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Irreducibility of Discrete Series Representations for Semisimple Symmetric Spaces

David A.Vogan, Jr.*

§ 1. Introduction

Let G be a connected reductive Lie group and H a symmetric subgroup. This means that there is an involution (an automorphism of order two) σ of G with the following properties: σ is trivial on H, and H contains the identity component of the fixed point set of σ. The quotient space G/H is a typical *reductive symmetric space.* (We allow G to be reductive instead of only semisimple to facilitate inductive arguments). Such a homogeneous space carries a G-invariant measure, so there is a unitary representation of G on $L^2(G/H)$. The representations of G on irreducible subrepresentations of G on $L^2(G/H)$ are called the *discrete series representations of G on G/H.* Write $L^2(G/h)_d$ for the sum of all these discrete series.

Building on work of Flensted-Jensen, Oshima and Matsuki in [Oshima-Matsuki] (1984) have given a detailed description of all discrete series representations of G/H. There is a parameter set $\mathscr{P}$ (roughly the characters of a certain compact torus satisfying some regularity and evenness conditions). For each X in $\mathscr{P}$, they construct a unitary representation $A(X)$ and an embedding of $A(X)$ in $L^2(G/H)_d$. Then they prove that

$$L^2(G/H)_d = \bigoplus_{X \in \mathscr{P}} A(X) \tag{1.1}$$

(What Flensted-Jensen did was to construct $A(X)$ and the embedding for "most" X.)

Our concern in this paper is with a small technical question: whether the representation $A(X)$ are irreducible. The most interesting question of this nature is a weaker one: whether (1.1) diagonalizes the invariant differential operators on G/H. That much is clear from the work of Oshima and Matsuki. In fact their proof is so compelling that (1.1) is clearly the "right" decomposition in some sense. Nevertheless, the irre-

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ducibility question is traditional, and it deserves a decent burial at least.

Theorem 1.2. *In the setting* (1.1), *the representations* $A(X)$ *are irreducible or zero.*

Several remarks are in order. First, this result is proved in [Oshima-Matsuki] (1984) when X is generic. The proof in general will proceed by reduction to the generic case. Second, Oshima and Matsuki give a (complicated) condition for deciding whether $A(X)$ is zero. Third, one would like to know whether all the $A(X)$ are inequivalent. The proof of Theorem 1.2 will probably decide this question as well, but I have not done the necessary calculations. (Examples indicate that the representations are inequivalent.) Fourth, one would like to extend the theorem to "limits of discrete series." Here again the ideas work, but the calculations have not been done. Finally, this result does *not* apply to "derived functor modules" more general than those in (1.1). Some of the ideas work, but the calculations break down (and the analogous result is false). Perhaps the most important way in which the special hypotheses of (1.1) are used is in establishing the dichotomy of Corollary 6.11.

The proof of Theorem 1.2 is based on the "translation principle" of Jantzen and Zuckerman. It is known that any $A(X)$ can be obtained from some $A(Y)$ (with Y generic) by tensoring $A(Y)$ with an appropriate finite-dimensional representation of G (and then localizing at a maximal ideal of the center of the enveloping algebra). This is proved here in Proposition 4.7. We recall in section 3 (essentially from [Vogan] (1986b)) some general hypotheses under which such a construction preserves irreducibility. Sections 5 and 6 describe ways to check these hypotheses.

Here is an outline of the contents. Section 2 recalls (1.1) in a suitable from, and gives a formulation of Theorem 1.2 which does not refer directly to G/H (Theorem 2.10). Section 3 discusses translation principles in general, and the translation of irreducibility we need. The main result is Corollary 3.11. Section 4 considers the related notion of coherent families. Section 5 contains several deeper results about the translation principle; the one we will apply directly is Theorem 5.11. The proof of Theorem 2.10 is in section 6. The reader in a hurry should pass from section 2 directly to section 6, and refer backwards as necessary.

Harmonic analysis on symmetric spaces suffers from an (unavoidably) complicated notation. I have used very little of it. After section 2, even H will not appear explicitly again; this allows us to use that letter for Cartan subgroups.

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§ 2. Definition of $A(X)$

Recall that G is a reductive Lie group in Harish-Chandra's class, and that σ is an involution. Fix a Cartan involution θ of G commuting with σ. Write K for the fixed points of θ, a maximal compact subgroup of G. The Lie algebra of G is called $\mathfrak{g}_0$, and its complexification is called $\mathfrak{g}$. Analogous notation is used for other Lie groups. We will occasionally use a non-degenerate symmetric bilinear form $\langle\ ,\ \rangle$ on $\mathfrak{g}_0$, invariant under G, θ, and σ. We may assume that it is negative definite on $\mathfrak{k}_0$, and positive definite on the -1 eigenspace of θ.

Fix an abelian subalgebra $\mathfrak{t}_0$ of $\mathfrak{k}_0$. Define

$$L = \text{centralizer of } \mathfrak{t}_0 \text{ in } G. \tag{2.1}$$

Then L is a θ-stable reductive subgroup of G. Fix a θ-stable parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}$ with Levi subalgebra $\mathfrak{l}$. (Such a subalgebra may be constructed as the set of non-negative eigenspaces of a generic element of $i\mathfrak{t}_0$.) Write $\mathfrak{u}$ for the nil radical of $\mathfrak{q}$, so that

$$\mathfrak{q} = \mathfrak{l} + \mathfrak{u}. \tag{2.2}$$

It is convenient at this point to recall the *metaplectic double cover* $L^\sim$ of L ([Vogan] (1987), Definition 5.7). This is the double cover which arises in the orbit method, where L appears as the isotropy group of an elliptic coadjoint orbit. A more elementary definition is that $L^\sim$ is attached to the square root of the determinant character of L on $\mathfrak{u}$. That is, we have the following things: a short exact sequence

$$1 \longrightarrow \{1, \zeta\} \longrightarrow L^\sim \longrightarrow L \longrightarrow 1; \tag{2.3a}$$

and a character $\rho(\mathfrak{u})$ of $L^\sim$ with the properties

$$\rho(\mathfrak{u})(\zeta) = -1, \tag{2.3b}$$

and

$$\rho(\mathfrak{u})(x^\sim)^2 = \det\,(Ad(x)|_{\mathfrak{u}}). \tag{2.3c}$$

This last equation is to hold for any element of L, and any preimage $x^\sim$ of x in $L^\sim$. These properties characterize $L^\sim$ up to unique isomorphism. Finally, recall that a *metaplectic representation* of $L^\sim$ is one which is -1 on ζ.

Obviously there is a natural bijection between representations of L and metaplectic representations of $L^\sim$, implemented by tensoring with the character $\rho(\mathfrak{u})$. Because of this, it is possible to avoid all mention of $L^\sim$. The disadvantage is that the bijection shifts infinitesimal characters by $\rho(\mathfrak{u})$, and the hypotheses of the theorems are most simply formulated in terms of the infinitesimal characters on $L^\sim$.

Recall next the Zuckerman functors

$$\mathscr{R}^i = (\mathscr{R}_{\mathfrak{q},\mathfrak{l}})^i. \tag{2.4}$$

These are covariant functors from the category of metaplectic $(\mathfrak{l}, (L\cap K)^\sim)$-modules to the category of $(\mathfrak{g}, K)$-modules. They are defined in [Vogan] (1987), Definition 6.20; except for a twist by $\rho(\mathfrak{u})$, they are the functors considered in Chapter 6 of [Vogan] (1981). One advantage of the twist is that these functors preserve infinitesimal character (in the Harish-Chandra parametrization). Some other useful properties may be found in Theorem 6.8 of [Vogan] (1987), which is due to Zuckerman.

Definition 2.5. In the setting above, fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{l}$. Any irreducible metaplectic $(\mathfrak{l}, (L\cap K)^\sim)$-module X has an infinitesimal character. In Harish-Chandra's parametrization, this will correspond to a weight λ in $\mathfrak{h}^*$, defined up to the Weyl group $W(\mathfrak{l}, \mathfrak{h})$. We say that X is *good* (or in the *good range*) if for each root α of $\mathfrak{h}$ in $\mathfrak{u}$

$$\mathrm{Re}\,\langle \alpha, \lambda\rangle > 0. \tag{G}$$

It is *integrally good* if for each such root $\langle \alpha^\vee, \lambda\rangle$ is not a negative integer or zero. (Equivalently, we require the condition (G) for each integral root in $\mathfrak{u}$.) It is *weakly good* (respectively *weakly integrally good*) if the corresponding weak inequalities hold.

Suppose now that $[\mathfrak{l}, \mathfrak{l}]$ acts by zero on X. Write $\mathfrak{z}$ for the center of $\mathfrak{l}$. We say that X is *fair* (or in the *fair range*) if for each root α of $\mathfrak{h}$ in $\mathfrak{u}$,

$$\mathrm{Re}\,\langle \alpha, \lambda|_{\mathfrak{z}}\rangle > 0. \tag{F}$$

It is *integrally fair* if the condition (F) holds for each integral root in $\mathfrak{u}$. It is *weakly fair* (respectively *weakly integrally fair*) if the weak inequalities hold.

Assuming that $[\mathfrak{l}, \mathfrak{l}]$ acts by zero, we will show that good implies fair. The restriction of λ to $\mathfrak{h}\cap[\mathfrak{l}, \mathfrak{l}]$ must be $\rho_{\mathfrak{l}}$, half the sum of a set of positive roots for $\mathfrak{h}$ in $\mathfrak{l}$. Write w for the long element of $W(\mathfrak{l}, \mathfrak{h})$. Then $w\rho_{\mathfrak{l}}$ is $-\rho_{\mathfrak{l}}$. Consequently

$$\lambda|_{\mathfrak{a}}=\tfrac{1}{2}(\lambda+w\lambda).$$

Condition (F) may now be written

$$\mathrm{Re}\langle\alpha+w\alpha,\ \lambda\rangle>0.$$

Because w (like every element of $W(\mathfrak{l},\mathfrak{b})$) permutes the roots of $\mathfrak{b}$ in $\mathfrak{u}$, this condition follows from condition (G).

This definition of "fair" is certainly not the most general one possible. The simplest generalization is to replace the assumption that $[\mathfrak{l},\mathfrak{l}]$ acts by zero by the weaker one that some weakly unipotent primitive ideal in $U([\mathfrak{l},\mathfrak{l}])$ annihilates X (see [Vogan] (1987), Definition 12.10). Definition 2.5 is adequate for our present purposes, however.

