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Involutes of fronts in the Euclidean plane

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Abstract

The notions of involutes (also known as evolvents) and evolutes were studied by C. Huygens. For a regular plane curve, an involute of it is the trajectory described by the end of stretched string unwinding from a point of the curve. Even if a regular curve, the involute of the curve have singularities. By using a moving frame of the front and the curvature of the Legendre immersion in the unit tangent bundle, we define an involute of the front in the Euclidean plane and discuss properties of them. We also consider about relationship between evolutes and involutes of fronts without inflection points. As a result, we observe that the evolutes and the involutes of fronts without inflection points are corresponding to the differential and the integral in classical calculus.

1 Introduction

The notions of involutes (also known as evolvents) and evolutes were studied by C. Huygens in his work [14] and studied in classical analysis, differential geometry and singularity theory of planar curves (cf. [6, 7, 11, 12, 13, 19]). For a regular plane curve, an involute of it is the trajectory described by the end of stretched string unwinding from a point of the curve. Alternatively, another way to construct the involute of a curve is to replace the taut string by a line segment that is tangent to the curve on one end, while the other end traces out the involute. The length of the line segment is changed by an amount equal to the arc length traversed by the tangent point as it moves along the curve. As a remarkable property of a regular curve without inflection points, the involute of the regular curve at a point has a $3/2$ cusp at the point.

On the other hand, the evolute of a regular plane curve is also classical object (cf. [6, 11, 12]). The evolute of a regular curve without inflection points is given by not only the locus of all its centres of curvature, but also an envelope of the normal lines of the regular curve. It is well-known that the relationship between involutes and evolutes of regular plane curves. The evolute of an involute is the original curve, less portions of zero or undefined curvature. The properties of evolutes discussed by using distance squared functions and the theories of Lagrangian, Legendrian singularity (cf. [2, 3, 4, 9, 17, 18, 20, 22]).

In this paper, we define involutes of curves with singular points which are called fronts. In section 2, we recall the definitions for the involute and the evolute of regular plane curves.

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Key Words and Phrases. involute, evolute, front, Legendre immersion, inflection point

Moreover, for a Legendre curve (a Legendre immersion) in the unit tangent bundle, we give a moving frame of the frontal (the front) and the curvature of the Legendre curve (the Legendre immersion) (cf. [8]). By using them, we define an involute of the front. We also recall the definition of the evolute of the front without inflection points (cf. [9]). We discuss on properties of involutes without inflection points, for example, the involute of the front without inflection points is also a front without inflection points. In section 3, we analyse singular points of the involute of the front without inflection points. Moreover, we give a relationship between singular points of the involute of the front and vertices. In section 4, we consider a relationship between evolutes and involutes of fronts without inflection points by using the curvature of Legendre immersions. Since the involute of the front without inflection points is also a front without inflection points, we can repeat the involute of the front. We give a formula of the n-th involute of the front. We introduce a special parametrisation for Legendre immersions without inflection points in section 5. By using the parametrisation, the evolute and the involute of the front without inflection points are corresponding to the differential and the integral in classical calculus. We give not only the relationship among the contact of Legendre immersions, evolutes and involutes, but also the same sharp of the front and the n-th evolute (respectively, the n-th involute) of the front under the same parametrisation.

We shall assume throughout the whole paper that all maps and manifolds are C^∞, unless the contrary is explicitly stated.

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2 Basic notations and definitions

We recall the definitions of the involute and the evolute of a regular curve (cf. [6, 11, 12]). Let I be an interval or $\mathbb{R}$. Suppose that $\gamma : I \to \mathbb{R}^2$ is a regular plane curve, that is, $\dot{\gamma}(t) = (d\gamma/dt)(t) \neq 0$ for all $t \in I$. If s is the arc-length parameter of γ, then $|\gamma'(s)| = \sqrt{\gamma'(s) \cdot \gamma'(s)} = 1$, where $\gamma'(s) = (d\gamma/ds)(s)$ and $\cdot$ is the inner product on $\mathbb{R}^2$. We denote $\boldsymbol{t}(s)$ by the unit tangent vector $\boldsymbol{t}(s) = \gamma'(s) = (d\gamma/ds)(s)$ and $\boldsymbol{n}(s)$ by the unit normal vector $\boldsymbol{n}(s) = J(\boldsymbol{t}(s))$ of $\gamma(s)$, where J is the anticlockwise rotation by $\pi/2$. Then we have the Frenet formula as follows:

$$\begin{pmatrix} \boldsymbol{t}'(s) \\ \boldsymbol{n}'(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) \\ -\kappa(s) & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{t}(s) \\ \boldsymbol{n}(s) \end{pmatrix},$$

where

$$\kappa(s) = \boldsymbol{t}'(s) \cdot \boldsymbol{n}(s) = \det(\gamma'(s), \gamma''(s))$$

is the curvature of γ.

Even if t is not the arc-length parameter, we have the unit tangent vector $\boldsymbol{t}(t) = \dot{\gamma}(t)/|\dot{\gamma}(t)|$, the unit normal vector $\boldsymbol{n}(t) = J(\boldsymbol{t}(t))$ and the Frenet formula

$$\begin{pmatrix} \dot{\boldsymbol{t}}(t) \\ \dot{\boldsymbol{n}}(t) \end{pmatrix} = \begin{pmatrix} 0 & |\dot{\gamma}(t)|\kappa(t) \\ -|\dot{\gamma}(t)|\kappa(t) & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{t}(t) \\ \boldsymbol{n}(t) \end{pmatrix},$$

where the curvature is given by

$$\kappa(t) = \frac{\dot{\boldsymbol{t}}(t) \cdot \boldsymbol{n}(t)}{|\dot{\gamma}(t)|} = \frac{\det(\dot{\gamma}(t), \ddot{\gamma}(t))}{|\dot{\gamma}(t)|^3}.$$

Note that the curvature $\kappa(t)$ is independent on the choice of a parametrisation.

In this paper, we consider involutes and evolutes of plane curves. For $t_0 \in I$, the *involute* $Inv(\gamma, t_0) : I \to \mathbb{R}^2$ *of a regular plane curve* $\gamma : I \to \mathbb{R}^2$ *at* t_0 is given by

$$Inv(\gamma, t_0)(t) = \gamma(t) - \left(\int_{t_0}^{t} |\dot{\gamma}(u)| du\right) \boldsymbol{t}(t).$$

The *evolute* $Ev(\gamma) : I \to \mathbb{R}^2$ *of a regular plane curve* $\gamma : I \to \mathbb{R}^2$ is given by

$$Ev(\gamma)(t) = \gamma(t) + \frac{1}{\kappa(t)}\boldsymbol{n}(t),$$

away from inflection points, that is, $\kappa(t) \neq 0$.

Note that if s is the arc-length parameter of γ, then the involute of γ at s_0 is given by

$$Inv(\gamma, s_0)(s) = \gamma(s) - (s - s_0)\boldsymbol{t}(s)$$

and the evolute of γ is given by

$$Ev(\gamma)(s) = \gamma(s) + \frac{1}{\kappa(s)}\boldsymbol{n}(s).$$

We give examples of an involute and an evolute of a regular curve, for more examples see in [6, 11, 12] etc.

Example 2.1 Let $\gamma : [-\pi, \pi) \to \mathbb{R}^2$ be a circle $\gamma(t) = (r\cos t, r\sin t)$ with radius $r > 0$. Then the involute of the circle at t_0 is

$$Inv(\gamma, t_0)(t) = (r\cos t + (t - t_0)r\sin t, r\sin t - (t - t_0)r\cos t),$$

and pictured the involute of the circle with $r = 1$ at $t_0 = 0$ as Figure 1.

Example 2.2 Let $\gamma : [0, 2\pi) \to \mathbb{R}^2$ be an ellipse $\gamma(t) = (a\cos t, b\sin t)$ with $a, b > 0$ and $a \neq b$. Then the evolute of the ellipse is

$$Ev(\gamma)(t) = \left(\frac{a^2 - b^2}{a}\cos^3 t, -\frac{a^2 - b^2}{b}\sin^3 t\right),$$

and pictured the evolute of the ellipse with $a = 3/2, b = 1$ as Figure 2.

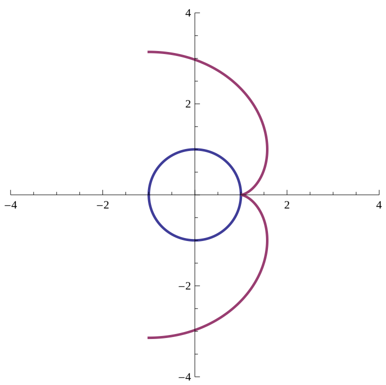


Figure 1. The involute of the circle at 0.

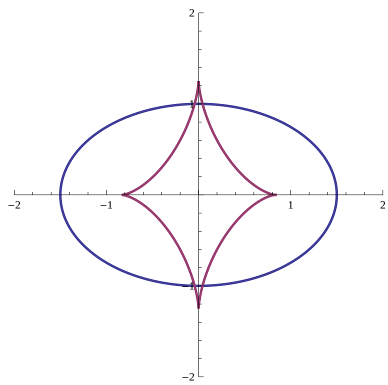


Figure 2. The evolute of the ellipse.

