J. Math. Soc. Japan
Vol. 20, No. 4, 1968

Irreducibility of certain unitary representations

By Shoshichi KOBAYASHI*)

(Received Dec. 15, 1967)

§ 1. Introduction.

The purpose of this paper is to prove the following theorem.

THEOREM. *Let M be an n-dimensional complex manifold and E a holomorphic complex vector bundle over M. Let G be a group of (holomorphic) automorphisms of E such that*

(1) *G is fibre-transitive, i.e., the induced action of G on M is transitive;*

(2) *If H is the isotropy subgroup of G (acting on M) at a point of M, then the natural representation of H on the fibre of E is irreducible;*

(3) *E admits a hermitian inner product invariant by G.*

Let F be the complex Hilbert space of square integrable holomorphic n-forms on M with values in E. Then the natural unitary representation of G on F is irreducible (provided that F is not trivial).

In my earlier note, I proved the theorem above in the special case where E is a trivial line bundle. The basic idea of the proof is already in that note [**1**]. We are not making any structural assumption on G such as semi-simplicity. In fact, we need not assume that G is a Lie group.

Assumption (3) is superfluous if H is compact. Assumption (2) is unnecessary if E is a complex line bundle.

The theorem above implies that if E is a homogeneous complex line bundle over a symmetric bounded domain $M=G/H$, then the natural unitary representation of not only G but also of its Iwasawa subgroup on F is irreducible.

Let M be a compact homogeneous complex manifold and G the group of holomorphic transformations. Assume that a compact subgroup K of G is already transitive on M. (This is the case if $\pi_1(M)$ is finite by a well known result of Montgomery.) Since M is compact, F is the space of all holomorphic n-forms with values in E. The theorem implies that if the isotropy subgroup of K is irreducible on the fibre of E, then the natural representation of K on F is irreducible.

*) Partially supported by NSF Grant GP-5798.

§ 2. Construction of reproducing kernel.

Let M, E and G be as in Theorem. Let g be a hermitian inner product in E invariant by G.

Let U be a coordinate neighborhood with local coordinate system $z^1, \cdots, z^n$ in M. Taking U sufficiently small, we may assume that the vector bundle E over U is isomorphic to $U \times \boldsymbol{C}^r$. Let f and f' be holomorphic n-forms on M with values in E. Then f and f' may be written locally in the following form:

$$f = f_U dz^1 \wedge \cdots \wedge dz^n \quad \text{and} \quad f' = f'_U dz^1 \wedge \cdots \wedge dz^n ,$$

where f_U and f'_U are holomorphic cross sections of E over U. The local inner product $\langle f, \bar{f}' \rangle$ is the $2n$-form on M defined by

$$\langle f, \bar{f}' \rangle = i^{n^2} g(f_U, \bar{f}'_U) dz^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^1 \wedge \cdots \wedge d\bar{z}^n .$$

Then $\langle f, \bar{f}' \rangle$ is defined independent of the choice of $z^1, \cdots, z^n$. The global inner product $(f, \bar{f}')$ is defined by

$$(f, \bar{f}') = \int_M \langle f, \bar{f}' \rangle .$$

Then F is, by definition, the Hilbert space of all holomorphic n-forms f on M with values in E such that $(f, \bar{f}) < \infty$. We shall assume that F is non-trivial.

Let $\bar{M}$ denote the complex manifold whose complex structure is conjugate to that of M. If J defines the complex structure of M, then $-J$ defines the complex structure of $\bar{M}$. If $z^1, \cdots, z^n$ is a local coordinate system for M, then $\bar{z}^1, \cdots, \bar{z}^n$ is a local coordinate system for $\bar{M}$. Let $\bar{E}$ be the holomorphic complex vector bundle over $\bar{M}$ which is complex conjugate to E; the transition functions for $\bar{E}$ are complex conjugate to those of E. Let $p: M \times \bar{M} \to M$ and $\bar{p}: M \times \bar{M} \to \bar{M}$ be the natural projections; both p and $\bar{p}$ are holomorphic mappings. If we set

$$E\bar{E} = (p^{-1}E) \otimes (\bar{p}^{-1}\bar{E}) ,$$

then $E\bar{E}$ is a holomorphic complex vector bundle over $M \times \bar{M}$ with fibre $\boldsymbol{C}^r \otimes \boldsymbol{C}^r$. Let $d: M \to M \times \bar{M}$ be the diagonal map; d is not holomorphic. The induced vector bundle $d^{-1}(E\bar{E})$ over M is not holomorphic. We define a certain real subbundle of $d^{-1}(E\bar{E})$. Let $E \circ \bar{E}$ be the real vector bundle over M whose fibres are spanned by elements $v \otimes \bar{v}$, where $v \in E$.

