Irreducibility of principal series representations for Hecke algebras of affine type

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To the memory of Professor Takuro Shintani

Introduction.

Principal series representations for Hecke algebras of affine type were first investigated by Matsumoto [10]. The purpose of this paper is to give a more detailed study (especially on irreducibility and cyclicity) of these representations.

We first recall the p-adic group case. Let G be a p-adic reductive group and let P be a minimal parabolic subgroup of G with Levi decomposition $P=M\cdot N$ (M is a Levi part of P; N is the unipotent radical of P). Put $X_{nr}(M)=\{\lambda\in\mathrm{Hom}(M, \boldsymbol{C}^{\times})\,|\,\lambda \text{ is trivial on } M_0\}$ where M_0 is the maximal compact subgroup of M. We denote by $\delta\in X_{nr}(M)$ the modulus character of P. For $\lambda\in X_{nr}(M)$, we define by

$$E_{\lambda}=\{f: G\to\boldsymbol{C}\ |\ \text{(i) } f \text{ is locally constant;}$$
$$\text{(ii) } f(gmn)=(\lambda\delta^{1/2})(m)f(g) \quad (g\in G,\ m\in M,\ n\in N)\},$$

the space of unramified principal series representation associated with λ. Then G acts on E_{λ} by left translations. This representation is studied by several authors (e.g. [4], [10]). Let B be an Iwahori subgroup of G and let $H(G, B)$ be the Hecke algebra of the pair (G, B). Naturally $H(G, B)$ acts on E_{λ}^{B}, the subspace of B-fixed vectors in E_{λ}.

Let $\tilde{W}$ be the (modified, in the sense of [3; 3.5]) affine Weyl group of G arising from Bruhat-Tits theory. We know that $\tilde{W}=W\cdot T$ (semi-direct product) with W the Weyl group of G and T the subgroup of translations. Then we can define $H(\tilde{W}, q)$, the Hecke algebra of $\tilde{W}$ associated with a quasi-multiplicative function q (for the definition, see below) by generators and relations. For each $\lambda\in\mathrm{Hom}(T, \boldsymbol{C}^{\times})$, Matsumoto defined a $H(\tilde{W}, q)$-module M_{λ}, which is called the principal series representation associated with λ. It is known that $H(G, B)\cong H(\tilde{W}, q)$ for the suitable choice of a quasi-multiplicative function q. Moreover, in the above situation, we have $E_{\lambda}^{B}\cong M_{\lambda}$ as Hecke algebra modules under the identification of T with M/M_0. Hence, the study of M_{λ} may be regarded as a natural generalization of the study of E_{λ}^{B} and, in view of [1] and [10], the study of E_{λ}.

In this paper, we give answers to the following questions in terms of a parameter λ:

(i) "When the $H(\widetilde{W}, q)$-module M_λ is irreducible?" (Theorem 2.2.)

(ii) "When a $H(W, q)$ (=the Hecke algebra of W)-*fixed* vector 1_λ of M_λ is a cyclic vector?" (Theorem 2.4.)

In the p-adic group case, as far as I know, the answer to (i) was first obtained by Muller [11] (for more general principal series representations, but only for split groups) by using Harish-Chandra's commuting algebra theorem [13]. (See for a similar proof [5].) But in contrast with the above mentioned proof, our method is more elementary even in the p-adic group case. In fact, our answer to (i) is easily deduced from the one to (ii) (Theorem 2.4). If we restrict ourself to the p-adic group case, Theorem 2.4 gives necessary and sufficient conditions for the cyclicity of a K-fixed vector in E_λ where K is a good maximal compact subgroup of G; this seems not to be known before. Incidentally, it should be noted that Theorem 2.4 and its proof have a close resemblance to a result of Kostant [8] (see also its reformulation due to Helgason [6]) in the real group case.

The organization of this paper is the following: In §1, we review Matsumoto's results [10] with slight generalizations and modifications for our later use. Then we prove the main results of this paper in §2. The content of §3 are devoted to applications of the main results. There we shall give answers to the followings:

(i) "When M_λ and L_λ (Lustzig's model for principal series representations) are isomorphic?" (Theorem 3.4.)

(ii) "When an eigenspace representation over G/K is irreducible?" (Theorem 3.8.)

This paper is a corrected and enlarged version of my manuscript "Cyclic vectors for unramified principal series representations of p-adic reductive groups" (unpublished). In writing this paper, I have profited much from the conversation with W. Casselman. (The notion $W_{(\lambda)}$ (2.2) was communicated by him in the p-adic split group case; cf. [12].) I would like to express my gratitude to W. Casselman.

§1. A review of Matsumoto's results.

In this section, we review Matsumoto's results. For the proofs we omit, see [10].

1.1. Let $\boldsymbol{G}$ be a connected complex reductive group and let $\boldsymbol{T}$ be a maximal torus of $\boldsymbol{G}$. Let $X(\boldsymbol{T})$ be the rational character group of $\boldsymbol{T}$. For a character $p \in X(\boldsymbol{T})$, we denote by t_p the translation by p (affine transformation) on

$X(\boldsymbol{T})\otimes\boldsymbol{R}$. Put $T=\{t_p \mid p\in X(\boldsymbol{T})\}$ $(\cong X(\boldsymbol{T}))$. Let W be the Weyl group of $(\boldsymbol{G}, \boldsymbol{T})$. Since W naturally acts on $\boldsymbol{T}$ (hence on $X(\boldsymbol{T})$), we can construct a semi-direct product $\widetilde{W}:=W\cdot T$ as a subgroup of the affine transformation group on $X(\boldsymbol{T})\otimes\boldsymbol{R}$. We call this $\widetilde{W}$ *the modified affine Weyl group of* $\boldsymbol{G}$.

Example 1.2. If $\boldsymbol{G}$ is a semisimple group of adjoint type, then $\widetilde{W}$ is an affine Weyl group in the usual sense (see below).

1.3. Let $\Delta\subset X(\boldsymbol{T})$ be the root system of $(\boldsymbol{G}, \boldsymbol{T})$. For simplicity, we assume henceforth that Δ is irreducible. We fix a system of simple roots Π, and a system of positive root Δ^+ of Δ. Put

$$T_{\text{root}}=\langle t_\alpha\ (\alpha\in\Delta)\rangle, \text{ the subgroup of } T \text{ generated by } t_\alpha\ (\alpha\in\Delta).$$

Then $W_{\text{aff}}:=W\cdot T_{\text{root}}$ is isomorphic to the affine Weyl group of the root system $\Delta^\vee$ (the set of coroots of Δ), with its generators (as a Coxeter group) S_{aff} given as follows: Put $S=\{w_\alpha\ (\alpha\in\Pi)\}$ where w_α denotes the reflection attached to α. We denote by $-\tilde{\alpha}^\vee$ the maximal root of $\Delta^\vee$. Then define S_{aff} to be $S\cup\{s_0\}$ where $s_0=w_{\tilde{\alpha}}t_{\tilde{\alpha}}$. Put $\Omega=\{w\in\widetilde{W} \mid w\cdot S_{\text{aff}}\cdot w^{-1}\subset S_{\text{aff}}\}$. It is known that $\widetilde{W}$ is the semi-direct product of Ω by W_{aff}, hence $\widetilde{W}=\Omega\cdot W_{\text{aff}}$. We extend the length function $l: W_{\text{aff}}\to\boldsymbol{N}\cup\{0\}$ to $\widetilde{W}$ by $l(xw)=l(w)$ $(x\in\Omega,\ w\in W_{\text{aff}})$.

