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INVOLUTIONS ON NUMERICAL CAMPEDELLI SURFACES

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Abstract. Numerical Campedelli surfaces are minimal surfaces of general type with vanishing geometric genus and canonical divisor with self-intersection 2. Although they have been studied by several authors, their complete classification is not known.

In this paper we classify numerical Campedelli surfaces with an involution, i.e., an automorphism of order 2. First we show that an involution on a numerical Campedelli surface S has either four or six isolated fixed points, and the bicanonical map of S is composed with the involution if and only if the involution has six isolated fixed points. Then we study in detail each of the possible cases, describing also several examples.

1. Introduction. Numerical Campedelli surfaces are minimal surfaces of general type with $p_g = 0$ (and so $q = 0$) and $K^2 = 2$. The first such example was presented by Campedelli [Cam] in 1932. Since then several authors (cf. [Mi], [Pe], [Re1], [Re2], [Ko], [Su], [Na2], [Ku], ...) have studied these surfaces, but our knowledge about them is far from being complete.

Since a classification of numerical Campedelli surfaces does not seem feasible at the moment, a possible approach is to restrict one's attention to the Campedelli surfaces which have some additional geometrical feature. This is what we do in the present paper, where we study the Campedelli surfaces which have an involution, i.e., an automorphism of order 2. This choice is motivated by work of Keum and Lee [KL] and of Calabri, Ciliberto and Mendes Lopes [CCM2], who have studied the same problem for numerical Godeaux surfaces, that is minimal surfaces of general type with $p_g = 0$ and $K^2 = 1$.

In order to put our work in perspective, we briefly recall here the main results of the paper [CCM2], which contains a complete classification of numerical Godeaux surfaces with an involution.

If S is a numerical Godeaux surface and σ is an involution of S, then σ has five isolated fixed points and:

- the bicanonical map of the surface factors through the natural projection onto the quotient surface S/σ;
- the quotient surface is either rational or birational to an Enriques surface;
- the possible quotient surfaces are classified and examples of each possibility in the list do exist;

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- if S/σ is rational, then the surface S can be obtained as a specialization of one of the surfaces in the list proposed by Du Val (see [Ci], cf. also [Bo]), by letting the branch locus acquire some singularities.

In the case of numerical Campedelli surfaces the situation is more involved, since the bicanonical map may not factor through the quotient map $S \to S/\sigma$. Indeed, we show that an involution on a numerical Campedelli surface S has either four or six isolated fixed points, and the bicanonical map of S factors through the quotient map $S \to S/\sigma$ if and only if the involution has six isolated fixed points. In the latter case the situation is very similar to the case of Godeaux surfaces. We have the following:

- the ramification divisor R on S is not 0, and its components can be described (see Section 3);
- the quotient surface S/σ is either birational to an Enriques surface or a rational surface;
- if S/σ is rational, then there are four possible cases which all have a precise description (see also Section 3). Each of the four cases actually occurs (cf. Section 5).

The analysis in Section 3 shows also that, if the bicanonical map of S is composed with the involution, then the 2-torsion of the surface S is nontrivial in three of the five possible cases.

If the bicanonical map is not composed with the involution, i.e., if the involution has four isolated fixed points, we show that the ramification divisor R is either 0 or constituted by one, two or three -2-curves. Note that if $R \neq 0$ then K_S is not ample.

In this case there are more possibilities for the quotient surface S/σ, as explained below:

- S/σ is of general type (a numerical Godeaux surface) if and only if the ramification divisor R is equal to 0;
- if R is irreducible, then S/σ is properly elliptic;
- if R has two or three components, then S/σ may be rational or birational to an Enriques surface or properly elliptic.

The case where S/σ is a numerical Godeaux surface appears in the examples constructed by Barlow in [Ba1] and [Ba2]. In Section 5, by specializing one of the examples of Barlow, we present examples for which the quotient surface is either an elliptic surface or birational to an Enriques surface. We do not know any instance in which the quotient surface is a rational surface for this case.

In Section 5 we also study a family of numerical Campedelli surfaces with torsion $\boldsymbol{Z}_3^2$, whose construction is attributed by J. Keum to A. Beauville and X. Gang. We show that every surface in this family has two involutions, one with four isolated fixed points and one with six isolated fixed points, whose quotients are respectively birational to a numerical Godeaux surface and a rational surface.

In Section 5 we study the involutions of numerical Campedelli surfaces with torsion $\boldsymbol{Z}_2^3$, the so-called "classical Campedelli surfaces". Using the description of these surfaces as a $\boldsymbol{Z}_2^3$-cover of $\boldsymbol{P}^2$ branched on 7 lines (cf. [Ku]), we show that these involutions are all composed with the bicanonical map.

The paper is organized as follows. In Section 2, using the results in [CCM2], we describe the general properties of numerical Campedelli surfaces with an involution, showing in particular that such an involution always has four or six isolated fixed points.

In Section 3 we study the case where the involution has six isolated fixed points and we describe in some detail each possibility. In Section 4 we study the case where the involution has four fixed points. Finally in Section 5 we describe the examples mentioned before, two of which were not (to our knowledge) previously known.

NOTATION AND CONVENTIONS. We work over the complex numbers. All varieties are projective.

Most of the notation is standard in algebraic geometry, hence we only recall here a few conventions that we use and that are maybe not universally accepted. We denote linear equivalence of divisors on a smooth variety by $\equiv$ and numerical equivalence by $\sim$. A divisor D on a smooth variety X is said to be *even* if its class is divisible by 2 in the group $\mathrm{Pic}(X)$.

An *involution* of a variety is a biregular automorphism of order 2. A map $f\colon X \to Y$ of projective varieties is said to be *composed with* an involution σ if $f \circ \sigma = f$. A $-n$*-curve* on a smooth surface is a curve C such that $C \simeq \boldsymbol{P}^1$ and $C^2 = -n$.

A singular point of type $[m, m]$ on a curve is a point of multiplicity m with an infinitely near point again of multiplicity m.

2. Involutions on a numerical Campedelli surface. Throughout the paper we make the following:

ASSUMPTION 2.1. S is a smooth minimal complex projective surface of general type with $K_S^2 = 2$, $p_g(S) = 0$ (thus also $q(S) = 0$). Such a surface S is called a *numerical Campedelli surface*.

Moreover, we assume that we are given an involution σ of S, namely an automorphism $\sigma\colon S \to S$ of order 2.

In this section we establish the notation and recall some known facts on involutions, giving all the statements in the special case of a numerical Campedelli surface. Our main reference is the paper [CCM2], which contains a detailed analysis of involutions on surfaces of general type with $p_g = 0$.

The fixed locus of the involution σ is the union of an effective divisor R and of k isolated points $p_1, \dots, p_k$. The effective divisor R, if not 0, is a smooth, possibly reducible, curve. Let $\pi\colon S \to \Sigma := S/\sigma$ be the quotient map, and set $B := \pi(R)$ and $q_i := \pi(p_i)$, $i = 1, \dots, k$. The surface Σ is normal and $q_1, \dots, q_k$ are ordinary double points, which are the only singularities of Σ. In particular, the singularities of Σ are canonical and the adjunction formula gives $K_S \equiv \pi^* K_\Sigma + R$.

Let $\varepsilon\colon V \to S$ be the blow-up of S at $p_1, \dots, p_k$ and let E_i be the exceptional curve over p_i, $i = 1, \dots, k$. Then σ induces an involution $\tilde\sigma$ of V whose fixed locus is the union of $R_0 := \varepsilon^*(R)$ and of $E_1, \dots, E_k$. Denote by $\tilde\pi\colon V \to W := V/\tilde\sigma$ the projection onto the quotient and set $B_0 := \tilde\pi(R_0)$, $N_i := \tilde\pi(E_i)$, $i = 1, \dots, k$. The surface W is smooth and the

N_i are disjoint -2-curves. Denote by $\eta\colon W \to \Sigma$ the map induced by ε. The map η is the minimal resolution of the singularities of Σ and there is a commutative diagram:

$$\begin{array}{ccc} V & \xrightarrow{\varepsilon} & S \\ {\scriptstyle\tilde\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ W & \xrightarrow{\eta} & \Sigma \end{array} \tag{2.1}$$

The map $\tilde\pi$ is a flat double cover branched on $\tilde B := B_0 + \sum_{i=1}^k N_i$. Hence there exists a divisor L on W such that $2L \equiv \tilde B$, namely $\tilde B$ is an even divisor.

REMARK 2.2. We have $p_g(V) = q(V) = 0$, since V is birational to S. Since V dominates W, we also have $p_g(W) = q(W) = 0$.

The number k of isolated fixed points is a very important invariant of the involution σ. As explained below, it determines whether the bicanonical map $\varphi\colon S \to \boldsymbol{P}^2$ is composed with σ.

PROPOSITION 2.3 ([CCM2], Proposition 3.3, (v) and Corollary 3.6). *One of the following two possibilities occurs*:

I) $k = 6$. *In this case φ is composed with σ.*

II) $k = 4$. *In this case φ is not composed with σ. More precisely, $\pi^*|2K_\Sigma + B|$ has dimension* 1, *namely it is a codimension* 1 *subsystem of* $|2K_S|$.

We set $D := 2K_W + B_0$. The divisor D will play an important role in our analysis of numerical Campedelli surfaces with an involution.

