Ionization layer at the edge of a fully ionized plasma

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A model is developed of the ionization layer which separates a thermal plasma close to full ionization from the space-charge sheath adjacent to the surface of an electrode or of an insulating wall. The multifluid description of the plasma is used. Asymptotic solutions are obtained for the cases in which the thickness of the ionization layer is much larger or much smaller than the mean free path for ion-atom collisions. The solution obtained for the latter case describes an interesting new regime which is in some aspects similar to the conventional diffusion regime, though essentially different from the diffusion regime in other aspects. Formulas are derived for the ion flux coming from the ionization layer to the edge of the space-charge sheath. Application of results to atmospheric-pressure argon and mercury plasmas is considered.
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I. INTRODUCTION

A nonequilibrium region separating a thermal plasma from the surface of an electrode or of an insulating wall includes a number of physically different subregions. A subregion adjacent to the surface is the space-charge sheath. Adjacent to the sheath is a subregion in which ionization equilibrium is established, i.e., in which a transition occurs from boundary conditions at the sheath edge to a charged particle density determined by the Saha equation. The latter subregion is usually referred to as the ionization layer. A theoretical description of this layer plays a central role in any theory of near-cathode phenomena in high-pressure arc discharges (e.g., Refs. [1–8]).

According to conventional concepts, the physics of the ionization layer is as follows. In the inner section of the layer where the density of the charged particles is small, dominating processes are ambipolar diffusion of the charged particles and ionization; recombination is a minor effect. As a distance from the edge of the space-charge sheath increases, the density of the charged particles grows and the recombination rate increases, while the effect of ambipolar diffusion decreases. At the ''edge'' of the ionization layer, the recombination rate becomes equal to the ionization rate and the increase of the charged particle density ceases: ionization equilibrium is attained.

The thickness of the ionization layer under conditions of practical interest may be not large as compared to the mean free path for ion-neutral collisions [9]. This means that coupling between the ions and the neutral particles in the layer is in a general case not strong enough and the diffusion description of the ion-neutral motion in the layer is not valid; one should rather employ a multifluid approach (see, e.g., Refs. [10,11]).

In many experimental situations, the plasma at the edge of the ionization layer is close to full ionization. (For example, plasma temperatures measured recently [12–17] in front of a thoriated-tungsten cathode of a 200-A atmospheric-pressure argon arc are in excess of 20 000 K, which corresponds to an equilibrium ionization degree exceeding 98%.) A question arises concerning a role of recombination in the ionization layer under such conditions. In other words, one can think of a model of the ionization layer in a fully ionized plasma disregarding recombination. In the framework of such a model, the increase of the charged particle density would cease at the edge of the ionization layer not because of recombination, but rather because the full ionization of the plasma has been reached.

Such a model is developed in the present paper. In order to give a simple introduction, we start with developing a theory of the ionization layer disregarding recombination in a diffusion approximation (Sec. II). The multifluid model is written down in Sec. III. Asymptotic solutions are obtained in Secs. IV and V, and discussed in Sec. VI. Application of the results to particular experimental situations is considered in Sec. VII.

A theory of the ionization layer in a partially ionized plasma was considered previously in Ref. [9]. A comparison carried out in Appendix A shows that the diffusion solution [9], if written in an appropriate form, also remains applicable to the case of a fully ionized plasma. On the other hand, a multifluid solution for a plasma close to full ionization was not found in Ref. [9].

II. DIFFUSION THEORY

We consider a quasineutral layer of a plasma containing one species of neutral atoms, singly charged positive ions of the same species, and electrons. Supposing that the thickness of the layer is much smaller than the transversal dimensions, one can write the system of governing equations in a one-dimensional form

$$\frac{d}{dy}(n_i v_i) = k_i n_i n_a, \quad n_i v_i + n_a v_a = 0, \tag{1}$$

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$$-k(T_h+T_e)\frac{dn_i}{dy}-\frac{n_i n_a kT_h}{(n_i+n_a)D_{\mathrm{ia}}}(v_i-v_a)=0, \tag{2}$$

$$n_i k(T_h+T_e)+n_a kT_h=p, \tag{3}$$

where the y axis is directed from the ''edge'' of the space-charge sheath into the plasma, n_i, n_a, v_i, and v_a are number densities and mean velocities of the ions and atoms, T_h and T_e are temperatures of heavy particles (ions and atoms) and electrons, p is the plasma pressure, k is the Boltzmann constant, k_i is the ionization rate coefficient, and D_{ia} is the binary-diffusion coefficient evaluated for a binary mixture constituted by the ion and neutral species. Variation of the electron temperature across the ionization layer is small [18] due to the high thermal conductivity of electrons, and is neglected. For simplicity, we also neglect variations of the heavy-particle temperature.

The first equation in Eq. (1) is the equation of conservation of ions written for the case when the dominating process of ionization is ionization by electron impact while recombination is insignificant. The second equation in Eq. (1) follows from conservation of nuclei, and is valid provided that there is no influx of the nuclei from the surface. Equation (3) follows from conservation of momentum of the plasma on the whole. Equation (2) is the transport equation for charged particles written in the diffusion approximation; the terms on the left-hand side describe, respectively, the pressure gradient of ions and electrons and the friction force due to elastic collisions between ions and neutral atoms. After v_a has been eliminated by means of the second equation in Eq. (1), Eq. (2) assumes the form of Fick's law for ambipolar diffusion,

$$n_i v_i=-(1+\beta)D_{\mathrm{ia}}\frac{dn_i}{dy}, \tag{4}$$

where $\beta=T_e/T_h$.

The coefficient D_{ia} in the framework of the first approximation in expansion in Sonine polynomials in the method of Chapman-Enskog (e.g., Refs. [19–21]) is given by the formula

$$D_{\mathrm{ia}}=\frac{3\pi}{32}\frac{C_{\mathrm{ia}}}{(n_i+n_\alpha)\overline{Q}_{\mathrm{ia}}^{(1,1)}}, \tag{5}$$

where $\overline{Q}_{\mathrm{ia}}^{(1,1)}$ is the average cross section for momentum transfer in ion-atom collisions, $C_{\mathrm{ia}}=(16kT_h/\pi m_i)^{1/2}$ is the mean relative speed of ions and atoms, and m_i is the mass of a heavy particle.

Boundary conditions for the considered equations are as follows. The density of the charged particles at the edge of the space-charge sheath should be set equal to zero in the diffusion approximation. At the edge of the ionization layer, the plasma is fully ionized and the charged particle density is $n_{i\infty}=p/k(T_h+T_e)$.

Substituting Eq. (4) into the first equation (1), and excluding n_a by means of Eq. (3), one obtains an equation for n_i

$$(1+\beta)n_{i\infty}D_{\mathrm{ia}}^{(0)}\frac{d}{dy}\left[\frac{1}{(1+\beta)n_{i\infty}-\beta n_i}\frac{dn_i}{dy}\right]=-k_i n_i(n_{i\infty}-n_i). \tag{6}$$

The quantity $D_{\mathrm{ia}}^{(0)}=D_{\mathrm{ia}}(n_i+n_a)kT_h/p$ introduced here does not depend on n_i or n_a, and may be considered as given. It may be interpreted as the diffusion coefficient of the ions in a gas of neutral atoms under the pressure p and the temperature T_h. In the framework of the diffusion theory, $D_{\mathrm{ia}}^{(0)}$ coincides with D_{ia} evaluated at the edge of the space-charge sheath.

