INVESTIGATING CONNECTIONS BETWEEN SOME SMARANDACHE SEQUENCES, PRIME NUMBERS AND MAGIC SQUARES

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In this paper we investigate some properties of Smarandache sequences of the 2nd kind and demonstrate that these numbers are near prime numbers. In particular, we establish that prime numbers and Smarandache numbers of the 2nd kind (a) may be computed from the similar analytical expressions, (b) may be used for constructing Magic squares 3×3 or Magic squares 9×9, consisted of 9 Magic squares 3×3.

Key words: prime numbers, Smarandache numbers of the 2nd kind, density of numerical sequences, Magic squares 3×3 and 9×9.

1 Introduction

We remind [2, 3], that in the general case *Magic squares* represent by themselves numerical or analytical square tables, whose elements satisfy a set of definite basic and additional relations. The basic relations therewith assign some constant property for the elements located in the rows, columns and two main diagonals of a square table, and additional relations, assign additional characteristics for some other sets of its elements.

Let it be required to construct Magic squares $n \times n$ in size from a given set of numbers. Judging by the mentioned general definition of Magic squares, there is no difficulty in understanding that the foregoing problem consists of the four interrelated problems

1. Elaborate the practical methods for generating the given set of numbers;
2. Look for a concrete family of n^2 elements, which would satisfy both the basic and all the additional characteristics of the Magic squares;
3. Determine how many Magic squares can be constructed from the chosen family of n^2 elements;
4. Elaborate the practical methods for constructing these Magic squares.

For instance, as we demonstrated in [5],

a) every $(n+1)$-th term a_{n+1} of Smarandache sequences of 1st kind may be formed by subjoining several natural numbers to previous terms a_n and also may be computed from the analytical expression

$$a_{\varphi(n)} = \sigma(a_n 10^{\psi(a_n)} + \xi\{\varphi(n)\}), \qquad (1)$$

where $\varphi(n)$, $\psi(a_n)$ and $\xi\{\varphi(n)\}$ are some functions; σ is an operator. In other words, for generating Smarandache sequences of 1st kind, the set of analytical formulae may be used (see the problem 1);

22232425262728	15161718192021	20212223242526
17181920212223	19202122232425	21222324252627
18192021222324	23242526272829	16171819202122

(1)

171819191817	101112121110	151617171615
121314141312	141516161514	161718181716
131415151413	181920201918	111213131211

(2)

17181920191817	10111213121110	15161718171615
12131415141312	14151617161514	16171819181716
13141516151413	18192021201918	11121314131211

(3)

Figure 1. Magic squares 3×3 from k-truncated Smarandache numbers of 1st kind.

b) it is impossible to construct Magic squares 3×3 from Smarandache numbers of 1st kind without previous truncating these numbers. Consequently, if the given set of numbers consists only of Smarandache numbers of 1st kind, then one releases from care on solving problems, mentioned above in items 2 – 4;

c) there is a set of analytical formulae available for constructing Magic squares 3×3 in size from k-truncated Smarandache numbers of 1st kind (examples of Magic squares 3×3, obtained by these formulae, are shown in figure 1). In this case the foregoing set of analytical formulae is also the desired practical method for constructing Magic squares 3×3 from k-truncated Smarandache numbers of 1st kind (see the problem 4).

The *main goal* of this paper is to investigate some properties of Smarandache sequences of the 2nd kind [6, 9] and to demonstrate that these numbers are near prime numbers. In particular, we establish in the paper, that prime numbers and Smarandache numbers of the 2nd kind

a) may be computed from the similar analytical expressions (see Section 2 and 3);

b) may be used for constructing Magic squares 3×3 or Magic squares 9×9, consisted of 9 Magic squares 3×3 (see Section 5 and 6).

2 Prime Numbers

We remind that in number theory [2, 10, 11] any positive integer (any natural number), simultaneously dividing positive integers a, b, ..., m, is called their *common divisor*. The largest of common divisors is called *greatest common divisor* and denoted by the symbol GCD(a, b, ..., m). The existence of GCD appears from the finiteness of the number of common divisors. The numbers a and b for which GCD(a, b) = 1 are called *relatively prime numbers*. The analytical formula available for counting the value of GCD(a, b) has form [6]

$$\mathrm{GCD}(a, b) = b\{1 - \mathrm{sign}(r)\} + k\,\mathrm{sign}(r), \quad r = a - b[a/b], \tag{2}$$

$$k = \mathop{\mathrm{MAX}}_{i=2}^{[b/2]}\{i(1-d)\}, \quad d = \mathrm{sign}\{a - i[a/i]\} + \mathrm{sign}\{b - i[b/i]\},$$

where the function $\mathrm{MAX}(a_1, a_2, \ldots, a_i)$ gives the greatest from numbers a_1, a_2, ..., a_i; $\mathrm{sign}(x) = |x|/x$ if $x \neq 0$ and $\mathrm{sign}(0) = 0$.

It is easy to prove, that any natural number larger than a unit, has no less than two divisors: the unit and itself. Any natural number $p > 1$, having exactly two divisors, is called *prime*. If the number of divisors is more than 2, then the number is called *composite* (for example, the number 11, having divisors 1 and 11, is the prime number, whereas the number 10, having the divisors 1, 2, 5 and 10, is the composite number). In this paper we shall consider the number 1 as the least prime number. The analytical formula, generating n-th prime number p_n, has form [6]

$$p_n = \sum_{m=0}^{(n+1)^2+1} \mathrm{sg}(n - 1 - \sum_{i=3}^{m}\chi_i), \quad \chi_i = \prod_{j=2}^{[\sqrt{i}]}\{\mathrm{sg}(i - j[i/j])\}, \tag{3}$$

where $p_2 = 2, p_3 = 3, p_4 = 5, \ldots$; $\mathrm{sg}(x) = 1$ if $x > 0$ and $\mathrm{sg}(x) = 0$ if $x \leq 0$.