One has the following properties (cf. [CCM2, §3] for the proofs):

PROPOSITION 2.4. (i) $\varepsilon^*(2K_S) = \tilde\pi^* D$.

(ii) *D is nef and big, and $D^2 = 4$.*

(iii) *$D + N_1 + \cdots + N_k$ is an even divisor.*

(iv) *If $k = 6$, then $-4 \le K_W^2 \le 0$, $K_W D = 0$.*

(v) *If $k = 4$, then $-2 \le K_W^2 \le 1$, $K_W D = 2$.*

REMARK 2.5. We will often apply Proposition 2.4, (i) as follows. Given a curve C of W, we can pull it back to a curve C' of V. If C' is not contained in the exceptional locus of ε, then we can push it down to a curve $\tilde C$ on S. Then $K_S \tilde C = DC$.

Assume that $K_W + D$ is not nef. Then one can show that there is an irreducible -1-curve E on W with $DE = 0$, $EN_i = 0$ for $i = 1, \dots, k$. By repeatedly blowing down such -1-curves, one obtains a sort of minimal model for the pair $(W, K_W + D)$. More precisely, we have the following

PROPOSITION 2.6 ([CCM2], Proposition 3.9). *There exists a birational morphism $f\colon W \to W'$, where W' is smooth, with the following properties*:

(i) *For $i = 1, \dots, k$ the curve $N_i' := f(N_i)$ is a -2-curve on W' and the curves $N_1', \dots, N_k'$ are disjoint.*

(ii) *There is a nef divisor D' on W' such that $f^*(D') = D$, ${D'}^2 = D^2$ and $K_{W'}D' = K_W D$.*

(iii) *$D'N'_i = 0$ for $i = 1, \dots, k$ and $D' + N'_1 + \cdots + N'_k$ is an even divisor.*

(iv) *$K_{W'} + D'$ is nef.*

REMARK 2.7. The proof of [CCM2], Proposition 3.9 actually shows more. Namely:

(i) since $K_W + D$ is effective and the curves contracted by f satisfy $E(K_W + D) < 0$, the components of the exceptional locus of f are contained in the fixed part of $|K_W + D|$;

(ii) if E is an irreducible component of the exceptional locus of f, then E gives a -2-curve on S. In particular, if K_S is ample then we have $W = W'$.

3. Involutions composed with the bicanonical map. This section studies case I) of Proposition 2.3, namely here we assume that $k = 6$ and the bicanonical map $\varphi \colon S \to \boldsymbol{P}^2$ is composed with σ.

In what follows we use freely the notation introduced in Section 2. By Proposition 2.4, in this case $-4 \le K_W^2 \le 0$ and $K_W D = 0$. This allows us to establish some properties of the ramification divisor R on S.

Using Proposition 2.4, and arguing as in the proof of Proposition 4.5 of [CCM2], one obtains the following:

PROPOSITION 3.1. *Let S be a numerical Campedelli surface with an involution σ, such that the bicanonical map φ is composed with σ. Then the divisorial part R of the fixed locus of σ satisfies*:

(i) $K_S R = 2$;

(ii) *$R^2 = 2K_W^2 + 2$ is even, and $-6 \le R^2 \le 2$.*

Furthermore, $R = \Gamma + Z_1 + \cdots + Z_h$, where Γ is a smooth curve with $K_S\Gamma = 2$ and $Z_1, \dots, Z_h$ are disjoint -2-curves, which are disjoint also from Γ. Here

(iii) *either Γ is irreducible, $0 \le p_a(\Gamma) \le 3$ and $\Gamma^2 = 2p_a(\Gamma) - 4$; or Γ has exactly two components $\Gamma_1 + \Gamma_2$, where each Γ_i, $i = 1, 2$, is either a rational curve with self-intersection -3 or an elliptic curve with self-intersection -1;*

(iv) *the number h of -2-curves $Z_1, \dots, Z_h$ satisfies*

$$h = p_a(\Gamma) - K_W^2 - 3 \ge 0\,;$$

(v) *if $\Gamma^2 = 2$, then $\Gamma \sim K_S$ and S has nontrivial torsion.*

In order to study in more detail these surfaces we consider the system $|D| := |2K_W + B_0|$ and its adjoint systems.

LEMMA 3.2. *Let $|K_W + D| = |M| + F$, where F is the fixed part. Then one has the following*

(i) $h^0(W, \mathcal{O}_W(M)) = 3$.

(ii) $MD = 4$.

(iii) *If $F \ne 0$, then every component E of F is such that $DE = 0$ and $E^2 < 0$.*

PROOF. Assertion (i) follows by the adjunction sequence for the general D, since W is regular by Remark 2.2 and $p_a(D) = 3$ by Proposition 2.4.

Let us prove part (ii). Since, by Proposition 2.4, D is nef and $D(M + F) = 4$, one has $MD \leq 4$.

Suppose by contradiction that $MD < 4$. We claim that in this case $|M|$ is not composed with a pencil, and so, in particular, $M^2 > 0$. Indeed, if $|M| = |2C|$ and $MD < 4$, then one would have $CD \leq 1$. But then $|C|$ would give a pencil $|\tilde{C}|$ on S such that $\tilde{C}K_S = 1$ (cf. Remark 2.5), which is impossible by the index theorem.

By a similar argument we verify that $MD \geq 2$.

Suppose that $MD = 2$. Then by the index theorem we obtain $M^2 = 1$ and $2M \sim D$. This is impossible, because we have $K_W D = 0$ by Proposition 2.4 and this implies $K_W M = 0$, contradicting the adjunction formula.

So we are left with the case $MD = 3$. We have $M^2 + MF = M(K_W + D)$. So $MD = 3$ means that MF is also odd and thus, because M is nef, $MF \geq 1$. Therefore $MK_W = M^2 + MF - 3 \geq M^2 - 2$.

On the other hand, the index theorem gives $M^2 D^2 \leq (MD)^2 = 9$ hence $M^2 \leq 2$. Since $|M|$ is not composed with a pencil, we have $M^2 = 1$ or $M^2 = 2$.

In the first case $\phi_M : W \to \boldsymbol{P}^2$ is a birational morphism, but this is impossible because $2g(M) - 2 = M^2 + K_W M \geq 0$.

In the second case the system $|M|$ gives a system $|\tilde{M}|$ on S with $\tilde{M}^2 \geq 4$ and $K_S\tilde{M} = 3$ (cf. Remark 2.5). By the adjunction formula we get $\tilde{M}^2 \geq 5$, contradicting the index theorem applied to K_S and $\tilde{M}$.

So we have shown that $MD = 4$. Now (iii) follows immediately from $DF = 0$ and the index theorem. □

Consider now the morphism $f: W \to W'$ and the divisor D' of Proposition 2.6.

PROPOSITION 3.3. (i) *One has* $-4 \leq K^2_{W'} \leq 0$.

(ii) *If* $K^2_{W'} = 0$, *then* W' *is an Enriques surface.*

(iii) *If* $K^2_{W'} < 0$, *then* W' *is rational.*

PROOF. We recall that $-4 \leq K^2_W \leq 0$ by Proposition 2.4, and so $K^2_{W'} \geq -4$. Since $D'(K_{W'} + D') = 4$, the index theorem implies that $(K_{W'} + D')^2 \leq 4$, or equivalently $K^2_{W'} \leq 0$.

The surface W' is either rational or birational to an Enriques surface by [CCM2, Corollary 3.7]. Since $K_{W'}D = 0$, if $K^2_{W'} = 0$, then $K_{W'} \sim 0$ and W' is an Enriques surface. If $K^2_{W'} < 0$, then $K_{W'}(K_{W'} + D) < 0$. Since $K_{W'} + D'$ is nef by Proposition 2.6, this implies that the Kodaira dimension of W' is negative, and therefore W' is rational. □

LEMMA 3.4. *If* $K^2_{W'} < 0$, *then* $|K_{W'} + D'|$ *has no fixed part.*

PROOF. Write, as usual, $|K_{W'} + D'| = F' + |M'|$, where F' is the fixed part. Since the morphism $f: W \to W'$ contracts only curves that are fixed for $|K_W + D|$ (cf. Remark 2.7), by Lemma 3.2 we see that $M'D' = 4$, $F'D' = 0$.

Notice that $F'K_{W'} = F'(K_{W'} + D') = F'M' + F'^2$, so $F'M'$ is even. Since both M' and $K_{W'} + D'$ are nef (cf. Proposition 2.6, (iii)), we have the following inequalities:

$$M'F' \leq M'F' + M'^2 = M'(K_{W'} + D') \leq (K_{W'} + D')^2 = K_{W'}^2 + 4 < 4\,.$$

It follows that $M'F' = 0$ or $M'F' = 2$. If $M'F' = 2$, then $M'^2 \leq 1$.

We start by seeing that $M'F' = 2$ does not occur. If $M'F' = 2$ and $M'^2 = 1$, then $K_{W'}M' = (K_{W'} + D')M' - 4 = 3 - 4 = -1$. Since $M'^2 = 1$, $|M'|$ is not composed with a pencil, the general curve of $|M'|$ is smooth and $\phi_{M'}\colon W' \to \boldsymbol{P}^2$ is a birational morphism. This is impossible because $p_a(M') = 1$.

