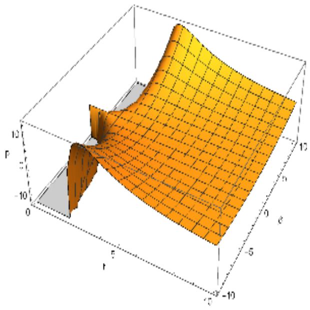


Figure 3: Ultra Relativistic Fluid for power law

$$\frac{-1}{4t^4}[-48f_0 + 4f_0t^2 - 2w_0\beta^2t^2 + w_0\beta^2t^4a_0^{2/\beta}(\alpha t)^{-2}] = p. \quad (23)$$

Now we find the exponential law solution (i.e $a(t) = e^{lt}$). We substitute $s = -2$, $m = -2$ and $n = 1$ for obtaining constraint equation but there is still exponential function involving here. For obtaining the constraint equation we choose $a_0 = e^{lt}$ and $\gamma = \beta$, we get

$$-\frac{3}{4}w_0\beta^2 + \frac{1}{2}w_0\beta^2l^2 + 9f0(-12l^2)^zl^2z - 9f0(-12l^2)^zl^2 = 0. \quad (24)$$

For obtaining the results, we can choose different values of parameters, used in constrained equation. We choose $f_0 = 1$, $w_0 = 1$, $l = \sqrt{3}/2$ and $z = 1$. We obtain the value of ρ and p by using the constrained equation.

$$-3l^2f_0(-12l^2)^zz + \frac{1}{2}w_0\beta^2l^2 + \frac{1}{4}w_0\beta^2 + 3l^2f_0(-12l^2)^z = \rho. \quad (25)$$

$$-\frac{1}{4}w0\beta^2 + \frac{1}{2}w_0\beta^2l^2 + 3l^2f_0(-12l^2)^zz - 3l^2f_0(-12l^2)^z = p. \quad (26)$$

Graphical analysis of power law shown in Figure 3, which shows that density has positive and decreasing behavior. Ultra relativistic universe for power law solution has similar behavior to dark energy universe and pressure has monotonic behavior i.e. negative as well as positive behavior. Similarly, energy density and pressure for exponential law solution is shown in figure 4, which shows that behavior is positive as well as monotonic.

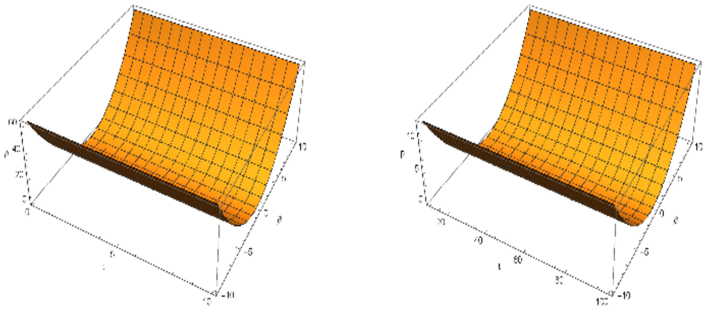


Figure 4: Ultra Relativistic Fluid for exponential law

3.3 Radiation Universe

For this case, we use $\omega = \frac{1}{3}$ for radiation universe and multiplying the field equation (6) and (7) becomes

$$[6\frac{\ddot{a}}{a} + 6(\frac{\dot{a}}{a})^2]f_R + 2[f + w(\phi)\dot{\phi}^2] - 9\frac{\dot{a}}{a}f_{R,0} - 3f_{R,00} = 0. \qquad (27)$$

