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IRREDUCIBLE COMPONENTS OF RIGID SPACES

by Brian CONRAD

Introduction.

Let k be a field complete with respect to a non-trivial non-archimedean absolute value and let X be a rigid analytic space over k. When $X = \mathrm{Sp}(A)$ is affinoid, it is clear what one should mean by the irreducible components of X: the analytic sets $\mathrm{Sp}(A/\mathfrak{p})$ for the finitely many minimal prime ideals $\mathfrak{p}$ of the noetherian ring A. However, the globalization of this notion is not as immediate. For schemes and complex analytic spaces, a global theory of irreducible decomposition is well-known ([EGA], 0_{I}, §2, [CAS], 9.2). Some extra commutative algebra is required in order to carry out the basic construction for rigid spaces.

In [CM], §1.2, Coleman and Mazur propose an elementary definition of irreducible components for rigid spaces (as well as a related notion that they call a "component part"). This definition has some technical drawbacks. For example, it is not clear that a global decomposition theory exists, and in order to prove that every point $x \in X$ lies on an irreducible component ([CM], 1.2.5) one is forced to go through an unexpectedly complicated argument which requires that X is separated and k is perfect. Also, this definition is sensitive to replacing X by its underlying reduced space.

There is a well-behaved definition of "normal" rigid spaces (analogous to normality for schemes and complex analytic spaces) and one can give

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the rigorous construction of the *normalization* of an arbitrary rigid space X. This is a certain finite surjective map $\widetilde{X} \to X$ with $\widetilde{X}$ normal, and (like in the case of complex analytic geometry) one can *define* the irreducible components of X to be the images of the connected components of $\widetilde{X}$. On the basis of this definition, we develop the theory of the irreducible components on arbitrary rigid spaces, and give shorter and more conceptual proofs of the results in [CM], 1.2, 1.3, on irreducible components, "component parts", and Fredholm series (in greater generality). Since we want results valid over ground fields such as $\mathbb{F}_p((t))$ which are not algebraically closed and not perfect, our methods differ somewhat from the methods over $\mathbb{C}$, which sometimes rely on the facts (whose analogues are false in rigid geometry over a non-perfect field) that reduced complex analytic spaces are smooth outside of a nowhere dense analytic set and all points on a complex analytic space are "rational". The main point is to use some non-trivial commutative algebra properties of the rings which arise in rigid geometry. We could sometimes get by with less algebra if we appealed to the work of Kiehl on analytically normal rings [K2]; however, the algebra results we use are interesting in their own right and provide a more natural point of view, so we base our exposition on these. Ultimately, we deduce everything we want from facts about normal rings and connectedness.

By analyzing behavior with respect to base change of the ground field, sometimes questions can be reduced to the case of an algebraically closed base field, where complex analytic methods are more often applicable. It should be noted that studying changes of the base field in rigid geometry is somewhat more subtle than in algebraic geometry, because power series (unlike polynomials) have infinitely many coefficients and we use completed tensor products in place of ordinary tensor products. Thus, the scheme-theoretic methods of [EGA], IV_2, §4.5ff, are sometimes insufficient.

When I mentioned to Coleman that a very simple and general development of the theory of irreducible components in rigid geometry can be given by using commutative algebra, he suggested that the details of the present article should be systematically worked out, due to its basic geometric significance. I would like to thank him for this suggestion. I am grateful to the Institute for Advanced Study for its hospitality, and to Siegfried Bosch, Johan deJong, Pierre Deligne, and Mark Kisin for some helpful discussions.

Notation and Terminology. Our notation from algebra is as follows. We denote by $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ the ring of integers, the field of rational numbers, and the field of real numbers respectively. If S is a ring, S_{red} is the quotient by the ideal of nilpotent elements and if S is a domain, $Q(S)$ is the fraction field. For basic facts about the theory of excellent rings, see [Mat1], Ch 13 and [EGA], IV_2, 7.8. We denote by $\overline{k}$ (resp. k_s, k_p) an algebraic closure (resp. a separable closure, a perfect closure) of a field k (usually equipped with the unique absolute value extending that on k if k is complete with respect to a given absolute value) and $\widehat{k}$ is the completion of a field k with respect to a given absolute value, again algebraically closed if k is (see [BGR], 3.4.1/3 for the non-archimedean case; the archimedean case is classical). For example, $\mathbb{C} = \overline{\mathbb{R}}$, $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ with respect to the p-adic absolute value for a prime p, and $\mathbb{C}_p = \widehat{\overline{\mathbb{Q}}}_p$.

A *complete field* k is a field k which is complete with respect to a non-trivial non-archimedean absolute value, and we say that a complete field k' is an *extension* of k if it is an extension of k as a field and if $k \to k'$ is a topological embedding. We often let R denote the valuation ring of k.

