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Investigating Solvability and Complexity of Linear Active Networks by Means of Matroids

BJØRN PETERSEN

Abstract—**The solvability and complexity problems of *linear active networks* are approached from a purely combinatorial point of view, using the concepts of *matroid theory*. Since the method is purely combinatorial, we take into account *the network topology alone*. Under this assumption necessary and sufficient conditions are given for the unique solvability of linear active networks. The complexity and the number of dc-eigenfrequencies are also given.**

The method enables you to decide if degeneracies are due to the topology alone, or if they are caused by special relations among network parameter values. If the network parameter values are taken into account, the complexity and number of dc-eigenfrequencies given by the method, are only upper and lower bounds, respectively.

The above conditions are fairly easily checked, and the complexity and number of dc-eigenfrequencies are found, using *polynomially bounded algorithms* (matroid partition and intersection algorithms).

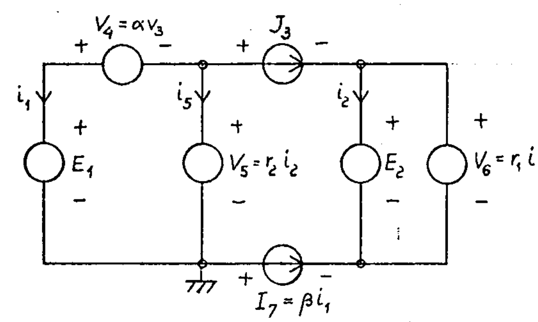

Fig. 1. The network N_1 of Example (1.1) and Example (6.3).

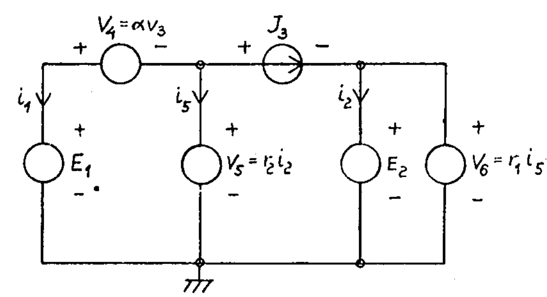

Fig. 2. The network N_2 of Example (1.2) and Example (6.4).

I. Introduction

In the theory of *RLC*-networks without controlled sources, combinatorial methods have been known for many years, which solve the following two essential problems.

1) *Decide if a certain network is solvable (i.e., if the network equations have a unique solution).*

2) *In case of a solvable network find a set of independent state variables.*

The graph theoretical tool is known as the normal tree method. This expression was originally used by Kuh and Rohrer [9] although the method is identical to that of Bryant [3]. The basic observation of this combinatorial method is, that *the topology alone* (i.e., the network graph and the type and position of the network elements) is sufficient to answer 1) and 2). These answers are *not* affected by *any* specific choice of the network parameter values.

If *RLC*-networks contain controlled sources then, in general, the topology and a specific choice of network parameter values must be taken into consideration answering 1) and 2). In most approaches this involves calculating the determinants of large matrices, although methods have been given to reduce the size of these matrices [4]. To determine the complexity, the normal tree method has been used to give a preliminary set of state variables [5], and conditions have been given for the increase or decrease of the complexity [6]. If a network is not solvable (in the following the network is then said to be singular) it is not possible however, by the above methods, to decide if the singularity is due to the topology or a specific choice of network parameter values.

In the following, examples are given, illustrating these topics.

Example (1.1)

The network N_1 of Fig. 1 is singular due to the topology alone (this is verified in Section VI). □

In most cases, however, singularity is also due to a specific choice of network parameter values. This is illustrated by the following example.

Example (1.2)

The network N_2 of Fig. 2 is solvable except for $\alpha = -1$ and/or $r_1 = 0$ and/or $r_2 = 0$ (this is verified in Section VI). □

In the same way the number of independent state variables, i.e., the complexity, may be restricted exclusively by the topology or by a specific choice of network parameter values.

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It has been the subject of many papers to develop methods making it possible to determine if singularity is due to the topology alone, and to give a better upper bound on the complexity, than the number of capacitors and inductors. In [11] a generalized tree is introduced to obtain sufficient conditions for capacitor voltages and inductor currents to be state variables. In [7], [14], and [21] the concept of two graphs is considered. The concept of two graphs was originally introduced by Mayeda [12], but [7], [14], and [21] cover a wider class of networks. In [7], [14], and [21] the network element descriptions are given, if not explicitly then essentially, by a hybrid immittance matrix. To each nonzero element in the hybrid immittance matrix a controlled and a controlling edge is introduced, which may increase the number of edges considerably, and two graphs, different from the network graph, is defined. In [7] and [21] necessary and sufficient conditions are given for solvability and complexity of linear active networks. Furthermore, [7] gives necessary and sufficient conditions for the existence of hybrid immittance descriptions. In [7] the conditions are given a matroid interpretation, which enables a unifying algorithmic approach, using the algorithm which finds a base in the union of matroids. In the present paper we will go one step further and use *matroids*, not only in the interpretation of the conditions, but in the mathematical model itself.

The constellation of electrical networks and matroids has been considered to be a very useful combinatorial tool in electrical network theory for some years [13]. The author was introduced to the subject through a seminar directed by Recski, Research Institute for Telecommunication, Budapest, during his visit to Denmark in the autumn of 1975. The basic idea of Recski [18] was the introduction of the *union* of matroids as a tool in the theory of linear active networks. The models of [19], however, treated current and voltage constrains separately, thus neglecting the algebraic constrains forced upon currents and voltages by, e.g., resistors. These models, therefore, did not allow the formulation of necessary and sufficient conditions for solvability and complexity.

The extension of the method of Recski [19], which is the subject of this paper, was conceived as a part of the author's M.Sc. thesis [15] during the spring of 1976. The basic idea of this model, is the introduction of *two copies* of the edge set of the network graph. The main new features is that the model introduce *no additional elements*, treat network element descriptions given by *any* algebraic matrix (not necessarily a hybrid immittance matrix; thus the model treat, e.g., *norators* and *nullators*), maintain a close relation between the network graph and the matroids treating the graphic constrains and introduce (by means of the union of matroids) the concept of *a network matroid* which give *compact* conditions, easily checked by polynomially bounded algorithms. Apart from necessary and sufficient conditions for solvability, complexity, and existence of hybrid immittance descriptions, we give necessary and sufficient conditions for the existence of *any* prescribed description. Furthermore, a modification proposed by the author [16], of the algorithm of [8], give the *zero* and *nonzero entries* of the corresponding matrices.

The paper contains more material than necessary for the explanation of the basic ideas in the mathematical model. It is the hope of the author, however, that this relatively detailed presentation will be appreciated by readers not familiar with matroid theory in advance. These readers are referred to the following section containing a brief introduction to matroid theory, and to an excellent expository paper by Wilson [22].

II. A Brief Introduction to Matroid Theory

A matroid M is a pair of sets: $M=(S,\mathfrak{B})$. The set S is called the *ground set* of the matroid, and $\mathfrak{B}$ is a set of subsets of the ground set, called *bases*, $B_1, B_2, \cdots, B_r$ with the following two properties: ($b1$) no base is properly contained in another base; ($b2$) if B_i and B_j are arbitrary bases, and $x_i \in B_i$, then an element $x_j \in B_j$ exists, such that $(B_i \setminus \{x_i\}) \cup \{x_j\}$ is an element of $\mathfrak{B}$ (i.e., is a base).

The set of base complements, denoted $\mathfrak{B}^*$, is the set of bases of a matroid $M^*=(S,\mathfrak{B}^*)$, called the *dual matroid* of M.

Trees (i.e., spanning trees) of graphs and maximal sets of independent columns of a matrix also have the properties ($b1$) and ($b2$). Thus matroid theory is the common *generalization* of *graph theory* and *linear algebra*. Notice, that graphs can be found for which the set of tree complements is *not* equal to the set of trees of another graph. These are the nonplanar graphs for which no dual graphs exist. In contrast to this the dual of a matroid *always* exists.

To give an idea of the concept of matroids, the following examples are given.