Putting the values in Eq. (27), we get

$$\frac{2a_0^{2/n}w_0 n^2\gamma^2 a^{s\beta+2\beta-\frac{2\beta}{n\gamma}}}{sn\gamma + 2n\gamma - 2} + 2w_0\beta^2\dot{a}^2 a^{m\beta+2\beta-2} + f_0(z+1)\Big(-\frac{6(\dot{a}^2 + a\ddot{a})}{a^2}\Big)^z\Big[6f_0\frac{\ddot{a}}{a} + 6(\frac{\dot{a}}{a})^2 + \frac{5z}{a^2\dot{a}^2 + a^3\ddot{a}}(6\dot{a}^3 - 3a\dot{a}^2\ddot{a} - 3a^2\dot{a}\dddot{a}) - \frac{3z^2}{a^2(a^2\dot{a}^2 + a^3\ddot{a})^2}(2\dot{a}^2 - a\dot{a}\ddot{a} - a^2\dddot{a})^2 + \frac{z}{a^2\dot{a}^2 + a^3\ddot{a}}(-3a^2\ddot{a}^2 + 24a\dot{a}^2\ddot{a} - 3a^3\ddddot{a} - 18\ddot{a}^2) - \frac{9}{(\dot{a}^2 + a\ddot{a})^2}(2a^2\dot{a}^3\ddot{a} - a^3\dot{a}^2\ddot{a}^2 - a^4\dot{a}\ddot{a}\dddot{a})\Big] + 2f_0[-\frac{6(\dot{a}^2 + a\ddot{a})}{a^2}]^{z+1} = 0 \qquad (28)$$

we choose the same parameters for obtaining the solutions like ultra relativistic fluid and we get the constraint equation

$$6f_0 + a_0^{2/\beta}w_0\beta^2\alpha^{-2} - 2w_0\beta^2 = 0 \qquad (29)$$

This constraint equation is very interesting because we obtained different results by choosing the values of f_0. For the sake of simplicity, we choose $f_0 = \beta^2$, $w_0 = 6$, $\alpha = 1$ and $a_0 = 1$ and we obtain the similar behavior

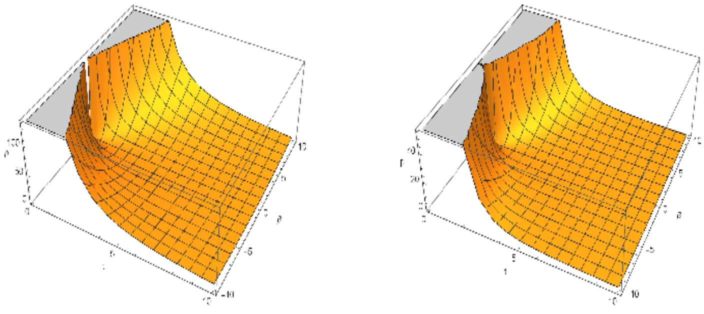


Figure 5: Power law solution of density and pressure for Radiation Universe

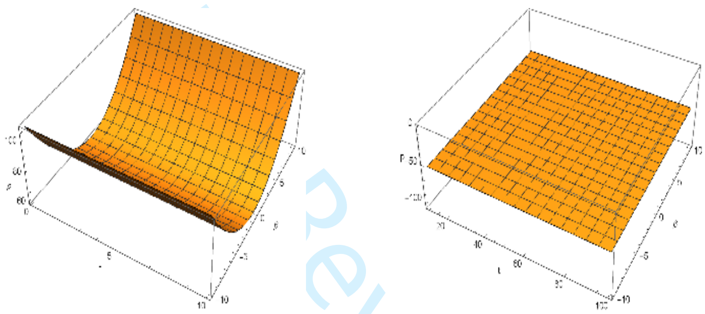


Figure 6: Exponential solution of density and pressure for Radiation Universe

of density and pressure like $w = \frac{1}{2}$. The Graphical behavior of this case is shown in Figure 5. Now we find the exponential law solution (i.e $a(t) = e^{lt}$). We fix $s = -2$ and $n = 1$ for constraint equation but there is still exponential function involving here. To avoid this problem we choose $a_0 = e^{lt}$, we get

$$-w_0\beta^2 + 2w_0\beta^2 l^2 - 12f0(-12l^2)^z + 12f0(-12l^2)^z l^2 z = 0. \qquad (30)$$

This is very complex constraint equation and we choose some parameters $f_0 = -1$, $w_0 = 1$, $z = 2$ and $l = \frac{1}{\sqrt{2}}$. The behavior of density with exponential law is monotonic But in case of pressure, the behavior is negative as well as flat, which is shown shown in Figure 6.