We use [BGR] as a reference for terminology, notation, and basic results in rigid geometry over a complete field k. In particular, T_n is the Tate algebra $k \ll x_1, \dots, x_n \gg$ of formal power series $f = \sum a_I x^I \in k[\![x]\!]$ over k in n variables for which $|a_I| \to 0$ as the "total degree" $|I| = i_1 + \dots + i_n \to \infty$; sometimes this is denoted $T_n(k)$ if several base fields are being considered. We define the unit ball $\mathbf{B}^n = \mathbf{B}^n_k$ to be $\mathrm{Sp}(T_n)$. An *analytic set* in a rigid space X is what is called an "analytic subset" in [BGR], 9.5.2; *i.e.* the zero locus of a coherent ideal sheaf.

If k'/k is an extension of complete fields, then for any (quasi-) separated X over k, the (quasi-)separated rigid space $X \times_k k'$ over k' is defined as in [BGR], 9.3.6, where it is denoted $X \widehat{\otimes}_k k'$. Of course, this base change functor is naturally "transitive" with respect to a tower of several extensions of the ground field, and is compatible with formation of fiber products and takes closed immersions to closed immersions. Moreover, for such base changes there is an analogous theory of an exact "pullback" functor on coherent sheaves (described in the obvious manner over affinoids) and when applied to ideal sheaves this returns the fact that base change takes closed immersions to closed immersions. One delicate point is determining whether the base change of an open immersion $i : U \hookrightarrow X$ is again an open immersion when i is not quasi-compact (the essential problem seems to be to show that the image of $U \times_k k'$ in $X \times_k k'$ is *admissible*). I do not know if this is true in general. The only case which we need is when U is a Zariski open in X, but this case is easy (and a more precise statement is true), as we will see in Lemma 3.1.1.

If X is a locally finite type scheme over $\mathrm{Spec}(k)$, we denote by X^{an} the "associated rigid analytic space" over k. The underlying set of X^{an} is the set of closed points of X, just like in the complex-analytic case. Some details concerning the construction and properties of this important functor are given in the appendix, due to lack of an adequate reference. If $\mathfrak{X}$ is a formal scheme over $\mathrm{Spf}(R)$ which is admissible in the sense of [BL1], §1, p.297, (*i.e.* "flat" and locally topologically of finite presentation over R), we denote by $\mathfrak{X}^{\mathrm{rig}}$ the "associated rigid analytic space" [BL1], §4.

When the valuation on k is discrete, so R is a discrete valuation ring, one can define (following Berthelot) the functor $\mathfrak{X} \rightsquigarrow \mathfrak{X}^{\mathrm{rig}}$ for the more general class FS_R of formal schemes $\mathfrak{X}$ over $\mathrm{Spf}(R)$ which are locally noetherian adic formal schemes for which the closed formal ("discrete") subscheme cut out by topologically nilpotent functions is locally of finite type over the residue field of R; that is, $\mathfrak{X}$ is covered by affines of type $\mathrm{Spf}(A)$ with A a quotient of one of the R-algebras $R \ll x_1, \dots, x_n \gg [\![y_1, \dots, y_m]\!]$. This is carefully explained in [deJ], §7]. Note that the excellence of the formal affines considered in [deJ], §7 follows from [V1], Prop 7, [V2], Thm 9 (the proof in [V2] only makes sense to the author when the ring C there is normal, which suffices for our needs). The functors $(\cdot)^{\mathrm{an}}$ and $(\cdot)^{\mathrm{rig}}$ have the usual functorial properties with respect to fiber products, open/closed immersions, finite maps, open coverings, and change of the base ring, and behave as expected on affines.

If A is a subset of B, we denote the set-theoretic complement of A in B by $B - A$. We use similar notation for the complement of a Zariski closed subset in a scheme, formal scheme, or rigid space, and we always regard this as an "open subspace" of the ambient space in the unique way possible. Given a morphism $f : X \to Y$ of "geometric objects" and $i : U \to X$ an "open immersion", we denote by $f|_U$ the composite $f \circ i$.

1. Commutative algebra.

1.1. Excellence results.

In order to construct the normalization of a rigid space, we recall some properties of the rings which arise in rigid analysis. If A is k-affinoid, $X = \mathrm{Sp}(A)$, and $x \in X$ corresponds to the maximal ideal $\mathfrak{m}$ of A, then there is a natural map $A_{\mathfrak{m}} \to \mathcal{O}_{X,x}$ of local rings. This is a local map of local noetherian rings which induces an isomorphism between completions [BGR], 5.2.6/1, 7.3.2/3, 7.3.2/7. Thus, in order to most conceptually relate properties of the structure sheaf $\mathcal{O}_{\mathrm{Sp}(A)}$ and the ring A, it is natural to hope that the stalks $\mathcal{O}_{\mathrm{Sp}(A),x}$ and the localizations of A at maximal ideals are excellent local rings; by [EGA], IV_2, 7.8.3 (v) this ensures that most interesting properties can be detected on the level of completions of these local rings, which are canonically "the same" in both cases. It should be noted that for what we need later, one could bypass some of our excellence arguments, instead using earlier work of Kiehl on normality [K2]. However, it seems more natural to systematically argue via excellence when one can do so.