Example (2.1)

Let H denote the graph of Fig. 3. If $\mathfrak{T}(H)$ denotes the set of trees of H, then

$$\mathfrak{T}(H)=\{\{a,c\},\{a,d\},\{a,e\},\{b,c\},\{b,d\},\{b,e\},\{c,e\},\{d,e\}\}.$$

It can be shown that $\mathfrak{T}(H)$ equals the set of bases of a matroid $M(H)$, with ground set $S=\{a,b,c,d,e\}$, called the *circuit matroid* (or *polygon matroid*) of H. The dual matroid of $M(H)$ is called the *cutset matroid* of H, and is denoted $M^*(H)$. Every subset of a base is called an *independent* set, e.g., $\{a\}$, $\{b\}$, $\{c\}$, and $\{d\}$ are independent sets. Notice, that the empty set is an independent set. The subsets of the ground set, which are not independent, are said to be *dependent*, e.g., $\{a,b,c\}$, $\{b,c,d,e\}$, and $\{c,d,e\}$ are dependent sets of $M(H)$. The minimal dependent sets are called the *circuits* of the matroid, e.g., $\{a,b\}$, $\{b,c,e\}$, and $\{c,d\}$ are circuits of $M(H)$. The circuits of the dual matroid is called the *cocircuits* (or *cutsets*) of the matroid, e.g., $\{a,b,e\}$, $\{c,d,e\}$, and $\{a,b,c,d\}$ are cocircuits of $M(H)$. The cardinality, i.e., the number of elements, of a base is called the *rank* of the matroid (the

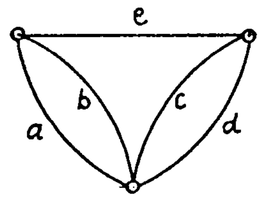

Fig. 3. The graph H of Example (2.1).

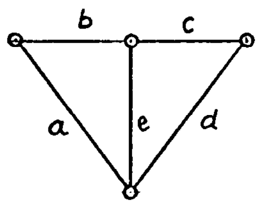

Fig. 4. The dual graph H^* of Example (2.1).

bases are equicardinal). The rank of $M(H)$ in this example equals two.

Let $\mathfrak{T}^*(H)$ denote the set of bases of the dual matroid of $M(H)$. Then

$$\mathfrak{T}^*(H)=\{\{b,d,e\},\{b,c,e\},\{b,c,d\},\{a,d,e\},\{a,c,e\},\{a,c,d\},\{a,b,d\},\{a,b,c\}\}$$

and $M^*(H)=(S,\mathfrak{T}^*(H))$. The set of bases $\mathfrak{T}^*(H)$, equals the set of trees of the graph H^* shown in Fig. 4, i.e., $\mathfrak{T}^*(H)=\mathfrak{T}(H^*)$. Thus $M^*(H)=(S,\mathfrak{T}^*(H))=(S,\mathfrak{T}(H^*))=M(H^*)$, i.e., $M(H^*)$ is the circuit matroid of H^* and the cutset matroid of H. It may be noticed that H^* is the dual of the graph H. This is always the case if H is *planar*. □

Example (2.2)

Let $\{a,b,c,d,e\}$ denote the column set of the matrix A of

$$A=\begin{array}{c} \\ \\ \end{array}\begin{array}{ccccc} a & b & c & d & e \end{array}$$
$$A=\begin{bmatrix} 3 & 0 & 0 & -2 & 0 \\ 9 & 0 & 0 & -6 & 0 \\ 6 & 0 & 0 & -4 & 0 \end{bmatrix}. \qquad (1)$$

If $\mathfrak{S}(A)$ denotes the set of maximal sets of independent columns of A, then $\mathfrak{S}(A)=\{\{a\},\{d\}\}$. It can be shown that $\mathfrak{S}(A)$ equals the set of bases of a matroid $M(A)$ with ground set $S=\{a,b,c,d,e\}$, called the *matric* matroid of A. In this example the rank of $M(A)$ equals one. □

Example (2.3)

Two arbitrary matroids are said to be *isomorphic*, if it is possible to establish a one-to-one correspondance between the ground sets of the matroids, such that a subset, in one of the matroids, is independent iff the corresponding subset is independent in the other matroid.

Let G denote the graph of Fig. 5 and $M(G)$ the circuit matroid of G. Then $M(A)$ of Example (2.2), is isomorphic

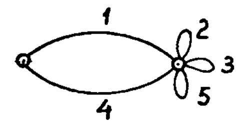

Fig. 5. The graph G, the circuit matroid of which is isomorphic to the matric matroid $M(A)$ of Example (2.2).

to $M(G)$, since the following one-to-one correspondence satisfies the condition: $a\leftrightarrow 1$, $b\leftrightarrow 2$, $c\leftrightarrow 3$, $d\leftrightarrow 4$, $e\leftrightarrow 5$.

If for an arbitrary matroid M, a graph G exists such that M is isomorphic to $M(G)$, the matroid M is said to be *graphic*. Thus the matroid $M(A)$ of Example (2.2) is graphic. □

Example (2.4)

The common ground set of $M(H)$ (Example (2.1)) and $M(A)$ (Example (2.2)) is $S=\{a,b,c,d,e\}$. If we take all possible unions of elements from $\mathfrak{T}(H)$ and $\mathfrak{S}(A)$, and denote by $\mathfrak{B}(U)$ the set of unions with maximal cardinality, then,

$$\mathfrak{B}(U)=\{\{a,b,c\},\{a,b,d\},\{a,b,e\},\{a,c,d\},\{a,c,e\},\{a,d,e\},\{b,d,e\},\{b,d,e\},\{c,d,e\}\}.$$

It can be shown that $\mathfrak{B}(U)$ equals the set of bases of a new matroid $M(U)$, with ground set S, obtained from $M(H)$ and $M(A)$, in the above way. The matroid $M(U)$ is called the *union* (*sum*) of $M(H)$ and $M(A)$, and is denoted $M(U)=(S,\mathfrak{B}(U))=M(H)\vee M(A)$.

In this example $M(U)$ can be shown not to be graphic. □

III. Dependencies Among Voltages and Currents

The idea of using *graphs* in electrical network theory arises from the fact that the basic algebraic Kirchhoff equations are formulated in graph theoretical terms. When no controlled sources are present the dependencies among the voltages are given by the *circuits* of the graph, and the dependencies among the currents are given by the *cutsets* of the graph. Since these dependencies are vital for the solvability and the complexity, graphs as a combinatorial tool have been used for many years. But *matroids* could have been used as well [1], [2]. Instead of the network graph, we could have used the circuit matroid of the graph or the cutset matroid of the graph. (In fact the reader may choose for himself, whether he wants to treat these graphic dependencies as *circuits* of matroids or as *cocircuits* (*cutsets*) of matroids). The main results of this paper are entirely based on consideration of *matroid cocircuits*, and their relation to certain matrices.

The all-important point in using matroids is that matroid theory combines and generalizes graph theory and linear algebra. Matroids are, therefore, the perfect tool for handling, and unifying, the graphic as well as the nongraphic, that is, algebraic dependencies encountered in linear active networks.

IV. The General Case and G-Solvable Networks

In order to determine if singularity is due to the network topology exclusively, the network parameter values must be *eliminated* from the treatment. This can be done by assuming all parameters to be *general*. Generality means that if a specific choice of the network parameter values would cause a network matrix under consideration to become singular, then the parameter values are considered to be different from those specific choices. A more precise definition of generality is given by the following definition.

Definition (4.1)

An electrical network is g-solvable iff at least one set of network parameter values exists for which the network is solvable.

Since matrices play an important role in the following sections, two more definitions concerning the generality are given.

Definition (4.2)

A matrix, containing general parameters, is g-nonsingular iff at least one set of parameter values exists for which the matrix is nonsingular (i.e., the determinant is nonzero).

Definition (4.3)

A set of columns of a matrix, containing general parameters, is g-linearly independent iff at least one set of parameter values exists for which the columns are linearly independent.

Notice, that ordinary nonsingularity (solvability) *implies* g-nonsingularity (g-solvability).

V. The Matroids of the Model

As mentioned in Section III *all* the dependencies will be treated by the *cocircuits* of appropriate matroids.

In the following S denotes the edge set of the network graph H. To each element $s \in S$ we associate two other elements s_i and s_v *corresponding* to the current and the voltage of the network element, respectively. The two associated sets will be denoted S_I and S_V, i.e., $S_I = \{s_i | s \in S\}$ and $S_V = \{s_v | s \in S\}$. The same notation will be used if subsets of S, S_I, and S_V are considered.

