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Inviscid limit for vortex patches

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Abstract. We investigate the inviscid limit for two dimensional incompressible fluids in the plane. We prove that, if the initial data are vortex patches with smooth boundaries, then the inviscid Eulerian dynamics is approached at a rate that is slower than the rate for smooth initial data. The circular patches provide lower bounds.

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1. Introduction

We study the difference between solutions $u^{(NS)}$ of the incompressible 2D Navier–Stokes equation and solutions $u^{(E)}$ of the incompressible 2D Euler equations. If the solutions start from identical, smooth initial data, then the difference is of the same order of magnitude as the kinematic viscosity. This fact [1, 2, 3, 4] holds in the whole plane or on the two dimensional torus, and is true for fixed intervals of time. In bounded domains such a global estimate is expected to fail, in general, because of boundary layers. Two dimensional turbulence theories [5, 6] that are based on the Euler equations are statistical theories for the vorticity field, which is assumed to be bounded and integrable, $\omega \in L^1 \cap L^\infty$. There exists a classical theory of existence and uniqueness in this setting [7]. It is therefore natural to study the inviscid limit in this phase space. The specific question we address here, that of the inviscid limit for vortex patches, was asked by R Krasny.

It is instructive to examine first an example, the circular patches. If the initial vorticity is radially symmetric,

$$\omega(x, 0) = a(x)$$

then, under the Euler evolution the vorticity does not change, $\omega(x,t) = a(x)$. For radially symmetric initial data the Navier–Stokes evolution coincides with the linear heat equation evolution. Thus

$$\omega^{(NS)} = G_{l(t)} * a$$

where

$$l(t) = \sqrt{\nu t}$$

with ν the kinematic viscosity,

$$G_l(x) = l^{-2} G\left(\frac{x}{l}\right)$$

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and G is a Gaussian. The difference $\omega = \omega^{(NS)} - \omega^{(E)}$ is thus given by

$$\omega(x,t) = \int \left(\delta_{l(t)y} a\right)(x) G(y) \mathrm{d}y$$

where

$$\left(\delta_z a\right)(x) = a(x-z) - a(x).$$

For smooth (radially symmetric) a the expected result is recovered by expanding in Taylor series: the first order term in the expansion vanishes because of the reflection symmetry of the Gaussian and the result is an $\mathrm{O}(l^2)$ estimate. For non-smooth initial radial vorticities, such as circular vortex patches, the result is different. For the latter, one can check that

$$\|\omega(\cdot,t)\|_{L^2} = \mathrm{O}(\sqrt{rl(t)})$$

is sharp (r is the radius of the vortex patch). Note that this is a $\nu^{\frac{1}{4}}$ estimate. The corresponding L^2 velocity difference is $\mathrm{O}(\sqrt{\nu})$.

In this paper we prove that the difference $u^{(NS)} - u^{(E)}$ for vortex patch initial data is in L^2 and satisfies an $\mathrm{O}(\sqrt{\nu})$ estimate. We have to restrict ourselves to initial vortex patches with smooth boundaries. Thus, the initial gradient of vorticity is a measure concentrated on a smooth curve. The inviscid limit is proved under the assumption

$$\int_0^T \|\nabla u^{(E)}(\cdot,t)\|_{L^\infty} \mathrm{d}t < \infty.$$

This estimate is true for vortex patches with smooth boundaries [8], but is not true in general, in particular if the vortex patch has corners.

For general vortex patch initial data we prove that the Navier–Stokes solutions are exponentially small outside a time dependent region that contains the Eulerian patch.