The next theorem explains the importance of the conditions in Definition 2.5. It combines the results of Zuckerman already mentioned with Theorem 7.1 and Proposition 8.17 of [Vogan] (1984).

Theorem 2.6. *In the setting of* (2.1)–(2.4), *fix an irreducible metaplectic* $(\mathfrak{l}, (L\cap K)^{\sim})$*-module* X. *Write* S *for the dimension of the* -1 *eigenspace of* θ *on* $\mathfrak{u}$.

a) *Suppose* X *is integrally good* (*Definition* 2.5). *Then* $\mathscr{R}^S(X)$ *is an irreducible* $(\mathfrak{g}, K)$*-module.*

b) *Suppose* X *is weakly integrally good. Then* $\mathscr{R}^S(X)$ *is an irreducible* $(\mathfrak{g}, K)$*-module or zero, and* $\mathscr{R}^j(X)=0$ *for* $j\neq S$.

c) *Suppose* X *is weakly good and unitary. Then* $\mathscr{R}^S(X)$ *is unitary.*

For the remaining results, assume that $[\mathfrak{l},\mathfrak{l}]$ *acts by zero on* X.

d) *Suppose* X *is weakly integrally fair. Then* $\mathscr{R}^j(X)=0$ *for* $j\neq S$.

e) *Suppose* X *is weakly fair and unitary. Then* $\mathscr{R}^S(X)$ *is unitary.*

For the purpose of describing irreducible unitary representations, it is important to answer the following question:

(2.7) In the setting of Theorem 2.6(e), when is $\mathscr{R}^S(X)$ irreducible?

In the remainder of the section, we will explain how Theorem 1.2 amounts to an answer to (2.7) in a special case. In sections 3, 4, and 5 we will present techniques for studying (2.7) in general. Section 6 describes their application to the special case.

Fix a maximal abelian subalgebra $\mathfrak{t}_0$ contained in the -1 eigenspace of σ on $\mathfrak{k}_0$; that is, a Cartan subspace for the compact symmetric space $K/K\cap H$. Oshima and Matsuki show that the discrete series of G/H is empty unless $\mathfrak{t}_0$ is also maximal abelian in the -1 eigenspace of σ on $\mathfrak{g}_0$. Define L to be the centralizer of this special $\mathfrak{t}_0$, exactly as in (2.1).

Definition 2.8. The θ-stable Levi factor L is said to be of *symmetric type in G* if it arises as above; that is, as the centralizer of a compact Cartan subspace for G/H. Notice that in this case

$$\mathfrak{l}=\mathfrak{l}\cap\mathfrak{h}\oplus\mathfrak{t};$$

this is the decomposition of $\mathfrak{l}$ into the eigenspaces of σ.

Suppose L is of symmetric type, and $\mathfrak{q}$ is a θ-stable parabolic with Levi factor $\mathfrak{l}$. We define the set $\mathscr{P}(\mathfrak{q})$ of *discrete series parameters in the chamber g*, as follows. $\mathscr{P}(\mathfrak{q})$ consists of all irreducible $(\mathfrak{l}, (L\cap K)^{\sim})$-modules X with these properties:

i) $[\mathfrak{l}, \mathfrak{l}]$ acts trivially on X;

ii) X is in the fair range (Definition 2.5); and

iii) the group $(L\cap H)^{\sim}$ acts on X by the restriction of the character $\rho(\mathfrak{u})$ (cf. (2.3)).

The first condition implies that X is finite-dimensional; if G is connected, it implies that X is one-dimensional. The first and third conditions imply that X is unitary. The third condition implies that X is metaplectic.

If X is in $\mathscr{P}(\mathfrak{q})$, we define the *discrete series representation with parameter X by*

$$A(X)=\mathscr{R}^{S}(X)$$

(cf. Theorem 2.6). This is a unitary $(\mathfrak{g}, K)$-module.

Finally (still assuming L to be of symmetric type) we will define the full set $\mathscr{P}$ of discrete series parameters. To do this, we fix representatives $\mathfrak{q}_1, \cdots, \mathfrak{q}_m$ for the conjugacy classes (under the normalizer of $\mathfrak{t}_0$ in K) of θ-stable parabolics with Levi factor $\mathfrak{l}$, (A convenient way to do this is to fix a set of positive (restricted) roots of $\mathfrak{t}$ in $\mathfrak{k}$, and to consider only parabolics compatible with these positive roots. In any case they are parametrized by the quotient of "little Weyl groups" $W(\mathfrak{g}, \mathfrak{t})/W(\mathfrak{k}, \mathfrak{t})$.) Then

$$\mathscr{P}=\bigcup\mathscr{P}(\mathfrak{q}_i).$$

Here now is an explicit version of the decomposition (1.1).

Theorem 2.9 ([Oshima-Matsuki] (1984)). *Suppose G is a connected reductive group in Harish-Chandra's class. Let σ be an involution of G, and H a subgroup between the fixed point set and its identity component.*

a) *Suppose there is no compact Cartan subspace for the symmetric space G/H. Then $L^2(G/H)_d=0$.*

b) *Suppose there is a compact Cartan subspace $\mathfrak{t}_0$ for G/H. With the notation of Definition* 2.8,

$$L^2(G/H)_d = \bigoplus_{X \in \mathscr{P}} A(X).$$

Oshima and Matsuki do not use the Zuckerman functors to construct $A(X)$, so this theorem still requires a little proof. It is convenient to postpone the argument to the end of section 4, when an appropriate translation principle will be available to help.

Finally, here is a precise (and strengthened) version of Theorem 1.2.

Theorem 2.10. *Suppose G is a reductive group in Harish-Chandra's class. Let L be a θ-stable Levi factor of symmetric type* (*Definition* 2.8), *and $\mathfrak{t}_0$ the corresponding Cartan subspace. Let X be an irreducible metaplectic* $(\mathfrak{l}, (L\cap K)^{\sim})$-*module. Assume that*

a) $[\mathfrak{l}, \mathfrak{l}]$ *acts trivially on X; and*

b) *X is in the fair range* (*Definition* 2.5).

Then the $(\mathfrak{g}, K)$-*module* $\mathscr{R}^S(X)$ (*cf. Theorem* 2.6) *is irreducible or zero.*

The proof of this result will occupy the rest of the paper. In (b), it is likely that "fair" can be replaced by "weakly fair." This would increase the length of the case-by-case part of the proof somewhat, but examples indicate that no essential new problems arise. (This would say that "limits of discrete series" for G/H are irreducible.) The assumption that L be of symmetric type is essential; it is not enough even to impose this condition only on the complexification.

§ 3. The translation principle: generalities

The ideas in this section are for the most part not new, but it is difficult to give good references to original sources for the precise formulations we need. As a substitute for such references, here are some historical remarks.

The term "translation principle" refers to the idea of studying infinite dimensional representations of reductive Lie algebras by investigating their tensor products with the (rich, complicated, and well-understood) family of finite-dimensional representations of G. The idea seems to originate in the work of Bernstein, Gelfand, and Gelfand on Verma modules (but I would not wish to have to defend this claim). The idea was extended greatly by Jantzen, with whom the term originated; his work is summarized in [Jantzen] (1979). Schmid and Hecht used the closely related idea of coherent families in their work on Blattner's conjecture (cf. [Schmid] (1977)). Zuckerman made the connection with the translation principle, and proved some analogues for Harish-Chandra modules of Jantzen's results ([Zuckerman] (1977)). At the same time, [Borho-Jantzen] (1977) applied the translation principle to ideal theory in the enveloping algebra.

Since then the ideas have been refined and extended substantially. Two important sources of the extension are the Kazhdan-Lusztig conjecture and the Beilinson-Bernstein localization theory. Each of these had implications for the translation principle. It was then natural to seek direct proofs of the implications. Such proofs suggested reformulations of the basic definitions, and these in turn led to further new results. I will not try to trace these developments; in addition to those people already mentioned, A. Joseph played a central part.

We turn now to the translation functors themselves. Our goal is Corollary 3.11, which gives an abstract criterion for a translation functor to take irreducibles to irreducibles.

Definition 3.1. Fix a homomorphism ϕ from $\mathscr{Z}(\mathfrak{g})$ (the center of $U(\mathfrak{g})$) to $\boldsymbol{C}$, and write $\mathscr{I}_\phi$ for the associated maximal ideal in $\mathscr{Z}(\mathfrak{g})$. If M is any $\mathfrak{g}$-module, set

${}_\phi M=\{m \in M \mid$ for some positive n, $(\mathscr{I}_\phi)^n m=0\}$. The functor taking M to ${}_\phi M$ is called *projection on the infinitesimal character* ϕ. If ϕ is attached by the Harish-Chandra homomorphism to a weight λ in a Cartan subalgebra, we may write $\mathscr{I}_\lambda$ and ${}_\lambda M$.

We say that M is $\mathscr{Z}(\mathfrak{g})$*-finite* if M is annihilated by an ideal of finite codimension in $\mathscr{Z}(\mathfrak{g})$. In that case,

$$M=\sum_{\phi} {}_\phi M.$$

Suppose ϕ is a character of $\mathscr{Z}(\mathfrak{g})$, and F is a finite dimensional representation of $\mathfrak{g}$. The *elementary translation functor* attached to these data is the functor T defined by

$$TM={}_\phi(F\otimes M).$$

A *translation functor* is a sum of composites of elementary translation functors.

If F is a representation of the group G (and not just the Lie algebra), then translation functors will act on $(\mathfrak{g}, K)$-modules.

The functors ${}_\phi(\cdot)$ are exact on the category of $\mathscr{Z}(\mathfrak{g})$-finite modules; they amount to localization functors there. The first important fact in the theory is that translation functors preserve this category. This follows from the following more precise result.