Let $\boldsymbol{Q}_f$ denote a fundamental cutset matrix (over the reals) of the directed graph H. From graph theory it is well known [20], that a subset of columns of $\boldsymbol{Q}_f$ is linearly independent iff the columns correspond to a circuit-free subgraph of H. Now the graphic dependencies among the currents due to KCL will be treated by the following matroid G_I.

Definition (5.1)

G_I is a matroid on the ground set S_I. A subset of S_I is defined to be independent in G_I iff the corresponding columns of $\boldsymbol{Q}_f$ are linearly independent.

It is implied by the above description that G_I *is isomorphic* to the *circuit matroid $M(H)$*. Thus the cocircuits of G_I treat KCL.

Similarly, let $\boldsymbol{B}_f$ denote a fundamental circuit matrix (over the reals) of the directed graph H. Now a subset of columns of $\boldsymbol{B}_f$ is linearly independent iff the columns correspond to a cutset-free subgraph of H [20]. The graphic dependencies among the voltages due to KVL will be treated by the following matroid G_V.

Definition (5.2)

G_V is a matroid on the ground set S_V. A subset of S_V is defined to be independent in G_V iff the corresponding columns of $\boldsymbol{B}_f$ are linearly independent.

It is implied by the above description that G_V *is isomorphic to the cutset matroid $M^*(H)$*, the dual of $M(H)$. Thus the cocircuits of G_V treat KVL.

Since the ground sets of G_I and G_V are disjoint, all graphic dependencies among voltages and currents due to KVL and KCL can be merged by the matroid operation: *direct sum*. The direct sum of two arbitrary matroids M_1 and M_2 on *disjoint* ground sets S_1 and S_2 is denoted $M_1 \oplus M_2$. A set X is independent in $M_1 \oplus M_2$ iff $X \cap S_1$ is independent in M_1 and $X \cap S_2$ is independent in M_2. All the above graphic dependencies will be treated by the following matroid G.

G is a matroid on the ground set $S_I \cup S_V$: $G = G_I \oplus G_V$.

In a similar way *all nongraphic*, that is, algebraic dependencies among voltages and currents, such as control equations, memoryless n-port descriptions and resistive element equations may be treated by a matroid.

Let $\boldsymbol{A}$ denote the matrix containing all the above mentioned algebraic equations, such that $\boldsymbol{A}\cdot(i_1, i_2, \cdots, i_{|S|}, v_1, v_2, \cdots, v_{|S|})^{\mathrm{T}} = \underline{0}$. In the general case $\boldsymbol{A}$ contains in each row, precisely one entry equal to 1 which is no network parameter, and the remaining nonzero elements represent different network parameters. Then the algebraic dependencies will be treated by the following matroid A.

Definition (5.3)

A is a matroid on the ground set $S_I \cup S_V$. A subset of $S_I \cup S_V$ is defined to be independent in A iff the corresponding columns of $\boldsymbol{A}$ are g-linearly independent.

More specifically, a subset S' of $S_I \cup S_V$ is independent in A iff the matrix formed by the corresponding columns in $\boldsymbol{A}$, contains at least one g-nonsingular maximal square submatrix $\boldsymbol{A}'$ (see (2)).

$$\boldsymbol{A} = \left[\qquad \overbrace{\boxed{\boldsymbol{A}'}}^{S'} \qquad \right]. \tag{2}$$

Finally, *all information* about the *dependencies* (i.e., Kirchhoff constraints as well as element and n-port constraints) in the network can be merged into one single matroid M using the matroid operation: union (or sum). The union of two arbitrary matroids M_1 and M_2 on the *same* ground set S_0 is denoted $M_1 \vee M_2$. By definition a subset $X \subseteq S_0$ is independent in $M_1 \vee M_2$ iff $X = X_1 \cup X_2$ where X_1 is independent in M_1 and X_2 is independent in M_2. Then,

M is a matroid on the ground set $S_I \cup S_V$: $M = G \vee A = (G_I \oplus G_V) \vee A$.

M is called the *network matroid.*

To check if a certain set is independent in M, it is not necessary to construct the whole set of bases of M. It is the advantage of using matroids, that polynomially bounded algorithms are available [8], [10], i.e., algorithms that terminate in *at most* a number of steps given by a *polynomial* in *the size of the problem* (i.e., in this case the number of elements of the ground set). These algorithms [8], [10], check independence in M, knowing the independent sets of G and A only. Normally, much storage space in a computer is required to check independence in G and A. For the type of problems considered here, however, this can be simplified. The simplification in G, is due to the fact, that the network graph H contains *all* information about G. In A the simplification is a consequence of Corollary 6.2 given in the following section. Thus the conditions in the following theorems concerning matroids, can fairly easily be checked.

VI. Networks Without Capacitors and Inductors

Let N denote the coefficient matrix of the total system of network equations (see (3)).

$$N\begin{bmatrix} i \\ v \end{bmatrix} = \underline{0} \tag{3}$$

where

$$N = \begin{array}{c} \overset{\longleftarrow 2|S| \longrightarrow}{\left[\begin{array}{c|c} Q_f & 0 \\ \hline 0 & B_f \\ \hline \multicolumn{2}{c}{A} \end{array}\right]} \end{array} \begin{array}{l} |S| \\ \\ |S|-g \end{array} = \begin{bmatrix} G \\ \hline A \end{bmatrix}.$$

Let E^1 denote the set of independent voltage sources and E_V^1 denote the corresponding subset of S_V. Similarly let I^1 denote the set of independent current sources and I_I^1 denote the corresponding subset of S_I. Now interchange columns of N to obtain the matrix N' (see (4)), where the columns of N_1 correspond to $E_V^1 \cup I_I^1$.

$$N' = [\, N_0 \;\vdots\; N_1 \,]. \tag{4}$$

If g is the number of independent sources then N_0 consists of $2|S|-g$ rows and columns and N_1 consists of $2|S|-g$ rows and g columns. Now the network is solvable iff the determinant of N_0 differs from zero. If the network parameters are assumed to be general, then it is the advantage of using matroids that the determinant need not be calculated. The question of g-solvability is answered by the following theorem.

Theorem (6.1)

Networks without capacitors and inductors are g-solvable iff $E_V^1 \cup I_I^1$ is a base of M^*.

Proof: "$\Rightarrow$": Expand the determinant of N_0 by $|S|-g$ columns of A, using the Laplace expansion formula. Since $\det N_0 \neq 0$, at least one term in the expansion, e.g., $\det G_0 \cdot \det A_0$, is different from zero (see (5)). The columns of A_0 are linearly independent, and thus the subset of $S_I \cup S_V$ corresponding to these columns form a base of A. Similarly the elements corresponding to the columns of G_0 form a base of G, and since the bases are disjoint, the union is a base of M. Thus $E_V^1 \cup I_I^1$ is a base of M^*.

$$N' = \begin{array}{c} \overset{|\leftarrow |S| \rightarrow| \qquad |\leftarrow g \rightarrow|}{\left[\begin{array}{c:c:c} G_0 & & \\ \hdashline & A_0 & N_1 \end{array}\right]} \\ |\longleftarrow 2|S| \longrightarrow| \end{array} \begin{array}{l} |S| \\ \\ |S|-g \end{array}. \tag{5}$$

"$\Leftarrow$": If $E_V^1 \cup I_I^1$ is a base of M^* then the rank $r_M = 2|S| - g$. Since $r_M \leq r_G + r_A \leq |S| + (|S|-g)$, $r_M = r_G + r_A$. Thus g-nonsingular column disjoint submatrices G_0 and A_0 exist (see (5)). The determinant of G_0 equals ± 1. Since A_0 is g-nonsingular, at least one set of network parameter values exists, such that A_0 is nonsingular. Choose the network parameter values of A_0 to equal such a set, and choose arbitrarily a set of finite values for the remaining network parameters. Now at least one term in the expansion of the determinant of A_0 differs from zero. Thus a permutation of the columns of A_0 exists such that the diagonal elements of the resulting matrix are nonzero. Some of these distinguished nonzero elements represent network parameters, others are the nonparameter element 1. Now imagine all distinguished nonzero elements to equal a common variable x, while all other elements remain unchanged. The determinant of N_0 thus is a polynomial in x. Furthermore the determinant differs from the zero polynomial, since the coefficient of $x^{(|S|-g)}$ equals $\det(G_0)$, i.e., ± 1. Thus an infinite number of finite and nonzero values x_0 exist, such that the determinant of N_0 differs from zero. Choose one of these values. If a distinguished nonzero element represents a network parameter, then that parameter assumes the value x_0. If the distinguished nonzero element is the nonparameter element 1, the values of the network parameters in that row are divided by x_0 (i.e., the row is scaled by x_0). Thus a set of network parameter values has been found such that the determinant of N_0 differs from zero, i.e., the network is g-solvable. □