We shall now define the kernel form $K(z, \bar{w})$ in the same manner as we define the kernel function of Bergman. Let $f_0, f_1, f_2, \cdots$ be a complete orthonormal basis for F. We set

$$K(z, \bar{w}) = i^{n^2} \sum_{j=0}^{\infty} f_j(z) \wedge \bar{f}_j(w) , \qquad (z, \bar{w}) \in M \times \bar{M} .$$

Then $K(z, \bar{w})$ is a holomorphic $2n$-form on $M \times \bar{M}$ with values in the holomorphic complex vector bundle $E\bar{E}$. It is easy to see that $K(z, \bar{w})$ does not depend on the choice of basis $f_0, f_1, f_2, \cdots$. By identifying M with the diagonal $d(M)$ of $M \times \bar{M}$ we can consider $K(z, \bar{z})$ as a real $2n$-form on M with values in the real vector bundle $E \circ \bar{E}$.

Every element of G induces a unitary transformation of F in a natural manner. Since $K(z, \bar{w})$ is defined independent of the orthonormal basis $f_0, f_1, f_2, \cdots$ chosen, $K(z, \bar{w})$ is invariant by G.

The most important fact we need is the reproducing property of $K(z, \bar{w})$:

$$(K(z, \bar{w}), f(w))_w = \int_{w \in M} \langle K(z, \bar{w}), f(w) \rangle = f(z) \qquad \text{for } f \in F .$$

We shall now explain the notations involved in the formula above. Let E_z and $\bar{E}_{\bar{w}}$ be the fibres of E and $\bar{E}$ at $z \in M$ and $\bar{w} \in \bar{M}$, respectively. The fibre of $E\bar{E}$ at $(z, \bar{w}) \in M \times \bar{M}$ is then given by $E_z \otimes \bar{E}_{\bar{w}}$. The inner product $g: E_w \times \bar{E}_{\bar{w}} \to \boldsymbol{C}$ induces a bilinear map

$$(E_z \otimes \bar{E}_{\bar{w}}) \times E_w \to E_z$$

in a natural manner. This bilinear map will be denoted by g_w. Let $z^1, \cdots, z^n$ and $w^1, \cdots, w^n$ be local coordinate systems in open sets U and V, respectively, of M. Let $\bar{w}^1, \cdots, \bar{w}^n$ be the conjugate local coordinate system in $\bar{V} \subset \bar{M}$. On V, $f \in F$ may be written as follows:

$$f(w) = f_V(w) dw^1 \wedge \cdots \wedge dw^n ,$$

where f_V is a holomorphic cross section of E over V. On $U \times \bar{V} \subset M \times \bar{M}$, $K(z, \bar{w})$ may be written as follows:

$$K(z, \bar{w}) = K_{U \times \bar{V}}(z, \bar{w}) dz^1 \wedge \cdots \wedge dz^n \wedge d\bar{w}^1 \wedge \cdots d\bar{w}^n ,$$

where $K_{U \times \bar{V}}$ is a holomorphic cross section of $E\bar{E}$ over $U \times \bar{V}$. Then, for a fixed $z \in M$, $\langle K(z, \bar{w}), f(w) \rangle$ is defined by

$$\begin{aligned} &\langle K(z, \bar{w}), f(w) \rangle \\ &\quad = g_w(f_V(w), K_{U \times \bar{V}}(z, \bar{w})) dw^1 \wedge \cdots \wedge dw^n \wedge d\bar{w}^1 \wedge \cdots \wedge d\bar{w}^n \wedge dz^1 \wedge \cdots \wedge dz^n . \end{aligned}$$

Integrating this with respect to w, we obtain the global inner product $(K(z, \bar{w}), f(w))_w$. The formula $(K(z, \bar{w}), f(w))_w = f(z)$ follows immediately from

$$K(z, \bar{w}) = i^{n^2} \sum_j f_j(z) \wedge \bar{f}_j(w) , \qquad (f_j, \bar{f}_k) = \delta_{jk} ,$$

$$f = \sum_j a_j f_j \qquad \text{where } a_j = (f, \bar{f}_j) .$$

§ 3. Proof of Theorem.