Excellence of affinoids is known, as we will explain, and our goal here is to use this to establish the excellence of the stalks of the structure sheaves on rigid spaces. The case of excellence of k-affinoids in characteristic 0 follows readily from applying [Mat1], Thm 102 to Tate algebras over k (note that regularity of Tate algebras in all characteristics is readily seen at rational points and is deduced in general by making finite extensions of the base field and using [BGR], 6.1.2/3, [Mat2], Thm 23.7; in a similar way we see that all maximal ideals of T_n have height n). In characteristic p, the excellence property for affinoids holds when $[k : k^p]$ is finite, by applying [Mat2], Thm 108 to Tate algebras. Excellence of affinoids in characteristic p was proven in complete generality (*i.e.* without restrictions on $[k : k^p]$) by Kiehl [K3].

Kiehl uses a critical regularity criterion [K3], Satz 2.2 whose proof wasn't published in readily available form. Due to the importance of this fact in Kiehl's proof, and since our proof of excellence of stalks $\mathcal{O}_{X,x}$ on a rigid space will depend on the excellence results for affinoids, we begin by first carefully explaining Kiehl's proof of [K3], Satz 2.2. We then also prove a faithful flatness result which will be useful later when we consider general change of the base field.

Before we give Kiehl's regularity criterion, note that since every affinoid is the quotient of a Tate algebra, which is regular, all affinoids are universally catenary [Mat2], Thms 17.8, 17.9. Thus, the essential issue for proving excellence of affinoids is the problem of proving regularity properties. More precisely, one needs only to prove that the Tate algebras are excellent, and so by [EGA], IV_2, 7.8.2(*iii*), [Mat2], Thms 24.4, 32.4, it is enough to show

- the regular locus in $\mathrm{Spec}(A)$ for an affinoid domain A contains a non-empty open set,
- the localizations of the Tate algebras at closed points have geometrically regular formal fibers.

A close look at the proof of [Mat2], Thm 32.5 shows that the formal fiber problem is reduced to checking that for any domain A finite over T_n, any $\mathfrak{r} \in \mathrm{Max}(A)$, and any $\mathfrak{p} \in \mathrm{Spec}(\widehat{A_\mathfrak{r}})$ with $\mathfrak{p} \cap A = (0)$, the local noetherian ring $(\widehat{A_\mathfrak{r}})_\mathfrak{p}$ is regular. The proof of these claims in characteristic p is what requires Kiehl's regularity criterion. Before stating this criterion, we define the *rank* $\rho(M)$ of a finite module M over a local ring $(A, \mathfrak{m})$ to be the size of a minimal generating set (*i.e.* the dimension of $M/\mathfrak{m}$ over $A/\mathfrak{m}$).

THEOREM 1.1.1 (Kiehl). — *Let T be a noetherian ring, S a regular noetherian T-algebra, C a finite type S-algebra. Assume that the S-module $\Omega^1_{S/T}$ is free with finite rank ρ. Choose $\mathfrak{q} \in \mathrm{Spec}(C)$ over $\mathfrak{p} \in \mathrm{Spec}(S)$. Assume that the $C_\mathfrak{q}$-module $\Omega^1_{C_\mathfrak{q}/T}$, which is necessarily finitely generated due to the first fundamental exact sequence, has rank at most $\rho + \dim C_\mathfrak{q} - \dim S_\mathfrak{p}$. Then $C_\mathfrak{q}$ is regular.*

Proof. — The assertion may seem a little unusual, since we are *not* claiming any map is smooth and therefore we would just expect to prove regularity of *quotients* via differentials, not regularity of more general algebras. Thus, we first want to put ourselves in this more familiar setting. The following proof is due to Kiehl. Choose a surjection

$$S' = S[X_1, \dots, X_n] \twoheadrightarrow C,$$

and let $\mathfrak{p}' \in \mathrm{Spec}(S')$ be the contraction of $\mathfrak{q}$, $\mathfrak{p}_0$ the contraction of all of our primes to T. Define $A = T_{\mathfrak{p}_0}$, $B = S'_{\mathfrak{p}'}$, $\overline{B} = C_\mathfrak{q}$. Then we have a local map of local noetherian rings $A \to B$, with B regular, and $\overline{B}$ a quotient of B.