DEFINITION 1.4. A function $q:\widetilde{W}\to\boldsymbol{C}^\times$ is said to be *quasi-multiplicative* if it satisfies

(i) $q(x)=1 \quad (x\in\Omega)$;

(ii) $q(ww')=q(w)q(w')$ if $l(ww')=l(w)+l(w')$ $(w, w'\in W)$.

1.5. Now we can define $H(\widetilde{W}, q)$, the Hecke algebra of $\widetilde{W}$ associated with a quasi-multiplicative function q. As a $\boldsymbol{C}$-vector space, $H(\widetilde{W}, q)=\bigoplus_{w\in W}\boldsymbol{C}\cdot e_w$ ($\{e_w\}$ is a basis of $H(\widetilde{W}, q)$). The multiplication law is given by

(i) $e_w\cdot e_{w'}=e_{ww'}$ if $l(ww')=l(w)+l(w')$ $(w, w'\in W)$;

(ii) $e_s^2=(q(s)-1)e_s+q(s)e_e$ $(s\in S_{\text{aff}})$.

It is known (see e.g. [1]) that $H(\widetilde{W}, q)\cong\boldsymbol{C}[\Omega]\otimes H(W_{\text{aff}}, q)$ where $\boldsymbol{C}[\Omega]$ is the group algebra of Ω. (Note that the multiplication on the right hand side is given by $(x\otimes e_w)(x'\otimes e_{w'})=xx'\otimes e_{(x')^{-1}(w)}\cdot e_{w'}$, $(x, x'\in\Omega,\ w, w'\in W_{\text{aff}})$.)

1.6. From now on, we shall fix quasi-multiplicative functions q and $q^{1/2}$ on $\widetilde{W}$ satisfying $(q^{1/2})^2=q$. Define the subsemigroup T^{++} of T by

$$T^{++}=\{t_p \mid p \text{ is a dominant character (relative to } \Pi) \text{ in } X(\boldsymbol{T})\}.$$

Then it is known ([10; (3.2.3)]) that $l(tt')=l(t)+l(t')$ for all $t, t'\in T^{++}$. Hence there exists a unique element $\delta^{1/2}\in \mathrm{Hom}\,(T, \boldsymbol{C}^{\times})$ which satisfies $\delta^{1/2}|_{T^{++}}=q^{1/2}|_{T^{++}}$ since T^{++} generates T. We note that the group $\mathrm{Hom}\,(T, \boldsymbol{C}^{\times})$ is canonically isomorphic (moreover W-equivariant) to $\boldsymbol{T}$ and we identify both groups hereafter.

DEFINITION 1.7. For $\lambda\in \boldsymbol{T}$, M_λ is the $\boldsymbol{C}$-vector space given by

$$M_\lambda=\{f: \widetilde{W}\to \boldsymbol{C}\,|\, f(wt)=(\lambda\delta^{1/2})(t)f(w)\quad (w\in W,\ t\in T)\}\,.$$

It is clear that $\dim M_\lambda=|W|$. For $s\in S_{\mathrm{aff}}$, we set

$$\alpha_s=\begin{cases}\beta & (s=w_\beta;\ \beta\in\Pi)\\ \tilde{\alpha} & (s\notin S).\end{cases}$$

THEOREM 1.8 ([10; (4.1.1)]). *Define the action* π_λ *of* $\{e_s\ (s\in S_{\mathrm{aff}})\}$ *and of* $\{e_x\ (x\in\Omega)\}$ *on* M_λ *by*

$$(\pi_\lambda(e_s)f)(wt)=\begin{cases}f(swt)+(q(s)-1)f(wt) & (w^{-1}(\alpha_s)>0)\\ q(s)f(swt) & (w^{-1}(\alpha_s)<0);\end{cases}$$

$$(\pi_\lambda(e_x)f)(wt)=f(xwt)\qquad (w\in W,\ t\in T;\ f\in M_\lambda).$$

Then π_λ *uniquely extends to the action of* $H(\widetilde{W}, q)$ *on* M_λ.

We call the $H(\widetilde{W}, q)$-module (via π_λ) M_λ *the principal series representation associated with* λ.

1.9. *Frobenius reciprocity.* We can embed commutative semigroup rings $A^+=\boldsymbol{C}[T^{++}]$ and $A^-=\boldsymbol{C}[(T^{++})^{-1}]$ in $H(\widetilde{W}, q)$ $(\sum a_t\cdot t\mapsto\sum a_t\cdot e_t)$ since $l(tt')=l(t)+l(t')$ for $t, t'\in T^{++}$ (or $(T^{++})^{-1}$). For $\mu\in\boldsymbol{T}$, let $\boldsymbol{C}_\mu$ be the 1-dimensional A^+ (or A^-)-module induced by μ.

PROPOSITION 1.10 (Frobenius reciprocity; 1st form). *Let* E *be a finite dimensional* $H(\widetilde{W}, q)$*-module. Then we have*

$$\mathrm{Hom}_{H(\widetilde{W},q)}(M_\lambda, E)\cong\mathrm{Hom}_{A^-}(\boldsymbol{C}_{(\lambda\delta^{1/2})^{-1}}, E)\,.$$

This is nothing but [10; (4.1.10)]. Since M_λ is the contragredient $H(\widetilde{W}, q)$-module of $M_{\lambda^{-1}}$ (see [10; (4.1.7)]), the following is easily deduced from (1.10).

PROPOSITION 1.11 (Frobenius reciprocity; 2nd form). *Let* E *be as above. Then we have*

$$\mathrm{Hom}_{H(\widetilde{W},q)}(E, M_\lambda)\cong\mathrm{Hom}_{A^+}(E, \boldsymbol{C}_{\lambda^{-1}\delta^{1/2}})\,.$$

COROLLARY 1.12 ([10; (4.2.4)]). *Let E be a finite dimensional irreducible $H(\widetilde{W}, q)$-module. Then there exists $\lambda\in \boldsymbol{T}$ (resp. $\lambda'\in \boldsymbol{T}$) such that E is isomorphic to a submodule of M_λ (resp. a quotient module of $M_{\lambda'}$).*

Moreover, it is known that λ and λ' as above are W-conjugate (see [10; (4.3.3)]).