Proposition 3.2 ([Kostant] (1975)). *Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, a weight λ in $\mathfrak{h}^*$, and a $\mathfrak{g}$-module M annihilated by $\mathscr{I}_\lambda$. Suppose F is a finite* *dimensional representation of* $\mathfrak{g}$, *with weights* $\mu_1, \cdots, \mu_d$ *(counted without multiplicity). Then* $F\otimes M$ *is annihilated by the product ideal*

$$\mathscr{I}_{\lambda+\mu_1}\cdots\mathscr{I}_{\lambda+\mu_d}.$$

In particular, the translation functors preserve the property of $\mathscr{Z}(\mathfrak{g})$*-finiteness.*

There is some evidence ([Vogan] (1979)) that this bound for the annihilator of $F\otimes M$ is best possible. It would be interesting to prove that.

Suppose now that M is a $\mathfrak{g}$-module, and (π, F) is a finite-dimensional representation of $\mathfrak{g}$. We want to analyze $F\otimes M$ as a $\mathfrak{g}$-module. To do that, let us recall the slightly subtle way in which the $\mathfrak{g}$-module structure arises. What obviously acts on $F\otimes M$ is the algebra

$$\operatorname{End}(F)\otimes U(\mathfrak{g}), \tag{3.3a}$$

by the formula

$$(E\otimes u)(f\otimes m)=Ef\otimes um. \tag{3.3b}$$

If X belongs to $\mathfrak{g}$, then we set

$$X(f\otimes m)=\pi(X)f\otimes m+f\otimes Xm.$$

Another way to phrase this is to consider the algebra homomorphism d from $U(\mathfrak{g})$ to $\operatorname{End}(F)\otimes U(\mathfrak{g})$, defined on elements of $\mathfrak{g}$ by

$$d(X)=\pi(X)\otimes 1+Id\otimes X. \tag{3.4}$$

Now the $\mathfrak{g}$-module structure arises from the simple action (3.3) and the complicated map (3.4).

To go further, we will introduce the hypothesis of $\mathscr{Z}(\mathfrak{g})$-finiteness, and work in a more general setting. Fix a connected complex reductive group G_C with Lie algebra $\mathfrak{g}$. Suppose we are given an (associative complex) algebra B, and the following additional structure: an algebra homomorphism

$$d\colon U(\mathfrak{g})\longrightarrow B; \tag{3.5a}$$

and an action (written Ad) of G on B by algebra automorphisms. The homomorphism d automatically makes B into a $U(g)$-bimodule. These structures are assumed to satisfy the following compatibility condition: Ad is locally finite, and its differential ad is related to the bimodule structure by

$$\operatorname{ad}(X)b=Xb-bX \qquad (X\in\mathfrak{g},\ b\in B). \tag{3.5b}$$

Finally, we assume that

(3.5c) B is a $U(\mathfrak{g})$-bimodule of finite length.

Before analyzing B, let us see how such algebras can arise in our setting.

Proposition 3.6 (cf. [Jantzen] (1983), Kapitel 6). *Let I be an ideal in $U(g)$ such that $I \cap \mathscr{Z}(\mathfrak{g})$ has finite codimension in $\mathscr{Z}(\mathfrak{g})$. Then the algebra $A = U(\mathfrak{g})/I$ satisfies the hypotheses of* (3.5).

Suppose in addition that F is a finite-dimensional representation of $\mathfrak{g}$, and that the adjoint action of $\mathfrak{g}$ on End (F) *exponentiates to G_C. Then the algebra $B = \mathrm{End}(F) \otimes A$ (with d defined in analogy with* (3.4) *and* Ad *in the natural way) satisfies the hypotheses of* (3.5).

It is a consequence of (c) that B is $(\mathscr{Z}(\mathfrak{g}) \times \mathscr{Z}(\mathfrak{g}))$-finite. The analogue of the decomposition in Definition 3.1 for B is

(3.7a) $$B = \sum_{\psi, \phi} {}_{\psi}B_{\phi}.$$

Here the sum runs over pairs of homomorphisms of $\mathscr{Z}(\mathfrak{g})$ into $\boldsymbol{C}$. The subscript on the left (respectively right) denotes the result of applying the functor of Definition 3.1 for the left (respectively right) action of $\mathfrak{g}$. Multiplication in the algebra is related to this decomposition by the rule

(3.7b) $${}_{\psi}B_{\phi} \cdot {}_{\chi}B_{\omega} \subset \delta_{\phi\chi} \cdot {}_{\psi}B_{\phi}.$$

Suppose now that N is (left) B-module. The left $\mathscr{Z}(\mathfrak{g})$-finiteness of B implies that N is $\mathscr{Z}(\mathfrak{g})$-finite, so

(3.7c) $$N = \sum_{\phi} {}_{\phi}N.$$

We have

(3.7d) $${}_{\psi}B_{\phi} \cdot {}_{\chi}N \subset \delta_{\phi\chi} \cdot {}_{\psi}N.$$

By elementary manipulations, we deduce

Proposition 3.8. *Let B be as in* (3.5), *and use the notation of* (3.7). *Suppose N is an irreducible B-module. Then each non-zero ${}_{\phi}N$ is an irreducible ${}_{\phi}B_{\phi}$-module.*

The next corollary uses the straightforward extension of the notion of translation functor to the case of bimodules.

Corollary 3.9. *Let A be an algebra as in* (3.5) *and M an A module. Fix data (ϕ, F) for an elementary translation functor T (Definition* 3.1) *and assume that the adjoint action of* $\mathfrak{g}$ *on* End (F) *exponentiates to G_C. Define*

$$B = \mathrm{End}\,(F) \otimes A$$

(*cf. Proposition* 3.6).

a) $TM = {}_\phi(F \otimes M)$.

b) *If M is an irreducible A-module, then TM is an irreducible ${}_\phi B_\phi$-module or zero.*

c) *Let $\mathscr{S}$ be the elementary translation functor for biomodules attached to the data* $((\phi, \phi), \mathrm{End}\,(F))$. *Then*

$${}_\phi B_\phi = \mathscr{S} A.$$

Parts (a) and (c) here are reformulations of definitions, and (b) is immediate from Proposition 3.8.

In the setting of the Corollary, the algebra ${}_\phi B_\phi$ comes equipped with a map

$$d\colon\ U(\mathfrak{g}) \longrightarrow {}_\phi B_\phi. \tag{3.10}$$

The Corollary says that irreducibility of translated modules is related to surjectivity of this map. Here is a result along those lines; it is in some sense the point of this section.

Corollary 3.11. *Let I be an ideal in $U(\mathfrak{g})$ meeting $\mathscr{Z}(\mathfrak{g})$ in an ideal of finite codimension. Let (ϕ, F) be data for an elementary translation functor T (Definition* 3.1). *Define the translation functor $\mathscr{T}$ for bimodules as in Corollary* 3.9. *Assume that the map*

$$d\colon\ U(g) \longrightarrow \mathscr{T}(U(\mathfrak{g})/I)$$

is surjective. Then T takes any irreducible $\mathfrak{g}$*-module annihilated by I to an irreducible* $\mathfrak{g}$*-module or zero.*

The surjectivity hypothesis in the Corollary is a statement about translation functors for certain bimodules. We will see how to approach it in section 5.

Because we are interested in irreducible $(\mathfrak{g}, K)$-modules (which need not be irreducible as $\mathfrak{g}$-modules), we need a slight refinement of Corollary 3.11.

Corollary 3.12. *In the setting of Corollary* 3.11, *assume in addition* *that (π, F) is a representation of G. Then T takes irreducible $(\mathfrak{g}, K)$-modules annihilated by I to irreducible $(\mathfrak{g}, K)$-modules.*

Proof. Write $A = U(\mathfrak{g})/I$, and $B = \mathrm{End}(F) \otimes A$. Let M be an irreducible $(\mathfrak{g}, K)$-module annihilated by I; write δ for the representation of K on M. Then the algebra of operators on M generated by A and the various $\delta(k)$ acts irreducibly. It follows that the algebra C generated by the action of $\mathrm{End}(F) \otimes A$ and the various $\mathrm{Id} \otimes \delta(k)$ acts irreducibly on $F \otimes M$. Since the operators $\pi(k)$ on F are invertible (and contained in End(F)), C is equal to the algebra generated by the action of B and the various $\pi(k) \otimes \delta(k)$. These latter elements are exactly those giving the action of K in the tensor product $(\mathfrak{g}, K)$-module structure. Consequently they preserve the decomposition of $F \otimes M$ by infinitesimal character. Therefore the algebra of operators on ${}_\phi(F \otimes M)$ generated by ${}_\phi B_\phi$ and the restrictions of the various $\pi(k) \otimes \delta(k)$ acts irreducibly. Since we are assuming that $U(\mathfrak{g})$ maps onto ${}_\phi B_\phi$, this is what we wanted to show. Q.E.D.

We will make use of a slight variant of these results as well.

Proposition 3.13 ([Vogan] (1986b), Proposition 6.5). *In the setting of Corollary* 3.11, *write J for the kernel of d. Assume that the annihilator in $U(\mathfrak{g})$ of*

$$\mathscr{T}(U(\mathfrak{g})/I)/d(U(\mathfrak{g}))$$

properly contains J. Suppose that M is an irreducible $\mathfrak{g}$-module with annihilator precisely equal to I. Then TM is an irreducible $U(\mathfrak{g})$-module with annihilator equal to J. The analogous assertion holds for $(\mathfrak{g}, K)$-modules.

The hypothesis says that the quotient has a large annihilator, and therefore that it is small; that is, that the map d is nearly surjective. Under this hypothesis, the proposition says that $U(\mathfrak{g})/J$ and $\mathscr{T}(U(\mathfrak{g})/I)$ have the same large irreducible modules.

§ 4. Coherent families

To make good use of the translation functors discussed in Section 3, we need a way to compute them effectively. This is provided by the conceptually more subtle (but technically less difficult) idea of coherent families. The version discussed here is taken from [Schmid] (1977), but related results may be found in older work (e.g. [Jantzen] (1974)).

Definition 4.1. Suppose G is a reductive group in Harish-Chandra's class, and H is a Cartan subgroup of G. A subgroup Λ of the group of one-dimensional characters of H is called *nice* (or (*G-nice*) if it has the following properties:

i) for each λ in Λ, there is a finite-dimensional irreducible representation F_λ of G of extremal weight λ; and

ii) the roots of H in $\mathfrak{g}$ belong to Λ.

A finite-dimensional representation of G is called *Λ-nice* if all its weights under H belong to Λ.