Notice that the proof of the necessity did not use the generality at all. More specifically the solvability follows from the fact that, under the conditions of Theorem (6.1) all voltages and currents $y^\top$ corresponding to elements not in $E_V^1 \cup I_I^1$, can be expressed as a unique linear combination of the voltages and currents $x^{\top}$ corresponding to elements of $E_V^1 \cup I_I^1$:

$$y^{\top} = -N_0^{-1} N_1 x^{\top}. \tag{6}$$

The following corollary is an immediate consequence of the proof of Theorem (6.1).

Corollary (6.2)

Let A_0 denote a maximal square submatrix of A. Then A_0 is g-nonsingular iff a permutation of the columns of A_0 can be found such that the resulting matrix has diagonal elements which are nonzero.

Similarly a subset of columns of A are g-linearly independent iff the matrix formed by these columns contains at least one maximal square submatrix, for which the desired permutation of columns can be found.

Now consider a bipartite graph, the vertices of which correspond to the rows and the columns of A, respectively, and the edges of which correspond to the nonzero elements of A. Then the algorithm which finds a maximal matching in a bipartite graph, can be used to find a maximal square submatrix with the desired properties, and a permutation of columns of A_0. Hence, the test for independence in A is fairly simple.

To give an idea of the matroid M, notice that the matroid $M-(E_V^1 \cup I_I^1)$ obtained from M by *deleting* $E_V^1 \cup I_I^1$, in case of regularity is the *free matroid*, i.e., $M-(E_V^1 \cup I_I^1)$ contains no circuits.

$$\begin{array}{c}
\begin{array}{cccccccccccccc} 1_i & 2_i & 3_i & 4_i & 5_i & 6_i & 7_i & 1_v & 2_v & 3_v & 4_v & 5_v & 6_v & 7_v \end{array} \\
N=\left[\begin{array}{ccccccc|ccccccc}
1 & 0 & 0 & 0 & 1 & 0 & -1 & & & & & & & \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & & & & \mathbf{0} & & & \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & & & & & & & \\
0 & 0 & 0 & 1 & -1 & 0 & 1 & & & & & & & \\
\hline
 & & & & & & & -1 & 0 & 0 & 1 & 1 & 0 & 0 \\
 & & & \mathbf{0} & & & & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\
 & & & & & & & 1 & -1 & -1 & -1 & 0 & 0 & 1 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\alpha & 1 & 0 & 0 & 0 \\
0 & -r_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & -r_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
-\beta & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
=\left[\begin{array}{c|c} \boldsymbol{Q}_f & \mathbf{0} \\ \hline \mathbf{0} & \boldsymbol{B}_f \\ \hline \multicolumn{2}{c}{\boldsymbol{A}} \end{array}\right].
\end{array} \tag{7}$$

Furthermore, the fundamental circuits of M with respect to the base $(S_I \cup S_V) \backslash (E_V^1 \cup I_I^1)$ and defined by the elements of the base complement $E_V^1 \cup I_I^1$, are related to the nonzero elements of $N_0^{-1} N_1$, in the following way. The element in the ith row and jth column of $N_0^{-1} N_1$ is nonzero, iff the element corresponding to y_i is an element of the fundamental circuit of M with respect to the base $(S_I \cup S_V) \backslash (E_V^1 \cup I_I^1)$ and defined by the element of the base complement $E_V^1 \cup I_I^1$ corresponding to x_j (see Fig. 6). The elements of these fundamental circuits of M are easily obtained using the algorithm which checks the condition of Theorem (6.1).

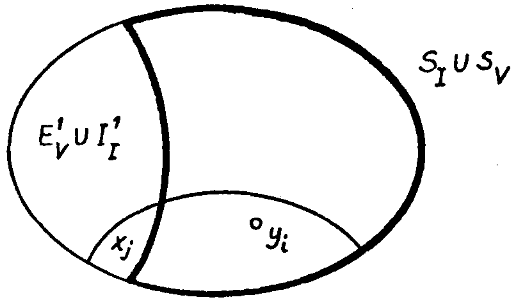

Fig. 6. The area bounded by heavy lines is the base $(S_I \cup S_V) \backslash (E_V^1 \cup I_I^1)$ of M.

The examples of Section I will now be considered more closely. In the following examples the total system of network equations and the associated matroids will be given. If the matroids are graphic, graphs will be given, such that the matroids are the circuit matroids of the graphs. It may be noticed that, if the graph H, corresponding to G_I, is planar, then the graph corresponding to G_V is the dual of H.

Example (6.3)

The network N_1, of Example (1.1), is shown in Fig. 1. The total system of network equations is given in (7). The submatrix N_0 is formed by the columns corresponding to 1_i, 2_i, 4_i, 5_i, 6_i, 7_i, 3_v, 4_v, 5_v, 6_v, 7_v.

The associated matroids are shown in Fig. 7 (in this case the matroids G_I, G_V, and A are graphic).

According to Theorem (6.1), the matrix N_0 is g-nonsingular iff $E_V^1 \cup I_I^1 = \{1_v, 2_v, 3_i\}$ is a base of M^*, i.e., the complementary set $B = \{3_v, 4_v, 5_v, 6_v, 7_v, 1_i, 2_i, 4_i, 5_i, 6_i, 7_i\}$ is a base of M. A reformulation of this condition is, that disjoint bases of G_I, G_V, and A exist, such that the union of these bases equals the complementary set B.

To find out whether such disjoint bases exist, $\{1_v, 2_v, 3_i\}$ is deleted from the matroids (deleted elements are denoted by crossed edges in Fig. 7). Since $\{6_v, 2_v\}$ is a cocircuit (cutset) in G_V, $\{6_v\}$ must be an element of B (the elements forced into B, are denoted by heavy edges in Fig. 7). Now, $\{6_v, 5_i\}$ is a cocircuit (cutset) of A, and $\{5_i\}$ must be an

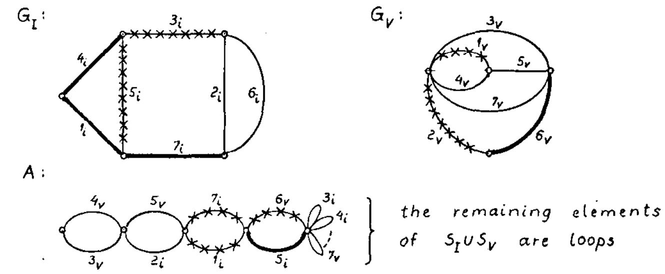

Fig. 7. The matroids G_I, G_V, and A associated with the network N_1 of Example (6.3) (Fig. 1).

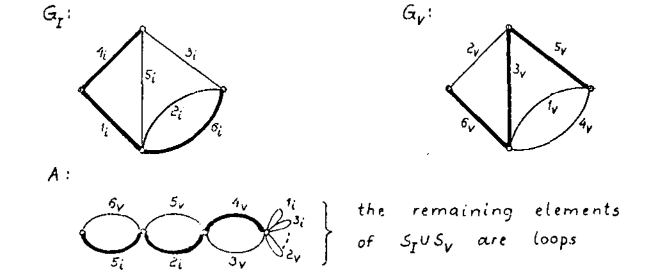

Fig. 8. The matroids G_I, G_V, and A associated with the network N_2 of Example (6.4) (Fig. 2).

element of B. When $\{3_i, 5_i\}$ is deleted from G_I, $\{4_i, 1_i, 7_i\}$ must be contained in B. So the disjoint bases, mentioned in the condition, do not exist, since $\{1_i, 7_i\}$ is a cocircuit (cutset) of A. Thus, according to Theorem (6.1), the determinant of N_0 equals zero for any choice of the network parameter values. □

Example (6.4)

The network N_2, of Example (1.2), is shown in Fig. 2. The associated matroids are given in Fig. 8 (in this case G_I, G_V, and A are graphic).