1.13. *Intertwining operators.* First we define the $\boldsymbol{c}$-function. For $\alpha\in\Delta$, put

$$q_\alpha^{1/2}:=q^{1/2}(s) \text{ if } w_\alpha \text{ is } W\text{-conjugate to } s\in S;$$

$$q_\alpha'^{1/2}:=q^{1/2}(s') \text{ if } w_\alpha t_\alpha \text{ is } \widetilde{W}\text{-conjugate to } s'\in S_{\text{aff}}.$$

Then define $\boldsymbol{c}_\alpha$ ($\alpha\in\Delta$), the meromorphic function on $\boldsymbol{T}$ by

$$\boldsymbol{c}_\alpha(\lambda)=\frac{(1-(q_\alpha^{1/2}q_\alpha'^{1/2})^{-1}\lambda(t_\alpha)^{-1})(1+(q_\alpha'^{1/2}/q_\alpha^{1/2})\lambda(t_\alpha)^{-1})}{1-\lambda(t_\alpha)^{-2}} \quad (\lambda\in\boldsymbol{T}).$$

Since $\boldsymbol{C}[X(\boldsymbol{T})]$ (=the ring of regular functions on $\boldsymbol{T}$) is a unique factorization domain, we can choose relatively prime elements $\boldsymbol{e}_\alpha$ and $\boldsymbol{d}_\alpha$ of $\boldsymbol{C}[X(\boldsymbol{T})]$ satisfying $\boldsymbol{c}_\alpha=\boldsymbol{e}_\alpha/\boldsymbol{d}_\alpha$. For example, if $q_\alpha^{1/2}=q_\alpha'^{1/2}$ and $q_\alpha\neq1$, we may take $\boldsymbol{e}_\alpha(\lambda)=1-q_\alpha^{-1}\lambda(t_\alpha)^{-1}$ and $\boldsymbol{d}_\alpha(\lambda)=1-\lambda(t_\alpha)^{-1}$. We note here that the situation $q_\alpha^{1/2}\neq q_\alpha'^{1/2}$ can occur only when $\langle\alpha^\vee, X(\boldsymbol{T})\rangle=2\boldsymbol{Z}$ ([10; (3.1.10)]).

For $s\in S$ with $s=w_\beta$ ($\beta\in\Pi$), define the linear map $A(s, \lambda)\colon M_\lambda\to M_{s.\lambda}$ by

$$(A(s, \lambda)f)(w)=\begin{cases} q_\beta^{-1}f(ws)+(\boldsymbol{c}_\beta(\lambda)-q_\beta^{-1})f(w) & (w(\beta)>0)\\ f(ws)+(\boldsymbol{c}_\beta(\lambda)-1)f(w) & (w(\beta)<0)\end{cases} \qquad (1.13.1)$$

$$(f\in M_\lambda;\ w\in W)$$

for λ with $\boldsymbol{d}_\beta(\lambda)\neq0$. Then we have

THEOREM 1.14 ([10; (4.3.2)]). *If $\boldsymbol{d}_\beta(\lambda)\neq0$,*

(i) $A(s, \lambda)\in\mathrm{Hom}_{H(\widetilde{W},q)}(M_\lambda, M_{s.\lambda})$.

(ii) $A(s, s.\lambda)A(s, \lambda)=\boldsymbol{c}_\beta(\lambda)\boldsymbol{c}_\beta(\lambda^{-1})\cdot\mathrm{Id}$ *where* Id *is the identity map on* M_λ.

PROPOSITION 1.15 ([10; (4.3.4)]). *Let $w=s_1\cdots s_k$ ($s_i\in S$) be a reduced expression of $w\in W$. If $\boldsymbol{d}_\alpha(\lambda)\neq0$ ($\alpha>0$, $w(\alpha)<0$), the operator*

$$A(w, \lambda):=A(s_1, s_2\cdots s_k.\lambda)A(s_2, s_3\cdots s_k.\lambda)\cdots A(s_k, \lambda)$$

is well-defined (independent of the choice of the reduced expression) and belongs to $\mathrm{Hom}_{H(\widetilde{W},q)}(M_\lambda, M_{w.\lambda})$.

Especially, if λ is regular (i.e., W_λ, the stabilizer of λ in W is trivial), $A(w, \lambda)$ is well-defined.

Let f_λ be the element of M_λ which satisfies $f_\lambda(e)=1$ and $f_\lambda(w)=0$ $(w\in W,\ w\neq e)$.

LEMMA 1.16 ([10; (4.3.4)]). *If* $\lambda\in \boldsymbol{T}$ *is regular,*

$$(A(w,\lambda)f_\lambda)(w^{-1})=1$$

and

$$(A(w,\lambda)f_\lambda)(x)=0 \qquad (x\neq w^{-1},\ l(x)\geqq l(w)).$$

Now we can prove the following proposition which is implicit in [10; (4.3.5)]. (In the p-adic group case, see [4].)

PROPOSITION 1.17. *If* $\lambda\in \boldsymbol{T}$ *is regular,* $M_\lambda\cong \bigoplus_{w\in W} \boldsymbol{C}_{(w.\lambda)^{-1}\delta^{-1/2}}$ *as* A^-*-modules.*

PROOF. Since $0\neq A(w^{-1}, w.\lambda)$ by (1.16), $\mathrm{Hom}_{A^-}(\boldsymbol{C}_{(w.\lambda)^{-1}\delta^{-1/2}}, M_\lambda)\neq 0$ for all $w\in W$ by (1.10). But $w.\lambda$ $(w\in W)$ are all distinct. This implies the above decomposition.

REMARK 1.18. It is known that $f_\lambda\in M_\lambda$ is an A^--eigenvector. More precisely, $\pi_\lambda(e_t)f_\lambda=(\lambda\delta^{1/2})^{-1}(t)f_\lambda$ for $t\in(T^{++})^{-1}$ ([10; (4.1.9)]). Hence the above argument shows that the element $A(w^{-1}, w.\lambda)f_{w.\lambda}\in M_\lambda$ gives a natural basis of the $\boldsymbol{C}_{(w.\lambda)^{-1}\delta^{-1/2}}$-component of M_λ.

1.19. We define $1_\lambda\in M_\lambda$ by $1_\lambda(w)=1$ for all $w\in W$. We note that 1_λ is characterized by the properties $1_\lambda(e)=1$ and

$$\pi_\lambda(e_s)1_\lambda=q(s)1_\lambda \qquad (s\in S). \tag{1.19.1}$$

(This can be easily proved by induction on $l(w)$.) The decomposition of 1_λ with respect to the natural basis defined in (1.18) is given as follows:

PROPOSITION 1.20. *If* $\lambda\in \boldsymbol{T}$ *is regular, we have*

$$1_\lambda=\sum_{w\in W}\boldsymbol{c}_w(\lambda)A(w^{-1}, w.\lambda)f_{w.\lambda} \tag{1.20.1}$$

where $\boldsymbol{c}_w(\lambda)=\prod_{\alpha>0,\, w(\alpha)>0}\boldsymbol{c}_\alpha(\lambda)$.