It is natural to introduce the dimensionless variables $\xi=y\sqrt{k_i n_{i\infty}/D_{\mathrm{ia}}^{(0)}}$ and $f=n_i/n_{i\infty}$ while treating Eq. (6). A first integral found with the use of the boundary condition $f(\infty)=1$ reads

$$(1+\beta)\left(\frac{1}{1+\beta-\beta f}\frac{df}{d\xi}\right)^2 =(1-f)\left[\frac{1+f}{\beta}+\frac{2}{\beta^2}-\frac{2(1+\beta)}{\beta^3(1-f)}\ln(1+\beta-\beta f)\right]. \tag{7}$$

Note that at large ξ where f is close to unity the quantity in the square brackets on the right-hand side of Eq. (7) is approximately equal to $1-f$, which ensures an exponential decay of $1-f$ at large ξ:

$$1-f=O\left(\exp\left(-\frac{\xi}{\sqrt{1+\beta}}\right)\right). \tag{8}$$

Equation (7) is to be solved with the boundary condition $f(0)=0$. Solutions for finite β should be found numerically; solutions for β small or large may be found analytically and read, respectively,

$$f=\frac{2\tanh\frac{\xi}{2}\left(2\tanh\frac{\xi}{2}+\sqrt{3}\right)}{\left(\tanh\frac{\xi}{2}+\sqrt{3}\right)^2},\quad f=\frac{\xi(\xi+2)}{\xi^2+2\xi+2}. \tag{9}$$

Before presenting the solutions, it is convenient to introduce a dimensionless distance from the sheath edge, $\eta=y/d$, in such a way that $f\approx\eta$ at small η. It follows from Eq. (7) that

$$d=\frac{1}{C_1}\sqrt{\frac{D_{\mathrm{ia}}^{(0)}}{k_i n_{i\infty}}}, \tag{10}$$

where

$$C_1=\left[\frac{(1+\beta)(2+\beta)}{\beta^2}-\frac{2(1+\beta)^2}{\beta^3}\ln(1+\beta)\right]^{1/2}. \tag{11}$$

The coefficient $C_1=C_1(\beta)$ defined by Eq. (11) is depicted by the solid line in Fig. 1. Note that its limit values for small and large β are finite: $C_1(0)=\sqrt{3}/3$ and $C_1(\infty)=1$. These values are shown in Fig. 1 by the dashed lines. For β of practical interest, $\beta\geqslant 1$, C_1 varies in a relatively narrow range from 0.674 to unity.

The dimensionless charged particle density f as a function of the dimensionless distance from the sheath edge, η, is shown in Fig. 2. The solutions for $\beta=1$ and 5 were obtained by means of a numerical integration of Eq. (7), and the so-

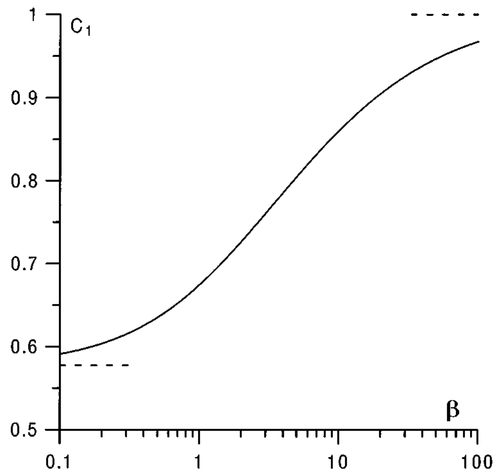

FIG. 1. Coefficient C_1 determining the ion flux from the ionization layer, calculated in the diffusion approximation.

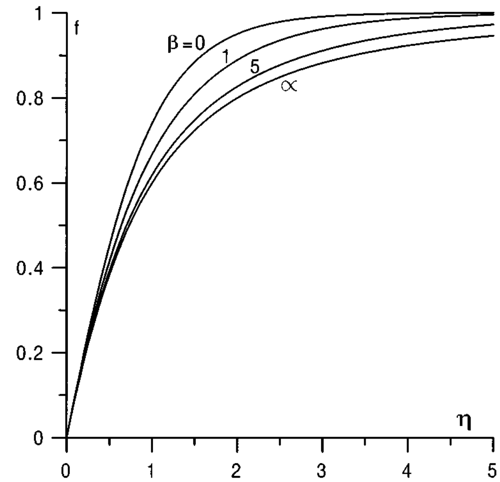

FIG. 2. Distribution of the charged particle density across the ionization layer, calculated in the diffusion approximation.

lutions for $\beta=0$ and ∞ were calculated by means of Eqs. (9). One can see that an increase of β results in a decrease of the normalized charged particle density for fixed η; however, this effect is weak: all the solutions are rather close between themselves. The length d used for normalization of abscissa in Fig. 2 represents a reasonable scale of thickness of the ionization layer: for example, in the range $y>2d$, the charged particle density deviates from the equilibrium value by no more than 20%.

Thus the diffusion model of the ionization layer disregarding recombination is complete. The density of the ion flux generated in the ionization layer and coming to the edge of the space-charge sheath is given by the formula

$$J_i=C_1(1+\beta)\sqrt{k_i D_{\mathrm{ia}}^{(0)} n_{i\infty}^3}. \tag{12}$$

III. MULTIFLUID MODEL

The aim of this section is to replace the conventional diffusion model (i.e., a model of one fluid with diffusing species) used in Sec. II with an approach regarding each species of the plasma as a separate fluid coexisting with the fluids made up of other species. To this end, one has to supplement Eq. (2) with terms accounting for ion inertia and momentum transfer from the neutral-atom species to the ion species due to ionization; Eq. (3) should be supplemented with terms accounting for dynamic pressure of the ion and atom species [9]

$$\frac{d}{dy}(n_i m_i v_i^2)=-k(T_h+T_e)\frac{dn_i}{dy}-\frac{n_i n_a k T_h}{(n_i+n_a)D_{\mathrm{ia}}}(v_i-v_a) + k_i n_i n_a m_i v_a, \tag{13}$$

$$m_i n_i v_i^2+m_i n_a v_a^2+n_i k(T_h+T_e)+n_a k T_h=p. \tag{14}$$

The boundary condition $n_i=0$ at the edge of the space-charge sheath should be replaced by the Bohm criterion (e.g., Refs. [10,22], and references therein; see also a discussion in Appendix B): $v_i=-v_s$, where $v_s=[k(T_h+T_e)/m_i]^{1/2}$.

Excluding v_a from the system (1), (13), and (14) and introducing dimensionless variables, one obtains

$$\alpha(1+\beta)\frac{d(fw^2)}{d\xi}=-\alpha(1+\beta)\frac{df}{d\xi}+\alpha^2 f w(f+\nu)+f^2 w, \tag{15}$$

$$\alpha(1+\beta)\frac{d(fw)}{d\xi}=-f\nu, \tag{16}$$

$$fw^2+\frac{f^2w^2}{\nu}+f+\frac{\nu}{1+\beta}=1, \tag{17}$$

where

$$w=-\frac{v_i}{v_s},\quad \nu=\frac{n_a}{n_{i\infty}},\quad \alpha=\frac{kT_h}{[m_i p k_i D_{\mathrm{ia}}^{(0)}]^{1/2}}. \tag{18}$$

The boundary conditions for Eqs. (15)–(17) read

$$w(0)=1,\quad f(\infty)=1. \tag{19}$$

After the problem (15)–(17) and (19) has been solved, one can find the ion flux coming from the ionization layer to the edge of the space-charge sheath, $J_i=n_{i\infty}v_s f_w$, where $f_w=f(0)$.

We shall need to know asymptotic behavior of a solution at large ξ. Retaining in Eqs. (15)–(17) terms of the first order in $1-f$, w, and ν, one obtains equations

$$-\alpha(1+\beta)\frac{df}{d\xi}+(1+\alpha^2)w=0,\quad \alpha(1+\beta)\frac{dw}{d\xi}=-\nu,$$

$$\frac{w^2}{\nu}+\frac{\nu}{1+\beta}=1-f. \tag{20}$$

A solution to these equations can be sought in an exponential form. One finds

$$\begin{Bmatrix} 1-f \\ w \\ \nu \end{Bmatrix} = \text{const}\times \begin{Bmatrix} 1 \\ \dfrac{\alpha\sqrt{1+\beta}}{1+\alpha^2} \\ \dfrac{\alpha^2(1+\beta)}{1+\alpha^2} \end{Bmatrix} \exp\left(-\frac{\xi}{\sqrt{1+\beta}}\right). \tag{21}$$

One can conclude that Eq. (8), which describes asymptotic behavior of a solution at large ξ in the framework of the diffusion theory, remains valid also in the multifluid model.