It is proved in the number theory [2, 10, 11], that any natural number larger than a unit can be represented as a product of prime numbers and this representation is unique (we assume that products, differing only by the order of cofactors, are identical). For solving the problem on decomposing the natural number a in simple cofactors, it is necessary to know all the prime numbers $p_a < \sqrt{a}$.

Let $m = [\sqrt{a}]$, where the notation $[b]$ means integer part from b. Then, for finding all the prime number p_a one may use the following procedure (*Eratosthenes sieve*) [2, 10, 11]:

1. Write out all the successive numbers from 2 to m and put $p = 2$;

2. In the series of the numbers 2, 3, 4, ..., m, cross out all the numbers having the form $p + kp$, where $k = 1, 2, \ldots$;
3. If, in the series of the numbers 2, 3, 4, ..., m, all the numbers larger than p have been crossed out, then pass to step 4. If there still remain the numbers larger than p, which have not been crossed out, then the first of these ones we denote by p_1. If $p_1^2 \geq m$, then pass to step 4. Otherwise, put $p = p_1$ and pass to step 2;
4. The end of the procedure: primes are all the numbers of the series 1, 2, 3, 4, ..., m, which have not been deleted.

If an arithmetical progression from n prime numbers is found then it should be known that [2, 10]

The difference of any arithmetical progression, containing n prime numbers larger than n, is divisible by all the prime numbers $\leq n$ (*Cantor theorem*).

From the series of the consecutive prime numbers one may reveal subsequences of numbers, possessed the different interesting properties. For instance

a) two prime numbers are called *reversed*, if each is obtained from other by reversing of its digits. If $p < 1\,000$ then such numbers are

1, 2, 3, 5, 7, 11, 13, 17, 31, 37, 71, 73, 79, 97, 101, 107, 113, 131, 149, (4)

151, 157, 167, 179, 181, 191, 199, 311, 313, 337, 347, 353, 359, 373,

383, 389, 701, 709, 727, 733, 739, 743, 751, 757, 761, 769, 787, 797,

907, 919, 929, 937, 941, 953, 967, 971, 983, 991.

b) among the numbers of (4) one may reveal the *symmetric* prime numbers:

1, 2, 3, 5, 7, 11, 101, 131, 151, 181, 191, 313, 353, 373, 383, 727, 757, (5)

787, 797, 919, 929;

c) two prime numbers are called *mirror-reversed*, if each is obtained from other by reflecting in the mirror, located above the number. If $p < 3\,000$ then such numbers are:

1, 2, 3, 5, 11, 13, 23, 31, 53, 83, 101, 131, 181, 227, 251, 311, 313, (6)

331, 383, 521, 557, 811, 823, 853, 881, 883, 1013, 1021, 1031, 1033,

1051, 1103, 1123, 1153, 1181, 1223, 1231, 1283, 1301, 1303, 1381,

1531, 1553, 1583, 1811, 1831, 2003, 2011, 2053, 2081, 2113, 2203,

2251, 2281, 2333, 2381, 2531, 2851.

3 Smarandache Numbers of the 2nd Kind

In this section we consider 4 different Smarandache sequences of the 2nd kind [6, 9] and demonstrate that the value of n-th numbers a_n in these sequences may be computed by the universal analytical formula {compare with formula (3)}

$$a_n = \sum_{m=0}^{U_n} \mathrm{sg}(n+2-b-\sum_{i=1}^{m}\chi_i), \tag{7}$$

where χ_i are the characteristic numbers for the described below Smarandache sequences of the 2nd type and $U_n = 10 + (n+1)^2$.

3.1 Pseudo-Prime Numbers

a) Smarandache P_1-*series*

$$1, 2, 3, 5, 7, 11, 13, 14, 16, 17, 19, 20, 23, 29, 30, 31, 32, 34, \ldots \tag{8}$$

contains the only such natural numbers, which are or prime numbers itself or prime numbers can be obtained from P_1-series numbers by a permutation of digits (for instance, the number 115 is the pseudo-prime of P_1-series because the number 151 is the prime).

It is clear from the description of P_1-series numbers that they may be generated by the following algorithm

1. Write out all the successive prime numbers from 1 to 13: 1, 2, 3, 5, 7, 11, 13 and put n=8; a_n = 13;
2. Assume $p = a_n + 1$.
3. Examine the number p. If p is a prime or a prime number can be obtained from a_n by a permutation of digits, then increase n by 1, put $a_n = p$ and go to step 2. Else increase p by 1 and go to the beginning of this step.

To convert the foregoing algorithm into a computer-oriented method (see problem 1 in Section 1), we are evidently to translate this description into one of special computer-oriented languages. There is a set of methods to realise such translation [6]. The most simplest among ones is to write program code directly from the verbal description of the algorithm without any preliminary construction. For instance, Pascal program identical with the verbal description of the algorithm under consideration are shown in Table 1. In this program the procedure *Pd*, the functions *PrimeList* and *PseudoPrime* are used for generating respectively permutations, primes numbers and pseudo-prime numbers; the meaning of the logical function *BelongToPrimes* is clear from its name.