Let F' be a closed subspace of F invariant by G and let F'' be its orthogonal complement in F. Assuming that both F' and F'' are non-trivial, we shall obtain a contradiction. Let $f'_0, f'_1, f'_2, \cdots$ (resp. $f''_0, f''_1, f''_2, \cdots$) be a complete orthonormal basis for F' (resp. F''). Since $K(z, \bar{w})$ does not depend on the choice of orthonormal basis for F, we have

$$K(z, \bar{w}) = K'(z, \bar{w}) + K''(z, \bar{w}),$$

where

$$K'(z, \bar{w}) = i^{n^2} \sum_j f'_j(z) \wedge \bar{f}'_j(w), \qquad K''(z, \bar{w}) = i^{n^2} \sum_k f''_k(z) \wedge \bar{f}''_k(w).$$

Both K' and K'' are holomorphic $2n$-forms on $M \times \bar{M}$ with values in the vector bundle $E\bar{E}$. Since F' and F'' are invariant by G, K' and K'' are invariant by G. In the preceding section, we defined the real vector bundle $E \circ \bar{E}$ over M. The fibre of $E \circ \bar{E}$ at $z \in M$ is equal to the subspace of $E_z \otimes \bar{E}_z$ spanned by elements $v \otimes \bar{v}$, $v \in E_z$. In other words, the fibre of $E \circ \bar{E}$ at z consists of "hermitian elements" of $E_z \otimes \bar{E}_z$. It is clear that $K(z, \bar{z})$, $K'(z, \bar{z})$ and $K''(z, \bar{z})$ are all $2n$-forms on M with values in $E \circ \bar{E}$.

Let H be the isotropy subgroup of G at $z \in M$. We have a natural representation of H on E_z, which is assumed to be irreducible. This representation gives rise to a representation of H on the fibre of $E \circ \bar{E}$ at z. Since H is irreducible on E_z, any two elements of the fibre of $E \circ \bar{E}$ at z which are invariant by H must coincide up to a constant factor. (We have used the fact that if H is an irreducible linear group acting on a vector space V, an H-invariant hermitian inner product on V is unique up to a constant factor.) Since both $K'(z, \bar{z})$ and $K(z, \bar{z})$ are "hermitian elements" of $E_z \otimes \bar{E}_z$ and are invariant by H, we obtain

$$K'(z, \bar{z}) = c \cdot K(z, \bar{z}),$$

where c is a constant. Since K' and K are invariant by G, this constant c does not depend on $z \times M$. Since both $K'(z, \bar{w})$ and $K(z, \bar{w})$ are holomorphic on $M \times \bar{M}$, we have

$$K'(z, \bar{w}) = c \cdot K(z, \bar{w}) \qquad \text{for } (z, \bar{w}) \in M \times \bar{M}.$$

Since both F' and F'' are assumed to be non-trivial, the constant c lies between 0 and 1, i.e., $0 < c < 1$.

If we use the formulas

$$K'(z, \bar{w}) = i^{n^2} \sum_j f'_j(z) \wedge \bar{f}'_j(w)$$

and

$$(f'_j, f''_k) = 0,$$

then

$$(K'(z, \bar{w}), f''_k(w))_w = 0.$$

On the other hand, if we use the relation $K'(z, \bar{w}) = c \cdot K(z, \bar{w})$ and the reproducing property of $K(z, \bar{w})$, then we obtain

$$(K'(z, \bar{w}), f''_k(w))_w = c(K(z, \bar{w}), f''_k(w)) = c \cdot f''_k(z).$$

Hence, $c = 0$. This is a contradiction.

University of California, Berkeley

Bibliography

[1] S. Kobayashi, On automorphism groups of homogeneous complex manifolds, Proc. Amer. Math. Soc., 12 (1961), 359-361.