PROOF. We note first that

$$A(s,\lambda)1_\lambda=\boldsymbol{c}_\beta(\lambda)1_{s.\lambda} \qquad \text{for} \quad s=w_\beta\in S \tag{1.20.2}$$

which is a direct consequence of the definition of $A(s,\lambda)$. Let w_0 be the longest element of W. By evaluating at w_0 and using (1.16), we see that the coefficient of $A(w_0^{-1}, w_0.\lambda)f_{w.\lambda}$ in the above decomposition is 1. In view of (1.14) (ii) and (1.20.2), we can easily prove (1.20.1) by downward induction on $l(w)$.

In the p-adic group case, see [4; 3.8].

1.21. For our later use, we remark here that there exists a non-zero homomorphism from M_λ to $M_{w.\lambda}$ for any $\lambda \in \boldsymbol{T}$, $w \in W$. In fact, let $w' \in W$ be the element of minimal length such that $w'.\lambda = w.\lambda$. Then it is easily seen that $A(w', \lambda)$ is well-defined (cf. [7; Addendum]).

§ 2. Irreducibility and cyclicity of principal series representations.

Matsumoto proved the following theorem.

THEOREM 2.1 ([10; (4.3.5)]). *Assume $\lambda \in \boldsymbol{T}$ is regular. Then M_λ is irreducible if and only if $\boldsymbol{c}(\lambda)\boldsymbol{c}(\lambda^{-1}) \neq 0$, where $\boldsymbol{c}(\lambda) = \prod_{\alpha>0} \boldsymbol{c}_\alpha(\lambda)$.*

In this section, we first give a refinement of the above result. For $\lambda \in \boldsymbol{T}$, put $W_\lambda = \{w \in W \mid w.\lambda = \lambda\}$. We define by $W_{(\lambda)}$ the normal subgroup of W_λ generated by $\{w_\alpha \mid \boldsymbol{d}_\alpha(\lambda) = 0 \ (\alpha \in \Delta^+)\}$. Now we can state one of the main results of this paper.

THEOREM 2.2. *For $\lambda \in \boldsymbol{T}$, M_λ is irreducible if and only if*

(i) $\boldsymbol{e}(\lambda)\boldsymbol{e}(\lambda^{-1}) \neq 0$; and
(ii) $W_\lambda = W_{(\lambda)}$,

where $\boldsymbol{e}(\lambda) = \prod_{\alpha>0} \boldsymbol{e}_\alpha(\lambda)$.

In the p-adic group case, as far as I know, this theorem seems to be essentially due to Muller [11]. But Muller's proof is not applicable to the general case (i. e. the Hecke algebra case). Incidentally our proof of Theorem 2.2 depends on the criterion for the existence of certain cyclic vectors (Theorem 2.4), which may be viewed as an analogue of the real group case (Kostant [8]; see also Helgason [6] for the reformulation in terms of $\boldsymbol{e}$-functions).

To prove (2.2), we first show

LEMMA 2.3. *For $\lambda \in \boldsymbol{T}$, M_λ is irreducible if and only if $1_{w.\lambda}$ is a cyclic vector of $M_{w.\lambda}$ (i.e., $\pi_{w.\lambda}(H(\widetilde{W}, q))1_{w.\lambda} = M_{w.\lambda}$) for each $w \in W$.*

PROOF. Note that M_λ is irreducible if and only if $M_{w.\lambda}$ is irreducible by (1.21). Hence we have only to prove the "if" part. Suppose $1_{w.\lambda}$ is cyclic for each $w \in W$. Let E be a non-trivial irreducible submodule of M_λ. By (1.12), there exists $w \in W$ and a surjective homomorphism $M_{w.\lambda} \to E$. As $1_{w.\lambda}$ is cyclic, its image in E is non-zero. Therefore (1.19.1) shows that $1_\lambda \in E$, which implies $E = M_\lambda$ by the cyclicity of 1_λ.

In view of $\boldsymbol{e}(\lambda)\boldsymbol{e}(\lambda^{-1}) = \prod_{\alpha \in \Delta} \boldsymbol{e}_\alpha(\lambda)$, (2.2) is a consequence of the above (2.3) and the following theorem.

THEOREM 2.4. *The vector* 1_λ *of* M_λ *is cyclic if and only if*

(i) $e(\lambda)\neq 0$; *and*
(ii) $W_\lambda = W_{(\lambda)}$.

This theorem gives an information about the composition series of M_λ and seems to be new even in the p-adic group case. (In [4], some weaker result is proved.) The proof of our Theorem 2.4 to be given below "resembles" that of the real group case (see [8]; especially the use of harmonic polynomials and the matrix P^γ).

In the rest of this section, we give a proof of Theorem 2.4.

LEMMA 2.5. *The vector* 1_λ *is cyclic if and only if* $\pi_\lambda(A^+)1_\lambda = M_\lambda$.

PROOF. The cyclicity of 1_λ is equivalent to the following condition:

(*) For any finite dimensional irreducible $H(\tilde{W}, q)$-module E and for any $\Phi\in \mathrm{Hom}_{H(\tilde{W},q)}(M_\lambda, E)$, $\Phi(1_\lambda)=0$ only if $\Phi=0$.

But we can embed E in M_μ for some $\mu\in \boldsymbol{T}$ by (1.11). Hence (1.10) shows that (*) is equivalent to

(**) For any $\mu\in\boldsymbol{T}$ and for any $\Psi\in \mathrm{Hom}_{A^+}(M_\lambda, \boldsymbol{C}_{\mu^{-1}\delta^{1/2}})$, $\Psi(1_\lambda)=0$ only if $\Psi=0$,

in other words, to the condition $\pi_\lambda(A^+)1_\lambda = M_\lambda$.

LEMMA 2.6. *The vector* 1_λ *is cyclic if and only if* $\pi_\lambda(A^-)1_\lambda = M_\lambda$.

PROOF. This follows from (2.5) since $\pi_\lambda(e_{w_0})1_\lambda = q(w_0)1_\lambda$ and $A^-\cdot e_{w_0} = e_{w_0}\cdot A^+$ ([10; (3.2.6)]).

REMARK 2.7. Let $W=\{w_1, \cdots, w_n\}$ $(n=|W|)$. Then (2.6) shows that 1_λ is cyclic if and only if $\det[\pi_\lambda(e_{t_i})1_\lambda(w_j)]_{1\le i,j\le n}\neq 0$ for some $t_1, \cdots, t_n \in (T^{++})^{-1}$.