If $M'F' = 2$ and $M'^2 = 0$, then $|M'|$ is composed with a pencil. Now $h^0(W', M') = 3$ and $q(W') = 0$ imply that $M' \equiv 2C$, where $|C|$ is a free pencil. Since $K_{W'}M' = (K_{W'} + D')M' - 4 = 2 - 4 = -2$, one has that $K_{W'}C = -1$, which contradicts the adjunction formula. So $M'F' = 2$ does not occur.

On the other hand, if $M'F' = 0$, then $F' = 0$. Otherwise, since $D'F' = 0$, then $F'^2 < 0$, implying that $F'(K_{W'} + D') = F'^2 + M'F' < 0$. This contradicts the fact that $K_{W'} + D'$ is nef. So $|K_{W'} + D'|$ has no fixed part. □

Next we examine separately each of the possibilities for $K_{W'}^2$, which ranges between -4 and 0 by Proposition 3.3.

3.1. The case $K_{W'}^2 = 0$. In this case the surface W' is an Enriques surface by Proposition 3.3.

PROPOSITION 3.5. *The system $|D'|$ is base point free and irreducible.*

PROOF. Write $D' := |M| + F$, where F is the fixed part. By Proposition 2.4 (i) and Proposition 2.6 (ii), the system $|M|$ pulls back on S to the moving part of $|2K_S|$. Since the bicanonical image of S is a surface by [Xi2], the general M is irreducible. In particular, M is nef and big and the Riemann-Roch theorem gives $3 = h^0(M) = M^2/2 + 1$, namely $M^2 = 4$. So we have: $4 = M^2 \leq M^2 + MF \leq D^2 = 4$, which implies $MF = F^2 = 0$. Hence $F = 0$ by the index theorem.

Now assume that $|D'|$ has base points. By Proposition 4.5.1 of [CD], there exists an effective divisor E on W' such that $E^2 = 0$, $ED' = 1$. By Remark 2.5, this gives a divisor $\tilde{E}$ on S with $K_S\tilde{E} = 1$ and $\tilde{E}^2 \geq 0$. The adjunction formula then gives $\tilde{E}^2 \geq 1$, but this contradicts the index theorem. □

COROLLARY 3.6. *The bicanonical system $|2K_S|$ is base point free.*

PROOF. The statement follows immediately by Proposition 3.5, since $|2K_S|$ is the pull back of $|D|$ to S by Proposition 2.4. □

PROPOSITION 3.7. *The torsion group* $\mathrm{Tors}(S)$ *of S has order* 4 *or* 8.

PROOF. Since the group $\mathrm{Tors}(S) = \mathrm{Tors}(V)$ has order at most 9 (cf. [BPHV, Chap. VII.10]), it is enough to show the existence of an étale cover of V of degree 4.

Let $p\colon K \to W$ be the étale double cover of W induced by the K3 cover of W'. Then we have a cartesian diagram:

$$\begin{array}{ccc} \tilde{K} & \xrightarrow{\tilde{p}} & V \\ {\scriptstyle\rho}\big\downarrow & & \big\downarrow{\scriptstyle\tilde{\pi}} \\ K & \xrightarrow{p} & W \end{array}$$

The map $\tilde{p}$ is an étale double cover, while ρ is a double cover branched on the inverse image $\varDelta$ of $B_0 + N_1 + \cdots + N_6$. The divisor $\varDelta$ is the disjoint union of a divisor $\varDelta_0$ with $\varDelta_0^2 = 8$ and of twelve -2-curves $\varGamma_1, \ldots, \varGamma_{12}$. Consider the natural map $\psi\colon \boldsymbol{Z}\varDelta_0 \oplus \boldsymbol{Z}\varGamma_1 \oplus \cdots \oplus \boldsymbol{Z}\varGamma_{12} \to H^2(K, \boldsymbol{Z}_2)$. The image of ψ is a totally isotropic subspace, and hence it has dimension at most 11, since $h^2(K, \boldsymbol{Z}_2) = 22$ and the intersection form on $H^2(K, \boldsymbol{Z}_2)$ is nondegenerate by Poincaré duality. Hence the kernel of ψ has dimension at least 2. By Lemme 2 of [Be], the surface $\tilde{K}$ has a connected étale double cover, and hence V has a connected étale cover of degree 4. □

REMARK 3.8. Examples of this situation can be found in [Na1]. Those examples have torsion group $\boldsymbol{Z}_2^3$ or $\boldsymbol{Z}_2 \times \boldsymbol{Z}_4$.

3.2. The case $K_{W'}^2 = -1$. By Proposition 3.3, W' is a rational surface. Denote $M' := K_{W'} + D'$ and recall that $|M'|$ has no fixed part by Lemma 3.4. One has $M'^2 = 3$, $K_{W'}M' = -1$. Since $|M'|$ is 2-dimensional, the general curve of $|M'|$ is irreducible. The system $|K_{W'} + M'|$ has dimension 1 by the adjunction sequence for the general M'.

LEMMA 3.9. *The linear system $|K_{W'} + M'|$ is a base point free pencil of nonsingular rational curves.*

PROOF. We claim that $K_{W'} + M'$ is nef.

Suppose otherwise. Then there exists an irreducible curve θ with $\theta(K_{W'} + M') < 0$. It follows that θ is a fixed component of $|K_{W'} + M'|$ and $\theta^2 < 0$. Since $\theta M' \geq 0$, because M' is nef, it follows that $\theta K_{W'} < 0$. Thus necessarily θ is a -1-curve and $M'\theta = 0$.

The divisor $G := M' - \theta$ is effective, since $|M'|$ has dimension 2, and we have $G^2 = 2$, $GD' = 3$. Then G gives a divisor $\tilde{G}$ on S such that $\tilde{G}^2 \geq 4$, $\tilde{G}K_S = 3$ (cf. Remark 2.5), and therefore $\tilde{G}^2 \geq 5$ by the adjunction formula. This contradicts the index theorem applied to K_S and $\tilde{G}$, showing that $K_{W'} + M'$ is nef.

Consider $|K_{W'} + M'| = |C| + F$, where F is the fixed part. The general M' is smooth and irreducible and $|K_{W'} + M'|$ restricts to the complete canonical system on M'. Hence the general M' does not meet F. So $M'F = 0$, and therefore $F^2 < 0$ if $F \neq 0$. Because $K_{W'} + M'$ is nef, $C(K_{W'} + M') = C^2 + CF \geq 0$ and $F(K_{W'} + M') = F^2 + CF \geq 0$. Since $(K_{W'} + M')^2 = 0$, we have equality in both cases.

But then, because C is nef, we must have $CF = 0$, implying also $F^2 = 0$ and so $F = 0$. So $|K_{W'} + M'| = |C|$ is a pencil of rational curves. □

PROPOSITION 3.10. (i) *There exists a fibration* $f: S \to \boldsymbol{P}^1$ *with* 3 *double fibres, such that the general fibre of* f *is hyperelliptic of genus* 3 *and* σ *induces on it the hyperelliptic involution.*

(ii) *The group* $\mathrm{Tors}(S)$ *contains a subgroup isomorphic to* $\boldsymbol{Z}_2^2$.

PROOF. Let $C := K_{W'} + M'$. By Lemma 3.9, $|C|$ is a free pencil of rational curves. Notice that $CN_i' = (2K_{W'} + D')N_i' = 0$ for every i by Proposition 2.6, so that the curves N_i' are contained in curves of $|C|$. Since $C \equiv 2K_{W'} + D'$ and $D' + N_1' + \cdots + N_6'$ is divisible by 2 in $\mathrm{Pic}(W')$ by Proposition 2.6, (iii), the divisor $C + N_1' + \cdots + N_6'$ is also divisible by 2. Let $Y \to W'$ be the double cover branched on $C + N_1' + \cdots + N_6'$, where $C \in |C|$ is general. The surface Y is smooth and the usual formulae for double covers give $\chi(Y) = 0$. Pulling back $|C|$ to Y, one obtains a fibration $h: Y \to \Gamma$, where Γ is a smooth curve and the general fibre of h is isomorphic to $\boldsymbol{P}^1$. Hence Y is a ruled surface with $q(Y) = 1$ and h is the Albanese pencil.

Arguing as in [DMP, Theorem 3.2], one shows that there exist effective divisors A_1, A_2, A_3 on W' such that, up to a permutation of the indices, the curves $2A_1 + N_1' + N_2'$, $2A_2 + N_3' + N_4'$, $2A_3 + N_5' + N_6'$ belong to $|C|$.

We have $CD' = 4$, and hence by Remark 2.5 the system $|C|$ gives a pencil $|\tilde{C}|$ on S with $K_S\tilde{C} = 4$. Since $CN_i' = 0$ for every i, we have $\tilde{C}^2 = 0$ and $|\tilde{C}|$ defines a fibration $f: S \to \boldsymbol{P}^1$ of hyperelliptic curves of genus 3. The curves of $|C|$ containing the N_i' give rise to double fibres of f.

Statement (ii) follows trivially from the existence of three double fibres of f. □

REMARK 3.11. In this case it is possible, using the same type of reasoning as in Corollary 7.6 of [CCM2], to show that S is a degeneration of surfaces with nonbirational bicanonical map originally described by Du Val as double planes (cf. [Ci]). Indeed, S is birationally equivalent to a double cover of $\boldsymbol{P}^2$ branched on a curve which is the union of three lines r_1, r_2, r_3 meeting in a point q_0 and of a curve of degree 13 with the following singularities:

- a 5-uple point at q_0;
- a point $q_i \in r_i$, $i = 1, 2, 3$, of type [4, 4], where the tangent line is r_i;
- three additional 4-uple points q_4, q_5, q_6 such that there is no conic through $q_1, \ldots, q_6$.