It is easily checked that $\{1_v, 2_v, 3_i\}$ is a base of M^*, i.e., $\{1_i, 2_i, 4_i, 5_i, 6_i, 3_v, 4_v, 5_v, 6_v\}$ is a base of M (the heavy edges of Fig. 8). Thus N_2 is g-solvable.

If specific choices of network parameter values are made, N_2 may become singular. To find these specific choices, the total system of network equations is considered (8).

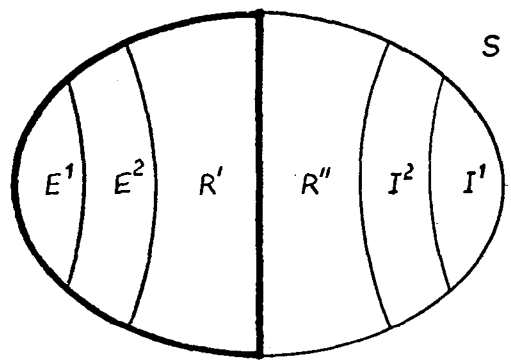

Fig. 9. The area bounded by heavy lines is the tree T of H.

equals $r_1 r_2(1+\alpha)$. Thus N_2 is singular iff $r_1=0$, $r_2=0$, or $\alpha=-1$. □

Theorem (6.1) has the following interesting corollary.

Corollary (6.5)

Networks with independent and controlled sources and resistors only, containing no circuits formed by voltage sources alone and no cutsets formed by current sources alone, are g-solvable.

Proof: Let E^2 and I^2 denote the set of controlled voltage- and current sources, respectively. If H denotes the network graph, then $E^1 \cup E^2$ contains no circuits of H. Similarly, $I^1 \cup I^2$ contains no cutset of H. Now, it is well known ([20], Theorem 6–10) that a tree $\top$ can be found such that $E^1 \cup E^2 \subseteq \top$ and $I^1 \cup I^2 \subseteq \top^*$, where $\top^*$ denotes the tree complement. Since G_I is isomorphic to $M(H)$ and G_V is isomorphic to $M^*(H)$, $(\top)_I$ and $(\top^*)_V$ are bases of G_I and G_V, respectively, (see Fig. 9). Thus $(\top)_I \cup (\top^*)_V$ is a base of G.

Let R' and R'' denote the resistors contained in $\top$ and $\top^*$, respectively. Choose from A the elements $D = E_V^2 \cup (R')_V \cup (R'')_I \cup I_I^2$.

The row of A, corresponding to an element of $E^2 \cup R \cup I^2$, contains a nonzero element in the column of A, corresponding to that specific element of D. Since the cardinality of D equals the number of rows of A, and since $|D|$ nonzero elements, in different rows and col-

$$
\begin{array}{c} \\ N= \end{array}
\begin{array}{c}
\begin{array}{cccccc|cccccc}
1_i & 2_i & 3_i & 4_i & 5_i & 6_i & 1_v & 2_v & 3_v & 4_v & 5_v & 6_v
\end{array}\\
\left[\begin{array}{cccccc|cccccc}
1 & 0 & 0 & 1 & 0 & 0 & & & & & & \\
0 & 1 & 0 & -1 & 1 & 1 & & & \mathbf{0} & & & \\
0 & 0 & 1 & -1 & 1 & 0 & & & & & & \\
\hline
 & & & & & & -1 & 1 & 1 & 1 & 0 & 0 \\
 & & \mathbf{0} & & & & 0 & -1 & -1 & 0 & 1 & 0 \\
 & & & & & & 0 & -1 & 0 & 0 & 0 & 1 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\alpha & 1 & 0 & 0 \\
0 & -r_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & -r_1 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right]
\end{array}
= \left[\begin{array}{c|c}
Q_f & \mathbf{0} \\
\hline
\mathbf{0} & B_f \\
\hline
\multicolumn{2}{c}{A}
\end{array}\right]. \tag{8}
$$

The submatrix N_0 is formed by the columns corresponding to $i_1, i_2, i_4, i_5, i_6, v_3, v_4, v_5, v_6$. The determinant of N_0

umns, have been found, D is a base of A (according to Corollary (6.2)). Since D and $\top_I \cup (\top^*)_V$ are disjoint,

$D \cup T_I \cup (T^*)_V$ is a base of M, and thus $E_V^1 \cup I_I^1$ is a base of M^*. Hence, according to Theorem (6.1), the network is g-solvable. □

Sufficient conditions, very much similar to those of Corollary (6.5), have been given by Ozawa [14, theorem 2].

The conditions of Corollary (6.5) are sufficient only. Example (1.2) gives a counterexample to the necessity. However, necessary conditions for solvability, have been given by Purslow [17]. The conditions of Purslow allow circuits formed by voltage sources, if the current in at least one of the elements of the circuit is a controlling quantity somewhere in the network. Similarly cutsets formed by current sources are allowed, if the voltage in at least one of the elements of the cutset is a controlling quantity. The conditions of Purslow are necessary only, which is shown in Example (1.1).

Sufficient conditions, similar to those of Corollary (6.5) can be given, if arbitrary memoryless n-ports are allowed in the network.

VII. N-Port Descriptions

From the networks treated in Section VI, the memoryless n-ports naturally arise, in the following way. Point out n arbitrary pairs of vertices, which will be considered as the ports of the n-port. Then insert an edge for each port. These edges are called port-edges, and the set of port edges is denoted by P. If the network contains independent sources, then for each source, the pair of vertices incident to the source are considered to be one of the n-ports.

Let x denote a n-tuple containing port voltages and/or port currents, and y denote another n-tuple containing the remaining port voltages and port currents. Then $y = Nx$ is called *a matrix description* of the n-port.

Example (7.1)

A 2-port N_3 is given in Fig. 10.

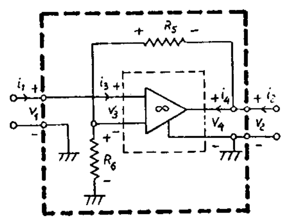

Fig. 10. The 2-port N_3 of Example (7.1).

The *norator-nullator* equivalent (considered here as a 2-port within N_3) is used for the ideal operational amplifier, and the equivalent network is shown in Fig. 11. The description of the norator-nullator pair is given in (9).

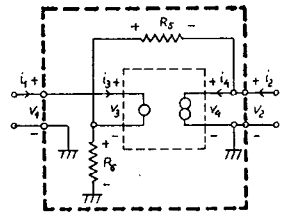

Fig. 11. The equivalent network N_3' of Example (7.1).

$$\begin{bmatrix} v_3 \\ i_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_4 \\ i_4 \end{bmatrix}. \qquad (9)$$

The total system of the network equations is given in (10).

$$\begin{array}{c} \\ N = \end{array}
\begin{array}{c}
\begin{array}{cccccccccccc} 1_i & 2_i & 3_i & 4_i & 5_i & 6_i & 1_v & 2_v & 3_v & 4_v & 5_v & 6_v \end{array} \\
\left[\begin{array}{cccccc|cccccc}
1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 1 \\
\hline
0 & 0 & 0 & 0 & -R_5 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & -R_6 & 0 & 0 & 0 & 0 & 0 & 1 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
\end{array}
= \left[\begin{array}{c|c} Q_f & 0 \\ \hline 0 & B_f \\ \hline \multicolumn{2}{c}{A} \end{array}\right]. \qquad (10)$$

A possible description of the 2-port is given in (11).

$$\begin{bmatrix} i_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \dfrac{R_5}{R_6} + 1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ i_2 \end{bmatrix}. \qquad (11)$$

□

The question of interest, is to find the types of matrix descriptions, if any at all. This question is answered by the following theorem.

Theorem (7.2)

There exists at least one description iff there exist subsets P^1, P^2 of P such that $|P^1| + |P^2| = |P|$ and $P_V^1 \cup P_I^2$ is a base of M^*.