2.8. In the case for regular λ, (1.20) and (2.6) show that 1_λ is cyclic if and only if $c(\lambda)\neq 0$. Hence it is convenient to consider the "generic module $M_\eta(R)$". Let R be the group ring $\boldsymbol{C}[T]$. (We identify R with $\boldsymbol{C}[X(\boldsymbol{T})]$, hence in particular, $\boldsymbol{e}_\alpha, \boldsymbol{d}_\alpha \in R$.) Let $Q(R)$ be the quotient field of R. For $\xi\in \mathrm{Hom}(T, R^\times)$, we can construct a $H(\tilde{W}, q)$-module (or $R\otimes H(\tilde{W}, q)$-module) $M_\xi(R)$ (a free R-module of rank $n=|W|$) as in (1.7). We note that an element $\lambda\in\boldsymbol{T}$ defines "specializations" $R\to\boldsymbol{C}$ and $M_\xi(R)\to M_{\lambda\circ\xi}$ ($H(\tilde{W}, q)$-homomorphism). Here $\lambda\circ\xi$ is the composite of ξ and λ. The intertwining operator $A(w, \xi)\in \mathrm{Hom}_{R\otimes H(\tilde{W},q)}(M_\xi(R), M_{w\xi}(R))\otimes Q(R)$ can be defined as in (1.11)-(1.14). In general, $A(w, \xi)$

$\in \mathrm{Hom}_{R\otimes H(\tilde{W},q)}(M_{\xi}(R), M_{w.\xi}(R))$. Now let η be the natural inclusion map $T \hookrightarrow \boldsymbol{C}[T]=R$. If $\lambda \in \boldsymbol{T}$ is regular, the image of $A(w, \eta)$ under the "specialization" by λ is $A(w, \lambda)$. Noting that we can define $1_{\xi}, f_{\xi} \in M_{\xi}(R)$ ($\xi \in \mathrm{Hom}(T, R^{\times})$) as in (1.15) and (1.19), we see that (1.20) implies the expression of 1_{η} by $\{A(w^{-1}, w.\eta)f_{w.\eta}\}$, a $Q(R)$-basis of $M_{\eta}(R)\otimes Q(R)$, i.e.,

PROPOSITION 2.9. *We have* $1_{\eta}=\sum_{w\in W}\boldsymbol{c}_w A(w^{-1}, w.\eta)f_{w.\eta}$. *Here* $\boldsymbol{c}_w=\prod_{\alpha>0, w(\alpha)>0}\boldsymbol{c}_{\alpha}\in Q(R)$.

2.10. Let $\{f_{\eta,w}\}_{w\in W}$ be a R-basis of $M_{\eta}(R)$ defined by $f_{\eta,w}(w')=\delta_{w,w'}$ (Kronecker's delta). In particular, $f_{\eta,e}=f_{\eta}$. Note that

$$\pi_{\eta}(e_t)1_{\eta}=\sum_{w\in W}(\pi_{\eta}(e_t)1_{\eta})(w)f_{\eta,w}\,.$$

Let J be the ideal of R generated by $\det[(\pi_{\eta}(e_{t_i})1_{\eta})(w_j)]$ ($1\leqq i, j\leqq n$; $t_1, \cdots, t_n \in (T^{++})^{-1}$). Put $V(J)=\{\mu\in \boldsymbol{T}\,|\,\mu$ is a common zero of $J\}$. Then, by (2.7), 1_{λ} is cyclic if and only if $\lambda\notin V(J)$. In view of (1.16), the transition matrix of base change of $M_{\eta}(R)\otimes Q(R)$, from $\{f_{\eta,w}\}_{w\in W}$ to $\{A(w^{-1}, w.\eta)f_{w.\eta}\}_{w\in W}$, is unitriangular for a suitable ordering. Therefore we have

$$\begin{aligned}\det[(\pi_{\eta}(e_{t_i})1_{\eta})(w_j)]&=\det[\boldsymbol{c}_{w_j}\cdot((w_j.\eta)^{-1}\delta^{-1/2})(t_i)]\\&=\prod_{w\in W}\boldsymbol{c}_w\prod_{i=1}^{n}\delta^{-1/2}(t_i)\det[w_j(t_i)^{-1}]\end{aligned}$$

and $J=\boldsymbol{c}^{n/2}\cdot\langle\det[w_i(t_j)]\,|\,t_1, \cdots, t_n\in T^{++}\rangle_{R\text{-ideal}}$ since $\prod_{w\in W}\boldsymbol{c}_w=\prod_{\alpha>0}\boldsymbol{c}_{\alpha}^{n/2}=\boldsymbol{c}^{n/2}$. Put $I=\langle\det[w_i(t_j)]\,|\,t_1, \cdots, t_n\in T^{++}\rangle_{R\text{-ideal}}$. We note that $I=\langle\det[w_i(t_j)]\,|\,t_1, \cdots, t_n\in T\rangle_{R\text{-ideal}}=\langle\det[w_i(\phi_j)]\,|\,\phi_1, \cdots, \phi_n\in R\rangle_{R\text{-ideal}}$. Let $\boldsymbol{f}_{\alpha}$ ($\alpha\in\Delta^+$) be the element of R defined by

$$\boldsymbol{f}_{\alpha}=\begin{cases}1-t_{\alpha}^{-1} & \text{if } \langle\alpha^{\vee}, X(\boldsymbol{T})\rangle=\boldsymbol{Z}\\ 1-t_{\alpha}^{-2} & \text{if } \langle\alpha^{\vee}, X(\boldsymbol{T})\rangle=2\boldsymbol{Z}\,.\end{cases}$$

By the definition, $\boldsymbol{d}_{\alpha}$ divides $\boldsymbol{f}_{\alpha}$ (see (1.13)).

LEMMA 2.11. *Any element of I is divisible by* $\boldsymbol{f}^{n/2}$ ($\boldsymbol{f}:=\prod_{\alpha>0}\boldsymbol{f}_{\alpha}$).

PROOF. Since $\boldsymbol{f}_{\alpha}$ ($\alpha\in\Delta^+$) are relatively prime (see [2]), it is sufficient to show that $\det[w_i(t_j)]$ is divisible by $\boldsymbol{f}_{\alpha}^{n/2}$ ($\alpha\in\Delta^+$; $t_1, \cdots, t_n\in T$). But, for $p\in X(\boldsymbol{T})$ and $w\in W$, $w(t_p)-w_{\alpha}w(t_p)=w(t_p)(1-t_{\alpha}^{-\langle\alpha^{\vee}, w(p)\rangle})$ is divisible by $\boldsymbol{f}_{\alpha}$. Therefore it can be easily seen that $\det[w_i(t_j)]$ is divisible by $\boldsymbol{f}_{\alpha}^{n/2}$ since $n/2=|\langle w_{\alpha}\rangle\backslash W|$.

