# Investigation of similar solutions for natural convection with distributed wall concentration

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### Abstract

Steady laminar natural convection flow over a semi-infinite vertical plate is examined in this paper. It is assumed that the concentration of a species along the plate follows some algebraic law with respect to chemical reaction. Similarity solutions may then be obtained for different orders of reaction. The fundamental parameters of this problem are the Schmidt number, Sc, and reaction order, n. Numerical results, based on the Fourth-order Runge-Kutta method, for Schmidt number ranging from 0.0 to 100.0 and reaction order from 0.0 to 1.5 are presented. When chemical reaction occurs, diffusion and velocity domains are seen to expand out from the plate. For large values of n, one may expect a smaller diffusion layer which, at fixed Schmidt number, is associated with increased velocity and reduced convection-layer.

### 1 Introduction

Natural convection flow exists in a variety of situations including porous media supported by surfaces. Such flows have application in a broad spectrum of engineering systems including geothermal reservoirs, building thermal insulation, direct-contact heat exchangers, solar heating systems, packed-bed catalytic reactors, nuclear waste disposal systems and enhanced recovery of petroleum resources.

This study is concerned mainly with the steady-state behaviour of the same binary system composed of a semi-infinite plate and ambient fluid, each initially at different concentrations of a given species, but between which a homogeneous, irreversible, isothermal chemical reaction of order n is assumed to occur. Similar studies have been pursued in the literature (Meadley and Transactions on Modelling and Simulation vol 21, © 1999 WIT Press, www.witpress.com, ISSN 1743-355X
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Rahman [4], Gebhart and Pera [1]). This study serves to highlight and thus gain more insight into the effects of chemical diffusion and reaction on natural convection flow.

#### 2 Mathematical Formulation

The focus of the study is on the essential nature of the flow and diffusion which occurs in the thin convection boundary-layer adjacent to the plate surface. For this, it is convenient to consider the idealised system composed of a semi-infinite plate set in a fluid of infinite extent. Such a model has been commonly used in previous work([1], [2], [4]) and permits the application of classical boundary-layer analysis in the mathematical formulation. In the analysis, steady two-dimensional laminar viscous flow over the semiinfinite vertical plate is examined. The constituents of the plate and the ambient fluid in which it is immersed undergo a homogeneous, isothermal, irreversible chemical reaction of n-th order, Gebhart et al. [2]. In the following, we present six different cases for the various reaction rates for the chemical reaction being investigated.

#### 2.1 Case I General n-th order Chemical reaction

Consider the transformations

$$\eta(X,Y) = Yb(x) \tag{1}$$

$$\psi(X,Y) = \nu a(X)f(\eta) \tag{2}$$

$$c(X,Y) = \frac{C - C_{\infty}}{C_0 - C_{\infty}}$$
(3)

$$e(X) = C_0 - C_\infty \tag{4}$$

and introduce them into the equations for natural convection to obtain the differential equations shown below.

$$f''' + \frac{g\beta^* e}{ab^3\nu^2}c + \frac{a_x}{b}f''f - \left(\frac{ab_x}{b^2} + \frac{a_x}{b}\right)f'^2 = 0$$
(5)

$$\frac{c''}{Sc} + \frac{a_x}{b}fc' - \frac{ae_x}{be}f'c - \frac{ke^{n-1}}{\nu b^2}c^n = 0$$
(6)

provided the following conditions hold:

$$\frac{a_x}{b} = C_1, \quad \frac{ab_x}{b^2} = C_2, \quad \frac{ae_x}{be} = C_3$$

$$\frac{g\beta^*e}{ab^3\nu^2} = C_4, \quad \frac{ke^{n-1}}{\nu b^2} = C_5$$

where  $C_1$ ,  $C_2$ ,  $C_3$ ,  $C_4$  and  $C_5$  are arbitrary constants. From these conditions, expressions for a(X), b(X) and e(X) need to be determined following the analysis of Gebhart et al [2]. Integration of these expressions lead to the following differential equations for n-th order cehnical reaction(Gebbhart et al. [2]):

$$f'''_{n} + (5-3n)ff'' + c - (4-2n)f'^{2} = 0$$
(7)

$$\frac{c''}{Sc} + (5-3n)fc' - 2f'c - pc^n = 0$$
(8)