3.4 Sub-Relativistic Universe

For this case, we use $\omega = \frac{1}{4}$ and multiplying the field equation (6) and (7) becomes

$$[7\frac{\ddot{a}}{a} + 8(\frac{\dot{a}}{a})^2]f_R + \frac{5}{2}f + \frac{3}{2}w(\phi)\dot{\phi}^2 - 11\frac{\dot{a}}{a}f_{R,0} - 4f_{R,00} = 0. \qquad (31)$$

Substituting these values in Eq. (31), we get

$$f_0(z+1)\left(-\frac{6(\dot{a}^2+a\ddot{a})}{a^2}\right)^{z+1}\left[\frac{7a\ddot{a}+8\dot{a}^2}{a^2}+\frac{19\dot{a}z}{6(a^2\dot{a}^2+a^3\ddot{a})}(12\dot{a}^2-6a\dot{a}\ddot{a}-6a^2\dddot{a})-\frac{z^2}{9a^2(a^2\dot{a}^2+a^3\ddot{a})^2}(12\dot{a}^2-6a\dot{a}\ddot{a}-6a^2\dddot{a})^2+\frac{2z}{3(a^2\dot{a}^2+a^3\ddot{a})}(-6a^2\ddot{a}^2+48a\dot{a}^2\ddot{a}-6a^3\ddddot{a}-6a\dot{a}-36\ddot{a}^2)-\frac{2z(3\dot{a}\ddot{a}+a\dddot{a})}{3a(\dot{a}^2+a\ddot{a})^2}(12\dot{a}^2\ddot{a}-6a^2\dddot{a})\right]+\frac{5a_0^{2/n}w_0n^2\gamma^2a^{s\beta+2\beta-\frac{2\beta}{n\gamma}}}{2(sn\gamma+2n\gamma-2)}+\frac{5}{2}f_0[-\frac{6(\dot{a}^2+a\ddot{a})}{a^2}]^{z+1}+\frac{3}{2}w_0\beta^2\dot{a}^2a^{m\beta+2\beta-2}=0. \quad (32)$$

We use some parameters i.e. $z=0$, $s=-2$, $m=-2$, $k=1$ and $n=1$. We are left with constraint equation involving some other parameters. For the sake of simplicity, we choose $\gamma=\beta$ and the constraint equation for power law is

$$-7f_0-\frac{5}{4}a_0^2w_0\beta^2\alpha^{-2}+\frac{3}{2}w_0\beta^2=0. \quad (33)$$

We choose $\alpha=1$, $w_0=1$, $a_0=\sqrt{2}$ and $f_0=\frac{-\beta^2}{7}$. There is one attracting fact that if we choose f_0 as negative then behavior of density is positive and pressure is negative, which is the validation of expansion of universe. If we choose f_0 as positive then density will be positive and give higher values instead of negative case and pressure is positive. In other words by the choice of f_0, density has same behavior for both cases but pressure has opposite behavior. The constraint equation for exponential law is

$$-15f0(-12l^2)^zl^2+15f0(-12l^2)^zl^2z+\frac{3}{2}w_0\beta^2l^2-\frac{5}{4}w_0\beta^2=0. \quad (34)$$

We choose $w_0=\frac{2}{3}$, $z=1$, $f_0=1$ and $l=\sqrt{\frac{5}{6}}$ and then draw the graphs of density and pressure for exponential case. In this case, density and pressure has the same behavior as shown in figure 7 and 8.

3.5 Stiff Fluid

For this case, we use $\omega=1$ and multiplying the field equations (6) and (7) becomes

$$[4\frac{\ddot{a}}{a}+2(\frac{\dot{a}}{a})^2]f_R-5\frac{\dot{a}}{a}f_{R,0}-f_{R,00}=0. \quad (35)$$