3.3. The case $K_{W'}^2 = -2$. As in the previous case we consider $M' := K_{W'} + D'$. Recall that $M'^2 = 2$ and $K_{W'}M' = -2$. Moreover, M' and D' are nef (cf. Proposition 2.6).

LEMMA 3.12. (i) *One has* $h^0(W', \mathcal{O}_{W'}(K_{W'} + M')) = 1$ *and, if* G *is the unique curve in* $|K_{W'} + M'|$, *then* $GM' = 0$.

Moreover, up to a permutation of the indices $\{1, \ldots, 6\}$, *one has the following*:

(ii) *There are two possible decompositions of* G:

a) $G = (2E_1 + N_5') + (2E_2 + N_6')$, *where* E_1, E_2 *are* -1*-curves such that* $E_1N_5' = E_2N_6' = E_1D' = E_2D' = 1$ *and the divisors* $(2E_1 + N_5')$ *and* $(2E_2 + N_6')$ *are disjoint, or*

b) *$G = 4E_1 + 3N'_5 + 2Z_1 + N'_6$, where E_1 is a -1-curve and Z_1 is a -2-curve such that $E_1N'_5 = Z_1N'_5 = Z_1N'_6 = E_1D' = 1$ and $E_1Z_1 = E_1N'_6 = 0$.*

(iii) *The divisor $N'_1 + \cdots + N'_4$ is even, and it is disjoint from G.*

PROOF. We will mimick the proof of Lemma 7.1 in [CCM2]. The first assertion follows from the long exact sequence obtained from

$$0 \to \mathcal{O}_{W'}(K_{W'}) \to \mathcal{O}_{W'}(K_{W'} + M') \to \mathcal{O}_{M'} \to 0$$

because W' is a rational surface by Proposition 3.3, (iii).

By definition of G, one has that $G^2 = GK_{W'} = -4$ and $GM' = 0$. Therefore, since M' is nef, each component θ of G is such that $\theta M' = 0$ and the intersection form on the components of G is negative definite. Since $G^2 = -4$, there exists an irreducible curve E_1 in G such that $E_1^2 < 0$ and $E_1G = E_1(K_{W'} + M') < 0$. Since $M'E_1 = 0$, one has that $E_1K_{W'} < 0$, thus E_1 is a -1-curve and $E_1G = -1$, $E_1D' = 1$. Recall that $D' \equiv M' - K_{W'}$ is nef, so the irreducible components of G are either -1-curves E such that $EG = -1$ and $ED' = 1$, or -2-curves Z such that $ZG = ZD' = 0$.

Since $D' + N'_1 + \cdots + N'_6$ is divisible by 2 and $E_1D' = 1$, E_1 must meet one of the -2-curves N'_i, say N'_5. Hence $N'_5(G - E_1) = -N'_5E_1 < 0$, so $N'_5 \leq G$ and moreover $E_1N'_5 = 1$, otherwise we would get $(E_1 + N'_5)^2 > 0$, a contradiction because the intersection form on the components of G is negative definite.

Similarly, $E_1(G - E_1 - N'_5) = -1$ implies that $2E_1 + N'_5 \leq G$.

Recall that $GK_{W'} = -4$, so either G contains another -1-curve E_2 or $4E_1 \leq G$. Assume the former case. Then, arguing as before, one sees that E_2 meets N'_i, for some i, and $2E_2 + N'_i \leq G$. If $i = 5$, then $(N'_5 + E_1 + E_2)^2 \geq 0$, a contradiction. So we may assume $i = 6$. Finally the negative definiteness implies that $E_1E_2 = 0$ and that case a) of statement (ii) occurs, because $(G - 2E_1 - 2E_2 - N'_5 - N'_6)^2 = 0$.

Assume now the latter case, i.e., $4E_1 \leq G$. Note that N'_5 is the only -2-curve contained in G that can intersect E_1. Indeed, if $Z \subset G$ is a -2-curve such that $E_1Z \geq 1$ and $Z \neq N'_5$, then $(2E_1 + N'_5 + Z)^2 \geq 0$, contradicting again the negative definiteness.

Since $E_1G = -1$ and $E_1(4E_1 + N'_5) = -3$, one has that $4E_1 + 3N'_5 \leq G$ and the components of $G' = G - 4E_1 - 3N'_5$ are -2-curves. Since $N'_5G = 0$ and $N'_5G' = 2$, G contains at least a -2-curve Z_1 with $Z_1N'_5 > 0$. Now $N'_5Z_1 = 1$, otherwise $(N'_5 + Z_1)^2 \geq 0$ gives a contradiction. Since $Z_1G = 0$, we have $2Z_1 \leq G'$. Recall that $Z_1D' = 0$ and $D' + N'_1 + \cdots + N'_6$ is even, and hence Z_1 meets another -2-curve N'_i, say N'_6. Then $N'_6(G - Z_1) = -N'_6Z_1 < 0$, so $N'_6 \leq G'$. Finally the negative definiteness implies that $N'_6Z_1 = 1$. Then we are in case (ii), b), because $(G - 4E_1 - 3N'_5 - 2Z_1 - N'_6)^2 = 0$.

It remains to prove that $N'_1 + \cdots + N'_4$ is divisible by 2 in $\mathrm{Pic}(W')$. Since $D' + N'_1 + \cdots + N'_6$ is even, one has that $2K_{W'} + D' + N'_1 + \cdots + N'_6 \equiv G + N'_1 + \cdots + N'_6$ is also even. Hence $G + N'_1 + \cdots + N'_6 \equiv 2(E_1 + E_2 + N'_5 + N'_6) + N'_1 + \cdots + N'_4$ is even in case a), and $G + N'_1 + \cdots + N'_6 \equiv 2(2E_1 + 2N'_5 + Z_1 + N'_6) + N'_1 + \cdots + N'_4$ is even in case b). In both cases, one sees that $N'_1 + \cdots + N'_4$ is even.

To finish the proof, it is enough to show that $N'_1, \ldots, N'_4$ are disjoint from E_1 and E_2 in case (ii), a) and from E_1 and Z_1 in case (ii), b). Arguing as before, this follows easily from the fact that the components of G and the curves $N'_1, \ldots, N'_4$ are orthogonal to the nef divisor M'. □

By Lemma 3.12, there exists a birational morphism $g\colon W' \to X$ such that X is a smooth rational surface, G is the exceptional divisor of g and $M' = -g^* K_X$. In particular, $-K_X$ is nef and big and $K_X^2 = 2$. In case (ii), a) of Lemma 3.12 the image of G consists of two points q_5 and q_6 and in case (ii), b) it is a single point q.

PROPOSITION 3.13. (i) *There exists a fibration $f\colon S \to \boldsymbol{P}^1$ with* 2 *double fibres, such that the general fibre of f is hyperelliptic of genus* 3 *and σ induces on it the hyperelliptic involution.*

(ii) *The group* $\mathrm{Tors}(S)$ *contains a subgroup isomorphic to* $\boldsymbol{Z}_2$.

PROOF. For $i = 1, \ldots, 4$, write Δ_i for the image of N'_i in X. By Lemma 3.12, $\Delta_1 + \cdots + \Delta_4$ is again an even set of disjoint -2-curves. By [CCM1, 1.1], there exist a free pencil $|C'|$ of rational curves of X and effective divisors A_1, A_2 such that, say, $2A_1 + \Delta_1 + \Delta_2$ and $2A_2 + \Delta_3 + \Delta_4$ belong to $|C'|$. The pull back $|C|$ of C' on W' satisfies $CD' = 4$, $CN'_i = 0$ for $i = 1, \ldots, 6$, and hence it gives a fibration $f\colon S \to \boldsymbol{P}^1$ as in statement (i). The curves $2A_1 + \Delta_1 + \Delta_2$ and $2A_2 + \Delta_3 + \Delta_4$ correspond to two double fibres of f.

Statement (ii) follows trivially from the existence of two double fibres of f. □

REMARK 3.14. As in the previous case, it is possible, again using the same type of reasoning as in Corollary 7.6 of [CCM2], to show that S is a degeneration of surfaces with nonbirational bicanonical map originally described by Du Val as double planes (cf. [Ci]). Indeed, S is birationally equivalent to a double cover of $\boldsymbol{P}^2$ branched on a curve of degree 14 which splits in two distinct lines r_1 and r_2 and a curve of degree 12 with the following singularities:

- the point $q_0 = r_1 \cap r_2$ of multiplicity 4;
- a point $q_i \in r_i$, $i = 1, 2$, of type [4, 4], where the tangent line is r_i;
- two further points q_3, q_4 of multiplicity 4 and two points q_5, q_6 of type [3, 3], such that there is no conic through $q_1, \ldots, q_6$.

The point q_6 is infinitely near to q_5, in case (ii), b) of Lemma 3.12.

3.4. The case $K_{W'}^2 = -3$. Denote $M' := K_{W'} + D'$. We have $M'^2 = 1$, $K_{W'}M' = -3$ and $|M'|$ is 2-dimensional. Since $|M'|$ has no fixed part by Lemma 3.4, the map $\phi_{M'}\colon W' \to \boldsymbol{P}^2$ is a birational morphism. It is an easy exercise to see that the branch curve is mapped to a plane curve of degree 10, which, as it is well known, has 6 singular points of type [3, 3] (possibly infinitely near).