2. The inviscid limit

We consider solutions of the two dimensional Navier–Stokes and Euler equations

$$\frac{\partial u}{\partial t} + u \cdot \nabla u + \nabla p = \nu \Delta u$$
$$\nabla \cdot u = 0.$$

The kinematic viscosity ν is a positive number in the case of the Navier–Stokes equations; it equals zero for the Euler equations. The vorticity

$$\omega(x,t) = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}$$

obeys the nonlinear advection–diffusion equation

$$\left(\partial_t + u \cdot \nabla - \nu \Delta\right) \omega = 0.$$

The vorticity equation can be viewed as the basic evolution equation. In this formulation the velocity is computed from the vorticity via the Biot–Savart law:

$$u = K * \omega$$

where

$$K(x) = \frac{1}{2\pi} \nabla^{\perp} \log\left(|x|\right).$$

Proposition 1. *Consider an initial vorticity*

$$a(x) = \omega(x, 0)$$

which belongs to the space $L^1 \cap L^\infty$ in the plane. Then the initial value problems for the Navier–Stokes and Euler equations have global solutions. The L^q norms of the vorticity are bounded by their initial data:

$$\|\omega(\cdot, t)\|_{L^q} \leqslant \|a\|_{L^q} \qquad 1 \leqslant q \leqslant \infty.$$

The associated velocities are bounded uniformly

$$\|u(\cdot, t)\|_{L^\infty} \leqslant U$$

where

$$U = (\|a\|_{L^1} \|a\|_{L^\infty})^{\frac{1}{2}}.$$

We will need the following notation. If $a \neq 0$ is a function in $L^1 \cap L^\infty$ then we associate to it the length scale ρ given by

$$\rho = \left(\frac{\|a\|_{L^1}}{\|a\|_{L^\infty}}\right)^{\frac{1}{2}}.$$

The velocity scale U encountered above,

$$U = (\|a\|_{L^1} \|a\|_{L^\infty})^{\frac{1}{2}}$$

will be used also in the following. If $R > 0$ we consider the annulus

$$A_R = \{x;\ R \leqslant |x| \leqslant 2R\}.$$

We will need the following result:

Proposition 2. *The velocities of the solutions of the Navier–Stokes and Euler equations with initial vorticity $a \in L^1 \cap L^\infty$ satisfy*

$$\|u\|^2_{L^2(A_R)} \leqslant C\|a\|^2_{L^1}\left(1 + \log_+\left(\frac{R}{\rho}\right)\right).$$

The constant C is an absolute constant. The pressures corresponding to these solutions satisfy

$$\|p\|_{L^2} \leqslant C\|a\|^2_{L^{\frac{4}{3}}}.$$

The pressures are normalized to have mean zero.

We consider now the case in which the initial vorticity, $a(x) = \omega(x, 0)$ is a constant multiple of the characteristic function of a bounded, simply connected domain D in the plane:

$$a(x) = \omega_0 \chi_D(x).$$

We will assume that the boundary ∂D of the initial patch is smooth ($C^{1,\mu}$ for some $\mu > 0$). We recall [8]

Proposition 3. *If the initial vorticity*

$$a(x) = \omega_0 \chi_D(x)$$

is a multiple of the characteristic function of a simply connected bounded domain D with $C^{1,\mu}$ *boundary,* $\mu > 0$ *then the global solution* $u^{(E)}$ *satisfies*

$$\int_0^T \|\nabla u^{(E)}(\cdot, t)\|_{L^\infty} \mathrm{d}t \leqslant \gamma$$

where the constant γ *depends only on the initial datum a and on T.*

The velocity is obtained from the vorticity by convolution with the Biot–Savart kernel. Such velocities are not square-integrable in the plane, they decay at infinity only as fast as $|x|^{-1}$. Our main result is:

Theorem 1. *Consider the velocity difference*

$$w(x, t) = u^{(NS)}(x, t) - u^{(E)}(x, t)$$

between a solution of the Navier–Stokes equation ($u^{(NS)}$) *and a solution of the Euler equation* ($u^{(E)}$)*. Assume that these have the same initial datum, corresponding to a vortex patch with smooth boundary*

$$a = \omega_0 \chi_D.$$

Then, the difference $w = u^{(NS)} - u^{(E)}$ *is square-integrable and obeys the estimate:*

$$\|w(\cdot, t)\|_{L^2}^2 \leqslant 2\nu t \|a\|_{L^2}^2 exp\left(\int_0^t 2\|\nabla u^{(E)}(\cdot, \tau)\|_{L^\infty} \mathrm{d}\tau\right).$$