Put $F=\{\phi/\boldsymbol{f}^{n/2}\,|\,\phi\in I\}$. Then we have $J=\boldsymbol{e}^{n/2}\cdot(\boldsymbol{f}/\boldsymbol{d})^{n/2}\cdot F$ ($\boldsymbol{e}=\prod_{\alpha>0}\boldsymbol{e}_{\alpha}$, $\boldsymbol{d}=\prod_{\alpha>0}\boldsymbol{d}_{\alpha}$). For $\lambda\in\boldsymbol{T}$, let $W_{[\lambda]}$ be the normal subgroup of W_{λ} generated by $\{w_{\alpha}\,|\,\boldsymbol{f}_{\alpha}(\lambda)=0\}$. (This is the maximal subgroup of W_{λ} generated by reflections.) Now we can state the key lemma.

LEMMA 2.12.[1] $V(F)=\{\lambda\in \boldsymbol{T} \mid W_{\lambda}\neq W_{[\lambda]}\}$.

It can be easily seen that Theorem 2.4 is an immediate consequence of Lemma 2.12. In fact, we have $W_{\lambda}\supset W_{[\lambda]}\supset W_{(\lambda)}$ and

$$W_{\lambda}\neq W_{[\lambda]}\Longleftrightarrow \lambda\in V(F) \qquad \text{(by (2.12))};$$
$$W_{[\lambda]}\neq W_{(\lambda)}\Longleftrightarrow \lambda\in V(\boldsymbol{f}/\boldsymbol{d}).$$

Thus the expression $J=\boldsymbol{e}^{n/2}\cdot(\boldsymbol{f}/\boldsymbol{d})^{n/2}\cdot F$ implies the theorem.

2.13. PROOF OF LEMMA 2.12.

Step. 1. We first show that 1 (the identity element of $\boldsymbol{T}$) is not contained in $V(F)$. Let $\mathfrak{t}$ be the Lie algebra of $\boldsymbol{T}$. Then for $p\in X(\boldsymbol{T})$, $(\exp(sZ))(t_p)=\exp(sdp(Z))$ $(s\in \boldsymbol{C};\ Z\in\mathfrak{t})$. Here $dp\in\mathfrak{t}^*=\mathrm{Hom}(\mathfrak{t}, \boldsymbol{C})$ is the differential of p. Note that

$$(\exp(sZ))(t_p)-(\exp(sZ))(t_0)=sdp(Z)+(\text{terms of higher degree in } s).$$

Here 0 is the zero weight, i.e., $t_0=e$. Hence, for any homogeneous element $\psi\in S^*(\mathfrak{t})$ (the algebra of polynomial functions on $\mathfrak{t}$), we can find an element $\phi\in R$ $(=\boldsymbol{C}[T])$ which satisfies $\phi(\exp(sZ))=s^{\deg\psi}\,\psi(Z)+(\text{terms of higher degree in } s)$.

LEMMA 2.14. (i) *For* $\psi_1, \cdots, \psi_n\in S^*(\mathfrak{t})$, $\det[w_i(\psi_j)]$ *is divisible by* $\prod_{\alpha>0}(d\alpha)^{n/2}$.

(ii) *Put* $dF=\langle\det[w_i(\psi_j)]/\prod_{\alpha>0}(d\alpha)^{n/2}\mid\psi_1, \cdots, \psi_n\in S^*(\mathfrak{t})\rangle_{S^*(\mathfrak{t})\text{-ideal}}$. *Then* $dF=S^*(\mathfrak{t})$.

PROOF. The proof of (i) is similar to (2.11) and is omitted. Let $S^*(\mathfrak{t})^W$ be the subalgebra of $S^*(\mathfrak{t})$ consisting of W-invariants and let H be the space of harmonic polynomials (see [14] for the definition). It is known that

$$S^*(\mathfrak{t})=S^*(\mathfrak{t})^W\cdot H\ (\cong S^*(\mathfrak{t})^W\otimes H) \qquad (2.14.1)$$

and $\dim H=n$ $(=|W|)$. Let $\{\psi_1^H, \cdots, \psi_n^H\}$ be a basis of H consisting of homogeneous polynomials. Then (2.14.1) implies that dF is a principal ideal of $S^*(\mathfrak{t})$ generated by $\det[w_i(\psi_j^H)]/\prod_{\alpha>0}(d\alpha)^{n/2}$. As a graded vector space, $H\cong H^*(\mathcal{B}, \boldsymbol{C})$ (the cohomology ring of the variety of Borel subgroups of $\boldsymbol{G}$). Hence its Poincaré polynomial $P_H(x)=\sum_k x^k\cdot\dim H_k$ (H_k is the homogeneous component of H of

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1) (Note added on September 11, 1981) After submitting this paper, I learned that R. Steinberg (On a theorem of Pittie, Topology **14** (1975), 173-177) had obtained a related result. This includes (2.12) if $\boldsymbol{G}$ is simply connected. Moreover, we can give another proof of (2.12) by using the Steinberg's result and by noting the fact that $I^2=I\cdot I$ is the *discriminant* of R over R^W (the subalgebra of R consisting of W-invariants).

degree k) is given by the following formula (see e. g. [16]):

$$P_H(x)=\prod_{i=1}^{l}\left(\frac{1-x^{m_i+1}}{1-x}\right) \quad (l \text{ is the semisimple rank of } G;\ m_1, \cdots, m_l \text{ are the exponents of } W \text{ [2]}).$$

But

$$\begin{aligned}\deg \det [w_i(\phi_j^H)] &\leqq \deg \phi_1^H \cdots \phi_n^H \\ &=\sum_k k\cdot \dim H_k \\ &=\frac{d}{dx}P_H(x)|_{x=1} \\ &=(1/2)\times(m_1+1)\cdots(m_l+1)\sum_i m_i \\ &=(1/2)nN \quad (n=|W|,\ N=|\Delta^+|;\ \text{see [16]}).\end{aligned}$$

On the other hand, $\deg \prod_{\alpha>0}(d\alpha)^{n/2}$ is clearly $(1/2)nN$. By (i), we have $\det[w_i(\phi_j^H)]/\prod_{\alpha>0}(d\alpha)^{n/2}\in \boldsymbol{C}$. Of course dF is not the zero ideal.

Let $\phi_1, \cdots, \phi_n$ be elements of R satisfying $\phi_i(\exp(sZ))=s^{\deg \phi_i^H}\phi_i^H(Z)+$(terms of higher degree in s). Then for a regular element Z of $\mathfrak{t}$,

$$\begin{aligned}&\det[w_i(\phi_j)](\exp(sZ))/\boldsymbol{f}(\exp(sZ))^{n/2} \\ &\qquad =\det[w_i(\phi_j^H)](Z)/\prod_{\alpha>0}(d\alpha)(Z)^{n/2}+(\text{terms involving } s).\end{aligned}$$

Taking the limit $s\to 0$, we see that $\det[w_i(\phi_j)]/\boldsymbol{f}^{n/2}$ does not vanish at 1 by (2.14). Thus we have proved that $1\notin V(F)$.