The following properties are well-known in the classical differential geometry of curves:

Proposition 2.3 *Let $\gamma : I \to \mathbb{R}^2$ be a regular curve and $t_0 \in I$.*

(1) *If t is a regular point of $Inv(\gamma, t_0)$, then $Ev(Inv(\gamma, t_0))(t) = \gamma(t)$.*

(2) *If t and t_0 are regular points of $Ev(\gamma)$ and not inflection points of γ, then $Inv(Ev(\gamma), t_0)(t) = \gamma(t) + (1/\kappa(t_0))\boldsymbol{n}(t)$.*

Even if γ is a regular curve, the point t_0 is a singular point of the involute $Inv(\gamma, t_0)$ and also the evolute $Ev(\gamma)$ may have singularities, see Figures 1 and 2. For a singular point of $Inv(\gamma, t_0)$ (respectively, $Ev(\gamma)$), $Ev(Inv(\gamma, t_0))(t)$ (respectively, $Inv(Ev(\gamma), t_0)(t)$) can not be defined.

In general, if γ is not a regular curve, then we can not define the involute and the evolute of the curve as above. In [9], however, we defined the evolute of the front without inflection points in the Euclidean plane, see Definition 2.10. In the present paper, we define an involute of the front in the Euclidean plane, see Definition 2.11. These are generalisations of evolutes and involutes of regular plane curves. In order to define an evolute and an involute of the front, we review on Legendre curves in the unit tangent bundle, the Frenet formula and the curvature of the Legendre curve ([8]).

We say that $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ is *a Legendre curve* if $(\gamma, \nu)^*\theta = 0$, where θ is a canonical contact 1-form on the unit tangent bundle $T_1\mathbb{R}^2 = \mathbb{R}^2 \times S^1$ (cf. [2, 3, 4]). This condition is equivalent to $\dot{\gamma}(t) \cdot \nu(t) = 0$ for all $t \in I$. Moreover, if (γ, ν) is an immersion, we call (γ, ν) *a Legendre immersion*. We say that $\gamma : I \to \mathbb{R}^2$ is *a frontal* (respectively, *a front* or *a wave front*) if there exists a smooth mapping $\nu : I \to S^1$ such that (γ, ν) is a Legendre curve (respectively, a Legendre immersion).

Let $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ be a Legendre curve. Then we have the Frenet formula of the frontal γ as follows. We put on $\boldsymbol{\mu}(t) = J(\nu(t))$. We call the pair $\{\nu(t), \boldsymbol{\mu}(t)\}$ a *moving frame of the frontal* $\gamma(t)$ in $\mathbb{R}^2$ and the Frenet formula of the frontal (or, the Legendre curve) which is given by

$$\begin{pmatrix} \dot{\nu}(t) \\ \dot{\boldsymbol{\mu}}(t) \end{pmatrix} = \begin{pmatrix} 0 & \ell(t) \\ -\ell(t) & 0 \end{pmatrix} \begin{pmatrix} \nu(t) \\ \boldsymbol{\mu}(t) \end{pmatrix},$$

where $\ell(t) = \dot{\nu}(t) \cdot \boldsymbol{\mu}(t)$. Moreover, there exists a smooth function $\beta(t)$ such that

$$\dot{\gamma}(t) = \beta(t)\boldsymbol{\mu}(t).$$

The pair (ℓ, β) is an important invariant of Legendre curves (or, frontals). We call the pair $(\ell(t), \beta(t))$ *the curvature of the Legendre curve* (with respect to the parameter t).

Definition 2.4 Let (γ, ν) and $(\widetilde{\gamma}, \widetilde{\nu}) : I \to \mathbb{R}^2 \times S^1$ be Legendre curves. We say that (γ, ν) and $(\widetilde{\gamma}, \widetilde{\nu})$ are *congruent as Legendre curves* if there exists a congruence C on $\mathbb{R}^2$ such that $\widetilde{\gamma}(t) = C(\gamma(t)) = A(\gamma(t)) + \boldsymbol{b}$ and $\widetilde{\nu}(t) = A(\nu(t))$ for all $t \in I$, where C is given by the rotation A and the translation $\boldsymbol{b}$ on $\mathbb{R}^2$.

We have the existence and the uniqueness for Legendre curves in the unit tangent bundle like as regular plane curves, see in [8].

Theorem 2.5 (The Existence Theorem) *Let $(\ell, \beta) : I \to \mathbb{R}^2$ be a smooth mapping. There exists a Legendre curve $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ whose associated curvature of the Legendre curve is (ℓ, β).*

Theorem 2.6 (The Uniqueness Theorem) *Let (γ,ν) and $(\widetilde{\gamma},\widetilde{\nu}) : I \to \mathbb{R}^2 \times S^1$ be Legendre curves whose curvatures of Legendre curves (ℓ,β) and $(\widetilde{\ell},\widetilde{\beta})$ coincide. Then (γ,ν) and $(\widetilde{\gamma},\widetilde{\nu})$ are congruent as Legendre curves.*

In fact, the Legendre curve whose associated curvature of the Legendre curve is (ℓ,β), is given by the form

$$\begin{aligned}
\gamma(t) &= \left(-\int \beta(t)\sin\left(\int \ell(t)dt\right)dt,\ \int \beta(t)\cos\left(\int \ell(t)dt\right)dt\right),\\
\nu(t) &= \left(\cos\int \ell(t)dt,\ \sin\int \ell(t)dt\right).
\end{aligned}$$

We also have the following Lemma:

Lemma 2.7 ([8]) *Let (γ,ν) and $(\widetilde{\gamma},\widetilde{\nu}) : I \to \mathbb{R}^2 \times S^1$ be congruent as Legendre curves. Then (γ,ν) and $(\widetilde{\gamma},\widetilde{\nu})$ have the same curvatures of Legendre curves (ℓ,β) and $(\widetilde{\ell},\widetilde{\beta})$ respectively.*

We give examples of Legendre curves.

Example 2.8 One of the typical example of a front (and hence a frontal) is a regular plane curve. Let $\gamma : I \to \mathbb{R}^2$ be a regular plane curve. In this case, we may take $\nu : I \to S^1$ by $\nu(t) = \boldsymbol{n}(t)$. Then it is easy to check that $(\gamma,\nu) : I \to \mathbb{R}^2 \times S^1$ is a Legendre immersion (a Legendre curve). By a direct calculation, the relationship between the curvature of the Legendre curve $(\ell(t),\beta(t))$ and the curvature $\kappa(t)$ is given by $\ell(t) = |\beta(t)|\kappa(t)$.

Example 2.9 Let n, m and k be natural numbers with $m = n + k$. Let $(\gamma,\nu) : I \to \mathbb{R}^2 \times S^1$ be

$$\gamma(t) = \left(\frac{1}{n}t^n, \frac{1}{m}t^m\right),\ \nu(t) = \frac{1}{\sqrt{t^{2k}+1}}\left(-t^k, 1\right).$$

It is easy to see that (γ,ν) is a Legendre curve, and a Legendre immersion when $k = 1$. We call γ is of *type* (n,m). For example, the frontal of type $(2,3)$ has the $3/2$ cusp (A_2 singularity) at $t = 0$, of type $(3,4)$ has the $4/3$ cusp (E_6 singularity) at $t = 0$, see Figure 3. By definition, we have $\boldsymbol{\mu}(t) = (1/\sqrt{t^{2k}+1})(-1, -t^k)$ and

$$\ell(t) = \frac{kt^{k-1}}{t^{2k}+1},\ \beta(t) = -t^{n-1}\sqrt{t^{2k}+1}.$$

Let $(\gamma,\nu) : I \to \mathbb{R}^2 \times S^1$ be a Legendre curve with the curvature of the Legendre immersion (ℓ,β). We say that $t_0 \in I$ is *an inflection point of the frontal* (or, *an inflection point of a Legendre curve* (γ,ν)) if $\ell(t_0) = 0$. Note that if t_0 is a regular point of γ, the definition of the inflection point coincides with the usual inflection point for regular curves. If a Legendre curve (γ,ν) does not have inflection points, then (γ,ν) is a Legendre immersion. In this paper, we consider a Legendre immersion without inflection points.

In [9], we have defined the evolute of the front without inflection points in the Euclidean plane by using parallel curves of the front. Here, we recall an alternative definition of the evolute of the front as follows, see Theorem 3.3 in [9].

Hereafter we assume that $(\gamma,\nu) : I \to \mathbb{R}^2 \times S^1$ is a Legendre immersion without inflection points. We denote the curvature of the Legendre immersion by (ℓ,β).

Definition 2.10 *The evolute $\mathcal{E}v(\gamma) : I \to \mathbb{R}^2$ of the front γ without inflection points is given by*

$$\mathcal{E}v(\gamma)(t) = \gamma(t) - \frac{\beta(t)}{\ell(t)}\nu(t).$$

The definition of the evolute $\mathcal{E}v(\gamma)$ of the front is a generalisation of the evolute $Ev(\gamma)$ of a regular curve γ. For properties of the evolute of the front see in [9].

We define the involute of the front as follows:

Definition 2.11 *The involute $\mathcal{I}nv(\gamma, t_0) : I \to \mathbb{R}^2$ of the front γ at $t_0 \in I$ is given by*

$$\mathcal{I}nv(\gamma, t_0)(t) = \gamma(t) - \left(\int_{t_0}^{t} \beta(u)du\right) \boldsymbol{\mu}(t).$$

For a regular plane curve $\gamma : I \to \mathbb{R}^2$, we consider $\boldsymbol{n}(t) = \nu(t)$ in Example 2.8. It follows that $\boldsymbol{t}(t) = -\boldsymbol{\mu}(t)$ and $|\dot{\gamma}(t)| = -\beta(t)$. Therefore, we have the following result.