Since $\Omega^1_{B/A} \simeq B \otimes_{S'} \Omega^1_{S'/T}$ and $S' = S[X_1, \dots, X_n]$, we get a split exact first fundamental sequence [Mat2], Thm 25.1

$$0 \to \Omega^1_{S/T} \otimes_S S' \to \Omega^1_{S'/T} \to \Omega^1_{S'/S} \to 0.$$

Using the freeness hypothesis on $\Omega^1_{S/T}$, it follows that $\Omega^1_{S'/T}$ is free of rank $n + \rho$ over S'. Thus, $\Omega^1_{B/A}$ is a finite B-module with rank $n + \rho$. Also, $\Omega^1_{\overline{B}/A} \simeq \Omega^1_{R_\mathfrak{q}/T}$, so

$$\rho(\Omega^1_{B/A}) - \rho(\Omega^1_{\overline{B}/A}) = n + \rho - \rho(\Omega^1_{R_\mathfrak{q}/T}) \geq n + \dim S_\mathfrak{p} - \dim C_\mathfrak{q}.$$

Since $S_{\mathfrak{p}} \to S'_{\mathfrak{p}'}$ is a local flat map, the dimension formula [Mat2], Thm 15(*ii*) gives

$$\begin{aligned}\dim S'_{\mathfrak{p}'} &= \dim S_{\mathfrak{p}} + \dim(S'/\mathfrak{p})_{\mathfrak{p}'} \\ &\leq \dim S_{\mathfrak{p}} + \dim \kappa(\mathfrak{p})[X_1, \ldots, X_n] \\ &= \dim S_{\mathfrak{p}} + n.\end{aligned}$$

Combining these relations, we get

$$\rho(\Omega^1_{B/A}) - \rho(\Omega^1_{\overline{B}/A}) \geq \dim B - \dim \overline{B}. \tag{1}$$

We will now show that for any local map of local noetherian rings $A \to B$ for which $\Omega^1_{B/A}$ is a finite B-module and B is regular, and any quotient $\overline{B} = B/\mathfrak{r}$ for which the inequality (1) holds, the ring $\overline{B}$ is regular. This clearly specializes to the theorem we want to prove. For ease of notation, let $M = \Omega^1_{B/A}$ and $k = B/\mathfrak{m}$, with $\mathfrak{m}$ the maximal ideal of B.

Applying $\otimes_{\overline{B}} k$ to the (right exact) second fundamental exact sequence, we get an exact sequence

$$(\mathfrak{r}/\mathfrak{r}^2) \otimes_{\overline{B}} k \xrightarrow{\delta} M/\mathfrak{m} \to \Omega^1_{\overline{B}/A} \otimes_{\overline{B}} k \to 0.$$

Choose $b_1, \ldots, b_t \in \mathfrak{r}$ whose image in $(\mathfrak{r}/\mathfrak{r}^2) \otimes_{\overline{B}} k$ is a basis for the kernel of $M/\mathfrak{m} \to \Omega^1_{\overline{B}/A} \otimes_{\overline{B}} k$. Define I to be the ideal generated by the b_i's in B. We will show that B/I is regular and that $\dim B/I \leq \dim \overline{B}$. Since we have a surjection $B/I \to \overline{B}$ and B/I is a local domain (as it is regular), the kernel will be forced to be 0, so $I = \mathfrak{r}$ and $\overline{B} = B/I$ will be regular, as desired.

Since the images $\delta(b_i \otimes 1) = \mathrm{d}b_i$ are k-linearly independent in $M/\mathfrak{m}$, it follows that the b_i are k-linearly independent in $\mathfrak{m}/\mathfrak{m}^2$. Thus, they form part of a regular system of parameters in the regular local ring B. This proves the regularity of B/I, as well as the equality $\dim B/I = \dim B - t$. But the choice of the b_i also shows that $t = \rho(\Omega^1_{B/A}) - \rho(\Omega^1_{\overline{B}/A})$, so $\dim B - t \leq \dim \overline{B}$. □

A special case of this theorem is:

COROLLARY 1.1.2 ([K3], Satz 2.2). — *Let T be a noetherian ring, S a regular noetherian T-algebra, C a finite S-algebra. Assume that $\Omega^1_{S/T}$ is free of finite rank ρ over S. Choose $\mathfrak{q} \in \mathrm{Spec}(C)$. Assume that $\rho(\Omega^1_{C_\mathfrak{q}/T}) \leq \rho$ and $C_\mathfrak{q}$ is flat over S. Then $C_\mathfrak{q}$ is regular.*

Proof. — Let $\mathfrak{p} \in \mathrm{Spec}(S)$ be the contraction of $\mathfrak{q}$, so $S_\mathfrak{p} \to C_\mathfrak{q}$ is local and flat, with a 0-dimensional fiber over the closed point. It follows from the dimension formula that $\dim C_{\mathfrak{q}} = \dim S_{\mathfrak{p}}$. Now use Theorem 1.1.1. □

We now deduce the desired excellence theorem.