In particular, there exists a constant γ *depending only on the initial vorticity and T, such that*

$$\|w(\cdot, t)\|_{L^2} \leqslant (2\nu t)^{\frac{1}{2}} \|a\|_{L^2} \mathrm{e}^{\gamma}$$

holds for $0 \leqslant t \leqslant T$.

Remarks

1. The result holds actually under the assumptions $a \in L^1 \cap L^\infty$ together with the requirement

$$\int_0^T \|\nabla u^{(E)}(\cdot, t)\|_{L^\infty} \mathrm{d}t < \infty.$$

It is however only for smooth vorticities or for vortex patches with smooth boundaries (or similar, non-constant vorticities) that we can prove *a priori* that the above quantity stays finite.

2. The result holds if the initial datum of the Navier–Stokes solution is slightly different than that of the Euler equation; however, one has to make sure that the difference has finite kinetic energy.

The next result is for general initial vortex patches. We write

$$\left[\left[\frac{x}{\delta}\right]\right] = \sqrt{1 + \frac{|x|^2}{\delta^2}}.$$

If the initial vorticity has compact support, one obtains

Theorem 2. *Assume that the initial vorticity* $a \in L^1 \cap L^\infty$ *has compact support, included in the disk*

$$\{x; |x| \leqslant L\}.$$

Then

$$\|\omega^{(NS)}(\cdot,t)-\omega^{(E)}(\cdot,t)\|_{L^\infty(Q_t)} \leqslant \|a\|_{L^\infty} e^{-\frac{UL}{\nu}}.$$

where

$$Q_t=\left\{x;|x|\geqslant C\left(L+\frac{\nu}{U}+Ut\right)\right\}$$

and C is an absolute constant.

The vorticity of the solution of the Euler equation vanishes in Q_t. Thus, this result is one of exponential convergence in Q_t, as the Reynolds number $\frac{UL}{\nu}$ tends to infinity. The assumption of compact support can be relaxed:

Proposition 4. *Let the initial vorticity for the Navier–Stokes or Euler equation $a\in L^1\cap L^\infty$. Then, for any $\delta>0$*

$$\|\omega(\cdot,t)e^{[[\frac{\cdot}{\delta}]]}\|_{L^q} \leqslant \|a(\cdot)e^{[[\frac{\cdot}{\delta}]]}\|_{L^q} exp\left(\frac{Ut}{\delta}+\frac{7\nu t}{\delta^2}\right)$$

holds for any q, $1\leqslant q\leqslant\infty$.

3. Proofs

We start with the proof of theorem 1. Consider the difference

$$w(x,t)=u^{(NS)}(x,t)-u^{(E)}(x,t)$$

between the solutions of the Navier–Stokes and, respectively, the Euler equation. The common initial vorticity is a vortex patch. The difference solves

$$\frac{\partial w}{\partial t}+u^{(NS)}\cdot\nabla w+w\cdot\nabla u^{(E)}=\nu\Delta u^{(NS)}-\nabla\left(p^{(NS)}-p^{(E)}\right).$$