Proposition 2.12 *For a regular curve $\gamma : I \to \mathbb{R}^2$ and any $t_0 \in I$, we have*

$$Inv(\gamma, t_0) = \mathcal{I}nv(\gamma, t_0).$$

Thus, the definition of the involute $\mathcal{I}nv(\gamma, t_0)$ of the front is a generalisation of the involute $Inv(\gamma, t_0)$ of a regular plane curve γ.

Remark 2.13 The evolute and the involute of the front are independent on the choice of a parametrisation. Moreover, if the set of regular points of γ is dense, then $\mathcal{E}v(\gamma)$ and $\mathcal{I}nv(\gamma, t_0)$ are uniquely determined by γ, namely, these are not depended on the choice of ν.

Proposition 2.14 *Let $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ be a Legendre immersion with the curvature of the Legendre immersion (ℓ, β) and without inflection points.*

(1) *The evolute $\mathcal{E}v(\gamma)$ is a front. More precisely, $(\mathcal{E}v(\gamma)(t), J(\nu(t)))$ is a Legendre immersion with the curvature*

$$\left(\ell(t),\ \frac{d}{dt}\left(\frac{\beta(t)}{\ell(t)}\right)\right).$$

(2) *The involute $\mathcal{I}nv(\gamma, t_0)$ is a front for any $t_0 \in I$. More precisely, $(\mathcal{I}nv(\gamma, t_0)(t), J^{-1}(\nu(t)))$ is a Legendre immersion with the curvature*

$$\left(\ell(t),\ \ell(t)\int_{t_0}^{t} \beta(u)du\right).$$

Proof. (1) By using the Frenet formula of the front, we have

$$\begin{aligned}
\dot{\mathcal{E}v}(\gamma)(t) &= \dot{\gamma}(t) - \frac{d}{dt}\left(\frac{\beta(t)}{\ell(t)}\right)\nu(t) - \frac{\beta(t)}{\ell(t)}\dot{\nu}(t) \\
&= \beta(t)\boldsymbol{\mu}(t) - \frac{d}{dt}\left(\frac{\beta(t)}{\ell(t)}\right)\nu(t) - \frac{\beta(t)}{\ell(t)}\ell(t)\boldsymbol{\mu}(t) \\
&= -\frac{d}{dt}\left(\frac{\beta(t)}{\ell(t)}\right)\nu(t) \\
&= \frac{d}{dt}\left(\frac{\beta(t)}{\ell(t)}\right)J(\boldsymbol{\mu}(t)).
\end{aligned}$$

Therefore, $\dot{\mathcal{E}v}(\gamma)(t) \cdot J(\nu(t)) = 0$. Since

$$\frac{d}{dt}(J(\nu(t))) = J(\dot{\nu}(t)) = J(\ell(t)\boldsymbol{\mu}(t)) = \ell(t)J(\boldsymbol{\mu}(t)) \neq 0,$$

$(\mathcal{E}v(\gamma)(t), J(\nu(t)))$ is a Legendre immersion with the curvature $(\ell(t), (d/dt)(\beta(t)/\ell(t)))$.

(2) We also have

$$\begin{aligned}
\dot{\mathcal{I}nv}(\gamma,t_0)(t) &= \dot{\gamma}(t) - \frac{d}{dt}\left(\int_{t_0}^{t} \beta(u)du\right)\boldsymbol{\mu}(t) - \left(\int_{t_0}^{t} \beta(u)du\right)\dot{\boldsymbol{\mu}}(t) \\
&= \beta(t)\boldsymbol{\mu}(t) - \beta(t)\boldsymbol{\mu}(t) + \left(\int_{t_0}^{t} \beta(u)du\right)\ell(t)\nu(t) \\
&= \ell(t)\left(\int_{t_0}^{t} \beta(u)du\right)\nu(t) \\
&= \ell(t)\left(\int_{t_0}^{t} \beta(u)du\right)J^{-1}(\boldsymbol{\mu}(t)).
\end{aligned}$$

Therefore, $\dot{\mathcal{I}nv}(\gamma,t_0)(t) \cdot J^{-1}(\nu(t)) = 0$. Since

$$\frac{d}{dt}(J^{-1}(\nu(t))) = J^{-1}(\dot{\nu}(t)) = J^{-1}(\ell(t)\boldsymbol{\mu}(t)) = \ell(t)J^{-1}(\boldsymbol{\mu}(t)) \neq 0,$$

$(\mathcal{I}nv(\gamma,t_0)(t), J^{-1}(\nu(t)))$ is a Legendre immersion with the curvature $(\ell(t), \ell(t)\int_{t_0}^{t}\beta(u)du)$. $\Box$

We give an example of an involute of a front. Examples of evolutes of fronts see in [9].

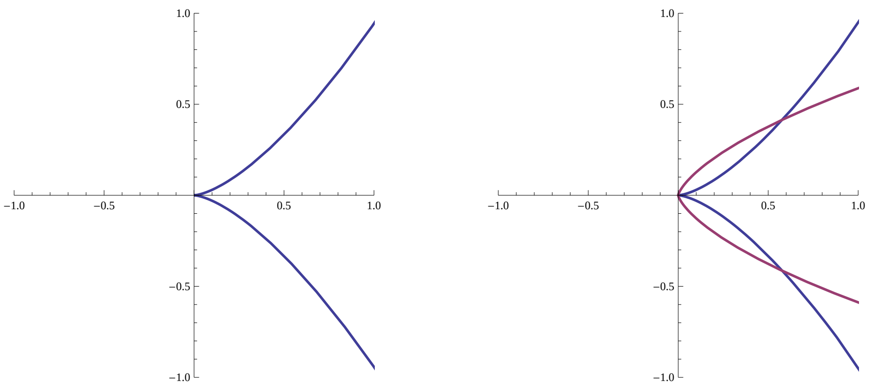
Example 2.15 Let $(\gamma,\nu) : \mathbb{R} \to \mathbb{R}^2 \times S^1$ be of type $(2,3)$ in Example 2.9,

$$\gamma(t) = \left(\frac{1}{2}t^2, \frac{1}{3}t^3\right), \ \nu(t) = \frac{1}{\sqrt{t^2+1}}(-t,1).$$

We have $\boldsymbol{\mu}(t) = (-1/\sqrt{t^2+1})(1,t)$, $\ell(t) = 1/(t^2+1)$ and $\beta(t) = -t\sqrt{t^2+1}$. It follows that the involute of the $3/2$ cusp at $t_0 \in \mathbb{R}$ is given by

$$\mathcal{I}nv(\gamma,t_0)(t) = \left(\frac{t^2}{6} - \frac{1}{3} + \frac{1}{3}\frac{(t_0^2+1)^{\frac{3}{2}}}{\sqrt{t^2+1}}, \ -\frac{t}{3} + \frac{1}{3}\frac{(t_0^2+1)^{\frac{3}{2}}}{\sqrt{t^2+1}}t\right).$$

Remark that the involute of the $3/2$ cusp at $t_0 = 0$ is diffeomorphic to the $4/3$ cusp at 0, see Figure 3 and Corollary 3.3 below.



The 3/2 cusp The involute of the 3/2 cusp at 0.

Figure 3.

3 Properties of involutes of fronts

Let $(\gamma,\nu): I \to \mathbb{R}^2 \times S^1$ be a Legendre immersion with the curvature of the Legendre immersion (ℓ,β) and without inflection points. We give properties of the involute of the front.

Proposition 3.1 *For any points $t_0, t_1 \in I$, $\mathcal{I}nv(\gamma,t_1)$ is a parallel curve of $\mathcal{I}nv(\gamma,t_0)$.*

Proof. By the definition of involutes, we have

$$\begin{aligned}
\mathcal{I}nv(\gamma,t_1)(t) &= \gamma(t) - \left(\int_{t_1}^{t} \beta(u)du\right)\boldsymbol{\mu}(t) \\
&= \gamma(t) - \left(\int_{t_0}^{t} \beta(u)du\right)\boldsymbol{\mu}(t) - \left(\int_{t_1}^{t_0} \beta(u)du\right)\boldsymbol{\mu}(t) \\
&= \mathcal{I}nv(\gamma,t_0)(t) + \left(\int_{t_0}^{t_1} \beta(u)du\right) J^{-1}(\nu(t)).
\end{aligned}$$

Since $J^{-1}(\nu(t))$ is the unit normal of $\mathcal{I}nv(\gamma,t_0)(t)$, $\mathcal{I}nv(\gamma,t_1)$ is a parallel curve of $\mathcal{I}nv(\gamma,t_0)$. $\Box$

We analyse singular points of the involute of the front.

Proposition 3.2 *Let $t_0 \in I$.*

(1) t_1 is a singular point of $\mathcal{I}nv(\gamma,t_0)$ if and only if $\int_{t_0}^{t_1} \beta(s)ds = 0$.

(2) Suppose that t_1 is a singular point of $\mathcal{I}nv(\gamma,t_0)$. Then $\mathcal{I}nv(\gamma,t_0)$ is diffeomorphic to the 3/2 cusp at t_1 if and only if $\beta(t_1) \neq 0$.