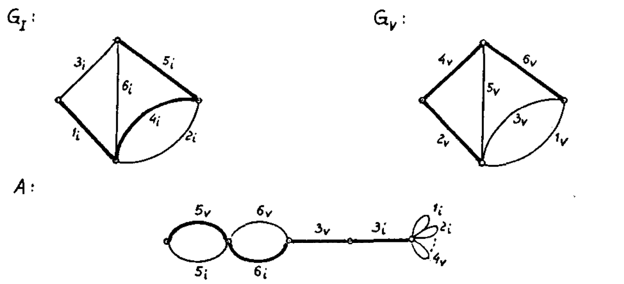

Fig. 12. The matroids G_I, G_V, and A associated with the 2-port N_3 of Example (7.1) (Fig. 10 or Fig. 11).

If a description exists, the elements of $P_V^1 \cup P_I^2$ *correspond to the elements of* $\boldsymbol{x}$.

The theorem can easily be proved, following the ideas in the proof of Theorem (6.1). The only thing which has to be noticed is that P^1 and P^2 need not be disjoint. If they are disjoint, $P_V^1 \cup P_I^2$ is said to be $I-V$-disjoint, and $\boldsymbol{y}=\boldsymbol{N}\boldsymbol{x}$ is called *a hybrid immittance description* (see, e.g., the description (11)).

The well-known open circuit impedance description and the short circuit admittance description, are both hybrid immittance descriptions. Theorem (7.2) is illustrated by the following example.

Example (7.3)

The 2-port of this example is the 2-port of Example (7.1), shown in Fig. 10. The matroids of the 2-port is shown in Fig. 12. It is easily verified that $\{1_v, 2_i\}$ and $\{2_v, 2_i\}$ are bases of M^* ($(S_I \cup S_V)\backslash\{1_v, 2_i\}$ is a base of M, which is illustrated by the heavy edges of Fig. 12). The set $\{1_v, 2_i\}$ corresponds to the description (11) and the set $\{2_v, 2_i\}$ corresponds to the description (12). Since a 2-port is considered, the description (12) is called *a chain description.*

$$\begin{bmatrix} v_1 \\ i_1 \end{bmatrix} = \begin{bmatrix} \dfrac{R_6}{R_5+R_6} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_2 \\ -i_2 \end{bmatrix}. \tag{12}$$

The zero entries of the descriptions (11) and (12) could have been predicted, since none of the sets $\{1_v, 1_i\}$, $\{1_v, 2_v\}$, $\{1_i, 2_i\}$, and $\{1_i, 2_v\}$ are bases of M^*. If, for example, the element of the second row and the first column in the description (12) had been nonzero, a description with $\boldsymbol{x}=(i_1, i_2)$ would have existed. Thus $\{1_i, 2_i\}$ would have been a base of M^*, which however, is not the case.

Furthermore, the element in the ith row and jth column of a description is nonzero (in the general case) iff the element corresponding to y_i is an element of the fundamental circuit of M with respect to the base $(S_I \cup S_V)\backslash(P_V^1 \cup P_I^2)$ and defined by the element of the base complement $P_V^1 \cup P_I^2$ corresponding to x_j (see Fig. 2). Theorem (7.2) has the following corollary.

Corollary (7.4)

There exists at least one hybrid immittance description iff there exist subsets P^1, P^2 *of* P *such that* $P^1 \cap P^2 = \varnothing$, $P^1 \cup P^2 = P$, *and* $P_V^1 \cup P_I^2$ *is a base of* M^*.

If a description exists, the elements of $P_V^1 \cup P_I^2$ *correspond to the elements of* $\boldsymbol{x}$.

Corollary (7.4) answers the following interesting questions. Is it possible to attach independent sources to the ports, such that the obtained network is g-solvable? If it is possible, how can it be done? Thus in Example (7.1) the only possible set of independent sources is a voltage source at port 1 and a current source at port 2, if the obtained network is to be g-solvable.

VIII. Networks Containing Capacitors and Inductors

The results and ideas of Section VI, now enable us to solve the interesting problem concerning the complexity of the network.

The following theorem gives the exact number in the general case, in the following denoted *g-complexity*, and an upper bound if specific choices of network parameter values are taken into consideration.

In the following L and C denote the set of inductors and capacitors, respectively. Subsets of L and C are denoted L^0 and C^0, respectively.

Theorem (8.1)

If $|C_V^0 \cup L_I^0|$ *is maximum with respect to* $E_V^1 \cup C_V^0 \cup (L\backslash L^0)_V \cup (C\backslash C^0)_I \cup L_I^0 \cup I_I^1$ *being a base of* M^* *then* $|C_V^0 \cup L_I^0|$ *is the g-complexity of the network, and the capacitor voltages and inductor currents corresponding to the elements of* $C_V^0 \cup L_I^0$ *form a set of independent state variables.*

Proof: First the statement on the complexity is proved. Let the number of reactive elements (i.e., inductors and capacitors) be k, and put the total system of network equations in the form of (13):

$$\begin{bmatrix} \boldsymbol{N} \\ \hline \boldsymbol{Z} \end{bmatrix} \cdot \begin{bmatrix} i \\ v \end{bmatrix} = \boldsymbol{T} \cdot \begin{bmatrix} i \\ v \end{bmatrix} = 0 \tag{13}$$

such that $\boldsymbol{Z}$ represents the Laplace transformed element descriptions of the inductors and capacitors (see (14)) where $\boldsymbol{L}$ and $\boldsymbol{C}$ represent diagonal matrices and $\boldsymbol{U}$ represents a unit matrix).

$$\overset{\longleftarrow 2|S| \longrightarrow}{}$$
$$\boldsymbol{Z} = \Bigg[\overbrace{\underbrace{\begin{matrix} -\boldsymbol{U} \\ \boldsymbol{0} \end{matrix}}_{C_I} \Bigg| \underbrace{\begin{matrix} \boldsymbol{0} \\ s\boldsymbol{L} \end{matrix}}_{L_I}}^{(C\backslash C^0)_I L_I^0} \Bigg| \begin{matrix} \boldsymbol{0} \\ \boldsymbol{0} \end{matrix} \Bigg| \overbrace{\underbrace{\begin{matrix} s\boldsymbol{C} \\ \boldsymbol{0} \end{matrix}}_{C_V}}^{C_V^0} \Bigg| \overbrace{\underbrace{\begin{matrix} \boldsymbol{0} \\ -\boldsymbol{U} \end{matrix}}_{L_V}}^{(L\backslash L^0)_V} \Bigg| \begin{matrix} \boldsymbol{0} \\ \boldsymbol{0} \end{matrix} \Bigg] \; \updownarrow k. \tag{14}$$

Now interchange columns of $\boldsymbol{T}$ to obtain the matrix $\boldsymbol{T}'$ (see (15)), where the columns of $\boldsymbol{T}_1$ correspond to $E_V^1 \cup I_I^1$. The columns of $\boldsymbol{T}_0$ form a $(2|S|-g)\times(2|S|-g)$ matrix. The determinant of $\boldsymbol{T}_0$ is a polynomial in s, the degree of which is the complexity of the network.

$$\boldsymbol{T}' = [\, \boldsymbol{T}_0 \mid \boldsymbol{T}_1 \,] = \left[\begin{array}{c|c} \boldsymbol{N}_0 & \\ \hline \boldsymbol{Z}_0 & \boldsymbol{T}_1 \end{array} \right]. \tag{15}$$

The determinant of $\boldsymbol{T}_0$ is now expanded using the Laplace expansion formula. The determinant of the matrices Z_0' formed by k columns of Z_0 (see (16)) is different from zero iff the corresponding subset of $S_I \cup S_V$ is $I-V$ disjoint, i.e., of the form $C_V^0 \cup (L \setminus L^0)_V \cup (C \setminus C^0)_I \cup L_I^0 \subset C_V \cup L_V \cup C_I \cup L_I$. The determinant of the matrices N_0' formed by $2|S|-(k+g)$ columns of N_0 (see (16)) is different from zero iff the columns correspond to a base of M.

$$T_0 = \begin{array}{c} \overset{2|S|-(k+g)}{\longleftrightarrow} \\ \left[\begin{array}{c:c} N_0' & \\ \hdashline & Z_0' \end{array}\right] \end{array} \updownarrow k \qquad . \tag{16}$$

The "only if" is due to the generality (Theorem (6.1)). Thus if we assume that the set $E_V^1 \cup C_V^0 \cup (L \setminus L^0)_V \cup (C \setminus C^0)_I \cup L_I^0 \cup I_I^1$ is a base of M^* then at least one term in the coefficient of $s^{|C_V^0 \cup L_I^0|}$ is different from zero, and since other nonzero terms will contain at least one capacitive or inductive element value corresponding to an element not in $C_V^0 \cup L_I^0$, the coefficient is, in the general case, different from zero. If, at the same time, $|C_V^0 \cup L_I^0|$ is *maximum* among the subsets of $C_V \cup L_I$ satisfying the above assumption, then $|C_V^0 \cup L_I^0|$ equals the degree of the polynomial in s. Hence, $|C_V^0 \cup L_I^0|$ is the g-complexity of the network.