Since G may be disconnected and its Cartan subgroups may be non-abelian, some care is required. For example, the representation in (i) is not necessarily unique. A consequence of (ii) in the definition is that if an irreducible representation F of G has one weight in Λ, then it is Λ-nice. In particular, its extremal weight spaces are one-dimensional, and F is irreducible under the Lie algebra $\mathfrak{g}_0$ of G.

Notice that the lattice generated by the roots of H in $\mathfrak{g}$ is nice.

The next definition uses the notion of virtual representation. The definition is discussed more completely in [Vogan] (1981) (Definition 7.2.5).

Definition 4.2. Suppose Λ is a nice set of characters of a Cartan subgroup H (Definition 4.1). Fix an element ζ of $\mathfrak{h}^*$, and write $\zeta+\Lambda$ for the set of formal symbols $\zeta+\lambda$ (with λ in Λ). A *coherent family of* $(\mathfrak{g}, K)$-*modules on* $(H, \zeta+\Lambda)$ is a function Θ on the set $\zeta+\Lambda$ with values in the Grothendieck group of the category $(\mathfrak{g}, K)$-modules of finite length. It must satisfy the following properties (for any λ in Λ):

i) if F is any finite-dimensional Λ-nice representation of G, then

$$\Theta(\zeta+\lambda)\otimes F=\sum_{\mu\in\Delta(F,H)}\Theta(\zeta+\lambda+\mu); \quad \text{and}$$

ii) the virtual representation $\Theta(\zeta+\lambda)$ has infinitesimal character $\zeta+d\lambda$.

In (i), $\Delta(F, H)$ denotes the set of weights of H in F, counted with multiplicity.

There are two important sources of coherent families.

Example 4.3. Suppose L is a θ-stable real Levi factor for the parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}$. (We make no further assumptions on $\mathfrak{q}$. In this paper it will usually be θ-stable; another interesting case is that of the complexification of a real parabolic subalgebra.) Fix a Cartan subgroup H of L, and a G-nice group Λ of characters of H (Definition 4.1). Then Λ is automatically L-nice. Let $L^\sim$ denote the metaplectic cover of L (attached to the square root of the determinant character of L on $\mathfrak{q}/\mathfrak{l}$—see [Vogan] (1987), Definition 5.7). Suppose Θ_L is a coherent family of metaplectic $(\mathfrak{l}, (L\cap K)^{\sim})$-modules based on $\zeta+\Lambda$ (Definition 4.2). Define

$$\Theta_G(\zeta+\lambda)=[\sum(-1)^i(\mathscr{R}_{\mathfrak{q}})^i](\Theta_L(\zeta+\lambda).$$

Here the Zuckerman functors $\mathscr{R}^i$ are defined as in [Vogan] (1987), Definition 6.20—cf. [Vogan] (1981), Chapter 6. Because of the existence of long exact sequences for the Zuckerman functors, the term in square brackets is well defined on the level of virtual representations. It is not difficult to see that Θ_G is a coherent family of virtual $(\mathfrak{g}, K)$-modules (cf. [Vogan] (1981), Lemma 7.2.9).

The preceding example shows how to make coherent families on big groups out of coherent families on small groups. We also need a way to construct coherent families from nothing; this is provided by the next example.

Example 4.4. Suppose H is a maximally split Cartan subgroup of G, and Λ is a nice set of characters of H (Definition 4.1). Fix a set Δ^+ of positive roots for $\mathfrak{h}$ in $\mathfrak{g}$. Let F be an irreducible finite-dimensional representation of G, of highest weight ξ. (The irreducible representation $\mathfrak{h}$ of H may not be one-dimensional). The differential of ξ maps $\mathfrak{h}$ to scalar operators; by abuse of notation, we regard $d\xi$ as an element of $\mathfrak{h}^*$. Finally, write ρ for half the sum of the positive roots (regarded as an element of $\mathfrak{h}^*$). We define a coherent family Θ based on $(d\xi+\rho)+\Lambda$, as follows. Fix λ in Λ. If $d\xi+\rho+d\lambda$ is singular, then we define

$$\Theta(\xi+\rho+\lambda)=0.$$

Otherwise, there is a unique element w of $W(\mathfrak{g}, \mathfrak{h})$ with the property that $w(\xi+\rho+d\lambda)$ is dominant. Write τ_w for the unique character of H which is a sum of roots, such that the differential of τ_w is $\rho-w\rho$. Consider the irreducible representation

$$\xi\otimes\lambda\otimes\tau_w$$

of H. It turns out to be an extremal weight of a unique finite-dimensional irreducible representation F of G. (The assumption that H is maximally split is used only here, to guarantee the unicity of F.) We define

$$\Theta(\xi+\rho+\lambda)=\varepsilon(w)F.$$

Here ε is the sign character on W.

The Weyl character formula implies that Θ is a coherent family. (Condition (ii) in Definition 4.2 is easy to check. When G is connected, condition (i) amounts to a standard formula for decomposing tensor products ([Humphreys] (1972), p. 142). The general case is similar, relying on the Weyl character formula for disconnected groups.)

Using these two examples and Theorem 2.6, we deduce immediately

Proposition 4.5. *In the setting of* (2.2)–(2.3), *let H be a maximally split Cartan subgroup of L, and let Λ be any G-nice group of characters of H. Fix a weight ζ in $\mathfrak{h}^*$ that is the infinitesimal character of a metaplectic $(\mathfrak{l}, (L\cap K)^{\sim})$-module Y on which $[\mathfrak{l}, \mathfrak{l}]$ acts trivally. Then there is a coherent family Θ based on $(H, \zeta+\Lambda)$ with the following property. Fix an element ξ of Λ that extends to a character of L, and assume that $\boldsymbol{C}_\xi \otimes Y$ is weakly fair* (*Definition* 2.5). *Then*

$$\Theta(\zeta+\xi)=\mathscr{R}^S(\boldsymbol{C}_\xi\otimes Y).$$

Here is the basic result which shows how coherent families can be used to compute translation functors. To make sense of it, recall that translation functors are exact (on $\mathscr{Z}(\mathfrak{g})$-finite $\mathfrak{g}$-modules), and therefore act on virtual representations.

Proposition 4.6. *Suppose Θ is a coherent family of* $(\mathfrak{g}, K)$*-modules on* $(H, \zeta+\Lambda)$ (*Definition* 4.2). *Fix data (ϕ, F) for an elementary translation functor T* (*Definition* 3.1), *and assume that F is a Λ-nice representation of G* (*Definition* 4.1). *Fix a weight γ in $\mathfrak{h}^*$ corresponding to ϕ under the Harish-Chandra homomorphism. Then*

$$T\Theta(\zeta+\lambda)=\sum_{\substack{\mu\in\Delta(F,H)\\ \zeta+d\mu+d\lambda\in W(\mathfrak{g},\mathfrak{h})\gamma}}\Theta(\zeta+(\lambda+\mu)).$$

As an illustration of how calculations of this kind work, we will show how to translate some derived functor modules.

Proposition 4.7. *In the setting of Definition* 2.5, *assume that X is a metaplectic* $(\mathfrak{l}, (L\cap K)^{\sim})$*-module in the weakly fair range. Fix a maximally split Cartan subgroup H of L, and a representative λ in $\mathfrak{h}^*$ for the infinitesimal character of X. Let F be an irreducible representation of G having a unique $\mathfrak{q}$-invariant line $\boldsymbol{C}_\xi$; here ξ is a character of L. Write T for the translation functor attached to the infinitesimal character λ* (*for G*) *and the representation F^** (*Definition* 3.1). *Then*

$$T(\mathscr{R}^S(X\otimes\boldsymbol{C}_\xi))\cong\mathscr{R}^S(X).$$

The key to the proof is a simple lemma about roots and weights.

Lemma 4.8. *Suppose* $\mathfrak{q}=\mathfrak{l}+\mathfrak{u}$ *is a parabolic subalgebra of the reductive Lie algebra* $\mathfrak{g}$, *and* $\mathfrak{h}$ *is a Cartan subalgebra of* $\mathfrak{l}$. *Assume that* $\lambda_{\mathfrak{z}}$ *in* $\mathfrak{h}^*$ *is the differential of a one-dimensional representation* X *of* $\mathfrak{l}$, *and that*

$$\operatorname{Re}\langle\alpha,\lambda_{\mathfrak{z}}\rangle\geq 0$$

for each root α *of* $\mathfrak{h}$ *in* $\mathfrak{u}$. *Let* F *be a finite-dimensional irreducible representation of* $\mathfrak{g}$ *having a* $\mathfrak{q}$*-invariant line on which* $\mathfrak{h}$ *acts by* ξ. *Write* λ *for the infinitesimal character of the representation* X *of* $\mathfrak{l}$. *Let* Z *be an irreducible constituent of* $F^*|_{\mathfrak{l}}$, *of highest weight* μ. *Assume that*

$$\mu+(\lambda+\xi)\in W(\mathfrak{g},\mathfrak{h})\lambda.$$

Then μ *is* $-\xi$, *and* Z *is the lowest weight space of* F^*.

We will give the proof in a moment.

Proof of Proposition 4.7. We will verify the equality in question on the level of virtual representations; looking at the argument a little more carefully would show that an isomorphism is actually produced. Write $\mathscr{R}$ for

$$\sum(-1)^{S-i}\mathscr{R}^i,$$

a map from metaplectic virtual $(\mathfrak{l},(L\cap K)^{\sim})$-modules to virtual (g,K)-modules. Theorem 2.6 allows us to replace $\mathscr{R}^S$ in the statement of the Proposition by $\mathscr{R}$. By [Vogan] (1981), Lemma 7.2.9 (b),

$$F\otimes\mathscr{R}Y\cong\mathscr{R}(Y\otimes F|_{\mathfrak{l}}).$$

It follows that $F\otimes\mathscr{R}(X\otimes C_\xi)$ is the sum of all the $\mathscr{R}(X\otimes C_\xi\otimes Z)$, with Z a constituent of $F|_{\mathfrak{l}}$. Applying the functor ${}_\lambda(\cdot)$ amounts to considering only those Z satisfying the hypothesis of Lemma 4.8. By the lemma,

$$\begin{aligned}T\mathscr{R}(X\otimes C_\xi)&\cong\mathscr{R}(X\otimes C_\xi\otimes(C_\xi)^*)\\&\cong\mathscr{R}(X).\end{aligned}$$

Q.E.D.