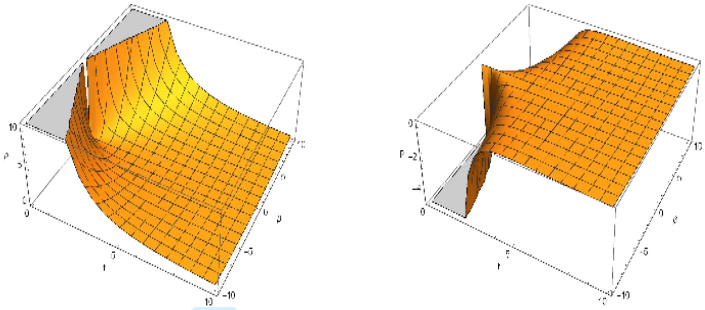


Figure 7: Sub-Relativistic Fluid for power law

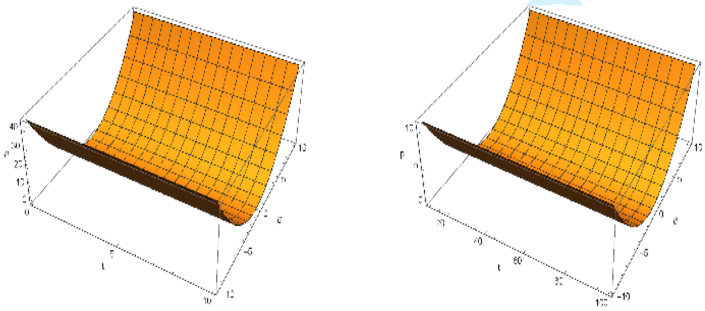


Figure 8: Sub-Relativistic Fluid for exponential law

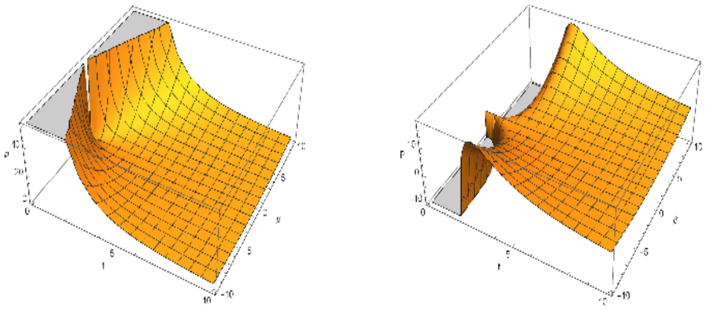


Figure 9: Stiff Fluid for power law

Substituting these values in Eq. (35), we get

$$f_0(z+1)(\dot{a}^2+a\ddot{a})^{-2}\left(-\frac{6(\dot{a}^2+a\ddot{a})}{a^2}\right)^z\left[\frac{4\ddot{a}(\dot{a}^2+a\ddot{a})^2}{a}+2(\frac{\dot{a}}{a})^2(\dot{a}^2+a\ddot{a})^2+\frac{5z}{6a}(\dot{a}^2 +a\ddot{a})(12\dot{a}^2-6a\dot{a}\ddot{a}-6a^2\dddot{a})+\frac{z}{6a^2}(\dot{a}^2+a\ddot{a})(-6a^2\ddot{a}^2+48a\dot{a}^2\ddot{a}-6a^3\ddddot{a}-6a\dot{a} -36\ddot{a}^2)-(\frac{z}{6a})^2(12\dot{a}^2-6a\dot{a}\ddot{a}-6a^2\dddot{a})^2-(\frac{z}{a})(2\dot{a}^2-a\dot{a}\ddot{a}-a^2\dddot{a})(\dot{a}^2+a\ddot{a})+ \frac{z\dot{a}}{3a^2}(\dot{a}^2+a\ddot{a})(12\dot{a}^2-6a\dot{a}\ddot{a}-6a^2\dddot{a})\right]+f_0[-\frac{6(\dot{a}^2+a\ddot{a})}{a^2}]^{z+1}+\frac{a_0^{2/n}w_0n^2\beta^2a^{m\beta+2\beta-\frac{2}{n}}}{mn\beta+2n\beta-2}=0 \quad (36)$$

The constraint equation for power law is

$$8f_0+a_0^2w_0\beta^2\alpha^{-2}=0 \quad (37)$$

By choosing different combinations of f_0 and w_0, it satisfies the constraint equation. But for simplicity, we choose $f_0=-\beta^2$, $w_0=8$, $a_0=1$ and $\alpha=1$. The graphical behavior is shown in figure 9. For exponential law solution, The constraint equation is

$$-15f0(-12l^2)^zl^2+15f0(-12l^2)^zl^2z+\frac{3}{2}w_0\beta^2l^2-\frac{5}{4}w_0\beta^2=0. \quad (38)$$

The constraint equation for stiff case is similar to Sub-Relativistic Universe, so it must have the same results for density and pressure as well.