IV. ASYMPTOTIC SOLUTION FOR $\alpha\to\infty$

We shall use the method of matched asymptotic expansions (e.g., Refs. [23–27]), and seek an outer asymptotic expansion of a solution of the problem (15)–(17) and (19) in the limit of large α in the form

$$f(\xi;\alpha,\beta)=f_1(\xi;\beta)+\cdots,$$

$$w(\xi;\alpha,\beta)=\alpha^{-1}w_1(\xi;\beta)+\cdots,$$

$$\nu(\xi;\alpha,\beta)=\nu_1(\xi;\beta)+\cdots. \tag{22}$$

Substituting this expansion into Eqs. (15)–(17), and neglecting terms of order of α^{-2} with respect to the leading terms, one obtains equations equivalent to the diffusion equations treated in Sec. II. Assuming that the first term of the outer asymptotic expansion obeys the diffusion boundary condition $f_1(0)=0$, one can conclude that this term is described by the formulas obtained in Sec. II. In particular, the asymptotic behavior of the functions f_1 , w_1, and ν_1 at $\xi\to 0$ is

$$f_1=C_1\xi+\cdots,\quad w_1=\frac{1}{\xi}+\cdots,\quad \nu_1\to 1+\beta. \tag{23}$$

The inner asymptotic expansion, applicable in the vicinity $\xi=O(\alpha^{-1})$ of the edge of the space-charge sheath, is

$$f(\xi;\alpha,\beta)=\alpha^{-1}f_2(\xi_2;\beta)+\cdots,$$

$$w(\xi;\alpha,\beta)=w_2(\xi_2;\beta)+\cdots,$$

$$\nu(\xi;\alpha,\beta)=\nu_2(\xi_2;\beta)+\cdots, \tag{24}$$

where $\xi_2=\alpha\xi$. Substituting this expansion into Eqs. (15)–(17) and neglecting terms of order of α^{-1} with respect to the leading terms, one obtains

$$(1+\beta)\frac{d(f_2w_2^2)}{d\xi_2}=-(1+\beta)\frac{df_2}{d\xi_2}+f_2w_2\nu_2, \tag{25}$$

$$\frac{d(f_2w_2)}{d\xi_2}=0,\quad \frac{\nu_2}{1+\beta}=1. \tag{26}$$

The boundary conditions for these equations at $\xi_2\to\infty$ follow from asymptotic matching,

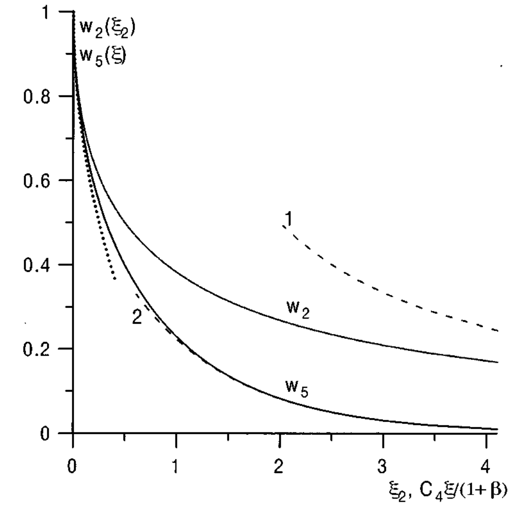

FIG. 3. Distribution of the normalized mean ion velocity in the inner section of the ionization layer. Dotted line: the square-root distribution characteristic for the Bohm criterion. (1) Second equation in Eqs. (27) (the diffusion solution). (2) Second equation in Eqs. (45) (the pseudodiffusion solution).

$$f_2\approx C_1\xi_2,\quad w_2\approx\frac{1}{\xi_2},\quad \nu_2\to 1+\beta. \tag{27}$$

Another boundary condition reads $w_2(0)=1$.

From Eqs. (26), one finds $f_2=C_1/w_2$ and $\nu_2=1+\beta$. Substituting these relations into Eq. (25) and solving the obtained equation with account of the above-described boundary conditions, one obtains

$$w_2=\frac{2}{2+\xi_2+\sqrt{\xi_2^2+4\xi_2}}. \tag{28}$$

This solution is shown in Fig. 3. Also shown is its asymptotic behavior at large and small ξ_2, which is governed, respectively, by the second equation in Eq. (27) and by the formula

$$w_2\approx 1-\sqrt{\xi_2}. \tag{29}$$

Note that the square-root behavior near the edge of the space-charge sheath described by Eq. (29) is characteristic for problems involving the Bohm criterion. Thus, the asymptotic solution for large values of α is complete. It follows, in particular, that asymptotic behavior of the function $f_w(\alpha,\beta)$ in the limit $\alpha\to\infty$ is

$$f_w(\alpha,\beta)\approx\frac{C_1(\beta)}{\alpha}. \tag{30}$$

V. ASYMPTOTIC SOLUTION FOR $\alpha\to 0$

We start with asymptotic estimates of various terms of the equations, supposing that a characteristic length scale of variation of the solution in terms of the variable ξ is unity, and orders of magnitude of f, w, and ν do not exceed unity. Evidently, the term on the right-hand side of Eq. (16) cannot be dominating (if it were, this would mean that f or ν is 0, which does not make sense). It follows that $\nu \leqslant O(\alpha w)$. This means, in particular, that the last term on the left-hand side of Eq. (17) is small.

The second term on the right-hand side of Eq. (15) may be represented as $\alpha^2 f w \nu + \alpha^2 f^2 w$. $\alpha^2 f w \nu$ does not exceed in order of magnitude α^2 with respect to the term on the left-hand side of Eq. (15). $\alpha^2 f^2 w$ is of order of α^2 with respect to the last term on the right-hand side. Thus, the second term on the right-hand side of Eq. (15) is small. Since the last term on the right-hand side of Eq. (15) cannot be dominating, one finds $fw \leqslant O(\alpha)$.

Returning to Eq. (17), one can see that the first term on the left-hand side of this equation is small. Since the difference $1-f$ is of order unity, the second term on the left-hand side also is of order unity and $\nu = O(f^2 w^2)$.

In order to apply the boundary condition $f(\infty)=1$, one should consider an asymptotic expansion in which $f=O(1)$. It follows from the above that $w \leqslant O(\alpha)$ in this expansion. The term on the left-hand side of Eq. (15) does not exceed in order of magnitude α^2 with respect to the first term on the right-hand side. Thus the only terms to be retained in Eq. (15) are the first and last terms on the right-hand side. No one of these terms can be dominating, therefore $w=O(\alpha)$. It follows that $\nu = O(\alpha^2)$. Thus an asymptotic expansion with $f=O(1)$ is

$$f(\xi;\alpha,\beta)=f_3(\xi_3;\beta)+\alpha^2 f_4(\xi_3;\beta)+\cdots,$$

$$w(\xi;\alpha,\beta)=\alpha w_3(\xi_3;\beta)+\alpha^3 w_4(\xi_3;\beta)+\cdots, \tag{31}$$

$$\nu(\xi;\alpha,\beta)=\alpha^2 \nu_3(\xi_3;\beta)+\alpha^4 \nu_4(\xi_3;\beta)+\cdots,$$

where $\xi_3=\xi-\Xi$, $\Xi=\Xi(\alpha,\beta)$ being an unknown quantity. The functions f_3, w_3, and ν_3 obey equations

$$-(1+\beta)\frac{df_3}{d\xi_3}+f_3^2 w_3=0, \quad (1+\beta)\frac{d(f_3 w_3)}{d\xi_3}=-f_3\nu_3,$$

$$\frac{f_3^2 w_3^2}{\nu_3}+f_3=1. \tag{32}$$

Excluding w_3 and ν_3 and solving the obtained equation, one finds

$$f_3=\frac{1}{1+C_2\exp(-C_3\xi_3)}, \tag{33}$$

where C_2 and C_3 are arbitrary constants.

Constant C_2 may be absorbed by redefining Ξ; hence one can set $C_2=1$ without losing generality. The constant C_3 should be determined with the use of the boundary condition $f(\infty)=1$. However, this boundary condition is ineffective in the approximation considered, and one should treat the second approximation.