Proof of Lemma 4.8. Because W preserves length, the main hypothesis of the lemma guarantees that $\mu+(\lambda+\xi)$ has the same length as λ. We will argue from the other conditions that $\mu+(\lambda+\xi)$ is at least as long as λ, with equality only if the desired conclusion holds.

Write $\mathfrak{h}_1$ for the intersection of $\mathfrak{h}$ with $[\mathfrak{l},\mathfrak{l}]$, and $\mathfrak{z}$ for the center of $\mathfrak{l}$; then

$$\mathfrak{h}=\mathfrak{z}+\mathfrak{h}_1, \tag{4.9}$$

an orthogonal direct sum. Because C_λ is one-dimensional, the restriction of λ to $\mathfrak{h}_1$ must be $\rho_{\mathfrak{l}}$, half the sum of some set of positive roots of $\mathfrak{h}$ in $\mathfrak{l}$. That is,

$$\lambda = \lambda_{\mathfrak{z}} + \rho_{\mathfrak{l}} \tag{4.10a}$$

(in accordance with (4.9)). Similarly, we write

$$\mu = \mu_{\mathfrak{z}} + \mu_{\mathfrak{l}}. \tag{4.10b}$$

Here $\mu_{\mathfrak{z}}$ must be a weight of $\mathfrak{z}$ on Z, and so on F. Consequently

$$\mu_{\mathfrak{z}} = -\xi + \sum n_\alpha \alpha|_{\mathfrak{z}}. \tag{4.11a}$$

The sum is over roots of $\mathfrak{h}$ in $\mathfrak{u}$, and the coefficients are non-negative integers. It follows that

$$(\mu + (\lambda + \xi))|_{\mathfrak{z}} = \lambda_{\mathfrak{z}} + \sum n_\alpha \alpha|_{\mathfrak{z}}. \tag{4.11b}$$

Using the dominance hypothesis on $\lambda_{\mathfrak{z}}$, we conclude that

$$|(\mu + (\lambda + \xi))_{\mathfrak{z}}| \geq |\lambda_{\mathfrak{z}}|. \tag{4.12a}$$

Equality holds if and only if all the n_α are zero; that is, if and only if Z is the lowest weight space of F^*.

On the other hand, the restriction of $\mu + (\lambda + \xi)$ to $\mathfrak{h}_1$ is $\mu_1 + \rho_{\mathfrak{l}}$. Since μ_1 is highest weight of a finite dimensional representation of $[\mathfrak{l}, \mathfrak{l}]$.

$$|\mu_1 + \rho_{\mathfrak{l}}| \geq |\rho_{\mathfrak{l}}|. \tag{4.12b}$$

Recall now that the hypothesis of the lemma guarantees that $\mu + (\lambda + \xi)$ and λ have the same length. In light of (4.10a) and (4.12), it follows that equality must hold in (4.12a). As explained after (4.12a), this implies the conclusion of the lemma. Q.E.D.

We conclude this section with a sketch of a proof of Theorem 2.9. Oshima and Matsuki prove a result like Theorem 2.9, but with $A(X)$ replaced by another $(\mathfrak{g}, K)$-module $B(X)$. $B(X)$ is defined (roughly) as the space of $\mathfrak{k}$-finite hyperfunction sections of a certain bundle (induced by X) on a space G^d/Q^d. Here G^d is another group with complexified Lie algebra $\mathfrak{g}$, and Q^d is a parabolic subgroup with complexified Lie algebra $\mathfrak{q}$. The sections are required to have support along a certain subvariety of G^d/Q^d. Because of recent work of Hecht, Milicic, Schmid, and Wolf, it is possible to find a natural isomorphism between $A(X)$ and $B(X)$; but an indirect argument is easier to sketch.

Choose a non-negative integer k so large that $X \otimes C_{2k\rho(\mu)}$ is in the good range (Definition 2.5). By Theorem 6.1 of [Schlichtkrull] (1983) (and its proof),

$$A(X \otimes C_{2k\rho(\mathfrak{u})}) \cong B(X \otimes C_{2k\rho(\mathfrak{u})}). \tag{4.13a}$$

(It is worth remarking that Schlichtkrull's argument apparently cannot be generalized to parameters not in the good range). Let F denote a finite-dimensional irreducible representation of G with a $\mathfrak{q}$-invariant line transforming by the character $2k\rho(\mathfrak{u})$ of L. (The product of the kth powers of the root vectors in $\mathfrak{u}$ generates such a representation in $S(\mathfrak{g})$.) Let T denote the elementary translation functor attached to F^* and the infinitesimal character of $A(X)$ (Definition 3.1). By Proposition 4.7,

$$TA(X \otimes C_{2k\rho(\mathfrak{u})}) \cong A(X). \tag{4.13b}$$

Since we have not defined $B(X)$ carefully, we cannot prove the corresponding assertion for $B(X)$ in detail. Here is a sketch, however. Write $C(X)$ for the full space of hyperfunction sections of the bundle on G^d/Q^d. Tensoring a space of sections with F^* is the same as tensoring the inducing bundle with F^*. Now a calculation analogous to the proof of Proposition 4.7 shows that

$$TC(X \otimes C_{2k\rho(\mathfrak{u})}) \cong C(X).$$

The isomorphisms involved are easy to write down, and one can see by inspection that they do not affect support. It follows that

$$TB(X \otimes C_{2k\rho(\mathfrak{u})}) \cong B(X). \tag{4.13c}$$

Now (4.13) implies that $A(X)$ is isomorphic to $B(X)$.

§ 5. The translation principle: theorems

In order to use Corollary 3.11, we need a detailed understanding of G_C-finite $U(\mathfrak{g})$-bimodules The main point, first systematically exploited in [Duflo] (1977), is that such bimodules are essentially Harish-Chandra modules for G_C. We will not recall the details of this idea; these may be found in [Jantzen] (1983), section 7.1 of [Vogan] (1981), or section 16 of [Vogan] (1986a). An important consequence is that the notion of parabolic induction may be applied to bimodules. Before describing it, we record a careful definition of the category of bimodules under consideration. (We omit a twist by a Chevalley automorphism that is often included to facilitate comparison of bimodules with actual Harish-Chandra modules).

Definition 5.1. Suppose G_C is a complex connected reductive Lie group with Lie algebra $\mathfrak{g}$. A *Harish-Chandra bimodule for G_C* is a $U(\mathfrak{g})$ bimodule B of finite length, endowed with a completely reducible locally finite holomorphic action (sometimes called Ad) of G_C. These two structures are related by the condition that the differential (sometimes called ad) of the G_C action should be given in terms of the bimodule structure as

$$\mathrm{ad}\,(X)b = Xb - bX.$$

Recall that examples of such bimodules are given by Proposition 3.6.

Definition 5.2. In the setting of Definition 5.1 suppose $\mathfrak{p}=\mathfrak{m}+\mathfrak{n}$ is a Levi decomposition of a parabolic subalgebra of $\mathfrak{g}$. Let C be a Harish-Chandra bimodule for M_C. The *induced bimodule* $\mathrm{Ind}_{\mathfrak{p}}(C)$ for G_C is defined as follows. Define a character $\rho_{\mathfrak{p}}$ of $\mathfrak{m}$ by

$$\rho_{\mathfrak{p}}(X) = \tfrac{1}{2}\,\mathrm{tr}\,(\mathrm{ad}\,(X)|_{\mathfrak{n}}) \qquad (X \in \mathfrak{m}).$$

Define a one-dimensional M_C bimodule $T_{\mathfrak{p}}$ by

$$\begin{aligned} Xt = tX = \rho_{\mathfrak{p}}(X)t \qquad & (X \in \mathfrak{m},\ t \in T_{\mathfrak{p}}) \\ \mathrm{Ad}\,(m)t = t \qquad & (m \in M_C,\ t \in T_{\mathfrak{p}}). \end{aligned}$$

Write $\mathfrak{p}^{\mathrm{op}}$ for the parabolic opposite to $\mathfrak{p}$. Make $C \otimes T_{\mathfrak{p}}$ into a $(\mathfrak{p}, \mathfrak{p}^{\mathrm{op}})$-bimodule by making $\mathfrak{n}$ act trivially on the left, and $\mathfrak{n}^{\mathrm{op}}$ trivially on the right. Define

$$J_{\mathfrak{p}}(C) = \mathrm{Hom}_{(\mathfrak{p},\mathfrak{p}^{\mathrm{op}})}(U(\mathfrak{g}) \otimes U(\mathfrak{g}),\ C \otimes T_{\mathfrak{p}}).$$

Here the Hom is defined using the left action of $\mathfrak{p}$ on the first $U(\mathfrak{g})$, and the right action of $\mathfrak{p}^{\mathrm{op}}$ on the second $U(\mathfrak{g})$. We make $J_{\mathfrak{p}}(C)$ into $U(\mathfrak{g})$-bimodule as follows: if u_1 and u_2 are in $U(\mathfrak{g})$, and j belongs to $J_{\mathfrak{p}}(C)$, then

$$(u_1 j u_2)(v_1 \otimes v_2) = j(v_1 u_1 \otimes u_2 v_2).$$

We can define an action ad of $\mathfrak{g}$ on $J_{\mathfrak{p}}$ by the formula in Definition 5.1. Finally, we define $\mathrm{Ind}_{\mathfrak{p}}(C)$ to be the subspace of $J_{\mathfrak{p}}(C)$ on which the action ad exponentiates to G_C. It is easy to show that, as a $\mathfrak{g}$-module under ad, $J_{\mathfrak{p}}(C)$ is isomorphic to

$$\mathrm{Hom}_{\mathfrak{m}}\,(U(\mathfrak{g}),\ C). \tag{5.3}$$

This formula shows that, as a representation of G_C, $\mathrm{Ind}_{\mathfrak{p}}(C)$ is just the representation holomorphically induced from M_C to G_C. It follows that $\mathrm{Ind}_{\mathfrak{p}}$ is an exact functor. Here are some additional standard properties.