In the case, when verbal descriptions are complex, babelized or incomplete, the translation of these descriptions into computer languages may be performed sometimes in two stages [7]: firstly, verbal descriptions of computational algorithms are translated into analytical ones and then analytical descriptions are translated into computer languages. To demonstrate how this scheme is realised in practice, let us apply it to the algorithm, generating P_1-series numbers.

Table 1. Pascal program 1 for generating Smarandache P_1-series

```
Type Ten=Array[1..10]Of Integer;
Procedure Pd(Var m4,n1,n:Integer;Var
nb3,nb4,nb5:Ten);
Label A28,A29,A30;  Var nt,k,m:Integer;
Begin
   If M4=1 Then
   Begin
      m4:=0;n:=n1;
      For k:=2 to n do
      Begin Nb4[k]:=0;  Nb5[k]:=1; End;
      Exit;
   End;
   k:=0; n:=n1;
A28:  m:=Nb4[n]+Nb5[n];Nb4[n]:=m;
      If m=n Then
      Begin  Nb5[n]:=-1;Goto A29; End;
      If Abs(m)>0 Then Goto A30;
      Nb5[n]:=1;Inc(k);
A29:  If n>2 Then
      Begin Dec(n);Goto A28; End;
      Inc(m);m4:=1;
A30:  m:=m+k;  nt:=nb3[m];
      nb3[m]:=nb3[m+1];  nb3[m+1]:=nt
End;

Const  Mn=10000;  MaxN:Integer=Mn;
Type int=Array[1..Mn]Of Integer;  pint=^int;
Var pl:pint;

Function PrimeList(Var MaxN:Integer):pint;
Var i,j,k:Integer;  p:pint;Ok:Boolean;
Begin
   GetMem(p,MaxN);  p^[1]:=2;i:=3;k:=1;
   While i<MaxN do
   Begin  {Is i prime or not ?}
      j:=3;Ok:=True;
      While Ok And (j<=Round(Sqrt(i))) do
      If i mod j=0 Then Ok:=False
      Else Inc(j,2);
      If Ok Then Begin Inc(k);p^[k]:=i;End;
      Inc(i,2);
   End;
   MaxN:=k;Primelist:=p;
End {PrimeList} ;

Function
BelongToPrimes(num:Integer):Boolean;
Var l,r,j:Integer;
Begin
   BelongToPrimes:=True; l:=1;r:=MaxN;
   Repeat
   j:=(l+r)shr 1;   If num<Pl^[j] Then r:=j
Else If num>Pl^[j] Then l:=j+1
   Else Exit;
   Until l=r;BelongToPrimes:=False;
End;

Function
PseudoPrime(Num:Integer):Boolean;
Var g,nb3,nb4,nb5:Ten;
  nd,m,r,mn,m4,n1,mm,i,j,d,k,n:Integer;
Begin
   PseudoPrime:=True;
   {Decomposition number num on digits}
   d:=Num;k:=0;
   Repeat
      Inc(k);  g[k]:=d mod 10;
      r:=d;  d:=d div 10;
   Until r div 10=0;
            {Examination whether numbers,
            composed from digits are prime}
   m4:=1; m:=0; n1:=k;
   For i:=1 to n1 do Nb3[i]:=g[i];
   Repeat
      Pd(m4,n1,n,nb3,nb4,nb5);  Inc(m);
      If m4=1 Then Break; mm:=1;d:=0;
      For i:=1 to n1 do
      Begin
         d:=d+nb3[i]*mm;  mm:=mm*10;
      End;
      If BelongToPrimes(d) Then Exit;
   Until False; PseudoPrime:=False;
End;

Var Ind,Num,i:Integer;  List:pint;
Begin  pl:=PrimeList(MaxN);
      {Generating list of primes up to MaxN}
   Ind:=0;Getmem(List,4*(MaxN shl 1));
   For Num:=10 to MN do
   If PseudoPrime(Num) Then
{If number is pseudoprime then add it to list}
   Begin Inc(Ind);List^[Ind]:=Num;  End;
    {Output generated numbers to ' Sp1' file,
                          10 values per row}
   Assign(Output,'Sp1');Rewrite(Output);
   WriteLn(Ind);
   For i:=1 to Ind do
   Begin
      Write(List^[i]:7);
      If i mod 10=0 Then WriteLn;
   End;
   Close(output)
End.
```