The functions f_4, w_4, and ν_4 are governed by the equations

$$-(1+\beta)\frac{df_4}{d\xi_3}+2f_3w_3f_4+f_3^2w_4=(1+\beta)\frac{d(f_3w_3^2)}{d\xi_3}-f_3^2w_3, \tag{34}$$

$$(1+\beta)\frac{d}{d\xi_3}(w_3f_4+f_3w_4)+\nu_3f_4+f_3\nu_4=0, \tag{35}$$

$$2\frac{f_3w_3^2}{\nu_3}f_4+2\frac{f_3^2w_3}{\nu_3}w_4-\frac{f_3^2w_3^2}{\nu_3^2}\nu_4+f_4=-f_3w_3^2-\frac{\nu_3}{1+\beta}. \tag{36}$$

Taking into account the asymptotic behavior of the functions f_3, w_3, and ν_3 at large ξ_3 that follows from Eq. (33) and making use of the fact that the functions f_4, w_4, and ν_4 must vanish at large ξ, one finds a limit form of Eqs. (34)–(36) for $\xi_3\to\infty$:

$$-(1+\beta)\frac{df_4}{d\xi_3}+w_4=-(1+\beta)C_3\exp(-C_3\xi_3),$$

$$(1+\beta)\frac{dw_4}{d\xi_3}+\nu_4=0, \tag{37}$$

$$\frac{2}{(1+\beta)C_3}w_4-\frac{1}{(1+\beta)^2C_3^2}\nu_4+f_4 =-(1+\beta)C_3^2\exp(-C_3\xi_3). \tag{38}$$

Excluding w_4 and ν_4, one obtains an equation for f_4,

$$\frac{1}{C_3^2}\frac{d^2f_4}{d\xi_3^2}+\frac{2}{C_3}\frac{df_4}{d\xi_3}+f_4=[1-(1+\beta)C_3^2]\exp(-C_3\xi_3). \tag{39}$$

A solution to this equation includes a secular term

$$\frac{C_3^2}{2}[1-(1+\beta)C_3^2]\xi_3^2\exp(-C_3\xi_3), \tag{40}$$

which should be removed. It follows that $C_3=1/\sqrt{1+\beta}$.

Now the functions f_3, w_3, and ν_3 are determined completely. The asymptotic behaviors of these functions at $\xi_3\to-\infty$ are

$$f_3\approx\exp\frac{\xi_3}{\sqrt{1+\beta}}, \quad w_3\approx\sqrt{1+\beta}\,\exp\left(-\frac{\xi_3}{\sqrt{1+\beta}}\right),$$

$$\nu_3\to 1+\beta. \tag{41}$$

Evidently, expansion (31) cannot satisfy the boundary condition $w(0)=1$. In order to apply this boundary condition, one should consider an expansion in which $w=O(1)$ and, consequently, $f\leqslant O(\alpha)$, $\nu=O(f^2)\leqslant O(\alpha^2)$. The third term on the left-hand side of Eq. (17) is small. The term on the right-hand side of Eq. (16) is small and this equation gives $fw=\text{const}$. If one assumes $f<O(\alpha)$, the last term on the right-hand side of Eq. (15) will be small and this equation will give $fw^2+f=\text{const}$, which, together with $fw=\text{const}$, results in a trivial (constant) solution. Hence, the assumption $f<O(\alpha)$ is inappropriate and one should assume $f=O(\alpha)$ instead. Thus, an expansion in which $w=O(1)$ reads

$$f(\xi;\alpha,\beta)=\alpha f_5(\xi;\beta)+\cdots,\quad w(\xi;\alpha,\beta)=w_5(\xi;\beta)+\cdots,$$
$$\nu(\xi;\alpha,\beta)=\alpha^2\nu_5(\xi;\beta)+\cdots, \tag{42}$$

with the functions f_5, w_5, and ν_5 being described by equations

$$(1+\beta)\frac{d(f_5w_5^2)}{d\xi}=-(1+\beta)\frac{df_5}{d\xi}+f_5^2w_5,$$
$$\frac{d(f_5w_5)}{d\xi}=0,\quad \frac{f_5^2w_5^2}{\nu_5}=1. \tag{43}$$

It follows from the second and third equations that $f_5=C_4/w_5$ and $\nu_5=C_4^2$, where C_4 is an arbitrary constant. Substituting these expressions into the first equation and solving it with the boundary condition $w_5(0)=1$, one obtains

$$\frac{w_5^2-1}{2}-\ln w_5=\frac{C_4}{1+\beta}\xi. \tag{44}$$

This solution is shown in Fig. 3. Also shown is its asymptotic behavior at small and large ξ, which is governed by the formulas

$$w_5\approx 1-\sqrt{\frac{C_4}{1+\beta}\xi},\quad w_5\approx\exp\left(-\frac{C_4}{1+\beta}\xi-\frac{1}{2}\right), \tag{45}$$

respectively. The first formula describes the square-root behavior characteristic for problems involving the Bohm criterion, similarly to Eq. (29).

It is interesting to compare the functions w_2 and w_5, which describe distributions of the normalized ion velocity in the inner section of the ionization layer for α large and small, respectively. To this end, the scales of abscissa in Fig. 3 are chosen in such a way that Eq. (29) be represented by the same line as the first equation in Eq. (45). Thus the plots of w_2 and w_5 are close between themselves at small values of abscissa. At larger values, their behavior is essentially different: w_5 decreases very fast and becomes quite close to its asymptotic representation for large arguments when the value of abscissa reaches 1, while w_2 decreases much slower, and approaches its asymptotic representation for large arguments also slowly.

Now one should consider an asymptotic expansion valid in between the regions of applicability of expansions (31) and (42). In this expansion, orders of magnitude of both f and w should be greater than α but smaller that unity. The term on the left-hand side of Eq. (15) is small; it follows that $fw=O(\alpha)$ and $\nu=O(\alpha^2)$. The third term on the left-hand side of Eq. (17) and the term on the right-hand side of Eq. (16) are small. Thus the equations describing the leading term of the considered expansion read

$$-\alpha(1+\beta)\frac{df}{d\xi}+f^2w=0,\quad \frac{d(fw)}{d\xi}=0,\quad \frac{f^2w^2}{\nu}=1. \tag{46}$$

All terms of these equations are contained both in Eqs. (32) and (43). Hence there is no need to consider a solution to these equations: expansions (31) and (42) can be matched directly. Indeed, one can check easily that such a matching is possible, and

$$C_4=\sqrt{1+\beta},\quad \Xi=\sqrt{1+\beta}\left(\ln\frac{1}{\alpha}-\frac{1}{2}-\ln\sqrt{1+\beta}\right). \tag{47}$$

Thus the asymptotic solution for small values of α is complete. In particular, the asymptotic behavior of the function $f_w(\alpha,\beta)$ in the limit $\alpha\to 0$ is

$$f_w(\alpha,\beta)\approx\alpha\sqrt{1+\beta}. \tag{48}$$

VI. DISCUSSION OF ASYMPTOTIC RESULTS

One can see that the parameter α is of order of the ratio of the length d, which can be considered as a scale of thickness of the ionization layer in a fully ionized plasma evaluated in the framework of a diffusion theory, to the mean free path for ion-atom collisions λ_{ia}. The solution that has been obtained for the case of large α (in other words, for the case $d\gg\lambda_{ia}$) involves two asymptotic zones: the outer zone $\xi=O(1)$ and the inner zone $\xi=O(\alpha^{-1})$. These zones may be identified in terms of the dimensional distance from the sheath edge, y, as $y=O(d)$ and $O(\lambda_{ia})$.

The conventional diffusion theory is applicable provided that a local length scale be much larger than λ_{ia}. Hence the outer solution obtained in the framework of the multifluid theory for the limit case $\alpha\to\infty$ must coincide, to a first approximation, with a solution in the framework of the diffusion theory, which is indeed the case. The first equation in Eqs. (26) shows that the ion flux is to a first approximation constant across the inner zone. In other words, the ion flux is generated primarily in the outer zone. Therefore, the ion flux coming to the edge of the space-charge sheath calculated in the framework of the multifluid theory for the limit case $\alpha\to\infty$ must coincide, to a first approximation, with that found in the framework of the diffusion theory. One can check that this is indeed the case: the ion flux calculated with the use of Eq. (30) coincides with that given by Eq. (12).