provided  $n \neq 1, \frac{3}{2}, \frac{5}{3}$ . We note that the order of reaction need not necessarily be an integer. The pyrolysis of acetaldehyde $(n = \frac{3}{2})$ , and the formation of phosgene from carbon monoxide and chlorine $(n = \frac{5}{2})$  are a few of the frequently encountered fractional reaction orders. The similarity transformations for these equations are given by:

$$\eta(X,Y) = \frac{Y}{X} \left\{ \frac{Gr_x}{6-4n} \right\}^{1/4} \tag{9}$$

$$\psi(X,Y) = \nu(6-4n) \left\{ \frac{Gr_x}{6-4n} \right\}^{1/4} f(\eta)$$
(10)

$$c(X,Y) = \frac{C - C_{\infty}}{C_0 - C_{\infty}} \tag{11}$$

$$e(X) = C_0 - C_{\infty} = N X^{\frac{1}{3-2n}}$$
(12)

#### 2.2 Case II Zeroth Order Chemical Reaction

From the n-th order general case, we can derive the zeroth order chemical reaction differential equations. The differential equations become:

$$f''' + 5ff'' + c - 4{f'}^2 = 0$$
(13)

$$\frac{c''}{Sc} + 5fc' - 2f'c - q = 0 \tag{14}$$

where

$$q = k \sqrt{\frac{6}{g\beta^* N^3}}$$

The similarity transformations for this set of equations are given by :

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$$\eta(X,Y) = \frac{Y}{X} \left\{ \frac{Gr_x}{6} \right\}^{1/4}$$
(15)

$$\psi(X,Y) = 6\nu \left\{\frac{Gr_x}{6}\right\}^{1/4} f(\eta) \tag{16}$$

$$c(X,Y) = \frac{C - C_{\infty}}{C_0 - C_{\infty}}$$
(17)

$$e(X) = NX^{1/3}$$
 (18)

# **2.3** Case III Chemical reaction of order n = 1/2

Again, from the general n-th order case, the differential equations for a chemical reaction of order  $n = \frac{1}{2}$  may be derived. They are shown below.

$$f''' + \frac{7}{2}ff'' + c - 3f'^2 = 0$$
<sup>(19)</sup>

$$\frac{c''}{Sc} + \frac{7}{2}fc' - 2f'c - Rc^{1/2} = 0$$
(20)

where

$$R = \frac{2k}{N\sqrt{g\beta^*}}$$

and the transformations are given by:

$$\eta(X,Y) = \frac{Y}{X} \left\{ \frac{Gr_x}{4} \right\}^{1/4}$$
(21)

$$\psi(X,Y) = 4\nu \left\{\frac{Gr_x}{4}\right\}^{1/4} f(\eta)$$
(22)

$$c(X,Y) = \frac{C - C_{\infty}}{C_0 - C_{\infty}}$$
(23)

$$e(X) = NX \tag{24}$$

### **2.4 Case IV Chemical reaction of order** n = 3/5

From the general n-th order case, the  $n = \frac{3}{5}$  order chemical reaction may be derived. The differential equations for this case are shown below.

$$f''' + \frac{16}{5}ff'' + c - \frac{14}{5}f'^2 = 0$$
(25)

$$\frac{c''}{Sc} + \frac{16}{5}fc' - 2f'c - Pc^{3/5} = 0$$
(26)

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The similarity transformations are:

$$\eta(X,Y) = \frac{Y}{X} \left\{ \frac{5Gr_x}{18} \right\}^{1/4}$$
(27)

$$\psi(X,Y) = \frac{18\nu}{5} \left\{ \frac{5Gr_x}{18} \right\}^{1/4} f(\eta)$$
(28)

$$c(X,Y) = \frac{C - C_{\infty}}{C_0 - C_{\infty}}$$
<sup>(29)</sup>

$$e(X) = NX^{5/9}$$
 (30)

# **2.5 Case V Chemical reaction of order** n = 6/5

Again from the n-th order chemical reaction, the case  $n = \frac{6}{5}$  may be derived. The differential equations for this are shown below.

$$f''' + \frac{7}{5}ff'' + c - \frac{8}{5}f'^2 = 0$$
(31)

$$\frac{c''}{Sc} + \frac{7}{5}fc' - 2f'c - Sc^{6/5} = 0$$
(32)