Next, the statement on the state variables is proved. For each base X of M^*, the voltages and currents corresponding to elements not in X, can be expressed as a linear combination of the voltages and currents corresponding to elements of X. Let us consider the elements of $C_I^0 \cup (L \setminus L^0)_I \cup (C \setminus C^0)_V \cup L_V^0$, only. The corresponding voltages and currents can be expressed as shown in (17).

$$\begin{bmatrix} y_0 \\ \hdashline y_1 \end{bmatrix} = \left[\begin{array}{c:c} F_{11} & F_{12} \\ \hdashline F_{21} & F_{22} \end{array}\right] \begin{bmatrix} x_0 \\ \hdashline x_1 \end{bmatrix} + \begin{bmatrix} H_{10} \\ \hdashline H_{20} \end{bmatrix} u \tag{17}$$

where x_0 is a vector formed by the voltages and currents corresponding to the elements of $C_V^0 \cup L_I^0$, denoted $x_0 \sim C_V^0 \cup L_I^0$.

$$x_1 \sim (C \setminus C^0)_I \cup (L \setminus L^0)_V \qquad y_0 \sim C_I^0 \cup L_V^0$$

$$y_1 \sim (C \setminus C^0)_V \cup (L \setminus L^0)_I \qquad u \sim E_V^1 \cup I_I^1.$$

Since $|C_V^0 \cup L_I^0|$ is maximum, *every symmetric* minor of F_{22} is singular. Otherwise, the vector x_0 could have been expanded by elements of y_1, contradicting the maximality of $|C_V^0 \cup L_I^0|$. (This observation is due to Recski). In particular F_{22} is singular. Thus it is possible by row operations in F_{21}, F_{22}, and H_{20}, to make the last row of F_{22} consist of zeros only. So the last coordinate y_{1n} of y_1, can be expressed as a linear combination of the coordinates of x_0 and u, in the following denoted by $y_{1n} = L_1(x_0, u)$. Now the derivative of y_{1n} is proportional to x_{1n}, and at the same time it can be expressed as a linear combination of the elements of x_0, x_1, u, and $\dot{u}$: $kx_{1n} = L_2(\dot{x}_0, \dot{u}) = L_3(x_0, x_1, u, \dot{u})$. The last equality is due to the fact that the derivative of each coordinate of x_0 is proportional to the *same* coordinate of y_0. Since the network parameters are general x_{1n} can be expressed as a linearcombination of the coordinates of x_0, x_1', u, and $\dot{u}$, where $(x_1', x_{1n}) = x_1$: $x_{1n} = L_4(x_0, x_1', u, \dot{u})$. A new system of equations is then obtained (see the following):

$$\begin{bmatrix} y_0 \\ \hdashline y_1' \end{bmatrix} = \left[\begin{array}{c:c} F_{11}' & F_{12}' \\ \hdashline F_{21}' & F_{22}' \end{array}\right] \begin{bmatrix} x_0 \\ \hdashline x_1' \end{bmatrix} + \begin{bmatrix} H_{10}' \\ \hdashline H_{20}' \end{bmatrix} u + \begin{bmatrix} H_{11} \\ \hdashline H_{21} \end{bmatrix} \dot{u}. \tag{18}$$

Every symmetric minor of F_{22}' is still singular and the process can be repeated. Finally the system of (19) is obtained.

$$y_0 = F_{11}^n x_0 + \sum_{j=0}^{n} H_j \overset{(j)}{u}. \tag{19}$$

Since $y_0 = D\dot{x}_0$, where D is a diagonal matrix, the state equations are obtained (see the following):

$$D\dot{x}_0 = y_0 = F_{11}^n x_0 + \sum_{j=0}^{n} H_j \overset{(j)}{u} \tag{20}$$

or

$$\dot{x}_0 = D^{-1} F_{11}^n x_0 + D^{-1} \sum_{j=0}^{n} H_j \overset{(j)}{u}.$$

The set of algebraic equations, obtained during the process, is a part of the system of equations expressing all remaining voltages and currents as linear combinations of the coordinates of x_0 and $\overset{(j)}{u}$. □

It appears from the proof of Theorem (8.1) that $\max |C_V^0 \cup L_I^0|$ is an *upper bound* on the complexity, i.e., $\sigma \leqslant \sigma_g = \max |C_V^0 \cup L_I^0|$. The upper bound is reached if the topology alone is taken into consideration. Specific choices of the network parameter values may *decrease* the complexity.

If $|C_V^0 \cup L_I^0|$ is minimum, instead of maximum, then a *lower bound* on the number of dc-eigenfrequencies δ is found, i.e., $\delta \geqslant \delta_g = \min |C_V^0 \cup L_I^0|$. Furthermore, the qualitative appearance of the determinant, considered as a polynomial in s, can in the general case be found.

The proof of Theorem (8.1) has the following immediate corollary.

Corollary (8.2)

RLC networks containing controlled sources are g-solvable iff subsets $C^0 \subseteq C$ and $L^0 \subseteq L$ exist such that $E_V^1 \cup C_V^0 \cup (L \setminus L^0)_V \cup (C \setminus C^0)_I \cup L_I^0 \cup I_I^1$ is a base of M^*.

A corollary similar to Corollary (6.5) can be given, if capacitors and inductors are allowed in the network.

Corollary (8.3)

RLC networks containing controlled sources are g-solvable if the set $E^1 \cup E^2$ does not contain any circuit, and the set $I^1 \cup I^2$ does not contain any cutset of the network graph.

The theorems concerning the complexity are now illustrated by some examples.

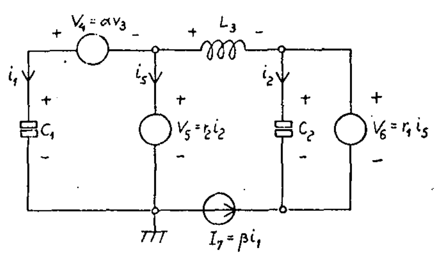

Fig. 13. The network N_4 of Example (8.4).

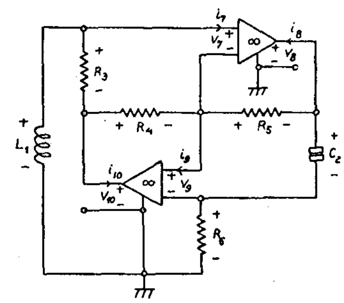

Fig. 15. The network N_5 of Example (8.5).

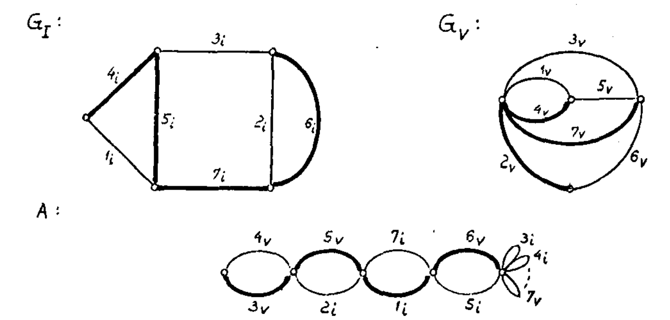

Fig. 14. The matroids G_I, G_V, and A associated with the network N_4 of Example (8.4) (Fig. 13).