Table 2. Pascal program 2 for generating Smarandache P_1-series

```
Const MaxG=5;
Var c,d,r:Array[1..MaxG]Of Integer;
   g:Integer;

Function Sg(x:Integer):Integer;
Begin {function returns unit if argument is
greater than zero}
   If x>0 Then Sg:=1 Else Sg:=0;
End;

Function Fact(x:Integer):LongInt;
Var i:Integer;f:LongInt;
Begin {function calculates factorial of
argument}
   f:=1;  For i:=1 to x do f:=f*i;  Fact:=f;
End;

Function Lg(x:Extended):Extended;
Begin {function returns decimal logarithm of
argument}
   Lg:=Ln(x)/Ln(10);
End;

Function
Power(x:Extended;Deg:Integer):Extended;
Var p:Extended;i:Integer;
Begin {function returns argument in 'deg'
power}
   p:=1;For i:=1 to Deg do p:=p*x;
   Power:=p;
End;

Function Mu(p,g:Integer):Integer;
Var m,q:Integer;
{this is an auxiliary function}
Begin   m:=1;
   For q:=1 to p do  m:=m*(g-q+1);  Mu:=m;
End;

Function GetPos(k,p:Integer):Integer;
Var i,f:Integer;
Begin
      {function returns location of element 'p'
               in 'k'th permutation of 'g' objects}
   c[p]:=(k div Mu(p,g)) mod 2;
   f:=(k div Mu(p-1,g))mod (g-p+1);
   d[p]:=p-1+(1-c[p])*f+c[p]*(g-p-f);
   r[p]:=d[p];
   For i:=p-1 downto 1 do
   r[p]:=r[p]-Byte(d[i]>=r[p]);  GetPos:=r[p];
End;
```

```
Function MXi(i:Integer):Integer;
Var k,q,p,s,Pro:Integer;
   Sum,c:Extended;
Begin
   {function returns unit if examined value 'i'}
 {belongs to set of Smarandache numbers}
   S:=0;g:=Trunc(Lg(i))+1;
   For k:=0 to Fact(g)-1 do
   Begin
   {Construction number 'c' from permutated
                          digits of number 'i'}
      sum:=0; For p:=1 to g do
      sum:=sum+(Int(i/Power(10,g-p))-
10*Int(i/Power(10,g-p+1)))/
      Power(10,GetPos(k,p));
      c:=Power(10,g-1)*sum;
      Pro:=1; {If 'c' is prime number}
      For q:=2 to Trunc(sqrt(c)) do
      Pro:=Pro*Sg(Round(c) mod q);
      s:=s+Pro;
   End;  Mxi:=Sg(s);
End;

Var xi,n,M:Integer;

Function BuildAn(n:Integer):Integer;
Var i,xi,a:Integer;
  m,Un,SumXi:LongInt;
Begin
            {function returns 'n'th element of
                      Smarandache sequence}
   a:=0;Un:=Sqr(LongInt(n));
   For m:=0 to Un do
   Begin
        {'SumXi' is quantity of Smarandache
    numbers which are less than number 'm'}
      SumXi:=0;   For i:=1 to m do
      SumXi:=SumXi+MXi(i);
       a:=a+sg(n-SumXi);
   End; BuildAn:=a;
End;

Begin {Output of the first 'M' Smarandache
numbers}
   M:=30;
   For n:=1 to M do Write(BuildAn(n):5);
   WriteLn;
End.
```

The analytical formula available for determining n-th number in the P_1-series is obtained from (7) when [6]

$$b = 2, \quad \chi_i = \mathrm{sg}\,\{ \sum_{k=0}^{g!-1} \prod_{q=2}^{[\sqrt{c}]} \mathrm{sg}\,(c - q[c/q]) \}, \tag{9}$$

and g, c and r_p are calculated by the formulae

$$g = [\lg i]+1, \quad c = 10^g \sum_{p=1}^{g} \left(\{ [i/10^{g-p}] - 10[i/10^{g-p+1}] \} / 10^{r_p} \right), \tag{10}$$

$$r_p = z_1, \quad d_p = p - 1 + f(1 - c_p) + c_p (g - p - f), \; c_p = |(-1)^{t_p} - 1|/2,$$

$$f = t_{p-1} - (g - p + 1)\,[t_{p-1}/(g - p + 1)], \quad t_p = [\,k / \prod_{q=1}^{p} (g - q + 1)\,],$$

$$z_1 = z_2 - \mathrm{sg}(1 + d_1 - z_2), \quad z_2 = z_3 - \mathrm{sg}(1 + d_2 - z_3), \; \ldots,$$

$$z_{p-2} = z_{p-1} - \mathrm{sg}(1 + d_{p-2} - z_{p-1}), \; z_{p-1} = d_p - \mathrm{sg}(1 + d_{p-1} - d_p).$$

Pascal program identical with the analytical description (9) – (10) of the algorithm, generating P_1-series numbers, takes the form, shown in Table 2.

It should be noted that most part of Pascal text of program 2 consists of formulae (9) – (10). In other words, translating analytical descriptions of computative algorithms into computer languages requires noticeably less efforts than the translation of verbal descriptions. Therefore, our conclusion is that

if it is possible, one should provide the verbal descriptions of computational algorithms with the analytical ones, constructed, for instanse, by using logical functions [5 – 7].

b) Smarandache P_2-*series*

14, 16, 20, 30, 32, 34, 35, 38, 50, 70, 74, 76, 91, 92, 95, 98, ... (11)

contains the only such natural numbers, which are the composite numbers itself, but the prime numbers can be obtained from P_2-series numbers by a permutation of digits. The analytical formula available for determining n-th number in the P_2-series has the same form as for P_1-series numbers, but in this case the value of χ_i from (9) is computed by the formula

$$\chi_i = (1 - w_0)\,\mathrm{sg}(\sum_{k=1}^{g!-1} w_k), \quad w_k = \prod_{q=2}^{[\sqrt{c}]} \mathrm{sg}\,(c - q[c/q]). \tag{12}$$

3.2 Some Modifications of Eratosthenes Sieve

a) Smarandache T_1-*series*

7, 13, 19, 23, 25, 31, 33, 37, 43, 47, 49, 53, 55, 61, 63, ... (13)

is obtained from the series of natural numbers by deleting all even numbers and all such odd numbers t_i that the numbers t_i+2 are primes. The analytical formula for the determination of n-th number in the T_1-series has the form (7) with

$$b = 2,\ \chi_i = (i - 2[i/2])\ \{1 - \prod_{k=2}^{[\sqrt{i+2}]+1} sg(i+2-k[(i+2)/k])\},\qquad (14)$$

b) Smarandache T_2-*series*

1, 3, 5, 9, 11, 13, 17, 21, 25, 27, 29, 33, 35, 37, 43, 49, ... (15)

This series may be obtained from the series of natural numbers by the following *step-procedure*:

On k-th step each 2^k-th numbers are deleted from the series of numbers constructed on $(k-1)$-th step.