THEOREM 1.1.3. — *Let X be a rigid space over k, $x \in X$. The local noetherian ring $\mathcal{O}_{X,x}$ is excellent.*

Proof. — We know that $\mathcal{O}_{X,x}$ is a local noetherian ring. Since we can locally get a closed immersion of X into a unit ball, we can express $\mathcal{O}_{X,x}$ as the quotient of the stalk at a point on the unit ball. These latter rings are regular, since they are noetherian with regular completions. Thus, $\mathcal{O}_{X,x}$ is a quotient of a regular local ring, so it is universally catenary [Mat2], Thms 17.8, 17.9. By [EGA], IV_2, 7.8.3(i), it remains to check that $\mathcal{O}_{X,x} \to \widehat{\mathcal{O}_{X,x}}$ has geometrically regular fiber rings. That is, for $\mathfrak{p} \in \mathrm{Spec}(\mathcal{O}_{X,x})$, we want $\widehat{\mathcal{O}_{X,x}} \otimes_{\mathcal{O}_{X,x}} \kappa(\mathfrak{p})$ to be geometrically regular over $\kappa(\mathfrak{p})$.

We first reduce to the case $\mathfrak{p} = (0)$ (so $\mathcal{O}_{X,x}$ is a domain) and $X = \mathrm{Sp}(A)$ with A a domain. Let $f_1, \ldots, f_r$ be generators of $\mathfrak{p}$. Without loss of generality $X = \mathrm{Sp}(A)$, with $x \in X$ corresponding to a maximal ideal $\mathfrak{m}$ in A. Since $\mathcal{O}_{X,x}$ is the direct limit of the localizations at x of the rings of the affinoid opens around x, if we choose X small enough then we can suppose that there are $h_j \in A_{\mathfrak{m}}$ mapping to $f_j \in \mathcal{O}_{X,x}$. Since $A_{\mathfrak{m}} \to \mathcal{O}_{X,x}$ is faithfully flat, $A_{\mathfrak{m}}/(h_j) \to \mathcal{O}_{X,x}/\mathfrak{p}$ is faithfully flat and therefore injective. This implies that the h_j's generate a prime ideal in $A_{\mathfrak{m}}$. This gives rise to a prime ideal $\mathfrak{q}$ in A contained in $\mathfrak{m}$, with $\mathfrak{q}_{\mathfrak{m}} = (h_j)$. Replace X by $\mathrm{Sp}(A/\mathfrak{q})$. This reduces us to the case with A a domain and $\mathfrak{p} = (0)$.

For a finite extension K of the fraction field F of $\mathcal{O}_{X,x}$, we want to show that the noetherian ring $\widehat{\mathcal{O}_{X,x}} \otimes_{\mathcal{O}_{X,x}} K$ is regular. First consider the special case $K = F$. Then $\widehat{\mathcal{O}_{X,x}} \otimes_{\mathcal{O}_{X,x}} K$ is the localization of $\widehat{\mathcal{O}_{X,x}}$ at $\mathcal{O}_{X,x} - \{0\}$. This is a localization of the localization of $\widehat{A_{\mathfrak{m}}} \simeq \widehat{\mathcal{O}_{X,x}}$ at $A_{\mathfrak{m}} - \{0\}$. That is, the ring of interest is a localization of the fiber of $A_{\mathfrak{m}} \to \widehat{A_{\mathfrak{m}}}$ over (0). But this is regular since the affinoid (domain) A is excellent!

Now let's reduce to the case $K = F$. Let S denote a *finite* $\mathcal{O}_{X,x}$-subalgebra in K whose fraction field is K. Suppose for a moment that we know that the $\mathcal{O}_{X,x}$-algebra S has the form $\mathcal{O}_{Y,y}$ for some rigid space Y equipped with a map $Y \to X$ taking $y \in Y$ to x. Since $\mathcal{O}_{X,x} \to S = \mathcal{O}_{Y,y}$

is then finite and local, with K the fraction field of S, we have

$$\widehat{\mathcal{O}_{X,x}} \otimes_{\mathcal{O}_{X,x}} K \simeq (\widehat{\mathcal{O}_{X,x}} \otimes_{\mathcal{O}_{X,x}} S) \otimes_S K \simeq \widehat{\mathcal{O}_{Y,y}} \otimes_{\mathcal{O}_{Y,y}} K.$$

By the special case just treated (applied to $y \in Y$), this ring is regular, as desired.

It remains to show that S in K has the form asserted above. We first make the general claim that for any rigid space X, $x \in X$, and finite $\mathcal{O}_{X,x}$-algebra S, there exists an affinoid neighborhood U around x and a finite map of rigid spaces $f : Y \to U$ with $f_*(\mathcal{O}_Y)_x \simeq S$ over $\mathcal{O}_{X,x}$. If this is known for S, then it follows for any quotient of S, so by writing S as a quotient of a ring of the form $\mathcal{O}_{X,x}[T_1, \dots, T_r]/(p_1, \dots, p_r)$ with $p_i \in \mathcal{O}_{X,x}[T_i]$ monic, we see that it suffices to treat rings of this latter type. It is enough to treat the case $S = \mathcal{O}_{X,x}[T]/(p)$ (and then take fiber products over a small neighborhood of x). Now suppose without loss of generality that $X = \mathrm{Sp}(A)$ and that there is a monic $q \in A[T]$ inducing p. Since $B = A[T]/p$ is finite over A, hence is affinoid, we can define $f : Y = \mathrm{Sp}(B) \to X$. There is a natural map $S \to f_*(\mathcal{O}_Y)_x$ between *finite* $\mathcal{O}_{X,x}$-modules, so to prove this is an isomorphism we can check modulo $\mathfrak{m}_x^n$ for all n. But this is clear.