Proposition 5.4 ([Vogan] (1981), Chapter 6). *In the setting of Definition* 5.2, $\mathrm{Ind}_{\mathfrak{p}}$ *is a covariant exact functor from Harish-Chandra bimodules for* M_C *to Harish-Chandra bimodules for* G_C. *Fix a Cartan subalgebra* $\mathfrak{h}$ *of* $\mathfrak{m}$, *and weights* λ *and* μ *in* $\mathfrak{h}^*$. *Let C be a Harish-Chandra bimodule for* M_C *with infinitesimal character* (λ, μ) (*in the Harish-Chandra darametrization*). *Then* $\mathrm{Ind}_{\mathfrak{p}}(C)$ *has infinitesimal character* (λ, μ).

All of our more serious theorems are based on the next result. All of the difficult ingredients in its proof were established by Kostant (for example in [Kostant] (1969).) Some were also found (in greater generality) by Zhelobenko ([Zhelobenko] (1974)). Understanding the importance of the formulation given here is another significant step, apparently first taken in [Duflo] (1977). Recall that a weight is called *dominant* if its inner product with a positive coroot is never a negative integer.

Theorem 5.5. *In the setting of Definition* 5.1, *suppose* $\mathfrak{b}=\mathfrak{h}+\mathfrak{n}$ *is a Borel subalgebra of* $\mathfrak{g}$. *Fix a dominant weight* λ *in* $\mathfrak{h}^*$, *and let* $\mathcal{J}_\lambda$ *be the corresponding maximal ideal in* $\mathscr{Z}(\mathfrak{g})$. *Write* I_λ (*or* $I_\lambda(\mathfrak{g})$) *for the ideal in* $U(\mathfrak{g})$ *generated by* $\mathcal{J}_\lambda$. *Definite a Harish-Chandra bimodule* R_λ (*or* $R_\lambda(\mathfrak{g})$) *by*

$$R_\lambda = U(\mathfrak{g})/I_\lambda$$

(*cf. Proposition* 3.6). *Then*

$$R_\lambda(\mathfrak{g}) = \mathrm{Ind}_{\mathfrak{b}}\,(R_\lambda(\mathfrak{h})).$$

More generally, suppose $\mathfrak{p}=\mathfrak{m}+\mathfrak{n}$ *is a parabolic subalgebra containing* $\mathfrak{b}$ (*and that* $\mathfrak{m}$ *contains* $\mathfrak{h}$). *Then*

$$R_\lambda(\mathfrak{g}) = \mathrm{Ind}_{\mathfrak{p}}\,(R_\lambda(\mathfrak{m})).$$

Notice that $R_\lambda(\mathfrak{h})$ is the one-dimensional bimodule on which H_C acts trivially, and

$$Xr = rX = \lambda(X)r \qquad (X \in \mathfrak{h},\ r \in R_\lambda(\mathfrak{h})).$$

The second claim in the theorem is an immediate consequence of the first and induction by stages.

Corollary 5.6. *Suppose* $\mathfrak{p}=\mathfrak{m}+\mathfrak{n}$ *is the Levi decomposition of a* *parabolic subalgebra of* $\mathfrak{g}$. *Let* $\mathfrak{h}$ *be a Cartan subalgebra of* $\mathfrak{m}$, *and fix a weight* λ *in* $\mathfrak{h}^*$. *Assume that for each root* α *of* $\mathfrak{h}$ *in* $\mathfrak{n}$, $\langle \alpha^\vee, \lambda \rangle$ *is not a negative integers. If* S *is any quotient of* $R_\lambda(\mathfrak{m})$ (*Theorem* 5.5), *then* $\mathrm{Ind}_\mathfrak{p}(S)$ *is a quotient of* $R_\lambda(\mathfrak{g})$.

This follows from the exactness of induction.

We turn now to some special results applicable to (2.7). The first is a version of a result that I learned from Bernstein and from Kashiwara (independently).

Proposition 5.7 ([Vogan] (1984), Proposition 16.8). *Suppose we are in the setting of Definition* 2.5 *and Proposition* 4.7; *use the notation there. Write* $\mathscr{T}$ *for the translation functor on bimodules associated to* (λ, λ) *and* $\mathrm{End}(F^*)$.

Write $R_\lambda(\mathfrak{l}; \mathfrak{l})$ *for the (one-dimensional) algebra of endomorphisms of* X *coming from the action of* $U(\mathfrak{l})$. *Define*

$$R_\lambda(\mathfrak{l}: \mathfrak{g}) = \mathrm{Ind}_\mathfrak{q}(R_\lambda(\mathfrak{l}: \mathfrak{l})),$$

a Harish-Chandra bimodule for G_C (*Definition* 5.2).

a) *If* λ *is in the weakly good range, then* $R_\lambda(\mathfrak{l}: \mathfrak{g})$ *is a quotient of* $U(\mathfrak{g})$. *The action of* $U(\mathfrak{g})$ *on* $\mathscr{R}^S(X)$ *factors through this quotient.*

b) *There is a natural isomorphism*

$$R_\lambda(\mathfrak{l}: \mathfrak{g}) \cong \mathscr{T} R_{\lambda + d\xi}(\mathfrak{l}: \mathfrak{g}).$$

c) $R_\lambda(\mathfrak{l}: \mathfrak{g})$ *has a natural algebra structure. This algebra acts on* $\mathscr{R}^S(X)$, *and the resulting* $(R_\lambda(\mathfrak{l}: \mathfrak{g}), K)$*-module is irreducible or zero.*

Proof. The first part of (a) follows from Corollary 5.6. For the second part (which requires more argument) we refer to [Vogan] (1984). Part (b) is analogous to Proposition 4.7, and may be proved in the same way. Part (c) follows from (a) (applied to an appropriate $\lambda + d\xi$), (b), and Corollary 3.9. Q.E.D.

It is not difficult to show that the ring $R_\lambda(\mathfrak{l}: \mathfrak{g})$ may be identified with the ring of global sections of a certain sheaf of twisted differential operators on G_C/Q_C. We will make no explicit use of this fact, however.

In the setting of the proposition, define

$$I_\lambda(\mathfrak{l}: \mathfrak{g}) = \mathrm{Ker}\,(d: U(\mathfrak{g}) \longrightarrow R_\lambda(\mathfrak{l}: \mathfrak{g})) \tag{5.8a}$$

$$A_\lambda(\mathfrak{l}: \mathfrak{g}) = U(\mathfrak{g})/I_\lambda(\mathfrak{l}: \mathfrak{g}). \tag{5.8b}$$

Corollary 5.9. *In the setting of Proposition* 5.7, *suppose that* $R_\lambda(\mathfrak{l}: \mathfrak{g})$

is a quotient of $R_\lambda(\mathfrak{g})$. *Then* $\mathscr{R}^S(X)$ *is an irreducible* $(\mathfrak{g}, K)$*-module or zero.*

The criterion of Corollary 5.9 is formulated in terms of induced representations of complex groups. This sounds like a reasonable way to approach Theorem 2.10. Unfortunately, the criterion is not always satisfied. What we will actually use is a reduction technique based directly on Proposition 5.7. That technique is sufficiently intricate that it is better to begin with an example.

Example 5.10. Suppose G_C is $\mathrm{Sp}(8, C)$, the group of linear transformation of C^{16} preserving the standard symplectic form. We can identify a fixed Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ with C^8, with basis linear functionals $\{e_i\}$. We choose a positive system so that the simple roots are

$$e_i - e_{i+1} \quad (i=1, 2, \cdots, 7), \quad 2e_8.$$

We consider the parabolic subalgebra $\mathfrak{q}=\mathfrak{l}+\mathfrak{u}$ with $\mathfrak{l}$ corresponding to the simple roots

$$e_1-e_2, \quad e_3-e_4, \quad e_5-e_6, \quad e_6-e_7, \quad e_7-e_8, \quad 2e_8.$$

We have

$$\mathfrak{l} \cong \mathfrak{gl}(2) \times \mathfrak{gl}(2) \times \mathfrak{sp}(4).$$

Consider the infinitesimal character

(a) $$\lambda=(2, 1, 1, 0, 4, 3, 2, 1).$$

Let F be the irreducible holomorphic representation of G of highest weight

(b) $$\xi=(3, 3, 0, 0, 0, 0, 0, 0).$$

Write T for the translation functor associated to (λ, F^*) (Definition 3.1) and $\mathscr{T}$ for the corresponding translation functor for bimodules (Corollary 3.9). Recall the notation (5.8)). We claim that

(c) if M is an irreducible $A_{\lambda+\xi}(\mathfrak{l}:\mathfrak{g})$ module, then TM is irreducible or zero.

The idea used in Corollary 5.9 will not work: it turns out that $A_{\lambda+\xi}(\mathfrak{l}:\mathfrak{g})$ is a proper subalgebra of $R_\lambda(\mathfrak{l}:\mathfrak{g})$. By Corollary 3.9, it is enough to show that

(c)′ $$\mathscr{T}A_{\lambda+\xi}(\mathfrak{l}:\mathfrak{g})=A_\lambda(\mathfrak{l}:\mathfrak{g}).$$

The difficulty is that we lack nice models of the algebras $A_{\lambda+\xi}(\mathfrak{l}:\mathfrak{g})$ and

$A_\lambda(\mathfrak{l}:\mathfrak{g})$.

To overcome it, we will find Harish-Chandra bimodules C_λ and $C_{\lambda+\xi}$ having the following properties:

(d1) $$\mathscr{S}C_{\lambda+\xi}\cong C_\lambda;$$

(d2) each of $C_{\lambda+\xi}$ and C_λ is generated by its unique G_C-fixed vector; and

(d3) there are bimodule maps

$$C_{\lambda+\xi}\longrightarrow R_{\lambda+\xi}(\mathfrak{l}:\mathfrak{g})$$
$$C_\lambda\longrightarrow R_\lambda(\mathfrak{l}:\mathfrak{g})$$

that are non-zero on the G_C-fixed vectors.

To construct these new bimodules, let $\mathfrak{p}$ and $\mathfrak{p}'$ be the standard parabolic subalgebras of $\mathfrak{g}$ with Levi factors

(e) $$\mathfrak{m}=\mathfrak{gl}(2)\times\mathfrak{gl}(4)\times\mathfrak{gl}(2).$$

(e)′ $$\mathfrak{m}'=\mathfrak{gl}(2)\times\mathfrak{gl}(2)\times\mathfrak{gl}(4).$$

It will be convenient to write

$$\mathrm{Ind}_\mathfrak{p}(\lambda)=\mathrm{Ind}_\mathfrak{p}(R_\lambda(\mathfrak{m}:\mathfrak{m})).$$

Put

(f) $$C_\lambda=\mathrm{Ind}_\mathfrak{p}((2,1),(4,3,2,1),(1,0)),$$

and similarly for $\lambda+\xi$. (The grouping of the coordinates is intended only as a reminder of what $\mathfrak{m}$ is.) Finally, put

(f)′ $$(C')_\lambda=\mathrm{Ind}_{\mathfrak{p}'}((2,1),(1,0),(4,3,2,1)),$$

and similarly for $\lambda+\xi$.