(3) Suppose that t_1 is a singular point of $\mathcal{I}nv(\gamma,t_0)$. Then $\mathcal{I}nv(\gamma,t_0)$ is diffeomorphic to the 4/3 cusp at t_1 if and only if $\beta(t_1) = 0$ and $\dot\beta(t_1) \neq 0$.

Proof. (1) By differentiate of the involute of the front, we have

$$\dot{\mathcal{I}}nv(\gamma,t_0)(t) = \ell(t)\left(\int_{t_0}^{t} \beta(u)du\right)\nu(t).$$

Since the assumption $\ell(t) \neq 0$ for all $t \in I$, we have the result.

(2) From the Frenet formula of the front, we have

$$\ddot{\mathcal{I}}nv(\gamma,t_0)(t) = \left(\dot\ell(t)\left(\int_{t_0}^{t} \beta(u)du\right) + \ell(t)\beta(t)\right)\nu(t) + \ell(t)^2\left(\int_{t_0}^{t} \beta(u)du\right)\boldsymbol{\mu}(t).$$

By (1), we obtain

$$\ddot{\mathcal{I}}nv(\gamma,t_0)(t_1) = \ell(t_1)\beta(t_1)\nu(t_1).$$

Moreover, we have

$$\dddot{\mathcal{I}}nv(\gamma,t_0)(t_1) = \left(2\dot\ell(t_1)\beta(t_1) + \ell(t_1)\dot\beta(t_1)\right)\nu(t_1) + 2\ell(t_1)^2\beta(t_1)\boldsymbol{\mu}(t_1).$$

Thus,

$$\det\left(\ddot{\mathcal{I}}nv(\gamma,t_0)(t_1), \dddot{\mathcal{I}}nv(\gamma,t_0)(t_1)\right) \;=\; 2\ell(t_1)^3\beta(t_1)^2 \neq 0$$

if and only if $\beta(t_1) \neq 0$. Therefore, we obtain (2).

(3) By (2),

$$\det\Big(\ddot{\mathcal{I}nv}(\gamma,t_0)(t_1), \dddot{\mathcal{I}nv}(\gamma,t_0)(t_1)\Big) = 0$$

if and only if $\beta(t_1) = 0$. Moreover,

$$\frac{d^4}{dt^4}\mathcal{I}nv(\gamma,t_0)(t_1) = \Big(3\dot{\ell}(t_1)\dot{\beta}(t_1) + \ell(t_1)\ddot{\beta}(t_1)\Big)\,\nu(t_1) + 3\ell(t_1)^2\dot{\beta}(t_1)\boldsymbol{\mu}(t_1)$$

and

$$\det\left(\frac{d^3}{dt^3}\mathcal{I}nv(\gamma,t_0)(t_1), \frac{d^4}{dt^4}\mathcal{I}nv(\gamma,t_0)(t_1)\right) \;=\; 3\ell(t_1)^3\dot{\beta}(t_1)^2,$$

under the conditions $\int_{t_0}^{t_1}\beta(u)du = 0$ and $\beta(t_1) = 0$. Hence, we have

$$\det\Big(\ddot{\mathcal{I}nv}(\gamma,t_0)(t_1), \dddot{\mathcal{I}nv}(\gamma,t_0)(t_1)\Big) = 0$$

and

$$\det\left(\frac{d^3}{dt^3}\mathcal{I}nv(\gamma,t_0)(t_1), \frac{d^4}{dt^4}\mathcal{I}nv(\gamma,t_0)(t_1)\right) \neq 0$$

if and only if $\beta(t_1) = 0$ and $\dot{\beta}(t_1) \neq 0$. It follows that $\mathcal{I}nv(\gamma,t_0)$ is diffeomorphic to the 4/3 cusp at t_1 (cf. [5, 15, 16]). Therefore, we obtain (3). $\Box$

By Proposition 3.2, we have the following Corollary:

Corollary 3.3 *Under the above notations, we have the following.*

(1) $\mathcal{I}nv(\gamma,t_0)$ *is diffeomorphic to the* 3/2 *cusp at* t_0 *if and only if* t_0 *is a regular point of* γ.

(2) $\mathcal{I}nv(\gamma,t_0)$ *is diffeomorphic to the* 4/3 *cusp at* t_0 *if and only if* γ *is diffeomorphic to the* 3/2 *cusp at* t_0.

Proof. (1) The point t_0 is a singular point of $\mathcal{I}nv(\gamma,t_0)$. Since the point t_0 is a regular point of γ if and only if $\beta(t_0) \neq 0$. By Proposition 3.2 (2), we have the result.

(2) Under the assumption $\ell(t) \neq 0$, γ is diffeomorphic to the 3/2 cusp at t_0 if and only if $\beta(t_0) = 0$ and $\dot{\beta}(t_0) \neq 0$. By Proposition 3.2 (3), we have the result. $\Box$

Remark 3.4 In this paper, we assume that the front does not have inflection points, though we can define the involute of the front with inflection points. In this case, the involute of the front is a frontal (cf. [10]). We can find other kinds of singularities of the involute, see [1, 10, 21].

Lemma 3.5 *If* $t_1 \in I \setminus \{t_0\}$ *is a singular point of* $\mathcal{I}nv(\gamma,t_0)$, *then there exists at least one singular point of* γ *in the open interval* (t_0,t_1) (*respectively,* (t_1,t_0)) *when* $t_0 < t_1$ (*respectively,* $t_1 < t_0$).

Proof. We show the case for $t_0 < t_1$. By the mean value theorem for integration, there exists a point $\xi \in (t_0,t_1)$ such that

$$\int_{t_0}^{t_1}\beta(u)du = \beta(\xi)(t_1 - t_0).$$

Since t_1 is a singular point of $\mathcal{I}nv(\gamma, t_0)$, we have $\int_{t_0}^{t_1} \beta(u)du = 0$. It follows that $\beta(\xi) = 0$, that is, ξ is a singular point of γ. $\square$

Next we discuss a relationship between singular points of an involute of the front and vertices. Let (γ, ν) be a Legendre immersion with the curvature of the Legendre immersion (ℓ, β) and without inflection points. We say that a point t_0 is a *vertex of the front* γ (or, *vertex of the Legendre immersion* (γ, ν)) if $(d/dt)(\beta/\ell)(t_0) = 0$, equivalently $(d/dt)\mathcal{E}v(t_0) = 0$. Note that if t_0 is a regular point of γ, the definition of the vertex coincides with the usual vertex for regular curves (cf. [9]).

In this paper, we say that a Legendre immersion $(\gamma, \nu) : [a, b] \to \mathbb{R}^2 \times S^1$ is *closed* if $(\gamma^{(n)}(a), \nu^{(n)}(a)) = (\gamma^{(n)}(b), \nu^{(n)}(b))$ for all $n \in \mathbb{N} \cup \{0\}$, where $\gamma^{(n)}(a)$, $\nu^{(n)}(a)$, $\gamma^{(n)}(b)$ and $\nu^{(n)}(b)$ means one-sided differential. If $(\gamma, \nu) : [a, b] \to \mathbb{R}^2 \times S^1$ is a closed Legendre immersion, then either a and b are regular points of γ or singular points of γ. When a and b are singular points of γ, we treat these singular points as one singular point of γ.

Note that if $(\mathcal{I}nv(\gamma, t_0), J^{-1}(\nu)) : [a, b] \to \mathbb{R}^2 \times S^1$ is a closed Legendre immersion, then (γ, ν) is also closed. By Lemma 3.5 and Proposition 3.11 in [9], we have the following Lemma.

Lemma 3.6 *Let $(\gamma, \nu) : [a, b] \to \mathbb{R}^2 \times S^1$ be a Legendre immersion without inflection points and at most finite number of singular points of γ. Suppose that $(\mathcal{I}nv(\gamma, t_0), J^{-1}(\nu))$ is a closed Legendre immersion and at most finite number of singular points of $\mathcal{I}nv(\gamma, t_0)$. Then*

$$\sharp\Sigma(\mathcal{I}nv(\gamma, t_0)) \leq \sharp\Sigma(\gamma) \leq \sharp V(\gamma, \nu), \tag{1}$$

where $\Sigma(\mathcal{I}nv(\gamma, t_0))$ (respectively, $\Sigma(\gamma)$) is the set of singular points of the involute $\mathcal{I}nv(\gamma, t_0)$ (respectively, γ) and $V(\gamma, \nu)$ is the set of vertices of (γ, ν).