The original construction proposed by Campedelli ([Cam]) is one of these surfaces. For a discussion of possible branch loci and relations with the 2-torsion of S see [St] and [W].

3.5. The case $K_{W'}^2 = -4$. We start by noticing that in this case $W' = W$, because $-4 \le K_W^2$ by Proposition 2.4, (iii).

Denote $M := K_W + D$. Recall that $|M|$ has no fixed part, by Lemma 3.4. Then $M^2 = 0$ and $h^0(W, M) = 3$ imply that $|M| = |2C|$, where $|C|$ is a pencil without base points. Since $K_W^2 = -4$ and $K_W D = 0$, we have $K_W C = -2$, and hence $|C|$ is a pencil of rational curves. Since $CN_i = 0, i = 1, \ldots, 6$ and $DC = 2$, C gives rise to a genus 2 fibration $\tilde{C}$ on S such that σ restricts to the hyperelliptic involution on the general C.

Notice that in this case the curve B_0 on W must be reducible, because by Proposition 3.1 $p_a(R) = -1$ and, of course, $p_a(B_0) = p_a(R)$. In fact, recalling that $D = 2K_W + B_0$, we obtain $B_0^2 = -12$, $K_W B_0 = 8$, and hence $p_a(B_0) = -1$.

REMARK 3.15. Conversely, assume that the numerical Campedelli surface S has a free pencil $|C|$ of curves of genus 2 and let σ be the involution of S that induces the hyperelliptic involution on the general C. Then the results of [Xi1, §1, 2] (cf. Remark 2.4, ibidem) show that we have $K_{W'}^2 = -4$ in this case.

REMARK 3.16. In this case by [Xi1, §2] the relative canonical map of S expresses S as a double cover of $\boldsymbol{F}_0 = \boldsymbol{P}^1 \times \boldsymbol{P}^1$ branched along a curve of degree $(6, 8)$, which in the general case has 6 distinct singular points of type $[3, 3]$.

4. Involutions not composed with the bicanonical map. In this section we consider case II) of Proposition 2.3, namely here we assume that $k = 4$ and the bicanonical map $\varphi\colon S \to \boldsymbol{P}^2$ is not composed with σ. We recall that by Proposition 2.4 in this case we have $D^2 = 4$, $K_W D = 2$, $-2 \leq K_W^2 \leq 1$.

LEMMA 4.1. *The curve B_0 decomposes as $B_0 = \Gamma_1 + \cdots + \Gamma_m$, where the Γ_i are disjoint irreducible -4-curves and $m = 1 - K_W^2$.*

PROOF. First of all, notice that $B_0 D = D^2 - 2DK_W = 0$. Let Γ be an irreducible component of B_0 and write $p^*\Gamma = 2\tilde{\Gamma}$. We have $D\Gamma = 0$, since D is nef, and thus $\varepsilon^* K_S \tilde{\Gamma} = 0$, since $p^* D = \varepsilon^*(2K_S)$ by Proposition 2.4. Since $\tilde{\Gamma}$ is disjoint from the exceptional locus of ε by construction, it follows that $\tilde{\Gamma}$ is a -2-curve. Hence $\Gamma^2 = -4$ and Γ is a smooth rational curve. Now let $m \geq 0$ denote the number of components of B_0. By the adjunction formula we have $K_W B_0 = 2m$. On the other hand, we can compute:

$$2m = K_W B_0 = K_W(D - 2K_W) = 2 - 2K_W^2 .$$

Finally, the components of B_0 are disjoint, since B_0 is smooth. □

COROLLARY 4.2. *If $K_W^2 \leq 0$, then K_S is not ample.*

PROOF. By Lemma 4.1, the branch divisor B of the map $\pi : S \to \Sigma$ contains at least a smooth rational curve Γ with $\Gamma^2 = -4$. Then the inverse image of Γ in S is a -2-curve and K_S is not ample. □

PROPOSITION 4.3. *We have the following possibilities*:

(i) *$K_W^2 = 1$, W is minimal of general type and $B_0 = 0$.*

(ii) *$K_W^2 = 0$, W is minimal and properly elliptic.*

(iii) *$K_W^2 = -1, -2$ and W is not of general type.*

PROOF. Recall that $-2 \leq K_W^2 \leq 1$ by Proposition 2.4. If $K_W^2 = 1$, then by Lemma 4.1 we have $B_0 = 0$ and $K_S = \pi^* K_\Sigma$, and hence $K_W = \eta^* K_\Sigma$ is nef and big and W is minimal of general type.

Next we show that if $K_W^2 \leq 0$, then W is not of general type. So assume by contradiction that W is of general type. Let $t\colon W \to W_1$ be the morphism to the minimal model and write $K_W = t^* K_{W_1} + E$, where $E > 0$. Since $DK_W = 2$ and D is nef, we have $Dt^* K_{W_1} \leq 2$. On the other hand, since $K_{W_1}^2 > 0$, the index theorem applied to D and $t^* K_{W_1}$ gives $Dt^* K_{W_1} \geq 2$. So we get $Dt^* K_{W_1} = 2$ and $D \sim 2t^* K_{W_1}$. This implies $B_0 + 2E \sim 0$, a contradiction, since $B_0 + 2E > 0$.

Assume now that $K_W^2 = 0$. By Lemma 4.1, B_0 is a smooth rational curve with $B_0^2 = -4$. By the exact sequence

$$0 \to H^0(2K_W) \to H^0(2K_W + B_0) \to H^0(\mathcal{O}_{B_0}), \tag{4.2}$$

we obtain $1 \leq h^0(2K_W) \leq 2$, and hence W has nonnegative Kodaira dimension. We have seen that W is not of general type, and hence it is minimal and it is either properly elliptic or Enriques. Since $K_W D = 2 \neq 0$, the latter case does not occur. This finishes the proof. □

REMARK 4.4. By Proposition 4.3, the desingularization W of the quotient surface S/σ may be a numerical Godeaux surface, an elliptic surface, birational to an Enriques surface or rational.

Unlike the previous case, in which one knows examples for all the possibilities for W, in this case we do not know any example for which W is rational. Barlow in [Ba1], [Ba2] presents examples of numerical Godeaux surfaces with four nodes double covered by numerical Campedelli surfaces and the new examples we present such that W is not a surface of general type are obtained by specializing one of these constructions (cf. §5).

It is possible to make a more detailed analysis of the cases with $K_W^2 \leq 0$, in the style of the previous section, but since the arguments are very lengthy and all the examples we know are obtained by specialization, we do not think worthwhile including it here.

5. Examples. In this section we study some families of numerical Campedelli surfaces with an involution, providing examples for the cases 3.2 to 3.5 in §3 and for the cases (i)–(iii) in Proposition 4.3.

EXAMPLE 1. *Numerical Campedelli surfaces with torsion* $\boldsymbol{Z}_2^3$.

These surfaces have two different descriptions: as the quotient by a free $\boldsymbol{Z}_2^3$-action of the intersection of four quadrics in $\boldsymbol{P}^6$ (cf. [Mi], [Re1]) and as $\boldsymbol{Z}_2^3$-covers of $\boldsymbol{P}^2$ branched on 7 lines (cf. [Ku]). We use the second description, which is more suitable for our purposes. Two special instances of surfaces in this family are the Burniat surface with $K^2 = 2$ and the classical Campedelli surface (cf. [Ku, §4]).

Set $G := \boldsymbol{Z}_2^3$ and let χ_1, χ_2, χ_3 be generators of G^*, the group of characters of G. By [Pa, Proposition 2.1 and Corollary 3.1], to give a normal G-cover $p\colon X \to \boldsymbol{P}^2$ it is enough to give an effective divisor D_g for every $0 \neq g \in G$ and line bundles L_1, L_2, L_3 on $\boldsymbol{P}^2$ such that the divisor $\Delta := \sum_{g\neq 0} D_g$ is reduced and the following relations are satisfied:

$$2L_i \equiv \sum_{g\neq 0} \varepsilon_i(g) D_g\,, \quad i = 1, 2, 3\,,$$

where we define $\varepsilon_i(g) = 0$ if $\chi_i(g) = 1$ and $\varepsilon_i(g) = 1$ if $\chi_i(g) = -1$.

Here we take the D_g to be distinct lines in $\boldsymbol{P}^2$ and we set $L_i := \mathcal{O}_{\boldsymbol{P}^2}(2)$, $i = 1, 2, 3$. Moreover we make the following assumptions on the configuration of the lines D_g:

1) at most three of the D_g pass through the same point;
2) if $D_{g_1}, D_{g_2}, D_{g_3}$ pass through the same point, then $g_1 + g_2 + g_3 \neq 0$.