Quantity f_w may be considered as the ion flux normalized by the quantity $n_{i\infty}v_s$, which may be treated as a characteristic value of the chaotic ion flux. The asymptotic behavior of the function $f_w(\alpha,\beta)$ for large and small α is described to a first approximation by Eqs. (30) and (48), respectively. In order to obtain a general idea of f_w for finite α, one can approximate the function $f_w(\alpha,\beta)$ by means of a rational fraction in α, with coefficients determined with the use of information available on asymptotic behavior of $f_w(\alpha,\beta)$ at $\alpha\to 0$ and $\alpha\to\infty$. In the simplest form, such a fraction must contain α in the numerator and a polynomial of the second degree in α in the denominator. Asymptotic formulas (30) and (48) allow us to find two of the three coefficients of this polynomial. In order to find the third one, we need to determine asymptotic behavior of the function $f_w(\alpha,\beta)$ at $\alpha\to 0$, or at $\alpha\to\infty$ to a second approximation. For brevity, we present the asymptotic behavior at $\alpha\to 0$ to a second approximation without a derivation,

$$f_w(\alpha,\beta)\approx\alpha\sqrt{1+\beta}-2\alpha^2(1+\beta). \tag{49}$$

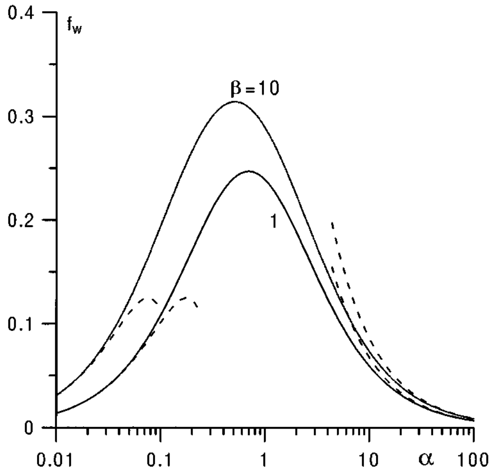

FIG. 4. The dimensionless ion flux from the ionization layer.

A rational fraction determined with the use of Eqs. (30) and (49) is

$$f_w(\alpha,\beta)=\frac{\alpha C_1\sqrt{1+\beta}}{C_1+2\alpha C_1\sqrt{1+\beta}+\alpha^2\sqrt{1+\beta}}. \qquad (50)$$

The function f_w calculated by means of this formula is depicted by the full lines in Fig. 4. The dashed lines represent asymptotic behavior described by Eqs. (30) and (49).

According to Eq. (30), $f_w=O(\alpha^{-1})$ in the case of large α, which means that the ion flux is much smaller than the chaotic flux in this case. This result conforms to the conventional diffusion concepts, according to which the mean velocity of the ions is of order of the thermal (chaotic) velocity times the Knudsen number $\lambda_{\rm ia}/d$. As α decreases, f_w increases; the ion flux becomes comparable to the chaotic flux when $\alpha=O(1)$. This, again, is quite understandable: collisions between ions and neutral atoms are not frequent, the ion-atom friction force is not large and cannot prevent acceleration of the ion fluid by the pressure gradient and by the ambipolar electric field to velocities of order of a thermal velocity. As α decreases further, f_w starts to decrease. When α becomes small, f_w is small, too: according to Eq. (48), $f_w=O(\alpha)$ for $\alpha\to 0$. Thus the ion flux in the case of small α is much smaller than the chaotic flux, as is in the diffusion limit case $\alpha\gg 1$. It will be shown below that the two cases are similar also in a number of other aspects, which is why the limit case of small α may be called a ''pseudodiffusion'' regime.

In order to clarify the physics of the pseudodiffusion regime, it is useful to construct a simplified model for this regime in a way similar to that in which a diffusion model was developed in Sec. II. It follows from the results obtained in Sec. V that a major variation of the charged particle density is localized and the ion flux to the edge of the space-charge sheath is formed in the region described by expansion (31). Inspection of Eqs. (32) indicates that Eqs. (13) and (14) in this region may be written to a first approximation as

$$-k(T_h+T_e)\frac{dn_i}{dy}+k_i n_i n_a m_i v_a=0,$$

$$m_i n_a v_a^2+n_i k(T_h+T_e)=p. \qquad (51)$$

Comparison of the first equation in Eqs. (51) with Eq. (2) indicates that, while ion inertia is insignificant in both cases, the pressure gradient is balanced in the pseudodiffusion case by momentum transfer from the neutral-atom species to the ion species due to ionization, rather than due to elastic collisions as is in the diffusion case. Comparison of the second equation in Eqs. (51) with Eq. (3) indicates that, while the dynamic pressure of the ion species is insignificant in both cases, a variation of the static pressure of the charged particles is balanced in the pseudodiffusion case by a variation of the dynamic pressure of the atom species, rather than by a variation of the static pressure of the atom species as is in the diffusion case.

Fick's law (4) remains valid in the pseudodiffusion case provided that the conventional diffusion coefficient $D_{\rm ia}$ (calculated in terms of a cross section of elastic collisions) is replaced by the combination $\tilde{D}_{\rm ia}=kT_h/m_i k_i n_i$ (which involves the ionization cross section). The equation governing a distribution of the charged particles in the pseudodiffusion case reads

$$\frac{d}{dy}\left(\frac{1}{n_i}\frac{dn_i}{dy}\right)=-\frac{1}{n_i(n_{i\infty}-n_i)}\left(\frac{dn_i}{dy}\right)^2. \qquad (52)$$

An essential difference between this equation and Eq. (6), which governs the charged-particle distribution in the diffusion case, is that Eq. (52) is invariant with respect to a linear transformation of the independent variable and does not contain a length scale. A solution to Eq. (52) may be written as

$$n_i=n_{i\infty}\left[1+\exp\left(-\frac{y-C_5}{C_6}\right)\right]^{-1}, \qquad (53)$$

where C_5 and C_6 are arbitrary constants.

This solution satisfies the boundary condition $n_i=n_{i\infty}$ at infinity without regard of C_5 and C_6, provided that $C_6>0$. Conversely, the boundary condition $n_i=0$ cannot be satisfied at finite y. The latter means that Eq. (52) cannot be used right up to the sheath edge, in contrast to Eq. (6). In perturbation theory terms one can say that Eq. (52) describes a ''shock layer'' positioned at $y=C_5$, while Eq. (6) describes a ''boundary layer'' at $y=0$.

Constant C_6 has the meaning of a scale of thickness of the shock layer. This constant cannot be determined in the approximation considered, which is a consequence of the fact that Eq. (52) does not contain any length scale. In order to determine C_6, one should either consider a second approximation, as was done in Sec. V, or employ another boundary condition. An appropriate boundary condition is supplied by Eq. (8): comparing this equation with a two-term asymptotic expansion of solution (53) for large values of the argument, one finds

$$C_6=\left[\frac{(1+\beta)D_{\rm ia}^{(0)}}{k_i n_{i\infty}}\right]^{1/2}. \qquad (54)$$

The constant C_5 determining a position of the shock layer remains indeterminate in the framework of the simplified approach considered. However, the ion flux from the ionization layer to the edge of the space-charge sheath is independent of C_5, and can be calculated as

$$J_i=\left[\frac{kpT_h}{m_i^2 k_i D_{\rm ia}^{(0)}}\right]^{1/2}. \qquad (55)$$

One can check that this formula conforms to Eq. (48). It should be emphasized that the ion flux depends on the ion-atom diffusion coefficient, in spite of the ion-atom momentum exchange due to elastic collisions being much smaller than the momentum exchange due to ionization. The reason is the above-discussed fact that the scale of thickness of the considered layer remains indeterminate in the first approximation or, in other words, that the limit $\alpha\to 0$ is singular.

Note that the scale of thickness of the shock layer determined by Eq. (54) is of order d, thus being much smaller than the mean free path for elastic ion-atom collisions $\lambda_{\rm ia}$. On the other hand, it is natural in this case to introduce a mean free path characterizing the momentum exchange due to ionization, which is related to the diffusion coefficient $\tilde{D}_{\rm ia}$ by the conventional formula $\tilde{\lambda}_{\rm ia}=\tilde{D}_{\rm ia}/C_{\rm ia}$. The mean velocity of ions in the shock layer is of order of the thermal velocity times the Knudsen number $\tilde{\lambda}_{\rm ia}/d$, similarly to the conventional diffusion concepts. The mean free path $\tilde{\lambda}_{\rm ia}$, being of order of $C_{\rm ia}/k_i n_{i\infty}$, is much smaller than d: $\tilde{\lambda}_{\rm ia}/d=O(\alpha)$. It follows that the mean ion velocity is much smaller than the thermal velocity, which explains why the ion flux in the pseudodiffusion case is much smaller than the chaotic flux. Note that, while the velocity of the ion fluid is much smaller than the thermal velocity both in the diffusion and pseudodiffusion cases, the velocity of the atom fluid is much larger than the thermal velocity in the pseudodiffusion case and much smaller than the thermal velocity in the diffusion case.