Proof. We show the first inequality. Suppose that $s_0, \ldots, s_n$ are singular points of $\mathcal{I}nv(\gamma, t_0)$ such that $a < s_0 < s_1 < \cdots < s_n < b$. By Lemma 3.5, there is at least one singular point of γ in the open interval (s_{i-1}, s_i) for each $i = 1, \ldots, n$. We show there is at least one singular point of γ in $(s_n, b] \cup [a, s_0)$. Since $(\mathcal{I}nv(\gamma, t_0), J^{-1}(\nu))$ and (γ, ν) are closed Legendre immersions, we have $\int_{t_0}^{b} \beta(u)du = \int_{t_0}^{a} \beta(u)du$, that is, $\int_{a}^{b} \beta(u)du = 0$. If $\beta(t) > 0$ (respectively, $\beta(t) < 0$) on $(s_n, b] \cup [a, s_0)$, then $\int_{t_0}^{t} \beta(u)du$ is a monotone increase function (respectively, monotone decrease function) on $(s_n, b]$ and $[a, s_0)$. Hence $0 = \int_{t_0}^{s_n} \beta(u)du < \int_{t_0}^{b} \beta(u)du$ and $\int_{t_0}^{a} \beta(u)du < \int_{t_0}^{s_0} \beta(u)du = 0$ (respectively, $0 = \int_{t_0}^{s_n} \beta(u)du > \int_{t_0}^{b} \beta(u)du$ and $\int_{t_0}^{a} \beta(u)du > \int_{t_0}^{s_0} \beta(u)du = 0$). This implies $\int_{a}^{b} \beta(u)du = \int_{t_0}^{b} \beta(u)du - \int_{t_0}^{a} \beta(u)du > 0$ (respectively, $\int_{a}^{b} \beta(u)du = \int_{t_0}^{b} \beta(u)du - \int_{t_0}^{a} \beta(u)du < 0$). This contradicts the fact $\int_{a}^{b} \beta(u)du = 0$. Therefore, there exists a point $\xi \in (s_n, b] \cup [a, s_0)$ such that $\beta(\xi) = 0$.

Next, we suppose that $s_0, \ldots, s_n$ are singular points of $\mathcal{I}nv(\gamma, t_0)$ such that $a = s_0 < s_1 < \cdots < s_n = b$. In this case, there are n singular points of the involute (note that we treat a and b as the one singular point). By Lemma 3.5, there is at least one singular point of γ in the interval (s_{i-1}, s_i) for each $i = 1, \ldots n$. Hence the inequality holds.

The second inequality is a direct conclusion of the proof of Proposition 3.11 in [9], see also Remark 3.7 below. $\square$

Remark 3.7 By definition, the set of vertices of (γ, ν) is the set of singular points of $\mathcal{E}v(\gamma)$. By Proposition 4.1, we can also prove the second inequality of (1) in Lemma 3.6 by the same method of the first inequality.

Remark 3.8 The inequality (1) in Lemma 3.6 also holds when $\sharp\Sigma(\mathcal{I}nv(\gamma, t_0))$ is infinite. In fact, if $\sharp\Sigma(\mathcal{I}nv(\gamma, t_0))$ is infinite, then $\sharp\Sigma(\gamma)$ is infinite. Moreover, if $\sharp\Sigma(\gamma)$ is infinite, then $\sharp V(\gamma, \nu)$ is infinite.

Remark 3.9 If $(\mathcal{I}nv(\gamma, t_0), J^{-1}(\nu)) : [a, b] \to \mathbb{R}^2 \times S^1$ is a closed Legendre immersion, then $\mathcal{I}nv(\gamma, t_0)(a) = \mathcal{I}nv(\gamma, t_0)(b)$ and hence $\int_a^b \beta(s)ds = 0$. It follows that γ must have a singular point. As a consequence, when γ is a regular curve, even if (γ, ν) is closed, $(\mathcal{I}nv(\gamma, t_0), J^{-1}(\nu))$ can not be a closed Legendre immersion.

We give the four vertices theorem of a front.

Proposition 3.10 *Let $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ be a Legendre immersion without inflection points. Suppose that $(\mathcal{I}nv(\gamma, t_0), J^{-1}(\nu))$ is a closed Legendre immersion.*

(1) *If $\mathcal{I}nv(\gamma, t_0)$ has at least four singular points, then (γ, ν) has at least four vertices.*

(2) *If $\mathcal{I}nv(\gamma, t_0)$ has at least two singular points which degenerate more than $3/2$ cusp, then (γ, ν) has at least four vertices.*

Proof. (1) This statement is obtained from the inequality in Lemma 3.6 directly.

(2) Suppose $\mathcal{I}nv(\gamma, t_0)$ has at least two singular points t_1 and t_2 which degenerate more than $3/2$ cusp. By Proposition 3.2, we have

$$\int_{t_0}^{t_i} \beta(u)du = 0, \quad \beta(t_i) = 0$$

for $i = 1, 2$. Thus t_1 and t_2 are singular points of γ. Moreover, by Lemma 3.5, there exists at least one singular point for each subset (t_1, t_2) and $I \setminus [t_1, t_2]$. Therefore, γ has at least four singular points. As a consequence, (γ, ν) has at least four vertices by Lemma 3.6. $\square$

4 Relationship between evolutes and involutes of fronts

In this section, we discuss relationships between evolutes and involutes of fronts. Let $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ be a Legendre immersion with the curvature (ℓ, β) and without inflection points. We give a justification of Proposition 2.3 with singular points.

Proposition 4.1 *For any $t_0 \in I$, we have the following.*

(1) *$\mathcal{E}v(\mathcal{I}nv(\gamma, t_0))(t) = \gamma(t)$.*

(2) *$\mathcal{I}nv(\mathcal{E}v(\gamma), t_0)(t) = \gamma(t) - (\beta(t_0)/\ell(t_0))\nu(t)$.*

Proof. (1) We denote the curvature of $(\mathcal{I}nv(\gamma, t_0)(t), J^{-1}(\nu(t)))$ by $(\ell_{-1}(t), \beta_{-1}(t))$. By the definition of the evolute,

$$\mathcal{E}v(\mathcal{I}nv(\gamma, t_0))(t) = \mathcal{I}nv(\gamma, t_0)(t) - \frac{\beta_{-1}(t)}{\ell_{-1}(t)} J^{-1}(\nu(t)).$$

By Proposition 2.14, we obtain

$$\begin{aligned} \mathcal{I}nv(\gamma, t_0)(t) - \frac{\beta_{-1}(t)}{\ell_{-1}(t)} J^{-1}(\nu(t)) &= \gamma(t) - \left(\int_{t_0}^{t} \beta(u)du\right) \boldsymbol{\mu}(t) + \left(\frac{\ell(t)\int_{t_0}^{t} \beta(u)du}{\ell(t)}\right) \boldsymbol{\mu}(t) \\ &= \gamma(t). \end{aligned}$$

(2) We denote the curvature of $(\mathcal{E}v(\gamma)(t), J(\nu(t)))$ by $(\ell_1(t), \beta_1(t))$. By the definition of the involute,

$$\mathcal{I}nv(\mathcal{E}v(\gamma), t_0)(t) = \mathcal{E}v(\gamma)(t) - \left(\int_{t_0}^{t} \beta_1(u)du\right) J(\boldsymbol{\mu}(t)).$$

By Proposition 2.14, we obtain

$$\begin{aligned} \mathcal{E}v(\gamma)(t) - \left(\int_{t_0}^{t} \beta_1(u)du\right) J(\boldsymbol{\mu}(t)) &= \gamma(t) - \frac{\beta(t)}{\ell(t)}\nu(t) + \left(\int_{t_0}^{t} \frac{d}{du}\left(\frac{\beta(u)}{\ell(u)}\right) du\right) \nu(t) \\ &= \gamma(t) - \frac{\beta(t_0)}{\ell(t_0)}\nu(t). \end{aligned}$$

□

For a given Legendre immersion $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$, we consider an existence condition of a Legendre immersion $(\widetilde{\gamma}, \widetilde{\nu}) : I \to \mathbb{R}^2 \times S^1$ such that $\mathcal{E}v(\widetilde{\gamma})(t) = \gamma(t)$ or $\mathcal{I}nv(\widetilde{\gamma}, t_0)(t) = \gamma(t)$ for some t_0. By using Proposition 4.1, we have the following Corollary.

Corollary 4.2 (1) *If* $(\widetilde{\gamma}(t), \widetilde{\nu}(t)) = (\mathcal{I}nv(\gamma, t_0)(t) + \lambda J^{-1}(\nu(t)), J^{-1}(\nu(t)))$ *for any* $t_0 \in I$ *and any* $\lambda \in \mathbb{R}$, *then* $\mathcal{E}v(\widetilde{\gamma})(t) = \gamma(t)$.

(2) *If* $(\widetilde{\gamma}(t), \widetilde{\nu}(t)) = (\mathcal{E}v(\gamma)(t), J(\nu(t)))$ *and* t_0 *is a singular point of* γ, *then* $\mathcal{I}nv(\widetilde{\gamma}, t_0)(t) = \gamma(t)$.

Proof. (1) Since $\widetilde{\gamma}(t)$ is a parallel curve of $\mathcal{I}nv(\gamma, t_0)(t)$, we have $\mathcal{E}v(\widetilde{\gamma})(t) = \mathcal{E}v(\mathcal{I}nv(\gamma, t_0))(t)$. It follows from Proposition 4.1 that $\mathcal{E}v(\widetilde{\gamma})(t) = \gamma(t)$.

(2) By Proposition 4.1 and $\beta(t_0) = 0$, we have $\mathcal{I}nv(\widetilde{\gamma}, t_0)(t) = \gamma(t)$. □

Conversely, we have the following result.