We now examine the singularities of X. By [Pa, Proposition 3.1], X is singular above a point $P \in \boldsymbol{P}^2$ if and only if P lies on three branch lines D_{g_1}, D_{g_2} and D_{g_3}. To resolve the singularity, let $\psi: \hat{\boldsymbol{P}} \to \boldsymbol{P}^2$ be the blow up of $\boldsymbol{P}^2$ at P, let E be the exceptional curve of ψ and consider the G-cover $\hat{p}: \hat{X} \to \hat{\boldsymbol{P}}$ obtained from p by base change and normalization. Write $g_0 := g_1 + g_2 + g_3$. By [Pa, §3], the components of the branch divisor of $\hat{p}$ are the following: $\hat{D}_g := \psi^* D_g$ if $g \neq g_0, \ldots, g_3$, $\hat{D}_{g_0} := \psi^* D_{g_0} + E$, $\hat{D}_{g_i} := \psi^* D_{g_i} - E$ for $i = 1, 2, 3$. The surface $\hat{X}$ is smooth above E and $\psi^{-1}(E)$ is a -2-curve. Hence X has a rational double point of type A_1 over P. We have $2K_X = \psi^*(\mathcal{O}_{\boldsymbol{P}^2}(1))$, and hence K_X is ample and X is the canonical model of a surface S of general type with $K_S^2 = 2$. By the projection formulae for abelian covers we have $|2K_X| = \psi^*|\mathcal{O}_{\boldsymbol{P}^2}(1)|$, and hence $h^0(X, 2K_X) = 3$, $\chi(S) = \chi(X) = 1$ and ψ is the bicanonical map of X.

Kulikov [Ku, Thm. 4.2] shows that the automorphism group of the general surface in this family coincides with the Galois group $G = \boldsymbol{Z}_2^3$ of the bicanonical map. The result that follows is a partial refinement of his, and gives evidence for the difficulty of finding an involution of a numerical Campedelli surface such that its bicanonical map is not composed with it.

PROPOSITION 5.1. *Let S be a numerical Campedelli surface with torsion $\boldsymbol{Z}_2^3$ and let σ be an involution of S. Then σ is in the Galois group $G = \boldsymbol{Z}_2^3$ of the bicanonical map of S.*

PROOF. Assume by contradiction that σ is an involution of S such that the bicanonical map $\varphi: S \to \boldsymbol{P}^2$ is not composed with σ. Since G is defined intrinsically, we have $\sigma G \sigma = G$ and σ induces an involution of $\boldsymbol{P}^2$ that we denote by $\bar{\sigma}$. Since the set of lines D_g contains at least 4 lines in general position, $\bar{\sigma}$ induces a nontrivial permutation of the D_g. Denote by h the automorphism of G defined by $h(g) = \sigma g \sigma$. Then we have $\sigma(D_g) = D_{h(g)}$, and it follows that h is a non trivial automorphism of G. Since h has order 2, we can find generators e_1, e_2, e_3 of G such that $h(e_i) = e_i$ for $i = 1, 2$ and $h(e_3) = e_3 + e_1$. Hence the lines $D_{e_1}, D_{e_2}, D_{e_1+e_2}$ are fixed for $\bar{\sigma}$, while D_{e_3} and $D_{e_3+e_1}$ are exchanged by $\bar{\sigma}$ and the same happens to $D_{e_3+e_2}$ and $D_{e_3+e_2+e_1}$. Then, taking also into account the combinatorial conditions on the configuration of the lines D_g, up to exchanging e_2 and $e_1 + e_2$, we can find homogeneous coordinates on $\boldsymbol{P}^2$ such that $\bar{\sigma}(x_0, x_1, x_2) = (x_0, x_1, -x_2)$ and such that:

$$D_{e_1} = \{x_1 = 0\}\,, \quad D_{e_2} = \{x_0 = 0\}\,, \quad D_{e_1+e_2} = \{x_2 = 0\}\,,$$
$$D_{e_3} = \{ax_0 + bx_1 + cx_2 = 0\}\,, \quad D_{e_3+e_1} = \{ax_0 + bx_1 - cx_2 = 0\}\,,$$
$$D_{e_3+e_2} = \{a'x_0 + b'x_1 + c'x_2 = 0\}\,, \quad D_{e_3+e_1+e_2} = \{a'x_0 + b'x_1 - c'x_2 = 0\}\,.$$

Since h maps the subgroup H of G generated by e_1 and e_3 to itself, σ induces an involution of the surface $Z := S/H$ that lifts $\bar{\sigma}$. On the other hand, the function field $\boldsymbol{C}(Z)$ is the quadratic extension of $\boldsymbol{C}(\boldsymbol{P}^2)$ obtained by adding the square root of $x_0(a'x_0 + b'x_1 + c'x_2)(a'x_0 + b'x_1 - c'x_2)/x_2^3$, and it is easy to check that the action of $\bar{\sigma}$ on $\boldsymbol{C}(\boldsymbol{P}^2)$ cannot be extended to an automorphism of order 2 of $\boldsymbol{C}(Z)$. Hence we have obtained a contradiction. □

We now study the involutions of G. There are different cases, according to the relative positions of the lines in Δ.

Case 1. The lines of Δ are in general position.

In this case K_S is ample and therefore $W = W'$ for any involution of S by Remark 2.7.

The divisorial part R of the fixed locus on S of any $0 \neq \sigma \in G$ is a paracanonical curve. Hence, the adjunction formula $K_S \equiv \pi^* K_\Sigma + R$ gives that $2K_\Sigma \equiv 0$ and Σ is an Enriques surface with 6 nodes. So this is an instance of case 3.1 of §3. Other examples of this case, with torsion $\boldsymbol{Z}_2 \times \boldsymbol{Z}_4$, appear in [Na1].

Case 2. The divisor Δ has one triple point P, lying on the lines D_{g_1}, D_{g_2}, D_{g_3}. Consider the involution $g_0 := g_1 + g_2 + g_3$. In this case the cover $\hat{p}\colon \hat{X} \to \hat{\boldsymbol{P}}$ is smooth and we have $S = \hat{X}$. The divisorial part of the fixed locus of g_0 on S is the disjoint union of the -2-curve that resolves the singularity of X and of a paracanonical curve. Hence one gets $K_W^2 = -1$. Since the only -2-curve of S is in the fixed locus of g_0, we have $W' = W$ and the surface W is rational by Proposition 3.3, namely this is an example of case 3.2. Indeed, it is easy to check that the lines through the point $P \in \boldsymbol{P}^2$ pull back to a pencil of rational curves on W, which in turn gives a free pencil of hyperelliptic curves of genus 3 with three double fibres on S.

Case 3. The divisor Δ has a triple point P_1, lying on the lines D_{g_1}, D_{g_2}, D_{g_3}, and another triple point P_2, lying on the lines D_{h_1}, D_{h_2} and D_{h_3}, with $g_1+g_2+g_3 = h_1+h_2+h_3 =: g_0$.

Arguing as in Case 2, one shows that the fixed locus of g_0 on S is the disjoint union of a paracanonical curve and of the two -2-curves that resolve the double points of X lying above P_1 and P_2. We have $W = W'$, $K_W^2 = -2$ and W is rational. Hence this is an example of Case 3.3.

Case 4. The divisor Δ has three triple points: P_1, lying on the lines D_{g_1}, D_{g_2}, D_{g_3}, P_2, lying on the lines D_{h_1}, D_{h_2} and D_{h_3}, and P_3, lying on the lines D_{f_1}, D_{f_2} and D_{f_3}. Moreover, we assume that $g_1 + g_2 + g_3 = h_1 + h_2 + h_3 = f_1 + f_2 + f_3 =: g_0$. We remark that the existence of such a configuration of lines is not difficult to verify.

Arguing as in Case 2, one shows that the fixed locus of g_0 on S is the disjoint union of a paracanonical curve and of the three -2-curves that resolve the double points of X lying above P_1, P_2 and P_3. We have $W = W'$, $K_W^2 = -3$ and W is rational. Hence this is an example of Case 3.4.

REMARK 5.2. One can check that Δ cannot have four triple points $P_1, \dots, P_4$ such that P_i lies on $D_{g_{1i}}$, $D_{g_{2i}}$, $D_{g_{3i}}$ with $g_{1i} + g_{2i} + g_{3i} = g_{1j} + g_{2j} + g_{3j} \neq 0$ for every

$i, j = 1, \dots, 4$. Hence, by Proposition 5.1 the cases 1–4 described above are essentially the only possibilities for an involution of a numerical Campedelli surface with torsion $\boldsymbol{Z}_2^3$.

EXAMPLE 2. *A family of numerical Campedelli surfaces with torsion $\boldsymbol{Z}_3^2$ and two involutions.*

This example has been kindly communicated to us by JongHae Keum, who attributes it to X. Gang and A. Beauville (cf. also Example 3.8 of [Cat]). For the classification of numerical Campedelli surfaces with torsion $\boldsymbol{Z}_3^2$ see [MP].

Consider $X := \boldsymbol{P}^2 \times \boldsymbol{P}^2$ with homogeneous coordinates $(x_0, x_1, x_2; y_0, y_1, y_2)$ and let two generators g_1 and g_2 of the group $G := \boldsymbol{Z}_3^2$ act on X as follows:

$$(x_0, x_1, x_2; y_0, y_1, y_2) \stackrel{g_1}{\mapsto} (x_1, x_2, x_0; y_1, y_2, y_0)\,;$$
$$(x_0, x_1, x_2; y_0, y_1, y_2) \stackrel{g_2}{\mapsto} (x_0, \omega x_1, \omega^2 x_2; y_0, \omega^2 y_1, \omega y_2)\,,$$

where ω is a primitive 3-rd root of 1. Consider the family of surfaces Y of X defined by the equations:

$$x_0 y_0 + x_1 y_1 + x_2 y_2 = 0\,,$$
$$(x_0^3 + x_1^3 + x_2^3)(y_0^3 + y_1^3 + y_2^3) + \lambda x_0 x_1 x_2 y_0 y_1 y_2 = 0\,.$$

For a general value of the parameter $\lambda \in \boldsymbol{C}$ the surface Y is smooth and simply connected with $K_Y^2 = 18$, $p_g(Y) = 8$, and the group G acts freely on it. Hence the quotient surface $S := Y/G$ is a numerical Campedelli surface with fundamental group equal to G.