A diffusion theory of the ionization layer in a partially ionized plasma was previously considered in Ref. [9]. A formula for the ion flux generated in the ionization layer written on the basis of the results [9] is given in Appendix A [Eq. (A1)]. For a plasma close to full ionization, this formula conforms to Eq. (12). Thus the case of a fully ionized plasma represents in the framework of the diffusion theory a regular limit of the general case of a partially ionized plasma.

On the other hand, a multifluid solution for a plasma close to full ionization was not found in Ref. [9]. The reason for this is clarified by the above asymptotic solutions, and is as follows. One can express ν in terms of f and w by analytically solving Eq. (17). A solution to this quadratic equation is twofold. It can be shown on the basis of the above asymptotic solutions that in the case of large α the proper branch is the one with plus, while in the case of small α the proper branch is that with minus. Thus one should deal in numerical calculations with both branches of the solution of Eq. (17) with eventual switching of branches during the calculations, while in the calculations of Ref. [9] only the branch with plus was considered. Note that a numerical solution with the switching of branches is not a simple task, and is not attempted here.

VII. IONIZATION LAYER IN ARGON AND MERCURY PLASMAS

In order to apply the above results to a particular experimental situation, one has to estimate the parameter α. With the help of Eq. (5), the function $\alpha(T_e,T_h)$ may be written as

$$\alpha=\left[\frac{2}{3}\,\frac{C_{\rm ia}\overline{Q_{\rm ia}^{(1,1)}}}{k_i}\right]^{1/2}. \qquad (56)$$

Thus one needs to know the ionization rate constant k_i and the average cross section for momentum transfer in elastic ion-atom collisions, $\overline{Q_{\rm ia}^{(1,1)}}$, in order to estimate α.

The ionization rate constant can be represented as a sum of the rate constant of direct ionization of atoms from the ground state and of the rate constant of stepwise ionization that occurs as a result of transitions between excited states of the atom due to collisions with electrons [28],

$$k_i=k_{\rm dir}+k_{\rm step}. \qquad (57)$$

The rate constant of direct ionization is given by the integral of the ionization cross section with the electron energy distribution function. The latter in a strongly ionized plasma considered in this work is governed by electron-electron collisions and is therefore close to Maxwellian. At thermal energies much smaller than the ionization energy I of the atom, the integral can be estimated as

$$k_{\rm dir}=c_i\left(\frac{8kT_e}{\pi m_e}\right)^{1/2}(I+2kT_e)\exp\left(-\frac{I}{kT_e}\right), \qquad (58)$$

where c_i is a derivative of the ionization cross section with respect to the electron energy, evaluated at the threshold.

The rate of stepwise ionization can be estimated both numerically and analytically; see, e.g., review [29]. A simple and reasonably accurate approach is provided by the so-called modified diffusion approximation (MDA) [28]. The process of stepwise ionization is considered in the framework of the MDA as a result of diffusion of bound electrons over energy levels of atoms. For a given T_e, a position in the energy spectrum exists where a diffusing bound electron spends most of the time (the so-called bottleneck). With the growth of T_e, the bottleneck shifts in the direction of lower energy levels.

For conditions when the role of radiation is unessential (which is the case if the electron number density exceeds 10^{21}–10^{22} m^{-3}) the value of $k_{\rm step}$ in the framework of the MDA is given by the interpolation equation

$$k_{\rm step}^{-1}=k_1^{-1}+k_2^{-1}\frac{4}{3\sqrt{\pi}}\Gamma\left(\frac{E}{kT_e};\frac{5}{2}\right), \qquad (59)$$

where E is the energy of the first excited state of the atom counted from the ionization threshold and $\Gamma(x;a)$ is the incomplete γ function. The quantities k_1 and k_2 represent limit values of $k_{\rm step}$ for the cases of high and low T_e, respectively (in the first case the bottleneck is positioned between the ground and the first excited states of the atom; in the second case it is positioned between the first excited state and the continuum).

TABLE I. Atomic parameters used in calculations of the ionization rate constant, of the average cross section for momentum transfer, and of equilibrium composition for argon and mercury plasmas.

	Ar	Hg
I, eV	15.75	10.44
E, eV	4.11	5.26
I_+, eV	27.63	18.76
g_1	1	1
g_+	6	2
g_{2+}	9	1
c_i, 10^{-22} m^2/eV	18	30
a, 10^{-10} m	7.0	11.9
b, 10^{-10} m	0.60	0.56

k_1 is given by the formula [28].

$$k_1 = \frac{4e^4\Lambda_1}{\Delta E}\left(\frac{2\pi}{m_e kT_e}\right)^{1/2} \exp\left(-\frac{\Delta E}{kT_e}\right), \tag{60}$$

where $\Delta E = I - E$, and Λ_1 is the Coulomb logarithm for the ground state (a function of $kT_e/\Delta E$). The latter can be approximated in the range $kT_e/\Delta E \gtrsim 0.07$ as $\Lambda_1 = 0.25(kT_e/\Delta E)^{1.2}$ [30]. The use of this approximation enables one to rewrite Eq. (60) as

$$k_1 = 4.3\times 10^{-14}\left(\frac{kT_e}{\text{Ry}}\right)^{0.7}\left(\frac{\Delta E}{\text{Ry}}\right)^{-2.2} \exp\left(-\frac{\Delta E}{kT_e}\right) \ \text{m}^3/\text{s}, \tag{61}$$

where Ry = 13.6 eV is the Rydberg constant.

k_2 calculated in the framework of the MDA is [28]

$$k_2 = \frac{8}{3}\left(\frac{2}{m_e}\right)^{1/2} \frac{e^4\bar{\Lambda}}{\text{Ry}^{3/2}} \frac{\Sigma_+}{g_1}\left(\frac{\text{Ry}}{kT_e}\right)^3 \exp\left(-\frac{I}{kT_e}\right), \tag{62}$$

where Σ_+ is the partition sum of the ion, g_1 is the statistical weight of the atomic ground state, and $\bar{\Lambda}$ is the mean value of the Coulomb logarithm for the excited atomic states. Taking into account that under the considered conditions the partition sum Σ_+ is approximately equal to the statistical weight g_+ of the ion ground state, and $\bar{\Lambda} \approx 0.2$, one can rewrite the last expression as

$$k_2 = 1.3\times 10^{-14}\frac{g_+}{g_1}\left(\frac{\text{Ry}}{kT_e}\right)^3 \exp\left(-\frac{I}{kT_e}\right) \ \text{m}^3/\text{s}. \tag{63}$$

As pointed out above, this value represents the rate constant of stepwise ionization in the limit of low T_e; it is related through the detailed balance (Saha) equation to the well-known expression for the electron-ion recombination rate constant involving the factor $T_e^{-9/2}$ (e.g., Ref. [31]).

Resonant charge exchange is a dominating mechanism of momentum transfer between singly charged ions and parent atoms. The energy-dependent momentum transfer cross section $Q_{\text{ia}}^{(1)}$ is related to the total charge exchange cross section Q_{ex} (which is measured in experiments) by the formula [32] $Q_{\text{ia}}^{(1)}(\varepsilon) \approx 2Q_{\text{ex}}(\varepsilon)$, where ε is the energy of collision. Taking into account the formula $Q_{\text{ex}}^{1/2}(\varepsilon) = a - b \ln\varepsilon$ [32], where a and b are constants, one can calculate $\overline{Q_{\text{ia}}^{(1,1)}}$ approximately as

$$\overline{Q_{\text{ia}}^{(1,1)}}(T_h) \approx 2[a - b \ln(2kT_h)]^2. \tag{64}$$

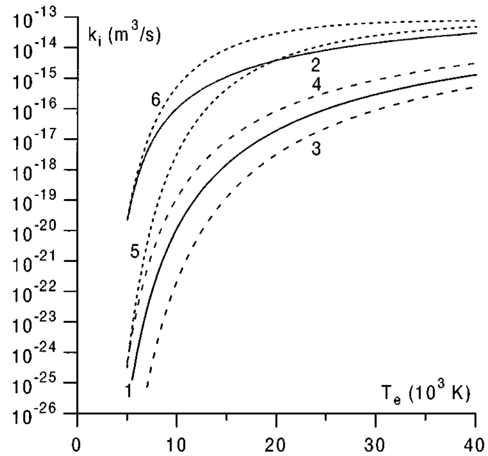

FIG. 5. Ionization rate constants in argon and mercury plasmas. (1) and (2): The total ionization rate constants. (3) and (4): The rate constants of direct ionization. (5) and (6): The rate constants calculated by means of Eq. (63). (1), (3), and (5): Ar. (2), (4), and (6): Hg.