Step 2. Let Q be a free $\boldsymbol{Z}$-module of finite rank. A function on Q is called *polynomial* if it can be expressed as the restriction of an element of $S^*(Q\otimes_Z \boldsymbol{C})$ to Q. Set $P=T\oplus\cdots\oplus T$ (direct sum of n-copies of T) and define the map D: $P\to R$ by $\boldsymbol{t}=(t_1, \cdots, t_n)\mapsto D(\boldsymbol{t})=\det[w_i(t_j)]/\boldsymbol{f}^{n/2}$. Then the argument in Step 1 implies that the function on P defined to be $\boldsymbol{t}=(t_1, \cdots, t_n)\mapsto D(\boldsymbol{t})(1)$ is a *non-zero* polynomial on P. Let $\lambda\in \boldsymbol{T}$ be an element of the center of G (i. e. $W_\lambda=W$). We can see easily that $D(t)(\lambda)=\prod_{i=1}^{n}\lambda(t_i)D(\boldsymbol{t})(1)$ for $\boldsymbol{t}=(t_1, \cdots, t_n)\in P$.

Step 3. Now we consider arbitrary $\lambda\in \boldsymbol{T}$. Put $\boldsymbol{M}=Z_G(\lambda)^\circ$ (the connected component of the centralizer of λ containing 1). The Weyl group of the pair $(\boldsymbol{M}, \boldsymbol{T})$ is canonically isomorphic to $W_{[\lambda]}$. Set $\Delta_\lambda^+=\Delta^+\cap\Delta_\lambda$, where Δ_λ is the root system of the pair $(\boldsymbol{M}, \boldsymbol{T})$. We put $L(R)=R\oplus\cdots\oplus R$ (direct sum of n-copies of R). The group W naturally acts on $L(R)$. Set $e_i=(0, \cdots, 0, \overset{i}{\check{1}}, 0, \cdots, 0)\in L(R)$. Put $L=\oplus \boldsymbol{C}\cdot e_i$. An element $\mu\in \boldsymbol{T}$ defines a "specialization" $L(R)\to L$ $(\sum\phi_i e_i\mapsto\sum\phi_i(\mu)e_i)$. We regard $t=(t_1, \cdots, t_n)$ $(t_i\in T)$ as an element of $L(R)$. Then we have

$$w_1(\boldsymbol{t})\wedge\cdots\wedge w_n(\boldsymbol{t})=\det[w_i(t_j)]e_1\wedge\cdots\wedge e_n\in\overset{n}{\wedge}L(R).$$

Let $\{x_1, \cdots, x_m\}$ be a representative set of $W_{[\lambda]}\backslash W$ $(m=|W_{[\lambda]}\backslash W|)$. The coefficients of $w_1'x_i(\boldsymbol{t})\wedge\cdots\wedge w_r'x_i(\boldsymbol{t})$ $(r=|W_{[\lambda]}|=n/m\,;\ W_{[\lambda]}=\{w_1', \cdots, w_r'\})$ with respect to the canonical basis $\{e_{k_1}\wedge\cdots\wedge e_{k_r}\}_{1\leq k_1<\cdots<k_r\leq n}$ of $\overset{r}{\wedge}L(R)$ are $\{\det[w_i'(t_{k_j})]\}_{1\leq k_1<\cdots<k_r\leq n}$, which are divisible by $\prod_{\alpha\in\Delta_\lambda^+}\boldsymbol{f}_\alpha^{r/2}$ (apply (2.11) for $\boldsymbol{M}$). We define $\boldsymbol{D}_i(\boldsymbol{t})\,(=\boldsymbol{D}_i(t_1, \cdots, t_n))$, the element of $\overset{r}{\wedge}L(R)\ (1\leqq i\leqq m)$ by

$$\boldsymbol{D}_i(\boldsymbol{t})=\Big(\prod_{\alpha\in\Delta_\lambda^+}\boldsymbol{f}_\alpha^{r/2}\Big)^{-1}w_1'x_i(\boldsymbol{t})\wedge\cdots\wedge w_r'x_i(\boldsymbol{t}),$$

i. e.

$$\boldsymbol{D}_i(\boldsymbol{t})=\sum_{1\leq k_1<\cdots<k_r\leq n}D_{\boldsymbol{M}}(x_i(t_{k_1}), \cdots, x_i(t_{k_r}))e_{k_1}\wedge\cdots\wedge e_{k_r}$$

where $D_{\boldsymbol{M}}(t_1', \cdots, t_r')=\det[w_i'(t_j')]/\prod_{\alpha\in\Delta_\lambda^+}\boldsymbol{f}_\alpha^{r/2}\in R$ $(t_1', \cdots, t_r'\in T)$. As λ is contained in the center of $\boldsymbol{M}$, we have

$$\boldsymbol{D}_i(\boldsymbol{t})(\lambda)=\sum_{1\leq k_1<\cdots<k_r\leq n}(x_i^{-1}.\lambda)(t_{k_1})\cdots(x_i^{-1}.\lambda)(t_{k_r})$$
$$\times D_{\boldsymbol{M}}(x_i(t_{k_1}), \cdots, x_i(t_{k_r}))(1)e_{k_1}\wedge\cdots\wedge e_{k_r}\quad\text{(see Step 2).}$$

If $x_i\in W_{[\lambda]}\backslash W_\lambda$, we may assume $x_i(\Delta_\lambda^+)=\Delta_\lambda^+$. Then it can be easily seen that $D_{\boldsymbol{M}}(x_i(t_1'), \cdots, x_i(t_r'))(\mu)=\pm D_{\boldsymbol{M}}(t_1', \cdots, t_r')(x_i^{-1}.\mu)$ for $\mu\in\boldsymbol{T}$ and $t_1', \cdots, t_r'\in T$. In particular, $D_{\boldsymbol{M}}(x_i(t_1'), \cdots, x_i(t_r'))(1)=\pm D_{\boldsymbol{M}}(t_1', \cdots, t_r')(1)$. Hence, the linear independence of characters over polynomials ([10; (3.4.2)]) shows that $\boldsymbol{D}_1(\boldsymbol{t})(\lambda)\wedge\cdots\wedge\boldsymbol{D}_m(\boldsymbol{t})(\lambda)\neq 0$ on P if and only if $W_\lambda=W_{[\lambda]}$. Noting that $\boldsymbol{f}_\alpha(\lambda)\neq 0$ $(\alpha\in\Delta^+\backslash\Delta_\lambda^+)$, we have $D(\boldsymbol{t})(\lambda)e_1\wedge\cdots\wedge e_n=(\text{non-zero constant})\times\boldsymbol{D}_1(\boldsymbol{t})(\lambda)\wedge\cdots\wedge\boldsymbol{D}_m(\boldsymbol{t})(\lambda)$. Thus we have seen that $D(\boldsymbol{t})(\lambda)=0$ as a function on P if and only if $W_\lambda\neq W_{[\lambda]}$. This completes the proof of Lemma 2.12.

§ 3. Applications.