We claim that the canonical map

$$\varphi : f_*(\mathcal{O}_Y)_x \to \prod_{f(y)=x} \mathcal{O}_{Y,y}$$

is an isomorphism. Assume this for a moment. In our case of interest we would then have that S is isomorphic to such a product. But S is a *domain*, so the product of local rings must be a single local ring. This completes the proof. To see that φ is an isomorphism, let $\{h_1, \dots, h_r\}$ be generators of $\mathfrak{m}_x$, so the $f^*(h_j)$'s cut out $f^{-1}(x)$ on Y. Using the proof of [BGR], 7.2.1/3, we can find disjoint open affinoids in Y around each of the points of $f^{-1}(x)$. Let V denote the disjoint union of these. By Lemma 1.1.4 below, there exists an $\varepsilon > 0$ such that V contains the locus on Y where all $|f^*(h_j)| \le \varepsilon$. Thus, V contains $f^{-1}(U_\varepsilon)$ where $U_\varepsilon = \{x \in X \mid |h_j(x)| \le \varepsilon\}$.

Choosing a small ε so that $\varepsilon = |a|$ for some $a \in k^\times$, U_ε is an affinoid. Replacing X by U_ε allows us to assume (using [BGR], pp. 345–346) that $B = \prod B_i$ with any two distinct $y, y' \in f^{-1}(x)$ in *different* $\mathrm{Sp}(B_i)$'s, so

$$f_*(\mathcal{O}_Y)_x \simeq B \otimes_A \mathcal{O}_{X,x} \simeq \prod (B_i \otimes_A \mathcal{O}_{X,x}),$$

where $B_i \otimes_A \mathcal{O}_{X,x}$ is a finite $\mathcal{O}_{X,x}$-algebra. Moreover, $(B_i \otimes_A \mathcal{O}_{X,x})/\mathfrak{m}_x \simeq B_i/\mathfrak{m}_x$ is equal to 0 if no $y \in f^{-1}(x)$ lies in $\mathrm{Sp}(B_i)$, and is a local ring if some (necessarily *unique*) $y \in f^{-1}(x)$ lies in $\mathrm{Sp}(B_i)$. If some $y \in f^{-1}(x)$ lies in $\mathrm{Sp}(B_i)$, we conclude that $B_i \otimes_A \mathcal{O}_{X,x}$ is a local ring. As long as the canonical map $\varphi_i : B_i \otimes_A \mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$ is an isomorphism, we recover the map φ as an isomorphism.

To see that φ_i is an isomorphism, we claim more generally that if $f : Y \to X$ is a finite map of affinoids and $f^{-1}(x) = \{y\}$ is a single point, then the natural map $f_*(\mathcal{O}_Y)_x \to \mathcal{O}_{Y,y}$ is an isomorphism. It suffices to show that every admissible open neighborhood of y contains one of the form $f^{-1}(U)$ for some admissible open U around x. This follows from Lemma 1.1.4 below. □

LEMMA 1.1.4 (Kisin). — *Let A be k-affinoid, $I = (f_1, \ldots, f_r)$ an ideal of A, $X = \mathrm{Sp}(A)$, $Z = \mathrm{Sp}(A/I)$, U an admissible open of X containing Z. There exists $\varepsilon > 0$ so that U contains $\{x \in X \mid |f_i(x)| \le \varepsilon\}$.*

Proof. — This is Lemma 2.3 in [Ki]. The proof uses Raynaud's theory of formal models [BL1], though one can give a direct argument purely in terms of rigid spaces at the cost of somewhat lengthening the proof. It should be noted that the definition of W_ε in the proof of Lemma 2.3 in [Ki] is slightly incorrect. In the notation there, it should be defined to be the neighorhood where one has $|f_i| \le \varepsilon$ for some i, not for all i. □

In our later considerations of change of the base field, we will want to know that if k'/k is an extension and A is k-affinoid, then $A \to k'\widehat{\otimes}_k A$ is faithfully flat (that this map is a topological embedding is somewhat simpler [BGR], 6.1.1/9). Kiehl's arguments in [K3] also seem to require this. Since the completed tensor product is not an ordinary tensor product, there is something to prove. This type of flatness assertion was studied in [BKKN], 1.4.4 by induction arguments. We give a more "intrinsic" proof, and check how codimensions behave. Before stating the lemma, we refer the reader to [BGR], 3.7.3 for a discussion of the basic facts concerning the normed module structure on finite modules over affinoids.