As an example, we consider an application of the above formulas to argon and mercury plasmas. Atomic parameters used in the calculations are given in Table I; also shown are the ionization potential of the singly charged ion, I_+, and the statistical weight of the ground state of the doubly charged ion, g_{2+}. Parameters I, E, I_+, g_1, g_+, and g_{2+} were taken from Ref. [33]. The values of c_i were obtained from the data on ionization cross sections for Ar and Hg given in Ref. [34]. The parameters a and b for Ar were taken in accordance with Ref. [35]; for Hg they were obtained by approximating the measured resonant charge exchange cross section [36] (ε is in eV).

Dependencies of the ionization rate constant k_i on T_e are shown by the full lines in Fig. 5. The values of k_{dir} are also shown. A major contribution to the ionization rate in the considered range of electron temperatures is due to stepwise ionization both in Ar and Hg. A contribution of direct ionization in argon is much greater than that in mercury, which is a consequence of different structures of energy levels: the first excited state in Ar is relatively close to continuum, $E \approx I/4$, while in Hg $E \approx I/2$. Note that in the whole range of T_e the values of k_i are close to k_1; that is, the ionization rate is governed mainly by transitions between the ground state and the first excited state. Also shown in Fig. 5 is the coefficient k_2, which corresponds to the ionization rate constant calculated with the use of the electron-ion recombination rate constant proportional to $T_e^{-9/2}$ and the Saha equation. One can see that such a calculation results in a considerable overestimation of the ionization rate in the conditions of practical interest.

The values of α as functions of T_e are shown in Fig. 6 for two values of the heavy-particle temperatures. The depen-

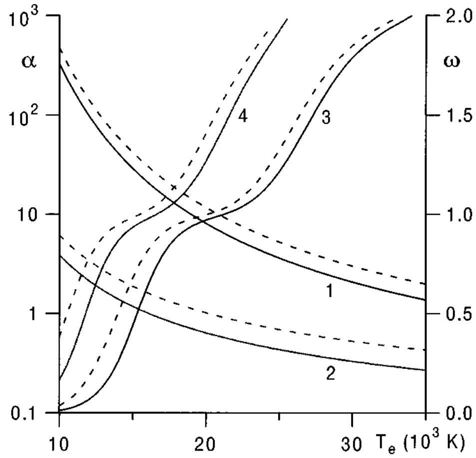

FIG. 6. (1) and (2): Parameter α in argon and mercury plasmas. (3) and (4): Equilibrium ionization degree in atmospheric-pressure argon and mercury plasmas. Full lines: $T_h = 3000$ K. Dashed lines: $T_h = 30\,000$ K. (1) and (3): Ar. (2) and (4): Hg.

dence of α on the heavy-particle temperature is rather weak. As α is proportional to $k_i^{-1/2}$, values of α in Hg are essentially smaller than those in Ar.

The theory developed in the preceding sections is applicable provided that the plasma at the edge of the ionization layer is close to full ionization while the percentage of multiply charged ions is small. It is convenient to define the ionization degree ω as a ratio of the electron number density to the total number density of the heavy particles; then the condition of applicability of the theory is that ω be close to unity. Values of the equilibrium ionization degree in atmospheric-pressure argon and mercury plasmas are also shown in Fig. 6. (In calculations, neutral atoms and singly and doubly charged ions were taken into account, and the atomic parameters shown in Table I were used.) Assuming that ω can differ from unity by no more than, say, 20%, one can see that at $T_h = 3000$ K the theory is applicable in the range $T_e = 17\,000–25\,000$ K for Ar and in the range $T_e = 13\,300–19\,500$ K for Hg. The ranges at $T_h = 30\,000$ K are $T_e = 16\,000–24\,000$ K for Ar and $T_e = 12\,300–18\,500$ K for Hg.

One can see from Fig. 6 that values of α for Ar in the above-mentioned ranges of T_e are essentially greater than unity. Hence, the ion flux from the ionization layer in an argon plasma to the edge of the space-charge sheath may be estimated by means of Eq. (12). In the case of a mercury plasma, the respective values of α are comparable to unity, and one can employ Eq. (50) as a first guess for the ion flux. Note that an increase of pressure will result in a shift of the range of values of electron temperature in which $\omega \approx 1$ to larger values, that is, in the direction of smaller α.

VIII. CONCLUDING REMARKS

A specific case important for applications in the theory of the ionization layer at the edge of a thermal plasma is represented by the case when the plasma is close to full ionization. The recombination in the ionization layer is negligible in this case, which allows one to construct a model which is mathematically simpler and more transparent physically than a general model for a plasma of an arbitrary ionization degree. In the framework of such a model, the increase of the charged particle density ceases at the edge of the ionization layer because the full ionization of the plasma has been attained, rather than because a balance between ionization and recombination has been reached.

A character of a solution in the framework of this model has been studied on the basis of asymptotic analysis for the cases $\alpha \gg 1$ and $\alpha \ll 1$, i.e., for the cases in which the thickness of the ionization layer is much larger or much smaller than the mean free path for elastic ion-atom collisions. Another interpretation of the physical sense of these limit cases is suggested by Eq. (56), according to which α^2 is of order of the ratio of the frequency of elastic ion-atom collisions to the ionization frequency.

An unexpected result of the analysis is that a regime occurring in the limit case $\alpha \ll 1$ is similar in a number of aspects to the conventional diffusion regime that realizes in the case $\alpha \gg 1$. Characteristic features of the former regime (which has been called the pseudodiffusion regime) have been found and discussed.

ACKNOWLEDGMENTS

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APPENDIX A: DIFFUSION FORMULA FOR THE ION FLUX GENERATED IN THE IONIZATION LAYER IN A PARTIALLY IONIZED PLASMA

A diffusion theory of the ionization layer in a partially ionized plasma was developed in Ref. [9]. The results of Ref. [9] for the ion flux cannot be directly applied to a plasma close to full ionization since they involve the diffusion coefficient of ions in the neutral gas evaluated at the edge in the ionization layer, which tends to infinity when the plasma approaches full ionization. However, these results can be transformed to give a formula which does not give rise to such a problem,

$$J_i = C_7 \left[(1+\beta) k_i D_{\mathrm{ia}}^{(0)} n_{i\infty}^2 \frac{p}{kT_h} \right]^{1/2}, \tag{A1}$$

where $C_7 = C_7(\beta, \gamma)$ is a coefficient determined by Eq. (37) of Ref. [9] (here $\gamma = n_{i\infty}/n_{a\infty}$).

One can check that the coefficient $C_7(\beta,\gamma)$ tends as $\gamma \to \infty$ to a finite limit coinciding with $C_1(\beta)$. It follows that Eq. (A1) for a plasma close to full ionization conforms to Eq. (12). In order to give an idea of values of the ionization degree at which the model of a fully ionized plasma is applicable in the diffusion limit, a comparison of the coefficients C_7 and C_1 is shown in Fig. 7. [The line $\beta = 0$ in this figure has been calculated by means of the formula $C_7(0,\gamma) = \sqrt{(2\gamma+3)/(6\gamma+6)}$.] One can see that the model of a fully ionized plasma already provides a good approximation for $\gamma \gtrsim 5$. For example, in the range $\beta \geq 1$, the coefficient C_1 differs from C_7 for $\gamma \gtrsim 5$ by no more than 1%.

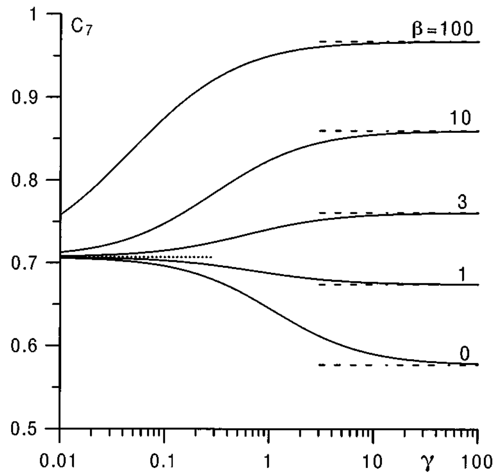

FIG. 7. Coefficients determining the ion flux generated in the ionization layer, calculated in the diffusion approximation. Full lines: $C_7 = C_7(\beta,\gamma)$, the model of a partially ionized plasma. Dashed lines: $C_7(\beta,\infty) = C_1(\beta)$, the limit of a fully ionized plasma. Dotted line: $C_7(\beta,0) = \sqrt{2}/2$, the limit of a weakly ionized plasma.