The analytical formula for the determination of n-th number in the T_2-series has the form (7) with

$$\chi_i = sg(\prod_{k=1}^{[\log i]+1} \{x_k - 2^k[x_k/2^k]\}),\ x_1 = i,\ x_{k+1} = x_k - [x_k/2^k],\qquad (16)$$

where $\log a$ is the logarithm of the number a to the base 2.

4 Algorithms for Solving Problems on Constructing Magic Squares 3×3 from Given Class of Numbers

Proposition 1. *A set of nine numbers is available for constructing Magic squares 3×3 only in the case if one succeeds to represent these nine numbers in the form of such three arithmetic progressions from 3 numbers whose differences are identical and the first terms of all three progressions are also forming an arithmetic progression.*

Proof. The general algebraic formula of Magic squares 3×3 is shown in figure 1(3) [2, 4]. The table 1(4) is obtained from table 1(3) by arranging its symbols. It is noteworthy that arithmetic progressions with the difference b are placed in the rows of the table 1(4), whereas ones, having the difference c, are located in its columns. Thus, the proof of Proposition 1 follows directly from the construction of tables 1(3) and/or 1(4).

1	2	3
4	5	6
7	8	9

(1)

$a+b+2c$	a	$a+2b+c$
$a+2b$	$a+b+c$	$a+2c$
$a+c$	$a+2b+2c$	$a+b$

(3)

1	2	4
3	5	7
6	8	9

(2)

a	$a+b$	$a+2b$
$a+c$	$a+b+c$	$a+2b+c$
$a+2c$	$a+b+2c$	$a+2b+2c$

(4)

Figure 1. To proofs of correctness of Proposition 1 and Algorithm 1:
(3) — the general algebraic formula of Magic squares 3×3; (4) — additional table of Magic squares 3×3; (1) ($c > 2b$) and (2) ($b < c < 2b$) — two possible arrangements of the nine increasing numbers in cells of the additional table (4).

By Proposition 1 and two possible arrangements of the nine increasing numbers in cells of the additional table 1(4), which are shown in figures 1(1) and 1(2), we may elaborate *algorithm 1* available for constructing Magic squares 3×3 from an arbitrarily given set of nine increasing numbers [2]:

1. Take two square tables 3×3 and arrange 9 testing numbers in them so as it is shown in figures 1(1) and 1(2).
2. Check whether three arithmetic progressions of Proposition 1 are in one of these square tables 3×3.

It should be noted, if the problem on constructing the Magic square 3×3 from the given set of nine increasing numbers has the solution, then this solution is always unique with regard for rotations and mappings.

For finding all Magic squares 3×3 from a given class of numbers with the number f in its central cell, one may use the following *algorithm 2* [2, 4]

a) write out the possible decompositions of the number $2f$ in the two summands of the following form:

$$2f = x_1(j) + x_2(j), \tag{17}$$

where j is the number of a decomposition and $x_1(j)$, $x_2(j)$ are the two numbers such that $x_1(j) < x_2(j)$ and both these numbers belong to the given class of numbers;

b) in the complete set of various decompositions (17), fix one, having, for instance, the number k and, for this decomposition, determine the number $d(k)$: $d(k) = f - x_1(j)$;

c) find all possible arithmetic progressions from 3 numbers with differences equal $d(k)$ among a set of numbers $\{x_1(j)\}$ without $x_1(k)$. If there are m such arithmetic progressions then there are m Magic squares 3×3 with the numbers $x_1(k)$ and $x_2(k)$ in its cells;

d) repeat items (b) and (c) for other values of k.

5 Magic Squares 3×3 and 9×9 from Prime Numbers

Proposition 2. *A Magic square 3×3 can be constructed from prime numbers only in the case if the parameters b and c of the general algebraic formula 1(3) and/or additional table 1(4) are the numbers multiple of 6.*

Proof. The truth of Proposition 2 follows from Proposition 1 and Cantor theorem of Section 2.

Corollaries from Proposition 2 [2]:

1. By using prime numbers one cannot construct a Magic square 3×3 with one of the cells containing numbers 2 or 3.

2. All nine prime numbers of a Magic square 3×3 are either numbers of the form $6k - 1$ or have the form $6k + 1$.

Proposition 3 [2]. *With regard for rotations and mappings, the last digits of the prime numbers may be arranged in the cells of the additional table of a Magic square 3×3 only in such variants, which are shown in figure 2.*

Proof. To prove the truth of Proposition 3, we need the two more easily verified properties of the additional table 1(4).

1. In this table the sums of the symbols of the central row, central column and both diagonals are identical and coincide with the Magic constant of the general algebraic formula 1(3).

2. An arithmetic progression, consisting of three numbers, occurs not only in the rows and columns but also in each diagonal of the additional table.

Now let us place a prime number, for instance, ending by 1, into the central cell of the additional table 1(4). It is clear, that in this case the last digits of all other prime numbers of the additional table of a Magic square 3×3 must be such that their sums in the central column, central row and both diagonals would terminate by 3. Thus, only certain arrangements of the last digits of prime numbers are possible in the remaining cells of the additional table and all such variants are shown in figure 2.