LEMMA 1.1.5. — *Let k'/k be an extension of complete fields, A a k-affinoid.*

1. *The natural map $A \to k'\widehat{\otimes}_k A = A'$ is faithfully flat and for any finite A-module M, the natural map $A' \otimes_A M \to A'\widehat{\otimes}_A M$ is a topological isomorphism.*

2. *For any $\mathfrak{m} \in \mathrm{Max}(A)$, there exist only finitely many $\mathfrak{m}' \in \mathrm{Spec}(A')$*

over $\mathfrak{m}$, all maximal, and there is a natural isomorphism of topological k'-algebras

$$k'\widehat{\otimes}_k\widehat{A_{\mathfrak{m}}} \simeq \prod_{\mathfrak{m}'} \widehat{(A')_{\mathfrak{m}'}}$$

(the left side completed tensor product is taken in the sense of [EGA] 0_{I}, 7.7.5*, where k' and k are given the discrete topology, and the topology on the right side is the semilocal one).*

3. *If $Z \hookrightarrow X$ is a closed immersion of quasi-separated rigid spaces over k and Z is of codimension at least d at all points (i.e. $\dim \mathcal{O}_{X,z} - \dim \mathcal{O}_{Z,z} \geq d$ for all $z \in Z$), then the same holds for the base change $Z' \hookrightarrow X'$ over k'.*

Proof. — First consider the faithful flatness claim. By [BGR], 6.1.1/12 we have $A'/\mathfrak{m} \simeq k'\widehat{\otimes}_k A/\mathfrak{m}$, and since $A/\mathfrak{m}$ is finite dimensional as a k-vector space, the natural map $k' \otimes_k (A/\mathfrak{m}) \to k'\widehat{\otimes}_k(A/\mathfrak{m})$ is an isomorphism. It follows that $A'/\mathfrak{m}$ is non-zero and finite over k', so there exists prime ideals in A' over $\mathfrak{m}$, all maximal and corresponding to the points of $\mathrm{Spec}(k' \otimes_k (A/\mathfrak{m}))$. Thus, to check faithful flatness, it is enough to check flatness [Mat2], Thm 7.3(*ii*). For an injection $M \to N$ of finite A-modules, we need to check that $A' \otimes_A M \to A' \otimes_A N$ is injective. By [BGR], 3.7.3/5,6, the natural map of A-modules $A' \otimes_A M \to A'\widehat{\otimes}_A M$ is a topological isomorphism (and likewise for N). Thus, it is enough (by functoriality) to check that

$$A'\widehat{\otimes}_A M \to A'\widehat{\otimes}_A N$$

is injective.

Since $M \to N$ is an injection of finite A-modules, it is necessarily a topological isomorphism of M onto a closed subspace of N as k-Banach spaces. Also, M and N are k-Banach spaces of countable type (*i.e.* have dense subspaces with countable k-dimension). By [BGR], 2.7.1/4, the map $M \to N$ therefore has a continuous k-linear section. Thus, the map $k'\widehat{\otimes}_k M \to k'\widehat{\otimes}_k N$ has a continuous k'-linear section, so it is injective. But by [BGR], 2.1.7/7, this is identified with the map $A'\widehat{\otimes}_A M \to A'\widehat{\otimes}_A N$. This completes the proof of the first part of the lemma.

Now we consider the assertion about fibers over a maximal ideal $\mathfrak{m}$ of A. We have already seen that the set Σ of primes $\{\mathfrak{m}'\}$ of A' over $\mathfrak{m}$ is non-empty, finite, and consists solely of maximal ideals. There is a natural map of topological k'-algebras (where k' and k have the *discrete* topology)

(2) $$k'\widehat{\otimes}_k\widehat{A_{\mathfrak{m}}} \to \prod_{\mathfrak{m}' \in \Sigma} \widehat{(A')_{\mathfrak{m}'}}$$

which we claim is a topological isomorphism. Since $A'/\mathfrak{m}$ is artinian, the right side is naturally identified with the $\mathfrak{m}A'$-adic completion of the noetherian ring A'. Meanwhile, the left side is defined to be

$$\varprojlim k' \otimes_k (A/\mathfrak{m}^n).$$

In the same way we saw above that $k' \otimes_k (A/\mathfrak{m}) \to A'/\mathfrak{m}A'$ is an isomorphism, we see that $k' \otimes_k (A/\mathfrak{m}^n) \to A'/\mathfrak{m}^n A'$ is an isomorphism for all n. Passing to the inverse limit, we get an isomorphism which is readily seen to coincide with the map (2).