Proposition 4.3 *Let* (γ, ν) *and* $(\widetilde{\gamma}, \widetilde{\nu}) : I \to \mathbb{R}^2 \times S^1$ *be Legendre immersions with the curvature of the Legendre immersions* (ℓ, β) *and* $(\widetilde{\ell}, \widetilde{\beta})$ *respectively, and without inflection points.*

(1) *If* $(\mathcal{E}v(\widetilde{\gamma})(t), J(\widetilde{\nu}(t)))$ *and* $(\gamma(t), \nu(t))$ *are congruent as Legendre immersions, then for any* $t_0 \in I$, $(\widetilde{\gamma}(t), \widetilde{\nu}(t))$ *and* $(\mathcal{I}nv(\gamma, t_0)(t) + (\widetilde{\beta}(t_0)/\widetilde{\ell}(t_0))J^{-1}(\nu(t)), J^{-1}(\nu(t)))$ *are congruent as Legendre immersions.*

(2) *Let* $t_0 \in I$. *If* $(\mathcal{I}nv(\widetilde{\gamma}, t_0)(t), J^{-1}(\widetilde{\nu}(t)))$ *and* $(\gamma(t), \nu(t))$ *are congruent as Legendre immersions, then* $(\widetilde{\gamma}(t), \widetilde{\nu}(t))$ *and* $(\mathcal{E}v(\gamma)(t), J(\nu(t)))$ *are congruent as Legendre immersions, and* t_0 *is a singular point of* γ.

Proof. (1) Suppose that $(\mathcal{E}v(\widetilde{\gamma})(t), J(\widetilde{\nu}(t)))$ and $(\gamma(t), \nu(t))$ are congruent as Legendre immersions. By Lemma 2.7 and Proposition 2.14, we have simultaneous equations:

$$\ell(t) = \widetilde{\ell}(t),\ \beta(t) = \frac{d}{dt}\left(\frac{\widetilde{\beta}(t)}{\widetilde{\ell}(t)}\right).$$

Thus, we have

$$\int_{t_0}^{t} \beta(u)du = \frac{\widetilde{\beta}(t)}{\widetilde{\ell}(t)} - \frac{\widetilde{\beta}(t_0)}{\widetilde{\ell}(t_0)}.$$

Since $\widetilde{\ell}(t)=\ell(t)$, we have

$$\widetilde{\beta}(t)=\ell(t)\int_{t_0}^{t}\beta(u)du+\left(\frac{\widetilde{\beta}(t_0)}{\widetilde{\ell}(t_0)}\right)\ell(t).$$

By Theorem 2.6, $(\widetilde{\gamma}(t),\widetilde{\nu}(t))$ and $(\mathcal{I}nv(\gamma,t_0)(t)+(\widetilde{\beta}(t_0)/\widetilde{\ell}(t_0))J^{-1}(\nu(t)),J^{-1}(\nu(t)))$ are congruent as Legendre immersions.

(2) Suppose that $(\mathcal{I}nv(\widetilde{\gamma},t_0)(t),J^{-1}(\widetilde{\nu}(t)))$ and $(\gamma(t),\nu(t))$ are congruent as Legendre immersions. By Lemma 2.7 and Proposition 2.14, we have simultaneous equations:

$$\ell(t)=\widetilde{\ell}(t),\ \beta(t)=\widetilde{\ell}(t)\int_{t_0}^{t}\widetilde{\beta}(u)du.$$

Thus, we have

$$\beta(t)=\ell(t)\int_{t_0}^{t}\widetilde{\beta}(u)du.$$

It follows that $\beta(t_0)=0$, that is, t_0 is a singular point of γ. Since $\ell(t)\neq 0$,

$$\int_{t_0}^{t}\widetilde{\beta}(u)du=\frac{\beta(t)}{\ell(t)}.$$

Take derivative both sides, we have $\widetilde{\beta}(t)=(d/dt)(\beta(t)/\ell(t))$. Therefore, we have

$$\widetilde{\ell}(t)=\ell(t),\ \widetilde{\beta}(t)=\frac{d}{dt}\left(\frac{\beta(t)}{\ell(t)}\right).$$

By Theorem 2.6, $(\widetilde{\gamma}(t),\widetilde{\nu}(t))$ and $(\mathcal{E}v(\gamma)(t),J(\nu(t)))$ are congruent as Legendre immersions. $\Box$

Let $(\gamma,\nu):I\to\mathbb{R}^2\times S^1$ be a Legendre immersion with the curvature (ℓ,β) and without inflection points. By Proposition 2.14, $(\mathcal{E}v(\gamma),J(\nu)):I\to\mathbb{R}^2\times S^1$ is also a Legendre immersion without inflection points. Therefore, we can repeat the evolute of the front. In [9], we give the form of the n-th evolute of the front, where n is a natural number. We denote $\mathcal{E}v^0(\gamma)(t)=\gamma(t)$ and $\mathcal{E}v^1(\gamma)(t)=\mathcal{E}v(\gamma)(t)$ for convenience. We define $\mathcal{E}v^n(\gamma)(t)=\mathcal{E}v(\mathcal{E}v^{n-1}(\gamma))(t)$ and

$$\beta_0(t)=\beta(t),\quad \beta_n(t)=\frac{d}{dt}\left(\frac{\beta_{n-1}(t)}{\ell(t)}\right),$$

inductively.

Theorem 4.4 ([9]) $(\mathcal{E}v^n(\gamma),J^n(\nu)):I\to\mathbb{R}^2\times S^1$ *is a Legendre immersion with the curvature* (ℓ,β_n)*, where the* n*-th evolute of the front is given by*

$$\mathcal{E}v^n(\gamma)(t)=\mathcal{E}v^{n-1}(\gamma)(t)-\frac{\beta_{n-1}(t)}{\ell(t)}J^{n-1}(\nu(t))$$

and J^n *is* n*-times of* J*.*

Moreover, by Proposition 2.14, $(\mathcal{I}nv(\gamma,t_0), J^{-1}(\nu)) : I \to \mathbb{R}^2 \times S^1$ is also a Legendre immersion without inflection points for any $t_0 \in I$. Therefore, we can repeat the involute of the front. We denote $\mathcal{I}nv^0(\gamma,t_0)(t) = \gamma(t)$ and $\mathcal{I}nv^1(\gamma,t_0)(t) = \mathcal{I}nv(\gamma,t_0)(t)$ for convenience. We define $\mathcal{I}nv^n(\gamma,t_0)(t) = \mathcal{I}nv(\mathcal{I}nv^{n-1}(\gamma,t_0),t_0)(t)$ and

$$\beta_{-1}(t) = \ell(t)\left(\int_{t_0}^{t} \beta(u)du\right), \quad \beta_{-n}(t) = \ell(t)\left(\int_{t_0}^{t} \beta_{-n+1}(u)du\right)$$

inductively. We give the form of the n-th involute of the front.

Theorem 4.5 $(\mathcal{I}nv^n(\gamma,t_0), J^{-n}(\nu)) : I \to \mathbb{R}^2 \times S^1$ *is a Legendre immersion with the curvature* (ℓ, β_{-n})*, where the n-th involute of the front* γ *at* t_0 *is given by*

$$\mathcal{I}nv^n(\gamma,t_0)(t) = \mathcal{I}nv^{n-1}(\gamma,t_0)(t) + \frac{\beta_{-n}(t)}{\ell(t)} J^{-n}(\nu(t))$$

and J^{-n} *is n-times of* J^{-1}.

Proof. We denote $\{\nu_{\mathcal{I}nv^{n-1}}(t), \boldsymbol{\mu}_{\mathcal{I}nv^{n-1}}(t)\}$ the moving frame of the $(n-1)$-th involute and $(\ell_{\mathcal{I}nv^{n-1}}(t), \beta_{\mathcal{I}nv^{n-1}}(t))$ the curvature of the $(n-1)$-th involute. By the definition of the involute,

$$\mathcal{I}nv^n(\gamma,t_0)(t) = \mathcal{I}nv^{n-1}(\gamma,t_0)(t) - \left(\int_{t_0}^{t} \beta_{\mathcal{I}nv^{n-1}}(u)du\right) \boldsymbol{\mu}_{\mathcal{I}nv^{n-1}}(t).$$

Since $\boldsymbol{\mu}_{\mathcal{I}nv}(t) = J^{-1}(\boldsymbol{\mu}(t))$, we have $\boldsymbol{\mu}_{\mathcal{I}nv^{n-1}}(t) = J^{-n+1}(\boldsymbol{\mu}(t)) = J^{-n+2}(\nu(t))$ inductively. Moreover, $\beta_{\mathcal{I}nv}(t) = \ell(t)\int_{t_0}^{t} \beta(s)ds = \beta_{-1}(t)$ by Proposition 2.14. Thus we have $\beta_{\mathcal{I}nv^{n-1}}(t) = \beta_{-n+1}(t)$ inductively. It follows that

$$\begin{aligned}
\mathcal{I}nv^n(\gamma,t_0)(t) &= \mathcal{I}nv^{n-1}(\gamma,t_0)(t) - \left(\int_{t_0}^{t} \beta_{-n+1}(u)du\right) J^{-n+2}(\nu(t)) \\
&= \mathcal{I}nv^{n-1}(\gamma,t_0)(t) - \frac{\ell(t)\int_{t_0}^{t} \beta_{-n+1}(u)du}{\ell(t)} J^{-n+2}(\nu(t)) \\
&= \mathcal{I}nv^{n-1}(\gamma,t_0)(t) - \frac{\beta_{-n}(t)}{\ell(t)} J^{-n+2}(\nu(t)) \\
&= \mathcal{I}nv^{n-1}(\gamma,t_0)(t) + \frac{\beta_{-n}(t)}{\ell(t)} J^{-n}(\nu(t)).
\end{aligned}$$