1	1	1
1	1	1
1	1	1

(1)

3	3	3
3	3	3
3	3	3

(4)

7	7	7
7	7	7
7	7	7

(7)

9	9	9
9	9	9
9	9	9

(10)

3	1	9
3	1	9
3	1	9

(2)

7	3	9
7	3	9
7	3	9

(5)

1	7	3
1	7	3
1	7	3

(8)

1	9	7
1	9	7
1	9	7

(11)

5	3	1
3	1	9
1	9	7

(3)

5	9	3
9	3	7
3	7	1

(6)

5	1	7
1	7	3
7	3	9

(9)

5	7	9
7	9	1
9	1	3

(12)

Figure 2. All possible arrangements of the last digits of the prime numbers in cells of the additional table 1(4).

Corollaries from Proposition 3 [2]:

1. Since 5 is a prime number having the form $6k - 1$, only the prime numbers of the form $6k - 1$ can be placed in cells of the additional table 1(4) with arrangements 2(3), 2(6), 2(9) and 2(12).

2. The arithmetic progression from three prime numbers a_k–30m, a_k, a_k+30m may be found among nine prime numbers of any Magic square 3×3, where the number a_k is located in the central cell of the Magic square and m is some integer number. Hence it appears that

no Magic square 3×3 may be constructed from prime numbers if $a_k < 30$.

Let us consider some results of [2], obtained for prime numbers by computer.

1. Magic squares 3×3, shown in figure 3, are the least ones, constructed only from prime numbers.

67	1	43	101	5	71	101	29	83	109	7	79
13	37	61	29	59	89	53	71	89	43	73	103
31	73	7	47	113	17	59	113	41	67	139	37
	(1)			(2)			(3)			(4)	

Figure 3. The least Magic squares 3×3, constructed only from prime numbers.

2. Let it be required to construct a Magic square 3×3 only from prime numbers with the number a_k in its central cell. This problem cannot be solved only for the following prime numbers a_k >30:

a) having the form 6k – 1: 41, 101; 53, 83, 113, 233; 47, 107, 197, 317; 569;

b) having the form 6k + 1: 31, 61, 181, 331; 43, 163, 223, 313, 433; 67, 97, 277, 457; 79, 199, 229, 439, 859.

3. The results of the item 2 make it possible to assume that, for any a_k larger than some prime number P_{max}, one can always construct a Magic square 3×3 with Magic sum $S = 3a_k$ and the prime numbers, ending by the same digit as the number a_k. P_{max} equals the following prime numbers:

a) having the form 6k – 1: 5081 (281); 3323 (683); 6257 (557); 3779 (359);

b) having the form 6k+1: 3931 (601); 3253 (523); 4297 (307); 7489 (769), where in brackets we indicate the least prime numbers a_k, for which one can construct a Magic square 3×3 with $S = 3a_k$ and the prime numbers, ending by the same digit as a_k.

4. Let it be required from prime numbers to construct a Magic square 9×9, which contains the number a_k in its central cell and consists of 9 Magic squares 3×3.

The example of the least Magic square 9×9, constructed only from prime numbers and consisted of 9 Magic squares 3×3, is shown in figure 4.

If a_k > 1019, then the problem on constructing Magic squares 9×9, discussed in this item, cannot be solved only for following prime numbers a_k:

1021, 1031, 1033, 1039, 1049, 1051, 1061, 1069, 1087, 1091, 1093, (18) 1097, 1109, 1117, 1123, 1129, 1153, 1171, 1181, 1193, 1201, 1213, 1217, 1229, 1231, 1237, 1249, 1259, 1279, 1283, 1303, 1307, 1321, 1327, 1439, 1453, 1481, 1483, 1489, 1511, 1531, 1543, 1567, 1783.

2531	17	1409	1097	71	863	2069	23	1091
197	1319	2441	443	677	911	83	1061	2039
1229	2621	107	491	1283	257	1031	2099	53
1433	29	821	1811	137	1109	2153	311	1367
149	761	1373	317	1019	1721	491	1277	2063
701	1493	89	929	1901	227	1187	2243	401
1487	431	1013	2339	173	1571	1307	11	839
503	977	1451	593	1361	2129	251	719	1187
941	1523	467	1151	2549	383	599	1427	131

Figure 4. The example of the least Magic square 9×9, constructed only from prime numbers and consisted of 9 Magic squares 3×3.

6 Magic Squares 3×3 and 9×9 from Smarandache Numbers of the 2nd Kind

6.1 Magic Squares 3×3 and 9×9 from P_1-Series Numbers

Let the notation $A_{C_j}(N)$ means the quantity of all C_j-series numbers, whose values are less than N, and the notation *P_0-series* means the prime numbers series.

Proposition 4. *For any natural number N the following inequality*

$$A_{P_1}(N) \geq A_{P_0}(N) \tag{19}$$

is fulfilled

Proof. The truth of Proposition 4 follows from the description of P_1-series numbers (see Section 3.1). Namely, P_0-series numbers is subset of P_1-series numbers at any N and agree with a set of P_1-series numbers only if $N \leq 13$.

Proposition 5. *P_1-series numbers are available for constructing Magic squares 3×3.*

Proof. The truth of Proposition 5 follows from Proposition 4 and that the prime numbers are available for constructing Magic squares 3×3 (see Section 5).

Solving the problems on constructing Magic squares 3×3 from P_1-series numbers by computer, we find that

1. Magic squares 3×3, shown in figure 5, are the least ones, constructed from P_1-series numbers.

47	5	35
17	29	41
23	53	11

(1)

50	11	35
17	32	47
29	53	14

(2)

53	11	41
23	35	47
29	59	17

(3)

50	17	38
23	35	47
32	53	20

(4)

Figure 5. The least Magic squares 3×3, constructed from P_1-series numbers.