To verify (d1), we proceed as in the proof of Proposition 4.7. Let Z be any irreducible constituent of the restriction of F^* to $\mathfrak{m}$, and μ its highest weight. Assume that

(g) $$\mu+((5,4),(4,3,2,1),(1,0))=w((2,1),(4,3,2,1),(1,0)),$$

for some w in the Weyl group. We must deduce that μ is $(-3,-3,0,\cdots,0)$; (d1) will follow. The Weyl group acts by permutations and sign changes. The sum of the coordinates on the right in (g) is therefore at most 14. The sum of the coordinates of μ (as of any weight of F^*) is at least -6, so the sum on the left is at least $-6+20$. It follows that the sums on both sides of (g) are 14, that w is a permutation, and that the coordinates of μ are non-positive integers whose sum is -6. Now μ is a highest weight for $\mathfrak{m}$, so it coordinates decrease in each of the three blocks of 2, 4, and 2 coordinates. Using this information, it is an elementary exercise to deduce that μ is $(-3, -3, 0, \cdots, 0)$.

The claim in (d2) for $\lambda+\xi$ is immediate from Corollary 5.6. For λ, we use the parabolic with Levi factor $\mathfrak{gl}(6)\times\mathfrak{sp}(2)$ and Corollary 5.6. The result is that the cyclicity we want is equivalent to a corresponding assertion about GL(6). At that level the induced bimodule is actually irreducible; this is a consequence of Proposition 12.2 of [Vogan] (1986a). (This the main step of the argument; everything else we are doing is either standard or faily easy.)

The maps wanted in (d3) will be constructed in two steps: first from C_λ to $(C')_\lambda$, then from $(C')_\lambda$ to $R_\lambda(\mathfrak{l}: \mathfrak{g})$ (and similarly for $\lambda+\xi$). Since $\mathfrak{p}'$ is contained in $\mathfrak{q}$, the second is just induced from the natural quotient map

$$R_{(4,3,2,1)}(\mathfrak{gl}(4): \mathfrak{sp}(4)) \longrightarrow R_{(4,3,2,1)}(\mathfrak{sp}(4): \mathfrak{sp}(4)).$$

The first map is induced from the parabolic with Levi factor $\mathfrak{gl}(2)\times\mathfrak{gl}(6)$. On the GL(6) level, we need a map

$$\mathrm{Ind}_{\mathfrak{gl}(4)\times\mathfrak{gl}(2)}((4, 3, 2, 1), (1, 0)) \longrightarrow \mathrm{Ind}_{\mathfrak{gl}(2)\times\mathfrak{gl}(4)}((1, 0), (4, 3, 2, 1)).$$

Corollary 5.6 guarantees that the term on the left is a quotient of $U(\mathfrak{gl}(6))$. The map arises by the action of $U(\mathfrak{gl}(6))$ on the GL(6)-fixed vector on the left. We only need to see that the ideal

$$I_{((4,3,2,1),(1,0))}(\mathfrak{gl}(4)\times\mathfrak{gl}(2): \mathfrak{gl}(6))$$

annihilates the bimodule on the right. This is a consequence of the theory of τ-invariants (see [Duflo] (1977) or [Jantzen] (1983), for example). Alternatively, one can apply some fairly straightforward intertwining operator theory. In either case we omit the details.

This completes the verification of the properties (d1), (d2), and (d3). Using the properties, we verify (c)$'$. By (d2) and (d3), $A_{\lambda+\xi}$ is a quotient of $C_{\lambda+\xi}$. It follows that $\mathscr{S}A_{\lambda+\xi}$ is a quotient of $\mathscr{S}C_{\lambda+\xi}$. By (d1), this implies that $\mathscr{S}A_{\lambda+\xi}$ is a quotient of C_λ. Using (d2), we deduce that $\mathscr{S}A_{\lambda+\xi}$ is generated by its unique G_C-fixed vector. Now (c)$'$ is immediate.

Here is a general theorem that can be proved using exactly the same method.

Theorem 5.11. *Suppose* $\mathfrak{g}$ *is* $\mathfrak{sp}(n, C)$. *Let* $\mathfrak{h}$ *be the standard Cartan* *subalgebra, identified with* $\mathbf{C}^n$ *as usual. Fix a positive integer* p *less than or equal to* $n/2$; *write* $r=n-2p$. *Let* q *be the standard parabolic subalgebra with Levi factor*

$$\mathfrak{l}=(\mathfrak{gl}(2))^p \times \mathfrak{sp}(r).$$

Suppose λ *is the infinitesimal character of a one-dimensional representation of* $\mathfrak{l}$ *in the fair range*:

$$\lambda=(m+1, m, \lambda_3 \cdots, \lambda_{2p}, r, r-1, \cdots, 1).$$

Assume that m *is less than* r *but greater than* 0. *Let* F *be the finite dimensional representation of* $\mathfrak{g}$ *of highest weight* $\xi=(r-m, r-m, 0, \cdots, 0)$. *Write* T *for the translation functor associated to* F^* *and* λ. *Suppose* M *is an irreducible* $(\mathfrak{g}, K)$*-module annihilated by* $I_{\lambda+\xi}(\mathfrak{l}: \mathfrak{g})$. *Then* TM *is an irreducible* $(\mathfrak{g}, K)$*-module or zero.*

Because of the notational problems, we leave to the reader the task of generalizing the argument in Example 5.10. The appropriate choice for $\mathfrak{m}$ is

$$\mathfrak{gl}(2)\times \mathfrak{gl}(r-m+1)\times(\mathfrak{gl}(2))^{p-1}\times \mathfrak{sp}(m-1).$$

We will need analogous results for two other classes of pairs $(\mathfrak{g}, \mathfrak{l})$:

$$(\mathfrak{so}(2n+\varepsilon), \mathfrak{so}(2r+\varepsilon)) \qquad (\varepsilon=0 \text{ or } 1);$$

and

$$(\mathfrak{gl}(n), (\mathfrak{gl}(1))^p \times \mathfrak{gl}(n-2p)\times \mathfrak{gl}(1))^p).$$

Of these results even the formulation will be left to the reader.

§ 6. Proof of Theorem 2.10

We begin with a reduction technique.

Proposition 6.1. *In the setting of Definition* 2.5, *suppose that* X *has infinitesimal character* λ. *Assume that there is* θ*-stable parabolic* $\mathfrak{p}=\mathfrak{m}+\mathfrak{n}$ *containing* $\mathfrak{q}$ *with the following properties*:

i) *for every root* α *of* $\mathfrak{h}$ *in* $\mathfrak{n}$, $\langle\alpha^\vee, \lambda\rangle$ *is not a negative integer*; *and*

ii) *the* $(\mathfrak{m}, (M\cap K)^\sim)$*-module* $(\mathscr{R}_{\mathfrak{m}\cap\mathfrak{q}})^s(X)$ *is irreducible or zero.*

(*In* (ii), s *is the dimension of the* -1 *eigenspace of* θ *on* $\mathfrak{m}\cap\mathfrak{u}$.) *Then* $\mathscr{R}^S(X)$ *is irreducible or zero.*

Proof. We use induction by stages ([Vogan] (1981), Proposition 6.3.6) for $\mathfrak{q}\subset\mathfrak{p}$. The hypotheses guarantee that the module on $\mathfrak{m}$ is irreducible and in the weakly integrally good range. Theorem 2.6(a) gives the conclusion. Q.E.D.

The next result give a sufficient condition for $\mathscr{R}^s(X)$ to vanish (and therefore to be irreducible or zero).

Proposition 6.2. *In the setting of* (2.1)–(2.4), *suppose there is a θ-stable parabolic $\mathfrak{p}=\mathfrak{m}+\mathfrak{n}$ containing $\mathfrak{q}$, with the property that M/L is compact. Let X be any finite-dimensional metaplectic $(\mathfrak{l}, (L\cap K)^{\sim})$-module with infinitesimal character λ. If λ is not regular and integral for $\mathfrak{m}$, then $\mathscr{R}^i X$ is zero for all i.*

Proof. Again one uses induction by stages. The compactness assumption implies that the -1 eigenspace of θ on $\mathfrak{u}\cap\mathfrak{m}$ is zero. By the generalized Blattner formula ([Vogan] (1981), Theorem 6.3.12) it follows that $(\mathscr{R}_{\mathfrak{m}\cap\mathfrak{q}})^s(X)$ is finite-dimensional. An irreducible finite-dimensional module has regular integral infinitesimal character; so the second assumption makes the derived functor module zero at the level of $\mathfrak{m}$. Q.E.D.

The value of this proposition is as a complement to a much deeper result (Theorem 6.5) below. Some technical preliminaries are needed for its formulation.

Definition 6.3. Suppose $(\mathfrak{g}, \mathfrak{h})$ is a reductive symmetric pair with Cartan subspace $\mathfrak{t}$. Write $\mathfrak{l}$ for the centralizer of $\mathfrak{t}$ in $\mathfrak{g}$. For each restricted root α of $\mathfrak{t}$ in $\mathfrak{g}$, write $\mathfrak{g}_\alpha$ for the root subspace (a representation of $\mathfrak{l}$). An element Z of $\mathfrak{g}_\alpha$ is called *generic* if $[Z, \mathfrak{g}_{-\alpha}]$ contains a non-zero element of $\mathfrak{t}$.

Proposition 6.4 ([Kostant-Rallis] (1971)). *In the setting of Definition* 6.3, *let $\alpha_1, \cdots, \alpha_r$ be a set of simple restricted roots, and $Z_1, \cdots, Z_r$ a set of generic root vectors* (*Definition* 6.3). *Write $\mathfrak{q}=\mathfrak{l}+\mathfrak{u}$ for the corresponding parabolic subalgebra. Then*

$$Z=\sum Z_i$$

is a representative of the largest nilpotent conjugacy class meeting $\mathfrak{u}$.