The similarity transformations are given by the following :

$$\eta(X,Y) = \frac{Y}{X} \left\{ \frac{5Gr_x}{6} \right\}^{1/4}$$
(33)

$$\psi(X,Y) = \frac{6\nu}{5} \left\{ \frac{5Gr_x}{6} \right\}^{1/4} f(\eta)$$
 (34)

$$c(X,Y) = \frac{C - C_{\infty}}{C_0 - C_{\infty}}$$
(35)

$$e(X) = NX^{5/3} (36)$$

## **2.6** Case VI Chemical reaction of order n = 3/2

When the reaction order is  $n = \frac{3}{2}$ , we must derive the differential equations from the original equations (5) and (6). An analysis similar to that in Case I is repeated to obtain the following differential equations:

$$f''' + c + C_1(ff'' - 2f'^2) = 0$$
(37)

$$\frac{c''}{Sc} + C_1(fc' - 4f'c) - rc^{3/2} = 0$$
(38)

where

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$$r = \frac{kN^{1/2}}{\nu \left\{\frac{g\beta^*N}{\nu^2}\right\}^{2/3}}$$

The similarity transformations are shown below.

$$\eta(X,Y) = Y\left\{\frac{g\beta^*N}{\nu^2}\right\}^{1/3} e^{\left\{\frac{g\beta^*N}{\nu^2}\right\}^{1/3}C_1X}$$
(39)

$$\psi(x,Y) = \nu e^{\left\{\frac{g\beta^*N}{\nu^2}\right\}^{1/3} C_1 X} f(\eta)$$
(40)

$$e(X) = N e^{4\left\{\frac{g\beta^* N}{\nu^2}\right\}^{1/3} C_1 X}$$
(41)

To aid in the graphical analysis, these cases have been arbitrarily chosen i.e.  $n = 0, \frac{1}{2}, \frac{3}{2}, \frac{3}{5}, \frac{6}{5}, \frac{5}{3}$ . These similarity transformations for the cases chosen are unique to this paper whereas the method of obtaining the transforms is not(Gebhart et al [2]). The boundary conditions for these 6 cases are given by:

$$\eta = 0 \quad : \quad f(0) = 0 \\ f'(0) = 0 \\ c(0) = 1$$
 (42)

$$\eta \to \infty \quad : \quad f'(\infty) = 0 \\ c(\infty) = 0$$
 (43)

#### **3** Numerical solutions

These sets of differential equations have been solved by using a fourth-order Runge-Kutta integration scheme. From the results, it was observed that decreasing the Schmidt number increases the velocity level and reduces the concentration diffusion region. The effect of the order of reaction on concentration was seen to vary as n varied from n = 0 to n = 1.2 at fixed Schmidt number, Sc = 0.01. Within the range (0,2), an increase of n(which effectively increases sensitivity of species depletion with change in concentration)tended to decrease the concentration diffusion region to about 50% of its value before increasing slightly. Overall, the concentration gradient was seen to increase. One can expect a smaller less distinct diffusion layer for larger orders of n. This was seen to agree with the analysis made by Meadley and Rahman [4]. On examination of the effect of rate of reaction on velocity for orders n = 0 to n = 1.5 at Sc = 0.01, it was seen that the maximum velocity decreases slightly before rising to a value above its initial level. On the other hand, the extent of the convection layer is reduced to a little over 50% of its original size. This is to be expected since the concentration of species on the plate increases as the length of the plate is traversed in the vertical direction. Flow reversal is also expected as n increases and the Schmidt number decreases.

## 4 Conclusions

This study has mainly been concerned with obtaining similarity solutions of natural convection flows induced by a semi-infinite vertical plate with distributed concentration along the plate wall. The basis of this theoretical work has been classical boundary-layer analysis.

It has been found that increasing the order of reaction n, increases the concentration gradient. For large values of n, a smaller diffusion layer is expected. Also, this increase in reaction order, n, at fixed Schmidt number is associated with increased velocity and reduced convection-layer. Detailed analysis results and discussions of this study can be found in (Mulolani [5]). Further study of this problem could investigate the effects of higher Schmidt numbers on the reaction rate and mass diffusion process. Such an investigation would necessarily need to incorporate a singular perturbation expansion in the equation for the concentration. A matched asymptotic expansion method will be needed to carry out this investigation.

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Figure 2: Zeroth order velocity profiles

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Figure 4: Velocity profiles for n = 1.2