Since

$$\begin{aligned}
\frac{d}{dt}\mathcal{I}nv^n(\gamma,t_0)(t) &= \frac{d}{dt}\mathcal{I}nv^{n-1}(\gamma,t_0)(t) - \beta_{-n+1}(t)J^{-n+2}(\nu(t)) - \beta_{-n}(t)J^{-n+2}(\boldsymbol{\mu}(t)) \\
&= \beta_{-n+1}(t)J^{-n+2}(\nu(t)) - \beta_{-n+1}(t)J^{-n+2}(\nu(t)) + \beta_{-n}(t)J^{-n}(\boldsymbol{\mu}(t)) \\
&= \beta_{-n}(t)J^{-n}(\boldsymbol{\mu}(t)), \\
\frac{d}{dt}J^{-n}(\nu(t)) &= J^{-n}(\dot{\nu}(t)) = J^{-n}(\ell(t)\boldsymbol{\mu}(t)) = \ell(t)J^{-n}(\boldsymbol{\mu}(t)),
\end{aligned}$$

$(\mathcal{I}nv^n(\gamma,t_0)(t), J^{-n}(\nu)(t))$ is a Legendre immersion with the curvature $(\ell(t), \beta_{-n}(t))$. □

Remark 4.6 We can consider n-th involutes of the front at different initial points. The difference is given by a parallel curve of the involute by Proposition 3.1. In this paper, we only consider the n-th involute of the front at the same initial point.

By Theorems 4.4 and 4.5, we have the following sequence of the Legendre immersions (the evolutes and the involutes) without inflection points,

$$\cdots \stackrel{\mathcal{I}nv}{\leftarrow} (\mathcal{I}nv^2(\gamma,t_0)(t), J^{-2}(\nu)(t)) \stackrel{\mathcal{I}nv}{\leftarrow} (\mathcal{I}nv(\gamma,t_0)(t), J^{-1}(\nu)(t)) \stackrel{\mathcal{I}nv}{\leftarrow}$$
$$(\gamma(t),\nu(t)) \stackrel{\mathcal{E}v}{\rightarrow} (\mathcal{E}v(\gamma)(t), J(\nu)(t)) \stackrel{\mathcal{E}v}{\rightarrow} (\mathcal{E}v^2(\gamma)(t), J^2(\nu)(t)) \stackrel{\mathcal{E}v}{\rightarrow} \cdots$$

and the corresponding sequence of the curvatures of the evolutes and the involutes,

$$\cdots \leftarrow (\ell(t),\beta_{-2}(t)) \leftarrow (\ell(t),\beta_{-1}(t)) \leftarrow (\ell(t),\beta(t)) \rightarrow (\ell(t),\beta_1(t)) \rightarrow (\ell(t),\beta_2(t)) \rightarrow \cdots . \quad (2)$$

5 The arc-length parameter for ν

Let $(\gamma,\nu): I \to \mathbb{R}^2 \times S^1$ be a Legendre immersion with the curvature of the Legendre immersion (ℓ,β) and without inflection points. If $\beta(t) \neq 0$ for all $t \in I$, we can choose the arc-length parameter s so that $|\gamma'(s)| = 1$. On the other hand, if $\ell(t) \neq 0$ for all $t \in I$, we can choose the special parameter s so that $|\nu'(s)| = 1$. It follows that $\nu(s)$ and also $\boldsymbol{\mu}(s)$ are the unit speed. By the same method for the are-length parameter of regular plane curves, one can prove the following:

Proposition 5.1 *Let $(\gamma,\nu): I \to \mathbb{R}^2 \times S^1$ be a Legendre immersion without inflection points, and let $t_0 \in I$. Then ν is parametrically equivalent to the unit speed curve*

$$\overline{\nu}: \overline{I} \to S^1; s \mapsto \overline{\nu}(s) = \nu \circ t(s),$$

under a change of parameter $t: \overline{I} \to I$ with $t(0) = t_0$ and with $t'(s) > 0$.

We call the above parameter s in Proposition 5.1 *the arc-length parameter for ν*. If t is the arc-length parameter for ν, then we have $|\ell(t)| = 1$ for all $t \in I$. Note that we may assume $\ell(t) = 1$ for all $t \in I$, if necessary, a change of parameter $t \mapsto -t$.

In this section, we suppose that $\ell(t) = 1$ for all $t \in I$. Then the second components of the curvatures of the evolutes and the involutes (2) are given by

$$\cdots \leftarrow \int_{t_0}^{t}\left(\int_{t_0}^{t}\beta(t)dt\right)dt \leftarrow \int_{t_0}^{t}\beta(t)dt \leftarrow \beta(t) \rightarrow \frac{d}{dt}\beta(t) \rightarrow \frac{d^2}{dt^2}\beta(t) \rightarrow \cdots . \quad (3)$$

As a result, we observe that the evolutes and the involutes of fronts are corresponding to the differential and the integral in classical calculus.

Next, we recall the notion of the contact between Legendre immersions (cf. [8]). Let $(\gamma,\nu): I \to \mathbb{R}^2 \times S^1; t \mapsto (\gamma(t),\nu(t))$ and $(\widetilde{\gamma},\widetilde{\nu}): \widetilde{I} \to \mathbb{R}^2 \times S^1; u \mapsto (\widetilde{\gamma}(u),\widetilde{\nu}(u))$ be Legendre immersions respectively and let k be a natural number. We say that (γ,ν) *and* $(\widetilde{\gamma},\widetilde{\nu})$ *have at least k-th order contact at $t = t_0, u = u_0$* if

$$(\gamma,\nu)(t_0) = (\widetilde{\gamma},\widetilde{\nu})(u_0),\ \frac{d}{dt}(\gamma,\nu)(t_0) = \frac{d}{du}(\widetilde{\gamma},\widetilde{\nu})(u_0), \cdots, \frac{d^{k-1}}{dt^{k-1}}(\gamma,\nu)(t_0) = \frac{d^{k-1}}{du^{k-1}}(\widetilde{\gamma},\widetilde{\nu})(u_0).$$

In general, we may assume that (γ,ν) and $(\widetilde{\gamma},\widetilde{\nu})$ have at least first order contact at any point $t=t_0, u=u_0$, up to congruence as Legendre immersions. We denote the curvatures of the Legendre immersions $(\ell(t),\beta(t))$ of $(\gamma(t),\nu(t))$ and $(\widetilde{\ell}(u),\widetilde{\beta}(u))$ of $(\widetilde{\gamma}(u),\widetilde{\nu}(u))$, respectively.

Theorem 5.2 ([8, Theorem 3.1]) *If (γ,ν) and $(\widetilde{\gamma},\widetilde{\nu})$ have at least $(k+1)$-th order contact at $t=t_0, u=u_0$ then*

$$(\ell,\beta)(t_0)=(\widetilde{\ell},\widetilde{\beta})(u_0),\ \frac{d}{dt}(\ell,\beta)(t_0)=\frac{d}{du}(\widetilde{\ell},\widetilde{\beta})(u_0),\ \cdots,\ \frac{d^{k-1}}{dt^{k-1}}(\ell,\beta)(t_0)=\frac{d^{k-1}}{du^{k-1}}(\widetilde{\ell},\widetilde{\beta})(u_0). \quad (4)$$

Conversely, if the condition (4) holds, then (γ,ν) and $(\widetilde{\gamma},\widetilde{\nu})$ have at least $(k+1)$-th order contact at $t=t_0, u=u_0$, up to congruence as Legendre immersions.

As a corollary of Theorem 5.2, we have the relationship among the contact of Legendre immersions, evolutes and involutes.

Corollary 5.3 *Under the above notations, we have the following.*

(1) *If (γ,ν) and $(\widetilde{\gamma},\widetilde{\nu})$ have at least $(k+1)$-th order contact at $t=t_0, u=u_0$, up to congruence as Legendre immersions, then $(\mathcal{E}v(\gamma),J(\nu))$ and $(\mathcal{E}v(\widetilde{\gamma}),J(\widetilde{\nu}))$ have at least k-th order contact at $t=t_0, u=u_0$, up to congruence as Legendre immersions.*

(2) *$(\mathcal{I}nv(\gamma,t_0),J^{-1}(\nu))$ and $(\mathcal{I}nv(\widetilde{\gamma},u_0),J^{-1}(\widetilde{\nu}))$ have at least $(k+1)$-th order contact at $t=t_0, u=u_0$, up to congruence as Legendre immersions if and only if (γ,ν) and $(\widetilde{\gamma},\widetilde{\nu})$ have at least k-th order contact at $t=t_0, u=u_0$, up to congruence as Legendre immersions.*

Finally, we consider what is the same sharp of fronts and the n-th evolutes (respectively, the n-th involutes) of the front under the same parametrisation. Namely, is there a Legendre immersion $(\gamma,\nu): I\to\mathbb{R}^2\times S^1$ such that (γ,ν) and $(\mathcal{E}v^n(\gamma),J^n(\nu)): I\to\mathbb{R}^2\times S^1$ (respectively, $(\mathcal{I}nv^n(\gamma,t_0),J^{-n}(\nu)): I\to\mathbb{R}^2\times S^1$) are congruent?