Here is the reason we care about the condition on Z in the proposition. In what follows, we write

$$\mathfrak{s}=-1 \text{ eigenspace of } \theta$$

Theorem 6.5. *In the setting of Definition* 2.8, *assume that* $\mathfrak{u}\cap\mathfrak{s}$

contains a representative of the largest nilpotent conjugacy class meeting $\mathfrak{u}$. *Let* X *be an irreducible metaplectic* $(\mathfrak{l}, (L\cap K)^{\sim})$*-module in the weakly fair range (Definition* 2.5) *on which* $[\mathfrak{l}, \mathfrak{l}]$ *acts trivially. Then* $\mathscr{R}^S(X)$ *is irreducible.*

A more general result may be found in [Bien] (1986), Corollary 2.2.6. We will give a proof, however, using ideas from [Borho-Brylinski] (1982). We also use terminology from that paper.

Proposition 6.6 ([Hesselink] (1978)). *In the setting of Definition* 6.3, *let* Q_C *be a parabolic subgroup with Levi factor* L_C. *Then the moment map*

$$\pi\colon T^*(G_C/Q_C)\longrightarrow \mathfrak{g}^*$$

is birational.

The proof is very easy (Hesselink does much more); the main point is that the nilpotent conjugacy class in Proposition 6.4 is even and naturally attached to Q_C.

Theorem 6.7 ([Borho-Brylinski] (1982) (Corollary 5.12). *Suppose* $\mathfrak{q}=\mathfrak{l}+\mathfrak{u}$ *is a parabolic subalgebra of* $\mathfrak{g}$, *and* λ *is the infinitesimal character of a one-dimensional representation of* $\mathfrak{l}$ *in the weakly fair range. Suppose that the moment map* π *for* G_C/Q_C *is birational. With notation* (5.8), *the associated variety of the bimodule*

$$R_\lambda(\mathfrak{l}\colon \mathfrak{g})/A_\lambda(\mathfrak{l}\colon \mathfrak{g})$$

is strictly smaller than the image of π. *Consequently the annihilator of the bimodule properly contains* $I_\lambda(\mathfrak{l}\colon \mathfrak{g})$.

Proposition 6.8 ([Borho-Brylinski] (1985), Corollary 1.9 and Proposition 2.8). *In the setting of Definition* 2.5, *suppose that* X *is an irreducible* $(\mathfrak{l}, [L\cap K]^{\sim})$*-module in the good range, and that* $[\mathfrak{l}, \mathfrak{l}]$ *acts by zero on* X. *Then the associated variety of* $\mathscr{R}^S(X)$ *is* $K_C\cdot(\mathfrak{u}\cap\mathfrak{s})$. *Consequently the associated variety of the annihilator of* $\mathscr{R}^S(X)$ *is* $G_C\cdot(\mathfrak{u}\cap\mathfrak{s})$.

In particular, the annihilator of $\mathscr{R}^S(X)$ *is equal to* $I_\lambda(\mathfrak{l}\colon \mathfrak{g})$ *if and only if* $\mathfrak{u}\cap\mathfrak{s}$ *contains a representative of the largest* G_C*-conjugacy class meeting* $\mathfrak{u}$.

This is a consequence of the $\mathscr{D}$-module construction of $\mathscr{R}^S(X)$ (on G_C/Q_C). We omit the details.

Applying the translation principle (Proposition 4.7), we can immediately extend the last assertion.

Proposition 6.9. *In the setting of Definition* 2.5, *suppose* X *is in the* *weakly fair range. Then the annihilator of* $\mathscr{R}^S(X)$ *is equal to* $I_\lambda(\mathfrak{l}: g)$ (*notation* (5.8)) *if and only if* $\mathfrak{u} \cap \mathfrak{s}$ *contains a representative of the largest* G_C*-conjugacy class meeting* $\mathfrak{u}$.

Proof of Theorem 6.5. We apply Proposition 3.13. The necessary hypotheses on rings are proved in Theorem 6.7, and those on modules in Proposition 6.9. Q.E.D.

We now need some effective way to check the hypothesis of Theorem 6.5.

Lemma 6.10. *In the setting of Definition* 2.8, *suppose* α *is a restricted root of* $\mathfrak{t}$ *in* $\mathfrak{g}$. *If* $\mathfrak{g}_\alpha$ *is not compact, then the* -1 *eigenspace of* θ *on* $\mathfrak{g}_\alpha$ *contains a generic element of* $\mathfrak{g}_\alpha$.

Proof. Write t_α for the coroot of α in $\mathfrak{t}$, and σ for the involution of $\mathfrak{g}$ under consideration. Then we can define a non-degenerate symmetric bilinear form B on $\mathfrak{g}_\alpha$ by

$$B(Y, Z) = \langle t_\alpha, [Y, \sigma Z] \rangle.$$

Since σ and θ commute, the Cartan involution restricted to $\mathfrak{g}_\alpha$ is orthogonal for the form B. Consequently B is still non-degenerate on each eigenspace of θ. Since the -1 eigenspace is assumed non-zero, we can choose an element Z in it such that $B(Z, Z)$ is non-zero. Then $[Z, \sigma Z]$ is non-zero; obviously it belongs to the -1 eigenspace of σ on $\mathfrak{l}$, which is $\mathfrak{t}$. Therefore Z is generic. Q.E.D.

Corollary 6.11. *In the setting of Definition* 2.8, *there are two mutually exclusive possibilities: some of the restricted simple root spaces are compact, or they are all non-compact.*

In the first case, there is a θ*-stable parabolic* $\mathfrak{p} = \mathfrak{m} + \mathfrak{n}$ *properly containing* q *such that* $\mathfrak{m}$ *is* σ*-stable and* M/L *is compact. In this case Proposition* 6.2 *applies* (*and may say that* $\mathscr{R}^S(X)$ *is zero*).

In the second case, the largest nilpotent conjugacy class meeting $\mathfrak{u}$ *has a representative in the* -1 *eigenspace of* θ *on* $\mathfrak{u}$. *In this case Theorem* 6.5 *applies* (*and* $\mathscr{R}^S(X)$ *is irreducible*).

Proof of Theorem 2.10. We proceed by induction on the dimension of $\mathfrak{g}$. By standard arguments, we may assume $\mathfrak{g}_0$ is simple. Write λ for a representative of the infinitesimal character of X.

Suppose first that there is a proper parabolic subalgebra $\mathfrak{p} = \mathfrak{m} + \mathfrak{n}$ containing $\mathfrak{q}$, with the following properties:

(6.12a) $\mathfrak{m}$ is the centralizer of a subspace of $\mathfrak{t}$; and

(6.12b) for every root α of $\mathfrak{h}$ in $\mathfrak{n}$, $\langle \alpha^\vee, \lambda \rangle$ is not a negative integer.

The first hypothesis guarantees that M is preserved by the involution σ. We may therefore apply our inductive hypothesis to the (smaller) algebra $\mathfrak{m}$, and conclude that $(\mathscr{R}_{\mathfrak{m}\cap\mathfrak{q}})^s(X)$ is irreducible or zero. By Proposition 6.1, $\mathscr{R}^S(X)$ is irreducible or zero.

We may therefore assume that

(6.13a) no parabolic subalgebra satisfying (6.12) exists.

This has the effect of forcing λ to be fairly small. By Corollary 6.11, we may as well assume that there is a parabolic $\mathfrak{p}=\mathfrak{m}+\mathfrak{n}$ with the properties specified there. By Proposition 6.2, we may assume that

(6.13b) λ is regular and integral for some $\mathfrak{m}\supset\mathfrak{l}$ corresponding to a simple restricted root.

This has the effect of forcing λ to be a little big.

At this point, it is necessary to make a list of all the λ satisfying (6.13), case by case. It is a happy coincidence that there are none for the exceptional groups. Unfortunately I know of no way to prove this except by brute force; my notes for the case of E_8 cover about twenty pages. (A similar amount of effort might of course produce a computer program which would make the calculation.)

For the classical groups, however, there are some cases remaining. For example, suppose $(\mathfrak{g}, \mathfrak{l})$ is $(\mathfrak{sp}(n), (\mathfrak{gl}(2))^p\times\mathfrak{sp}(n-2p))$ (cf. Theorem 5.11)). Write r for $n-2p$. Because $[\mathfrak{l}, \mathfrak{l}]$ acts trivially on X, λ must be of the form

(6.14a) $$(m_1+1, m_1, m_2+1, m_2, \cdots, m_p, r, r-1, \cdots, 1).$$

The hypothesis that X be in the fair range amounts to

(6.14b) $$m_1+\tfrac{1}{2}>m_2+\tfrac{1}{2}>\cdots>m_p+\tfrac{1}{2}>0.$$

Hypothesis (6.13a) means that

(6.14c) m_1 is an integer less than r.

Hypothesis (6.13b) is

(6.14d) for some i, $m_i-m_{i+1}-1$ is a positive integer.

Taken together, (6.14b) and (6.14d) imply

$$m_1 > 1. \tag{6.14e}$$

We now have all the hypotheses for Theorem 5.11. Write C_ξ for the one-dimensional representation of L of which the highest weight is $r-m$ times the root e_1+e_2, and F for the representation of G of highest weight ξ. Then

$$\lambda+\xi=(r+1, r, \cdots);$$

all coordinates after the first are less than or equal to r. We can apply Proposition 6.2 to $X\otimes C_\xi$ and the parabolic with Levi factor $\mathfrak{gl}(2)\times \mathfrak{sp}(n-2)$. By inductive hypothesis, we get irreducibility (or vanishing) on $\mathfrak{m}$; so Proposition 6.2 says that $\mathscr{R}^S(X\otimes C_\xi)$ is irreducible or zero. Now apply the translation functor T associated to F^* and λ. This gives $\mathscr{R}^S(X)$ (Proposition 4.7), which is therefore irreducible or zero by Theorem 5.11.

Similar discussions may easily be given for the other classical groups; of course they invoke the variation on Theorem 5.11 mentioned at the end of section 5. This completes the proof of Theorem 2.10.

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Department of Mathematics
Massachusetts Institute of Technology
Cambridge, Massachusetts 02139
U. S. A.