3.6 Dust universe

For this case, we use $\omega = 0$ and solving the field equations (6) and (7) becomes

$$-3\frac{\ddot{a}}{a}f_R - \frac{1}{2}[f - w(\phi)\dot{\phi}^2] + 3\frac{\dot{a}}{a}f_{R,0} = 0. \quad (39)$$

$$[\frac{\ddot{a}}{a} + 2(\frac{\dot{a}}{a})^2]f_R + \frac{1}{2}[f + w(\phi)\dot{\phi}^2] - 2\frac{\dot{a}}{a}f_{R,0} - f_{R,00} = 0. \quad (40)$$

From eq. (39) and (40), it is observed that exist two solutions. Firstly, we have

$$[-2\frac{\ddot{a}}{a} + 2(\frac{\dot{a}}{a})^2]f_R + w(\phi)\dot{\phi}^2 + \frac{\dot{a}}{a}f_{R,0} - f_{R,00} = 0. \quad (41)$$

We observe that behaviour of eq. (41) is similar to dark energy universe. Secondly, we have

$$[4\frac{\ddot{a}}{a} + 2(\frac{\dot{a}}{a})^2]f_R - 5\frac{\dot{a}}{a}f_{R,0} - f_{R,00} = 0. \quad (42)$$

In (42), its behavior is similar to stiff universe.

4 Conclusion

In this article, $f(R, \phi)$ theory is discussed. We have considered Friedmann-Robertson-Walker metric for the analysis. Moreover, six cases of equation of state which are dark energy universe, dust universe, radiation universe, ultra relativistic universe, sub-relativistic universe and stiff universe models are discussed. Due to complex behavior of field equations, we used two different techniques. First is power law technique $a(t) = \alpha t^k$, where α is any constant and k be any real number. It is also observed that we obtain trivial solution for $k = 0$. Second technique is exponential law technique $a(t) = e^{lt}$, where l be any real number.

We developed field equations of dark energy universe by using the said techniques and investigated the behavior of density and pressure. It is noticed that density is positive and decreasing behavior for power law. Pressure has also the same results for this case. It is interesting that while discussing the solutions with power law technique, we find some singularities. Figure 2 shows the graphical analysis of density and pressure for exponential technique. This case of dark energy case is quite interesting because density and pressure has totally opposite behavior. Density is positive and pressure is negative, which is the validation of expansion of universe.

In ultra relativistic universe, the energy density has decreasing behavior for power law, which is similar to the dark energy universe but pressure in ultra relativistic universe has some different behavior. For exponential case, density and pressure has monotonic behavior. In this case, density is similar but pressure is in opposite direction as compared to Dark energy universe. Density and pressure for power law in Radiation Universe has decreasing behavior but it has some interesting result of pressure for exponential law. In Radiation Universe, the behavior of pressure is flat and negative, which is the validation of universe expanding. In sub-Relativistic Universe, density and pressure for exponential law has the similar graphical behavior as Ultra Relativistic Universe with some different values. Power law solution for this case has some different results as compared to previous discussed all the universes. Pressure was positive for Dark energy universe, Ultra Relativistic Universe and Radiation Universe for power law technique but it is negative for Sub-Relativistic Universe as well as Stiff universe which is again the validation of expansion of universe. The constraint equation of Stiff Universe for exponential law is exactly same as Sub-Relativistic Universe, that's why the graphical behavior for this case are same. Dust Universe has two types of solutions and both cases has similar results to the preceding Universes. In first Case, Dust universe has the same equation as Acceleration expansion universe, while in second case its behavior is like stiff universe. So Dust case has no different graphical behavior.

The main feature of this work is the 3-dimensional analysis in which some model parameters are fixed and some are chosen according to the constraint equations. The results may be varied because we have some options to choose the parameters. It is observed that density is always positive for power law as well as exponential law in all types of universe but pressure has dual nature. Some cases has the positive pressure and some has the negative pressure which is the validation of expansion of universe. Furthermore, it is concluded that some cases of equation of state have independent results and some are dependent on other models. The work can be extended by taking different model of $f(R, \phi)$ gravity as well as checking the behavior of the discussed $f(R, \phi)$ gravity model by varying the parameters which are considered fixed in the present study.

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