The surface Y is mapped to itself also by the involution $\tilde{\sigma}_1$ of X defined by

$$(x_0, x_1, x_2; y_0, y_1, y_2) \stackrel{\tilde{\sigma}_1}{\mapsto} (y_0, y_1, y_2; x_0, x_1, x_2)\,.$$

The involution $\tilde{\sigma}_1$ satisfies the following relations

$$\tilde{\sigma}_1 g_1 = g_1 \tilde{\sigma}_1\,, \quad \tilde{\sigma}_1 g_2 = g_2^2 \tilde{\sigma}_1\,, \tag{5.1}$$

and hence G and $\tilde{\sigma}_1$ generate a group G_0 of order 18, the involution $\tilde{\sigma}_1$ induces an involution σ_1 of S and we have $Y/G_0 = S/\sigma_1$.

The fixed locus of $\tilde{\sigma}_1$ on Y consists of 12 points and the same is true for $\tilde{\sigma}_1 g_2$ and $\tilde{\sigma}_1 g_2^2$, since these involutions are conjugated to $\tilde{\sigma}_1$. Consider now an element of G_0 of the form $\tilde{\sigma}_1 g$, where $g \in G \setminus \langle g_2 \rangle$. The relations (5.1) imply that $(\tilde{\sigma}_1 g)^2$ is a nonzero element of G, and hence in particular $\tilde{\sigma}_1 g$ has no fixed points on Y. It follows that σ_1 has 4 fixed points on S and the quotient surface $T := S/\sigma_1$ is a numerical Godeaux surface. By [Ba2, §0], the fundamental group of T is isomorphic to $\boldsymbol{Z}_3$. Hence we have an example in which the bicanonical map is not composed with the involution and the quotient surface is of general type, that is Case (i) of Proposition 4.3.

Consider now the involution $\tilde{\sigma}_2$ defined by

$$(x_0, x_1, x_2; y_0, y_1, y_2) \stackrel{\tilde{\sigma}_2}{\mapsto} (x_0, x_2, x_1; y_0, y_2, y_1)\,.$$

For every $g \in G$ one has the relation $\tilde{\sigma}_2 g = g^{-1}\tilde{\sigma}_2$. Hence the group G_0 generated by G and $\tilde{\sigma}_2$ has order 18. G_0 contains nine elements of order 2, that form a conjugacy class. The surface Y is mapped to itself by $\tilde{\sigma}_2$ and $\tilde{\sigma}_2$ induces an involution of S that we denote by σ_2. We have $Y/G_0 = S/\sigma_2$.

The fixed locus of $\tilde{\sigma}_2$ on the threefold $\{x_0y_0+x_1y_1+x_2y_2 = 0\} \subset X$ consists of 3 disjoint rational curves: $\Gamma_1 = \{(0, 1, -1; a, b, b) \mid (a, b) \in \boldsymbol{P}^1\}$, $\Gamma_2 = \{(a, b, b; 0, 1, -1) \mid (a, b) \in \boldsymbol{P}^1\}$, $\Gamma_3 = \{(a, b, b; -2b, a, a) \mid (a, b) \in \boldsymbol{P}^1\}$. It is not difficult to check that Γ_1 and Γ_2 are contained in Y, while Γ_3 meets the general Y at 6 distinct points. Since K_Y is the restriction of $\mathcal{O}_{\boldsymbol{P}^2\times\boldsymbol{P}^2}(1, 1)$ to Y, we have $K_Y\Gamma_i = 1$, for $i = 1, 2$ and Γ_1, Γ_2 are -3-curves on Y. Hence the fixed locus of σ_2 on S is the union of 6 isolated points and two -3-curves, and thus $K_W^2 = -4$. If Y is smooth, then K_Y and K_S are ample and we have $W = W'$ by Remark 2.7. So this surface is also an example of Case 3.5.

We are now going to show that the involution σ_2 of S is actually induced by a genus 2 pencil, as explained in 3.5. Consider the pencil of hypersurfaces of X spanned by $x_0x_1x_2$ and $x_0^3 + x_1^3 + x_2^3$ and denote by $|F|$ the restriction of this pencil to Y. The fixed part of $|F|$ is the union of the curves in the orbit of Γ_1 under the group G. Then we can write $|F| = Z + |C|$, where the Z is the disjoint union of nine -3-curves and $|C|$ has no fixed part. On the surface Y we have $F^2 = 27$, $K_Y F = 27$. Using this and $F\Gamma = 0$ for every component Γ of Z, one gets $C\Gamma = 3$, $K_Y C = 18$, $C^2 = 0$. Every element of $|C|$ is mapped to itself by G_0. Hence $|C|$ induces a genus 2 pencil $|C'|$ of S such that every element of $|C'|$ is mapped to itself by σ_2. Finally, using $\Gamma_2 F = 3$ and the fact that the G-orbits of Γ_1 and Γ_2 are disjoint, we get $C\Gamma_1 = C\Gamma_2 = 3$. So the general C' meets the fixed locus of σ_2 at 6 points, and hence σ_2 restricts to the hyperelliptic involution on C'.

EXAMPLE 3. *Numerical Campedelli surfaces with an involution with which the bicanonical map is not composed and such that the quotient is not of general type.*

Here we provide examples for Cases (ii) and (iii) of Proposition 4.3. These examples are obtained by specializing a construction due to Barlow ([Ba1]). We start by recalling briefly her construction.

Consider the space $\boldsymbol{P}^6$ with homogeneous coordinates $(x_1, \ldots, x_7)$ and the automorphisms of $\boldsymbol{P}^6$ defined as follows:

$$(x_1, \ldots, x_7) \stackrel{t}{\mapsto} (\zeta x_1, \zeta^2 x_2, \ldots, \zeta^7 x_7)\,,$$
$$(x_1, \ldots, x_7) \stackrel{a}{\mapsto} (x_3, x_6, x_1, x_4, x_7, x_2, x_5)\,,$$

where ζ is a primitive 8-th root of 1. The automorphism t has order 8, the automorphism a has order 2 and one has

$$ata = t^3\,.$$

Hence a and t generate a subgroup G of order 16 of $\mathrm{Aut}(\boldsymbol{P}^6)$. Consider the intersection Y of the following four quadrics of $\boldsymbol{P}^6$:

$$F_0 := b(x_1x_7 + x_3x_5) + ax_4^2 + fx_2x_6\,,$$

$$F_2 := cx_1^2 + dx_3x_7 + ex_4x_6 + hx_5^2\,,$$
$$F_4 := k(x_2^2 + x_6^2) + gx_1x_3 + mx_5x_7\,,$$
$$F_6 := cx_3^2 + dx_1x_5 + ex_4x_2 + hx_7^2\,.$$

Barlow proves that for a general choice of the coefficients $a, b, c, d, e, f, g, h, k, m$ the following are true:

- Y is a smooth surface mapped to itself by G;
- the subgroup $\mathbf{Z}_8$ of G generated by t acts freely on Y;
- the involution a has 8 isolated fixed points on Y.

It follows easily from the properties above that the quotient surface $S := Y/\mathbf{Z}_8$ is a numerical Campedelli surface with torsion $\mathbf{Z}_8$. In addition, the involution a of Y induces an involution σ of S with four isolated fixed points. The quotient surface $\Sigma := S/\sigma$ has four nodes and its minimal desingularization W is a minimal surface of general type with $K_W^2 = 1$ and $p_g(W) = 0$, namely a numerical Godeaux surface. Barlow also shows that $\pi_1(W) = \mathbf{Z}_2$.

Let Γ be the group of automorphisms of $\mathbf{P}^6$ of the form $\mathrm{Diag}(1, \lambda, 1, \mu, \nu, \lambda, \nu)$ for $\lambda, \mu, \nu \in \mathbf{C}^*$. The elements of Γ commute with a and t and act on the family of surfaces Y, and hence the family of numerical Campedelli surfaces S that we obtain has at most 4 moduli.

We are going to specialize this construction by letting S acquire one or two ordinary double points which are fixed by σ and whose images in Σ are quotient singularities of type $(1/4)\,(1,1)$. Passing to the minimal desingularization S' of S we obtain an involution whose fixed locus consists of four isolated points and of the -2-curves that resolve the singularities of Y. In the case of one double point we get an example of Case (ii) of Proposition 4.3. In particular, the minimal desingularization W of Σ is a properly elliptic surface. In the case of two double points we have an example of Case (iii) of Proposition 4.3 and we will show that W is a nonminimal Enriques surface.