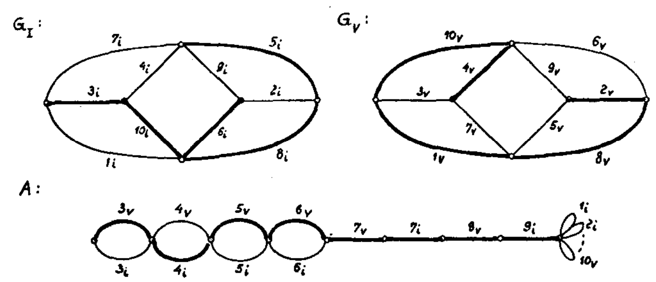

Fig. 16. The matroids G_I, G_V, and A associated with the network N_5 of Example (8.5) (Fig. 15).

Example (8.4)

The network N_4 is shown in Fig. 13. The total system of network equations is shown in (21).

$$
\begin{array}{c}
\begin{array}{cccccccccccccc} 1_i & 2_i & 3_i & 4_i & 5_i & 6_i & 7_i & 1_v & 2_v & 3_v & 4_v & 5_v & 6_v & 7_v \end{array}\\
\left[\frac{N}{Z}\right]=\left[\begin{array}{ccccccc|ccccccc}
1 & 0 & 0 & 0 & 1 & 0 & -1 & & & & & & & \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & & & & \mathbf{0} & & & \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & & & & & & & \\
0 & 0 & 0 & 1 & -1 & 0 & 1 & & & & & & & \\
\hline
 & & & & & & & -1 & 0 & 0 & 1 & 1 & 0 & 0 \\
 & & & \mathbf{0} & & & & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\
 & & & & & & & 1 & -1 & -1 & -1 & 0 & 0 & 1 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\alpha & 1 & 0 & 0 & 0 \\
0 & -r_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & -r_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
-\beta & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
-1 & 0 & 0 & & \mathbf{0} & & & sC_1 & 0 & 0 & & & \mathbf{0} & \\
0 & -1 & 0 & & & & & 0 & sC_2 & 0 & & & & \\
0 & 0 & sL_3 & & & & & 0 & 0 & -1 & & & &
\end{array}\right]
=\left[\begin{array}{c|c} Q_f & \mathbf{0} \\ \hline \mathbf{0} & B_f \\ \hline \multicolumn{2}{c}{A} \\ \hline \multicolumn{2}{c}{Z} \end{array}\right].
\end{array}
\tag{21}
$$

The associated matroids are given in Fig. 14 (in this case graphic). It can be verified that $\{1_v, 2_i, 3_i\}$ and $\{1_v, 2_v, 3_v\}$ are bases of M^* and that $\{1_v, 3_i\}$ and $\{1_v, 2_v\}$ are maximum sets. $((S_I \cup S_V)\backslash\{1_v, 2_i, 3_i\}$ is a base of M, which is illustrated by the heavy edges of Fig. 14). Thus the g-complexity is two. If specific choices of network parameters are taken into account the complexity may decrease. For instance, if $C_2/L_3 = r_1 r_2/\alpha$ the complexity equals zero. □

Example (8.5)

The network N_5 is shown in Fig. 15. The matroids of the network is shown in Fig. 16.

It can easily be verified that $\{1_i, 2_i\}$ and $\{1_v, 2_v\}$ are bases of M^* $((S_I \cup S_V)\backslash\{1_i, 2_i\}$ is a base of M, which is illustrated by the heavy edges of Fig. 16). Furthermore

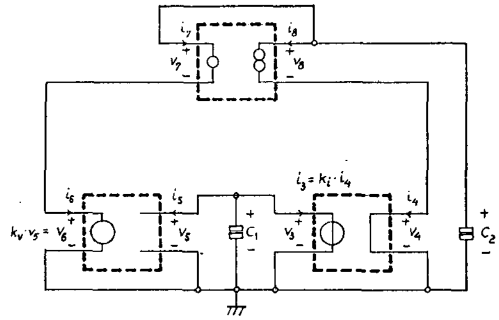

Fig. 17. The network N_6 of Example (8.6).

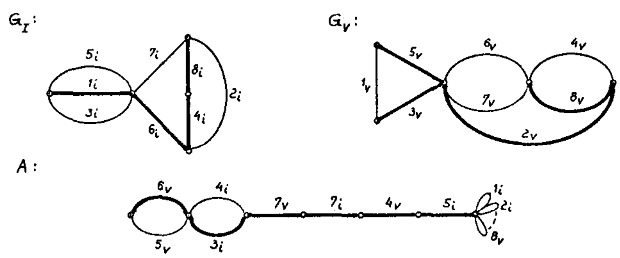

Fig. 18. The matroids G_I, G_V, and A associated with the network N_6 of Example (8.6) (Fig. 17).

both are maximum and minimum sets. Thus either the complexity and the number of dc-eigenfrequencies both equal one, or the network is singular. Now the network determinant is found to equal $-s(C_2R_3R_5R_6+L_1R_4)$. Thus if $R_j>0(j=3,4,5,6)$ the network is solvable. □

Example (8.6)

The network N_6 is shown in Fig. 17. The matroids of the network is shown in Fig. 18.

It is easily verified that $\{1_v,2_i\}$ and $\{1_i,2_v\}$ are bases of M^*, and $\{1_v\}$ or $\{2_v\}$ are maximum sets $((S_I\cup S_V)\backslash\{1_v,2_i\}$ is a base of M, which is illustrated by the heavy edges of Fig. 18). Thus the g-complexity is one. Furthermore $\{1_v\}$ or $\{2_v\}$ are minimum sets as well. Thus either the complexity and the number of dc-eigenfrequencies both equal one, or the network is singular. The latter occurs if $k_ik_v=C_1/C_2$ (in this case, the currents i_3 and i_4 are not unique.) □

IX. Conclusion

In the general case, necessary and sufficient conditions for unique solvability of linear active networks have been given. Pure graph theoretical sufficient conditions have also been found. Necessary and sufficient conditions for the existence of n-port descriptions were given. A set of independent state variables has been found, and the complexity as well as the number of dc-eigenfrequencies were given.

If the network parameter values are taken into consideration, the necessary and sufficient conditions of the general case, are necessary only. If these conditions *are* met, then degeneracies are caused by special relations among the network parameter values. Numerical calculations are then *unavoidable*. The combinatorial tool is matroids. The conditions of the theorems can be checked using polynomially bounded algorithms. A detailed description of the algorithms will be given in a forthcoming paper. The algorithms have been implemented in Fortran IV on an IBM 370/165 computer (e.g., the execution time of Example (6.3) is 0.007 s, and the execution time of Example (8.5) is 0.020 s).

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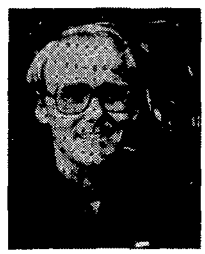

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Realization Theory of Discrete-Time Nonlinear Systems: Part I — The Bounded Case

EDUARDO D. SONTAG

Abstract—**A state-space realization theory is presented for a wide class of discrete time input/output behaviors. Although in many ways restricted, this class does include as particular cases those treated in the literature (linear, multilinear, internally bilinear, homogeneous), as well as certain nonanalytic nonlinearities. The theory is conceptually simple, and matrix-theoretic algorithms are straightforward. Finite-realizability of these behaviors by state-affine systems is shown to be equivalent both to the existence of high-order input/output equations and to realizability by more general types of systems.**

Introduction

THIS WORK deals with some aspects of realization theory of deterministic nonlinear discrete-time systems. The realization theory of *linear* systems is by now a successful part of system theory, which has resulted in a deep understanding of behavior and has permitted the application of state-space methods of analysis and synthesis. It may be reasonable to expect, then, that a corresponding theory will eventually derive analogous benefits for nonlinear systems.

For the most part, this paper presents a "linearized" realization theory via systems which are linear in state variables but arbitrarily nonlinear in inputs, *state-affine* systems. While such systems are highly restrictive vis à vis general nonlinear models, they do include those for which a detailed realization theory has been developed, in particular, linear, internally bilinear, and multilinear systems. The importance of S-A representations in the analysis of certain nuclear reactors, heat-transfer processes, and population models, among others, has been made explicit by various authors (see, for instance, [34]); other applications being currently explored are in the areas of image processing and in stochastic filtering. Moreover, in some

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