2. Let it be required from P_1-series numbers to construct a Magic square 3×3 with the number a_k in its central cell.

If $a_k > 35$, then this problem cannot be solved only for the following P_1-series numbers: 38, 43, 47, 50 and 61.

3. Let it be required from P_1-series numbers to construct a Magic square 9×9, which contains the number a_k in its central cell and consists of 9 Magic squares 3×3.

Magic square 9×9, shown in figure 6, is the least such one, constructed from P_1-series numbers.

We note, that

a) in the Magic square 9×9, shown in figure 6, the numbers 215, 35, 143, 59, 203, 119, 227 and 47 may be replaced respectively by 203, 47, 143, 71, 191, 119, 215 and 59;

413	101	329	137	20	92	383	2	269
197	281	365	38	83	128	104	218	332
233	461	149	74	146	29	167	434	53
215	35	143	293	11	278	380	17	374
59	131	203	179	194	209	251	257	263
119	227	47	110	377	95	140	497	134
317	5	188	323	272	320	182	14	125
41	170	299	302	305	308	50	107	164
152	335	23	290	338	287	89	200	32

Figure 6. The least Magic square 9×9, constructed from P_1-series numbers and consisted of 9 Magic squares 3×3.

b) if $a_k > 194$, then the problem on constructing Magic squares 9×9, discussed in this point, cannot be solved only for following 10 P_1-series numbers a_k: 196, 197, 199, 211, 214, 217, 223, 229, 232 and 300.

6.2 Magic Squares 3×3 and 9×9 from P_2-Series Numbers

Proposition 6. *For any natural number N the following inequality*

$$A_{P_2}(N) < A_{P_1}(N) \qquad (20)$$

is fulfilled

Proof. The truth of Proposition 6 follows from the description of P_2-series numbers (see Section 3.2). Namely, P_2-series numbers may be obtained by deleting all prime numbers from P_1-series numbers.

It follows from Proposition 6 that, although we know about the availability of P_o- and P_2-series numbers for constructing Magic squares 3×3, we cannot state that P_2-series numbers are also available for constructing Magic squares 3×3. To clear up this situation, let us consider our results, obtained for P_2-series numbers by computer.

1. Magic squares 3×3, shown in figure 7, are the least ones, constructed from P_2-series numbers.

152	14	110
50	92	134
74	170	32

(1)

164	50	143
98	119	140
95	188	74

(2)

203	20	134
50	119	188
104	218	35

(3)

215	20	140
50	125	200
110	230	35

(4)

Figure 7. The least Magic squares 3×3, constructed from P_2-series numbers.

2. Let it be required from P_2-series numbers to construct a Magic square 3×3 with the number a_k in its central cell.

If a_k = 92, 125, 441, 448, 652, 766 or 928, then this problem has a single solution.

If a_k > 125, then this problem cannot be solved only for the following P_2-series numbers:

130, 142, 143, 145, 152, 160, 166, 169, 172, 175, 176, 190, 196, 232, (21)

238, 289, 292, 298, 300, 301, 304, 319, 325, 382, 385, 391, 478, 517.

3. Let it be required from P_2-series numbers to construct a Magic square 9×9, which contains the number a_k in its central cell and consists of 9 Magic squares 3×3.

If a_k = 473, then there are 609 the least Magic squares 9×9 with mentioned properties (the example of such Magic square is shown in figure 8).

If a_k > 473, then the problem on constructing Magic squares 9×9, discussed in this item, cannot be solved only for two P_2-series numbers a_k: 478 and 517.

1007	140	578	374	278	344	830	74	632
146	575	1004	302	332	362	314	512	710
572	1010	143	320	386	290	392	950	194
785	32	413	728	20	671	902	14	692
38	410	782	416	473	530	326	536	746
407	788	35	275	926	218	380	1058	170
740	92	470	872	236	734	533	203	377
164	434	704	476	614	752	215	371	527
398	776	128	494	992	356	365	539	209

Figure 8. The example of the least Magic square 9×9, constructed from P_2-series numbers and consisted of 9 Magic squares 3×3.

6.3 Magic Squares 3×3 and 9×9 from T_1-Series Numbers

Proposition 7. *There exists such natural number N_0 that for any natural $N > N_0$ the following inequality*

$$A_{T_1}(N) > A_{P_0}(N) \qquad (22)$$

is fulfilled

Proof. As it follows from the description of T_1-series numbers (see Section 3.2), this series numbers may be obtained from series odd natural numbers by deleting all such odd numbers, which are prime numbers decreased by 2. Thus, we have the following relation

$$A_{T_1}(N)=(N-1)/2 - A_{P_0}(N) \quad \text{or} \quad A_{T_1}(N)/A_{P_0}(N) = \{(N-1)/2\}/A_{P_0}(N)-1 \qquad (23)$$

where the term $(N-1)/2$ is the quantity of all odd natural numbers, whose values are less than N. Since [8]

$$A_{P_0}(N) = N/\{\ln(N)+1\} \pm N/\ln^2(N), \qquad (24)$$

we obtain from (23) and (24) that

$$A_{T_1}(N)/A_{P_0}(N) \approx \ln(N)/2 - 1 > 2 \ \text{ for any } N > 500. \qquad (25)$$

Thus, Proposition 7 is true, if N_0, for instance, equals 500.

Proposition 8. *T_1-series numbers are available for constructing Magic squares 3×3.*

Proof. The truth of Proposition 8 follows from Proposition 7 and that the prime numbers are available for constructing Magic squares 3×3.