3.1. Let $(W_{\mathrm{aff}}, S_{\mathrm{aff}})$ be the Coxeter system of the affine Weyl group of type $\Delta^\vee$. We can identify W_{aff} with the modified affine Weyl group of $\boldsymbol{G}_{\mathrm{ad}}$, a connected semisimple group of adjoint type associated with Δ (see (1.2)-(1.3)). A Coxeter subsystem (W', S') of $(W_{\mathrm{aff}}, S_{\mathrm{aff}})$ is called *special* if it satisfies $W_{\mathrm{aff}}=W'\cdot T_{\mathrm{root}}$. Put $W_{\mathrm{sc}}:=W\cdot T_{\mathrm{weight}}$. Here T_{weight} is the group of translations by weights of Δ (hence $W_{\mathrm{sc}}\supset W_{\mathrm{aff}}$). We note that W_{sc} is canonically identified with the modified affine Weyl group of the simply connected covering group $\boldsymbol{G}_{\mathrm{sc}}$ of $\boldsymbol{G}_{\mathrm{ad}}$. As in (1.3), there exists a finite subgroup Ω of W_{sc} satisfying $W_{\mathrm{sc}}=\Omega\cdot W_{\mathrm{aff}}$. It is known that all special subsystems are conjugate under Ω.

Now we consider the case where $\widetilde{W}=W_{\mathrm{aff}}$ and the quasi-multiplicative function $q^{1/2}\colon W_{\mathrm{aff}}\to\boldsymbol{C}^\times$ is constant on S_{aff} until (3.5). For $H(\widetilde{W}, q)$-module (π, E), a vector $v\in E$ is called *special* if it has the property

$$\pi(e_s)v=q(s)v \qquad (s\in S')$$

for some special subsystem (W', S'). Then we have

PROPOSITION 3.2. *For $\lambda\in\mathrm{Hom}(T_{\mathrm{root}}, \boldsymbol{C}^{\times})$, M_λ is generated by all of its special vectors if and only if $\boldsymbol{e}(\lambda)\neq 0$.*

PROOF. Let $\lambda^{\sim}$ be an element of $\mathrm{Hom}(T_{\mathrm{weight}}, \boldsymbol{C}^{\times})$ such that $\lambda^{\sim}|_{T_{\mathrm{root}}}=\lambda$. We extend $q^{1/2}$ to $(q^{\sim})^{1/2}$, the quasi-multiplicative function on W_{sc} by $(q^{\sim})^{1/2}(xw)=q^{1/2}(w)\,(x\in\Omega,\ w\in W_{\mathrm{aff}})$. Consider the $H(W_{\mathrm{sc}}, q^{\sim})$-module $M_{\lambda^{\sim}}$. By (2.4), $1_{\lambda^{\sim}}$ generates $M_{\lambda^{\sim}}$ if and only if $\boldsymbol{e}(\lambda^{\sim})\neq 0$ since $W_{\lambda^{\sim}}$ is always a reflection group ([15]). Noting that $\boldsymbol{e}(\lambda^{\sim})=\boldsymbol{e}(\lambda)$, we can see that $\pi_{\lambda^{\sim}}(\boldsymbol{C}[\Omega])1_{\lambda^{\sim}}$ generates $M_{\lambda^{\sim}}$ as a $H(W_{\mathrm{aff}}, q)$-module if and only if $\boldsymbol{e}(\lambda)\neq 0$ (see (1.5)). But it is clear that $\pi_{\lambda^{\sim}}(\boldsymbol{C}[\Omega])1_{\lambda^{\sim}}$ coincide with the linear span of all special vectors of $M_{\lambda^{\sim}}$. Since $M_{\lambda^{\sim}}\cong M_\lambda$ as $H(W_{\mathrm{aff}}, q)$-modules, the proof of (3.2) is complete.

3.3. In [9], Lusztig defined a new model of principal series representations for $H(W_{\mathrm{aff}}, q)$, which will be denoted by $L_\lambda\,(\lambda\in\mathrm{Hom}(T_{\mathrm{root}}, \boldsymbol{C}^{\times}))$ in this paper. It is known that $\dim L_\lambda=|W|$. Moreover he defined a non-zero intertwining homomorphism $p_\lambda: L_\lambda\to M_\lambda$ ([9; 8.11]). We do not go into details of the definition of L_λ. For our purpose, it is sufficient to note the following fact which is a direct consequence of the definition: The image of L_λ under p_λ, $p_\lambda(L_\lambda)$ is generated by all special vectors of M_λ. Thus, in view of (3.2), we have proved

THEOREM 3.4. *The intertwining homomorphism $p_\lambda: L_\lambda\to M_\lambda$ is an isomorphism if and only if $\boldsymbol{e}(\lambda)\neq 0$.*

Of course, L_λ is irreducible if and only if M_λ is irreducible.

Example 3.5. In the rank 1 case $(\Delta=\{\pm\alpha\})$, p_λ is an isomorphism except when $\lambda(t_\alpha)=q^{-1}$. This agrees with Lusztig's observation.

3.6. We apply the results (2.2), (2.4) to the p-adic group case. For the notation used below, see Introduction and [7]. In [7], the eigenspace representation $P_{K,G}(\omega_\lambda)$ associated with an algebra homomorphism $\omega_\lambda: H(G, K)\to \boldsymbol{C}$ $(\lambda\in X_{nr}(M))$ and a non-zero intertwining homomorphism, the Poisson integral $\mathcal{P}_\lambda^\infty: E_\lambda\to P_{K,G}(\omega_\lambda)^\infty$ (or $\mathcal{P}_\lambda: E'_{\lambda^{-1}}\to P_{K,G}(\omega_\lambda)$) are defined for a p-adic reductive group G. Choose a modified affine Weyl group $\widetilde{W}$ and a positive real valued quasi-multiplicative function $q^{1/2}$ satisfying $H(G, B)\cong H(\widetilde{W}, q)$. Then [7; 3.2] (see also its Addendum) and (2.4) show

THEOREM 3.7. *The Poisson integral $\mathcal{P}_\lambda^\infty: E_\lambda\to P_{K,G}(\omega_\lambda)^\infty$ (or $\mathcal{P}_\lambda: E'_{\lambda^{-1}}\to P_{K,G}(\omega_\lambda)$) is an isomorphism if and only if*

(i) $e(\lambda^{-1})\neq 0$; *and*
(ii) $W_{\lambda}=W_{\langle\lambda\rangle}$.

Since $\dim P_{K,G}(\omega_{\lambda})^{B}\geqq |W|$ ([7; 2.8]), (2.2) and (3.7) imply the following theorem.

THEOREM 3.8. *The eigenspace representation* $P_{K,G}(\omega_{\lambda})^{\infty}$ *is irreducible if and only if*
(i) $e(\lambda)e(\lambda^{-1})\neq 0$; *and*
(ii) $W_{\lambda}=W_{\langle\lambda\rangle}$.

Compare (3.8) with [6; 12.2].

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