The fixed locus of a on $\mathbf{P}^6$ consists of the $\mathbf{P}^3$ defined by

$$x_1 - x_3 = x_2 - x_6 = x_5 - x_7 = 0$$

and of the $\mathbf{P}^2$ defined by

$$x_1 + x_3 = x_2 + x_6 = x_5 + x_7 = x_4 = 0\,.$$

In [Ba1] it is shown that the general Y intersects the $\mathbf{P}^3$ in 8 points and it does not intersect the $\mathbf{P}^2$. Let $P_1 \in \mathbf{P}^2$ be the point $(1, 1, -1, 0, 1, -1, -1)$ and let $P_2 := t^4(P_1) \in \mathbf{P}^2$ the point $(1, -1, -1, 0, 1, 1, -1)$. Let $P_3, \ldots, P_8$ denote the remaining points in the orbit of P_1 under the action of $\mathbf{Z}_8$. The surfaces Y that contain $P_1, \ldots, P_8$ are defined by four quadrics as follows:

$$F_0 := b(x_1x_7 + x_3x_5) + ax_4^2 - 2bx_2x_6\,,$$
$$F_2 := cx_1^2 + dx_3x_7 + ex_4x_6 - (c+d)x_5^2\,,$$
$$F_4 := k(x_2^2 + x_6^2) + gx_1x_3 + (2k - g)x_5x_7\,,$$

$$F_6 := cx_3^2 + dx_1x_5 + ex_4x_2 - (c+d)x_7^2 .$$

It is easy to verify that the tangent space to the general Y at P_1 has dimension 3 and a acts on it as multiplication by -1. Since the points $P_1, \dots, P_8$ form an orbit under the action of t and Y is mapped to itself by t, the singularities of Y at $P_1, \dots, P_8$ are isomorphic.

REMARK 5.3. The orbit of P_1 under the action of Γ is dense in the $\boldsymbol{P}^2$ fixed by a. It follows that if a surface Y intersects this $\boldsymbol{P}^2$ in a point P, then Y is singular at P and the subspace of the tangent space to Y at P on which a acts as multiplication by -1 has dimension at least 3. In addition, if P satisfies $x_1x_2x_5 \neq 0$ and Y is general among the surfaces through P, then the tangent space to Y at P has dimension 3 and a acts on it as multiplication by -1.

We claim that for a general choice of the parameters a, b, c, d, e, g, k the surface Y satisfies the following conditions:

1) the subgroup generated by t acts freely on Y;
2) Y meets the $\boldsymbol{P}^3$ fixed by a in 8 points and it meets the $\boldsymbol{P}^2$ fixed by a in P_1 and P_2;
3) Y has an ordinary double point in $P_1, \dots, P_8$ and it is smooth elsewhere.

Conditions 1)–3) are open, and hence it is enough to check them for one surface Y. Let Y_0 be the surface corresponding to the following choice of parameters:

$$a = e = -1\,, \quad b = c = d = g = k = 1\,.$$

Using a computer program (we have used Singular), one checks the following:

- Y_0 does not intersect the spaces $H_1 := \{x_1 = x_3 = x_5 = x_7 = 0\}$ and $H_2 := \{x_2 = x_4 = x_6 = 0\}$ fixed by t^4, and hence condition 1) is satisfied;
- Y_0 intersects the $\boldsymbol{P}^3$ fixed by Y at 8 points;
- the scheme of singular points of Y_0 has dimension 0 and degree 8.

Since we already know that Y is singular at $P_1, \dots, P_8$, the last condition above implies 3). The fact that Y_0 meets the $\boldsymbol{P}^2$ fixed by a only at P_1, P_2 is now a consequence of Remark 5.3. Hence Conditions 1)–3) are satisfied by Y_0 and therefore they are satisfied by the general Y that has nonempty intersection with the $\boldsymbol{P}^2$ fixed by a. For such a surface Y, the quotient surface $S := Y/\boldsymbol{Z}_8$ has an ordinary double point at the image point P of $P_1, \dots, P_8$ and it is smooth elsewhere. Hence S is the canonical model of a numerical Campedelli surface. Let S' be the minimal resolution of S, let Z be the exceptional curve and let σ' be the involution of S' induced by a. Since a acts on the tangent space to Y at P_1 as multiplication by -1, the fixed locus of σ' on S' consists of the curve Z and of 4 isolated fixed points. Hence we have $K_W^2 = 0$ by Lemma 4.1 and W is minimal and properly elliptic by Proposition 4.3. Applying the argument used in [Ba1], one can show that the fundamental group of W is $\boldsymbol{Z}_2$.

Since the elements of Γ with $\lambda = \nu = 1$ act on the family of surfaces Y passing through P_1, the family of Campedelli surfaces with one node that we have constructed has at most 3 moduli.

We are now going to degenerate the construction further, letting S acquire two double points, and thus obtain an example with $K_W^2 = -1$. Set $Q_1 := (1, 2, -1, 0, 4, -2, -4)$ and $Q_2 := t^4 Q_1 = (1, -2, -1, 0, 4, 2, -4)$ and denote by $Q_3, \dots, Q_8$ the remaining points in the orbit of Q_1 under the action of t. The surfaces Y through $P_1, \dots, P_8$ and $Q_1, \dots, Q_8$ are defined by the following four quadrics:

$$F_0 := b(x_1x_7 + x_3x_5) + ax_4^2 - 2bx_2x_6\,,$$
$$F_2 := cx_1^2 - \frac{5}{4}cx_3x_7 + ex_4x_6 + \frac{1}{4}cx_5^2\,,$$
$$F_4 := 5(x_2^2 + x_6^2) + 8x_1x_3 + 2x_5x_7\,,$$
$$F_6 := cx_3^2 - \frac{5}{4}cx_1x_5 + ex_4x_2 + \frac{1}{4}cx_7^2\,.$$

By Remark 5.3 every surface as above is singular at $P_1, \dots, P_8, Q_1, \dots, Q_8$. Let now Y_0 be the surface corresponding to the following choice of parameters:

$$a = -1\,, \quad b = 1\,, \quad c = 4\,, \quad e = -1\,.$$

Also in this case, we have used the computer program Singular to check that Y_0 has the following properties:

- the automorphism t acts freely on Y_0;
- Y_0 intersects the $\boldsymbol{P}^3$ fixed by a at 8 points;
- Y_0 intersects the $\boldsymbol{P}^2$ fixed by a at Q_1, Q_2, P_1, P_2;
- the scheme of singular points of Y_0 has dimension 0 and degree 16, and thus Y_0 has a node at $P_1, \dots, P_8, Q_1, \dots, Q_8$ and is smooth elsewhere.

Since these properties are open, they hold for the general Y passing through P_1 and Q_1. The quotient surface $S := Y/\boldsymbol{Z}_8$ has two nodes which are fixed by a and K_S is ample. Let S' be the minimal desingularization of S, let Z_1 and Z_2 be the exceptional curves on S' and let σ be the involution of S induced by a. The fixed locus of σ on S consists of 4 isolated points and of the curves Z_1 and Z_2 (cf. Remark 5.3). Hence we have $K_W^2 = -1$ and this is an example of Case (iii) of Proposition 4.3.

As in the case of one node, one can use the same argument as in [Ba1] to show that $\pi_1(W) = \boldsymbol{Z}_2$. Hence W is not rational and, by Proposition 4.3, it is birational either to an Enriques surface or to a properly elliptic surface. We are going to see that in fact W is birational to an Enriques surface.

The intersection of Y with the hypersurface $x_4^2 = 0$ is a bicanonical curve which descends to a bicanonical curve $2C \subset S$ passing through the nodes of S. Pulling back to S', we obtain a bicanonical curve $2C' = 2Z_1 + 2Z_2 + 2G'$, where G' is effective.

By the adjunction formula, there is an effective divisor G on W such that $G \sim_{\mathrm{num}} K_W$ and such that the pull back of G to S' is G'. Let $t\colon W \to \bar{W}$ be the morphism onto the minimal model and let E be the exceptional curve of t. We have $GE = K_WE = -1$, hence $G = E + G_0$, where $G_0 \geq 0$ and $G_0 \sim_{\mathrm{num}} t^*K_{\bar{W}}$. Assume that W is properly elliptic and denote by F a general fibre of the elliptic fibration of W. Then there is $\alpha \in \boldsymbol{Q}$, $\alpha > 0$, such that $G_0 \sim_{\mathrm{num}} \alpha F$. For $i = 1, 2$ let Γ_i be the image of Z_i in W. The curves Γ_1 and Γ_2 are -4-curves, and hence we have $4 = K_W(\Gamma_1 + \Gamma_2) = E(\Gamma_1 + \Gamma_2) + G_0(\Gamma_1 + \Gamma_2)$. By construction, the curve R_0 does not meet the nodal curves $N_1, \dots, N_4$ of W contained in the branch divisor

B_0 of the double cover $V \to W$. Hence $EB_0 = E(\Gamma_1 + \Gamma_2)$ and $G_0B_0 = E(\Gamma_1 + \Gamma_2)$ are both even. Moreover, we have $EB_0 > 0$, since otherwise E would pull back on S' to the disjoint union of two -1-curves, contradicting the minimality of S', and $G_0B_0 > 0$, since otherwise $|F|$ would pull back on S' to a pencil of elliptic curves, contradicting the fact that S' is of general type. Hence we have $EB_0 = G_0B_0 = 2$ and the pull back of E on S' is either a -2-curve or the union of two -2-curves meeting in a point. This is not possible, since Z_1 and Z_2 are the only -2-curves of S' by construction. Hence we have reached a contradiction and the only possibility is that W is birational to an Enriques surface.

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