Let us consider our results, obtained for T_1-series numbers by computer.

1. Magic square 3×3, shown in figure 9(1), is the least one, constructed from T_1-series numbers.

49	7	37
19	31	43
25	55	13

(1)

83	13	63
33	53	73
43	93	23

(2)

117	19	89
47	75	103
61	131	33

(3)

185	31	141
75	119	163
97	207	53

(4)

Figure 9. Examples of Magic squares 3×3, constructed from T_1-series numbers.

2. Let it be required from T_1-series numbers to construct a Magic square 3×3 with the number a_k in its central cell.

If $a_k = 53$, 75 or 119, then this problem has a single solution {see figure 9(2 – 4)}.

If $a_k > 31$, then this problem cannot be solved only for two T_1-series numbers: 33 and 47.

3. Let it be required from T_1-series numbers to construct a Magic square 9×9, which contains the number a_k in its central cell and consists of 9 Magic squares 3×3.

If $a_k = 181$, then there are 118 the least Magic squares 9×9 with mentioned properties (the example of such Magic square is shown in figure 10).

If $a_k > 181$, then the problem on constructing Magic squares 9×9, discussed in this item, can be solved for all T_1-series numbers a_k.

319	247	301	55	7	49	317	93	241
271	289	307	31	37	43	141	217	293
277	331	259	25	67	19	193	341	117
127	79	121	205	151	187	283	203	273
103	109	115	163	181	199	243	253	263
97	139	91	175	211	157	233	303	223
215	61	159	443	169	363	123	13	83
89	145	201	245	325	405	33	73	113
131	229	75	287	481	207	63	133	23

Figure 10. The example of the least Magic square 9×9, constructed from T_1-series numbers and consisted of 9 Magic squares 3×3.

If $a_k = 181$, then there are 118 the least Magic squares 9×9 with mentioned properties (the example of such Magic square is shown in figure 10).

If $a_k > 181$, then the problem on constructing Magic squares 9×9, discussed in this item, can be solved for all T_1-series numbers a_k.

6.4 Magic Squares 3×3 and 9×9 from T_2-Series Numbers

Proposition 9. *There exists such natural number N_0 that for any natural $N > N_0$ the following inequality*

$$A_{T_2}(N) > A_{P_0}(N) \tag{26}$$

is fulfilled

Proof. As it follows from the description of T_2-series numbers (see Section 3.2), this series numbers may be obtained from series natural numbers by deleting all 2^k-th numbers on each k-th step of step-procedure (sieve). Thus, we have the following relation

$$A_{T_2}(N) \approx N - N\prod_{k=1}^{[\log(n)]} 1/2^k \approx N(1-2/\{\log(N)\,(\log(N)+1)\}) \approx \tag{27}$$

$$\approx N(1-2.9/\{\ln(N)\,(1.44\ln(N)+1)\}).$$

We obtain from (24) and (27) that

$$A_{T_1}(N)/A_{P_0}(N) \approx \ln(N) > 2 \;\text{ for any }\; N > 20. \tag{28}$$

Thus, Proposition 9 is true, if N_0, for instance, equals 20.

Proposition 10. *T_2-series numbers are available for constructing Magic squares 3×3.*

Proof. The truth of Proposition 10 follows from Proposition 9 and that the prime numbers are available for constructing Magic squares 3×3.

Our computations give the following results:

1. Magic squares 3×3, shown in figure 11, are the least ones, constructed from T_2-series numbers.

29	1	21
9	17	25
13	33	5

(1)

33	5	25
13	21	29
17	37	9

(2)

51	1	29
5	27	49
25	53	3

(3)

43	1	33
17	27	37
21	53	11

(4)

43	5	33
17	27	37
21	49	11

(5)

Figure 11. The least Magic squares 3×3, constructed from T_2-series numbers.

2. Let it be required from T_2-series numbers to construct a Magic square 3×3 with the number a_k in its central cell.

If $a_k > 27$, then this problem cannot be solved only for two T_2-series numbers: 37 and 49.

3. Let it be required from T_1-series numbers to construct a Magic square 9×9, which contains the number a_k in its central cell and consists of 9 Magic squares 3×3.

395	11	299	265	17	153	373	5	237
139	235	331	33	145	257	69	205	341
171	459	75	137	273	25	173	405	37
249	29	197	325	1	259	397	9	269
113	165	217	129	195	261	97	225	353
133	301	81	131	389	65	181	441	53
321	13	221	401	21	313	251	27	187
85	185	285	157	245	333	91	155	219
149	357	49	177	469	89	123	283	59

Figure 12. The example of the least Magic square 9×9, constructed from T_2-series numbers and consisted of 9 Magic squares 3×3.

If a_k = 195, then there are 6 the least Magic squares 9×9 with mentioned properties (the example of such Magic square is shown in figure 12).

If a_k > 195, then the problem on constructing Magic squares 9×9, discussed in this point, cannot be solved only for the following P_2-series numbers a_k:

$$197, 201, 205, 213, 213, 217, 221, 225, 229, 237, \tag{29}$$

$$245, 249, 257, 261, 269.$$

7 Concluding Remarks

As it is demonstrated in this paper, preliminary theoretical analysis of number-theoretic and combinatorial problems is always useful. In particular, the results of this analysis are able sometimes to provide investigators with valuable information, facilitating considerably the solution of all such of practical tasks, which are enumerated in Section 1. We hope, that the technique of theoretical analysis, elaborated in the paper, will become useful tool of investigators, occupied in the considered problems.

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