Theorem 5.4 *Under the above notations, we have the following.*

(1) *Legendre immersions (γ,ν) and $(\mathcal{E}v^n(\gamma),J^n(\nu))$ are congruent as Legendre immersions if and only if the curvature of the Legendre immersion $(\gamma(t),\nu(t))$ is given by*

$$\ell(t)=1,\ \beta(t)=\sum_{k=0}^{n-1}c_k e^{\lambda_n^k t},$$

where c_k is a constant, λ_n is a primitive n-th root of unity, $\lambda_n^k=\cos(2\pi k/n)+i\sin(2\pi k/n)$ for $k=0,\dots,n-1$ and i is the imaginary unit.

(2) *Legendre immersions (γ,ν) and $(\mathcal{I}nv^n(\gamma),J^{-n}(\nu))$ are congruent as Legendre immersions if and only if (γ,ν) is given by*

$$\gamma(t)=(a,b),\ \nu(t)=(\cos t,\sin t),$$

up to congruence as Legendre immersions, where $a,b\in\mathbb{R}$.

Proof. (1) By Lemma 2.7, $\ell(t)=1$ and the sequence of curvatures of the evolutes (3), we have

$$\beta(t)=\frac{d^n}{dt^n}\beta(t).$$

The linear ordinary differential equation can be solved, and the general solution is given by

$$\beta(t) = \sum_{k=0}^{n-1} c_k e^{\lambda_n^k t},$$

where c_k is a constant, λ_n is a primitive n-th root of unity, $\lambda_n^k = \cos(2\pi k/n) + i\sin(2\pi k/n)$ for $k = 0, \dots, n-1$ and i is the imaginary unit. By Theorem 2.6, the converse is holded.

(2) By Lemma 2.7, $\ell(t) = 1$ and the sequence of curvatures of the involute (3), we have

$$\beta(t) = \int_{t_0}^{t} \cdots \left(\int_{t_0}^{t} \beta(t)dt \right) \cdots dt,$$

n-times integrations for $\beta(t)$. This is equivalent to the conditions

$$\beta(t) = \frac{d^n}{dt^n}\beta(t), \quad \frac{d^i}{dt^i}\beta(t_0) = 0 \quad (0 \le i \le n-1).$$

It follows from (1) that $c_k = 0$ for $k = 0, \dots, n-1$, namely, $\beta(t) = 0$ for all $t \in I$. By Theorem 2.5, we obtain

$$\gamma(t) = (a, b), \ \nu(t) = (\cos t, \sin t),$$

up to congruence as Legendre immersions, where $a, b \in \mathbb{R}$. By a direct calculation, we have the converse. $\square$

We give examples for the cases $n = 1, 2$ and 3 in Theorem 5.4 (1).

Example 5.5 (1) The case of $n = 1$ in Theorem 5.4 (1). Since $\ell(t) = 1$ and $\beta(t) = \dot{\beta}(t)$, we have $\beta(t) = ce^t$, where $c \in \mathbb{R}$. It follows that

$$\gamma(t) = \left(\frac{c}{2}e^t(\cos t - \sin t), \ \frac{c}{2}e^t(\cos t + \sin t) \right), \ \nu(t) = (\cos t, \ \sin t),$$

up to congruent. We draw the front $\gamma(t)$ for $c = 1$ in Figure 4 left.

(2) The case of $n = 2$ in Theorem 5.4 (1). Since $\ell(t) = 1$ and $\beta(t) = \ddot{\beta}(t)$, we have $\beta(t) = c_1 e^t + c_2 e^{-t}$, where $c_1, c_2 \in \mathbb{R}$. It follows that

$$\begin{aligned} \gamma(t) &= \left(\frac{c_1}{2}e^t(\cos t - \sin t) + \frac{c_2}{2}e^{-t}(\cos t + \sin t), \ \frac{c_1}{2}e^t(\cos t + \sin t) + \frac{c_2}{2}e^{-t}(\sin t - \cos t) \right), \\ \nu(t) &= (\cos t, \ \sin t), \end{aligned}$$

up to congruent. We draw the front $\gamma(t)$ for $c_1 = 1$ and $c_2 = -1$ in Figure 4 centre. In this case, 0 is a singular point of γ.

(3) The case of $n = 3$ in Theorem 5.4 (1). Since $\ell(t) = 1$ and $\beta(t) = \dddot{\beta}(t)$, we have

$$\beta(t) = c_1 e^t + c_2 e^{-t/2} \cos \frac{\sqrt{3}}{2}t + c_3 e^{-t/2} \sin \frac{\sqrt{3}}{2}t$$

as a smooth general solution, where $c_1, c_2, c_3 \in \mathbb{R}$. By a direct calculation, we have

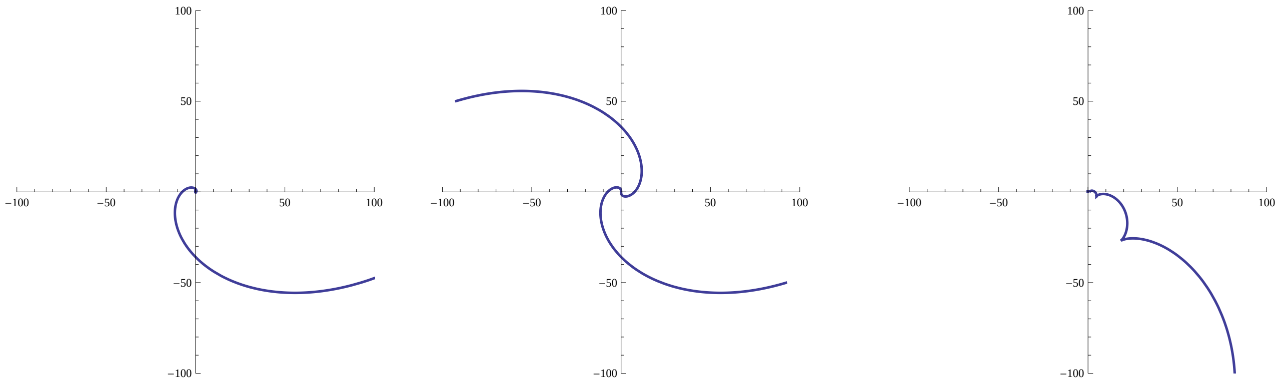
$$\gamma(t) = (\gamma_1(t), \gamma_2(t)), \ \nu(t) = (\cos t, \sin t),$$

where

$$\begin{aligned}
\gamma_1(t) &= \frac{c_1}{2}e^t(\cos t - \sin t) \\
&\quad -c_2e^{-t/2}\Bigg(\frac{1}{2(2+\sqrt{3})}\left(-\frac{1}{2}\sin(\frac{\sqrt{3}}{2}+1)t - (\frac{\sqrt{3}}{2}+1)\cos(\frac{\sqrt{3}}{2}+1)t\right) \\
&\qquad -\frac{1}{2(2-\sqrt{3})}\left(-\frac{1}{2}\sin(\frac{\sqrt{3}}{2}-1)t - (\frac{\sqrt{3}}{2}-1)\cos(\frac{\sqrt{3}}{2}-1)t\right)\Bigg) \\
&\quad -c_3e^{-t/2}\Bigg(-\frac{1}{2(2+\sqrt{3})}\left(-\frac{1}{2}\cos(\frac{\sqrt{3}}{2}+1)t + (\frac{\sqrt{3}}{2}+1)\sin(\frac{\sqrt{3}}{2}+1)t\right) \\
&\qquad +\frac{1}{2(2-\sqrt{3})}\left(-\frac{1}{2}\cos(\frac{\sqrt{3}}{2}-1)t + (\frac{\sqrt{3}}{2}-1)\sin(\frac{\sqrt{3}}{2}-1)t\right)\Bigg),
\end{aligned}$$

$$\begin{aligned}
\gamma_2(t) &= \frac{c_1}{2}e^t(\cos t + \sin t) \\
&\quad +c_2e^{-t/2}\Bigg(\frac{1}{2(2+\sqrt{3})}\left(-\frac{1}{2}\cos(\frac{\sqrt{3}}{2}+1)t + (\frac{\sqrt{3}}{2}+1)\sin(\frac{\sqrt{3}}{2}+1)t\right) \\
&\qquad +\frac{1}{2(2-\sqrt{3})}\left(-\frac{1}{2}\cos(\frac{\sqrt{3}}{2}-1)t + (\frac{\sqrt{3}}{2}-1)\sin(\frac{\sqrt{3}}{2}-1)t\right)\Bigg) \\
&\quad +c_3e^{-t/2}\Bigg(\frac{1}{2(2+\sqrt{3})}\left(-\frac{1}{2}\sin(\frac{\sqrt{3}}{2}+1)t - (\frac{\sqrt{3}}{2}+1)\cos(\frac{\sqrt{3}}{2}+1)t\right) \\
&\qquad +\frac{1}{2(2-\sqrt{3})}\left(-\frac{1}{2}\sin(\frac{\sqrt{3}}{2}-1)t - (\frac{\sqrt{3}}{2}-1)\cos(\frac{\sqrt{3}}{2}-1)t\right)\Bigg),
\end{aligned}$$

up to congruent. We draw the front $\gamma(t)$ for $c_1 = 0, c_2 = 0$ and $c_3 = 1$ in Figure 4 right.



(1) $n = 1$ and $c = 1$ (2) $n = 2$, $c_1 = 1$ and $c_2 = -1$ (3) $n = 3$, $c_1 = c_2 = 0$ and $c_3 = 1$.
Figure 4.

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