Thus Eq. (A1) may be used for a weakly to fully ionized plasma. It is convenient for practical applications to rewrite this equation with the help of Eq. (5) as

$$J_i = C_7 n_{i\infty}\left[\frac{3\pi(1+\beta)k_i C_{\mathrm{ia}}}{32\bar{Q}_{\mathrm{ia}}^{(1,1)}}\right]^{1/2}. \tag{A2}$$

APPENDIX B: BOHM CRITERION AS A BOUNDARY CONDITION FOR HYDRODYNAMIC EQUATIONS

A brief comment in this respect seems appropriate, given the recent controversy [37–40]. In the present paper, the Bohm criterion is invoked along the lines (Ref. [10], pp. 26–28) as a boundary condition for equations of a quasineutral plasma taking into account ion inertia (Sec. III), when the solution has a singular point. In Sec. II, in which the conventional diffusion description of the plasma is considered, the Bohm criterion is not used, and the boundary condition for diffusion equations at the edge of the space-charge sheath is that of zero charge-particle density, $n_i = 0$. No explicit limitation on the space-charge sheath being collisionless or collisional is imposed in either section.

This approach may seem to contradict the reasoning of Refs. [38,40], according to which the Bohm criterion is applicable at the edge of the space-charge sheath provided that the sheath is collisionless. It is easy to see, however, that this approach involves implicit limitations, which when taken into account are totally consistent with Refs. [38,40].

Consider first the case when ion inertia is essential. Assuming that the respective term [the one on the left-hand side of Eq. (13)] is not much smaller than the pressure-gradient term, one finds that the mean velocity of the ions is comparable to the thermal velocity. Assuming that the ion inertia term is not much smaller than the term accounting for the ion-atom friction, one finds that a local macroscopic length scale is comparable to the mean free ion-atom path, λ_{ia}. Since the plasma is assumed to be quasineutral on the length scale considered, this implies that λ_{ia} is much larger than the local Debye length. Hence the space-charge sheath is collisionless and the Bohm criterion applies at its edge, thus providing an appropriate boundary condition for the equations of a quasineutral plasma accounting for ion inertia.

Now consider the case when the bulk plasma is controlled by diffusion. A local macroscopic length scale is much larger than λ_{ia}, and the space-charge sheath may be collisionless as well as collisional. Treatments of a situation with a collisional sheath [40–43] indicate that the Bohm criterion is not appropriate, and the proper boundary condition at the sheath edge for diffusion equations describing a quasineutral plasma is $n_i = 0$. If the sheath is collisionless, the Bohm criterion applies at its edge. However, the diffusion equations cannot be extended right up to the sheath edge in this situation: as pointed out in Ref. [40], an intermediate (transitional) layer exists, separating the diffusion-controlled bulk plasma from the sheath. The thickness of this layer is of order of λ_{ia}, and it may be called a Knudsen layer. This layer appears in the present analysis as the inner zone considered in Sec. IV, and is described by solution (28). Thus a boundary condition for diffusion equations describing the bulk plasma should be established at the edge of the Knudsen layer rather than at the sheath edge, and the treatment of Sec. IV indicates that the proper condition is $n_i = 0$. One can conclude that it is an appropriate boundary condition for diffusion equations regardless of whether the sheath is collisional or collisionless.

[1] K. C. Hsu and E. Pfender, J. Appl. Phys. **54**, 3818 (1983).
[2] C. Delalondre and O. Simonin, Colloq. Phys. **51**, C5/199 (1990).
[3] R. Morrow and J. J. Lowke, J. Phys. D **26**, 634 (1993).
[4] X. Zhou and J. Heberlein, Plasma Sources Sci. Technol. **3**, 564 (1994).
[5] M. S. Benilov and A. Marotta, J. Phys. D **28**, 1869 (1995).
[6] A. Kaddani, C. Delalondre, O. Simonin, and H. Minoo, in *Proceedings of the 3rd European Congress on Thermal Plasma Processes (Aachen, 19-21 Sept. 1994)*, edited by D. Neuschütz (VDI Verlag, Düsseldorf, 1995), pp. 167–175.
[7] B. Rethfeld, J. Wendelstorf, T. Klein, and G. Simon, J. Phys. D **29**, 121 (1996).
[8] J. J. Lowke and J. C. Quartel, Aust. J. Phys. **50**, 539 (1997).
[9] M. S. Benilov, J. Phys. D **28**, 286 (1995).
[10] R. N. Franklin, *Plasma Phenomena in Gas Discharges* (Clarendon, Oxford, 1976).
[11] D. A. Frank-Kamenetskii, *Diffusion and Heat Transfer in Chemical Kinetics* (Plenum, New York, 1969).
[12] J. Haidar and A. J. D. Farmer, J. Phys. D **26**, 1224 (1993).
[13] S. Pellerin, K. Musio, B. Pokrzywka, and J. Chapelle, J. Phys. D **27**, 522 (1994).
[14] J. Haidar and A. J. D. Farmer, J. Phys. D **27**, 550 (1994).
[15] J. Haidar, J. Phys. D **28**, 2494 (1995).
[16] X. Zhou and J. Heberlein, Plasma Chem. Plasma Process. **16**, 229 (1996).

[17] B. Pokrzywka *et al.*, J. Phys. D **29**, 2644 (1996).
[18] M. S. Benilov, J. Phys. D **30**, 3353 (1997).
[19] S. Chapman and T. G. Cowling, *The Mathematical Theory of Non-Uniform Gases*, 3rd ed. (Cambridge University Press, Cambridge, 1970).
[20] J. O. Hirschfelder, C. F. Curtiss, and R. B. Bird, *Molecular Theory of Gases and Liquids* (Wiley, New York, 1964).
[21] J. H. Ferziger and H. G. Kaper, *Mathematical Theory of Transport Processes in Gases* (North-Holland, Amsterdam, 1972).
[22] K.-U. Riemann, J. Phys. D **24**, 493 (1991).
[23] M. van Dyke, *Perturbation Methods in Fluid Mechanics* (Academic, New York, 1964).
[24] J. D. Cole, *Perturbation Methods in Applied Mathematics* (Blaisdell, Whaltham, 1968).
[25] A. H. Nayfeh, *Perturbation Methods* (Wiley, New York, 1973).
[26] A. H. Nayfeh, *Introduction to Perturbation Techniques* (Wiley, New York, 1981).
[27] A. H. Nayfeh, *Problems in Perturbation* (Wiley, New York, 1985).
[28] L. M. Biberman, V. S. Vorob'ev, and I. T. Yakubov, *Kinetics of Nonequilibrium Low-Temperature Plasma* (Plenum, New York, 1987).
[29] J. A. M. van der Mullen, Phys. Rep. **191**, 109 (1990).
[30] L. M. Biberman, V. S. Vorob'ev, and I. T. Yakubov, Proc. IEEE **59**, 555 (1971).
[31] E. Hinnov and J. G. Hirschberg, Phys. Rev. **125**, 795 (1962).
[32] E. A. Mason and E. W. McDaniel, *Transport Properties of Ions in Gases* (Wiley, New York, 1988).
[33] A. A. Radzig and B. M. Smirnov, *Reference Data on Atoms, Molecules, and Ions* (Springer, Berlin, 1985).
[34] H. Tawara and T. Kato, At. Data Nucl. Data Tabl. **36**, 167 (1987).
[35] R. S. Devoto, Phys. Fluids **16**, 616 (1973).
[36] B. M. Palyukh and L. A. Sena, Zh. Éksp. Teor. Fiz. **20**, 481 (1950).
[37] H.-B. Valentini, Phys. Plasmas **3**, 1459 (1996).
[38] K. U. Riemann and P. Meyer, Phys. Plasmas **3**, 4751 (1996).
[39] H.-B. Valentini, Phys. Plasmas **3**, 4753 (1996).
[40] R. N. Franklin and J. Snell, J. Phys. D **30**, L45 (1997).
[41] C. H. Su and S. H. Lam, Phys. Fluids **6**, 1479 (1963).
[42] I. M. Cohen, Phys. Fluids **6**, 1492 (1963).
[43] J. L. Blank, Phys. Fluids **11**, 1686 (1968).