We take a non-negative, smooth cutoff function χ, identically equal to 1 for $|x|\leqslant 1$ and to 0 for $|x|\geqslant 2$. We multiply the equation for w by $w(x,t)\chi_R^2$ where $\chi_R(x)=\chi(\frac{x}{R})$ and $R>0$. Integrating in space we obtain:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\chi_R^2|w|^2\,\mathrm{d}x=I+II+III+IV$$

where

$$I=\frac{1}{2}\int\left(u^{(NS)}\cdot\nabla\chi_R^2\right)|w|^2\,\mathrm{d}x,$$

$$II=-\int\chi_R^2\left(w\cdot\nabla u^{(E)}\right)w\,\mathrm{d}x,$$

$$III=\int\left(p^{(NS)}-p^{(E)}\right)w\cdot\nabla\chi_R^2\,\mathrm{d}x$$

and

$$IV=\nu\int\Delta u^{(NS)}w\chi_R^2\,\mathrm{d}x.$$

We estimate the first term by

$$|I|\leqslant C\frac{1}{R}\left(\int_{A_R}|u^{(NS)}|^2|w|^2\mathrm{d}x\right)^{\frac{1}{2}}\|\chi_R w\|_{L^2}.$$

Using the results of propositions 1 and 2, more precisely the facts

$$\|w\|_{L^\infty} \leqslant \|u^{(NS)}\|_{L^\infty} + \|u^{(E)}\|_{L^\infty} \leqslant 2U$$

and

$$\|u^{(NS)}\|_{L^2(A_R)} \leqslant \Gamma(R)$$

with

$$\Gamma(R) = C\|a\|_{L^1}\sqrt{1+\log_+\left(\frac{R}{\rho}\right)}$$

we obtain

$$|I| \leqslant CUR^{-1}\Gamma(R)\|\chi_R w\|_{L^2}.$$

The smoothness of the vortex patch boundary is needed only in the estimate for the second term:

$$|II| \leqslant \|\nabla u^{(E)}\|_{L^\infty}\|\chi_R w\|^2.$$

The third term is estimated by

$$|III| \leqslant C\frac{1}{R}\|p^{(NS)} - p^{(E)}\|_{L^2}\|\chi_R w\|_{L^2}$$

which, in view of propositions 1 and 2, is bounded by

$$|III| \leqslant \frac{G}{R}\|\chi_R w\|_{L^2}$$

with

$$G = C\|a\|^2_{L^{\frac{4}{3}}}.$$

Before estimating the last term we rewrite it:

$$IV = -\nu\int \nabla u^{(NS)} :: \nabla(\chi_R^2 w)\,\mathrm{d}x$$

where $M :: N = \mathrm{Tr}MN^*$. Using the fact that $u^{(NS)} = w + u^{(E)}$ we obtain

$$IV \leqslant -\frac{\nu}{2}\int \chi_R^2|\nabla w|^2\,\mathrm{d}x + \nu\int|\nabla u^{(E)}|^2\,\mathrm{d}x + C\frac{\nu}{R^2}\int_{A_R}|w|^2$$
$$+C\frac{\nu}{R}\|\nabla u^{(E)}\|_{L^2(A_R)}\|\chi_R w\|_{L^2}.$$

It follows that

$$IV \leqslant \nu\|a\|^2_{L^2} + C\frac{\nu}{R^2}(\Gamma(R))^2 + C\frac{\nu}{R}\|a\|_{L^2}\|\chi_R w\|_{L^2}.$$

The proof of the theorem is completed by using the Gronwall lemma, then passing to the limit $R \to \infty$ and using the Fatou lemma.

We prove now proposition 4. We consider the function

$$\Phi(x) = e^{[[\frac{x}{\delta}]]}.$$

It satisfies:

$$|\frac{\nabla\Phi}{\Phi}| \leqslant \frac{1}{\delta},$$
$$0 \leqslant \frac{\Delta\Phi}{\Phi} \leqslant \frac{3}{\delta^2}$$

and

$$0 \leqslant \nabla \cdot \left(\frac{\nabla\Phi}{\Phi}\right) \leqslant \frac{2}{\delta^2}.$$

If ω solves the Navier–Stokes or Euler equations, then the function $\Phi\omega$ solves

$$(\partial_t + v \cdot \nabla - \nu\Delta)(\Phi\omega) = (\Phi\omega)\left[v \cdot \left(\frac{\nabla\Phi}{\Phi}\right) - \nu\frac{\Delta\Phi}{\Phi}\right]$$

where

$$v = u + 2\nu\frac{\nabla\Phi}{\Phi}$$

and u is the Navier–Stokes or Euler velocity. In order to simplify the exposition we take even powers $q = p + 1$. We multiply by $q\,(\Phi\omega)^p$ and integrate. Strictly speaking, one has not proved yet that the integrals converge, so one has to pass through an additional process of approximating Φ by bounded functions which agree with Φ on larger and larger regions. We obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int(\Phi\omega)^q\,\mathrm{d}x \leqslant \int(\Phi\omega)^q\left\{q\left[v \cdot \left(\frac{\nabla\Phi}{\Phi}\right) - \nu\frac{\Delta\Phi}{\Phi}\right] + \nabla \cdot v\right\}\mathrm{d}x$$

and hence

$$\|\Phi\omega(\cdot,t)\|_{L^q} \leqslant \|\Phi a\|_{L^q} exp\left[\frac{U}{\delta} + \frac{7\nu}{\delta^2}\right]t.$$

Theorem 2 follows from proposition 4 by choosing judiciously $\delta = c\frac{\nu}{U}$. The proof of proposition 1 is classical; we mention only that the bound $\|u\|_{L^\infty} \leqslant U$ follows directly from the Biot–Savart law. The result regarding the pressures in proposition 2 follows from the fact that the mean-free pressures are given by

$$p = R_i R_j\left(u_i u_j\right)$$

where $R_i = (-\Delta)^{-\frac{1}{2}}\frac{\partial}{\partial x_i}$ are the Riesz transforms, and from classical inequalities for the Riesz transforms. The bounds on the velocity in $L^2(A_R)$ are easy consequences of the Biot–Savart law. One has

$$|u(x)| \leqslant \frac{1}{2\pi}\int\frac{|\omega(y)|}{|x-y|}dy$$

and hence

$$|u(x)| \leqslant I_1(x) + I_2(x) + I_3(x)$$

where I_1 is the integral corresponding to $0 \leqslant |x-y| \leqslant \rho$, I_2 corresponds to $\rho \leqslant |x-y| \leqslant R$ and I_3 to $|x-y| \geqslant R$. Clearly

$$\|I_3\|^2_{L^2(A_R)} \leqslant C\|\omega\|^2_{L^1}$$

and because $\|I_1\|_{L^\infty} \leqslant C\|\omega\|_{L^\infty}\rho$, $\|I_1\|_{L^1} \leqslant C\|\omega\|_{L^1}\rho$, it follows that

$$\|I_1\|^2_{L^2} \leqslant C\|\omega\|_{L^1}\|\omega\|_{L^\infty}\rho^2.$$

In order to estimate I_2 one considers an L^2 test function ϕ and obtains

$$\left|\int\phi I_2\mathrm{d}x\right| \leqslant \left(\log_+\left(\frac{R}{\rho}\right)\right)^{\frac{1}{2}}\|\phi\|_{L^2}\|\omega\|_{L^1}.$$

The result follows then from proposition 1.

4. Conclusions

We proved that the velocity difference $w = u^{(NS)} - u^{(E)}$ between solutions of the Navier–Stokes and Euler equations converges to zero in L^2 if the initial datum is a vortex patch with smooth boundaries. The convergence rate is $O(\sqrt{\nu})$, and is slower than the rate $O(\nu)$ of convergence for smooth initial data. The circular patches provide lower bounds. We do not know whether a rate of convergence exists for vortex patches with non-smooth boundaries. We do not know whether a rate of convergence for vorticity in L^2 exists, even for smooth vortex patch boundaries. What we do know and prove for arbitrary vortex patch initial data is that the convergence is exponential outside a time depending domain which contains the Euler patch.

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