Finally, to handle the last part of the lemma, we may assume that $X = \mathrm{Sp}(A)$ is affinoid and $Z = \mathrm{Sp}(A/I)$. Let $A' = k'\widehat{\otimes}_k A$, $I' = k'\widehat{\otimes}_k I$, so I' is an ideal of A' and $(A/I)\widehat{\otimes}_k k' \simeq A'/I'$ [BGR], 6.1.1/12. Since affinoid rings are catenary, the closed immersion $Z \hookrightarrow X$ is of codimension at least d at all points of Z if and only if every minimal prime over I has height at least d, which is to say that the codimension of $\mathrm{Spec}(A/I)$ in $\mathrm{Spec}(A)$ is at least d. Since $A \to A'$ is faithfully flat, it suffices to show that if $Z \hookrightarrow X$ is a closed immersion of locally noetherian schemes of codimension at least d and $X' \to X$ is a faithfully flat quasi-compact base change with X' a locally noetherian scheme, then the induced closed immersion $Z' \hookrightarrow X'$ after base change has codimension at least d. We immediately reduce to the case where $X = \mathrm{Spec}(A)$, $Z = \mathrm{Spec}(A/\mathfrak{p})$, $X' = \mathrm{Spec}(A')$ with $\mathfrak{p}$ a prime ideal of A with height d. Since $A \to A'$ is faithfully flat, $\mathfrak{p}A' \neq A'$. We want to show that for any minimal prime $\mathfrak{p}'$ of A' over $\mathfrak{p}A'$, $\mathfrak{p}'$ has height at least d. By the going-down theorem for the faithfully flat map $A \to A'$, $\mathfrak{p}'$ lies over $\mathfrak{p}$. Thus, $A_{\mathfrak{p}} \to A'_{\mathfrak{p}'}$ is a faithfully flat local map of local noetherian rings, so by the dimension formula or the going-down theorem, $\dim A'_{\mathfrak{p}'} \geq \dim A_{\mathfrak{p}} = d$. □

It should be noted that in the above lemma, the maximals in A' which lie over maximals in A are in some sense quite rare; this corresponds to the overwhelming presence of points in $\mathbf{B}^n_{\mathbb{C}_p}$ which have coordinates that are not algebraic over $\mathbb{Q}_p$.

1.2. Normality.

Let $\mathbf{P}$ denote any of the following standard homological properties of noetherian rings: reduced, normal, regular, Gorenstein, Cohen-Macaulay, complete intersection, R_i, S_i [Mat2]. If A is a noetherian ring, we denote by $\mathbf{P}(A)$ the statement that the property $\mathbf{P}$ is true for the ring A. It is well-known that $\mathbf{P}(A)$ is true if and only if $\mathbf{P}(A_{\mathfrak{m}})$ is true for all maximal ideals $\mathfrak{m}$ of A, and for a local A, $\mathbf{P}(A)$ is equivalent to $\mathbf{P}(\widehat{A})$ for the above properties $\mathbf{P}$ aside from normality, reducedness, R_i, and S_i; equivalence between $\mathbf{P}(A)$ and $\mathbf{P}(\widehat{A})$ for local noetherian A holds for all of our $\mathbf{P}$'s if the local ring A is excellent [EGA], IV_2, 7.8.3(v). If X is a rigid space, scheme, or formal scheme, we define the *non-*$\mathbf{P}$ *locus* of X to be $\{x \in X \mid \mathbf{P}(\mathcal{O}_{X,x}) \text{ fails}\}$, and $\mathbf{P}(X)$ denotes the statement that the non-$\mathbf{P}$ locus on X is empty.

For a k-affinoid A, with $\mathfrak{m} \in \mathrm{MaxSpec}(A)$ corresponding to $x \in X$, the local map $A_{\mathfrak{m}} \to \mathcal{O}_{X,x}$ induces an isomorphism between completions, so $\mathbf{P}(\mathrm{Sp}(A))$ is equivalent to $\mathbf{P}(A)$. By a similar argument, we see that in the rigid case (as in the case of excellent schemes which locally admit closed immersions into regular schemes) the non-$\mathbf{P}$ locus of X is a Zariski-closed set (use [Mat2], Exercises 21.2, 24.3 for the Gorenstein and complete intersection cases, and use excellence for the rest [EGA], IV_2, 7.8.3(iv)). Moreover, if X is a locally finite type scheme over $\mathrm{Spec}(k)$, then $\mathbf{P}(X^{\mathrm{an}})$ is equivalent to $\mathbf{P}(X)$, since we may compare completions at closed points and use the fact that schemes locally of finite type over a field are excellent. In fact, the analytification of the non-$\mathbf{P}$ locus on X (given its reduced structure, say) is the non-$\mathbf{P}$ locus on X^{an}.

The case of formal schemes is a little more delicate. Although this seems to be implicit in [deJ], §7 and is presumably well-known, we give some details due to lack of a reference.

LEMMA 1.2.1. — *Suppose R is a complete discrete valuation ring. Let $\mathfrak{X}$ be in FS_R.*

1. *The non-*$\mathbf{P}$ *locus in $\mathfrak{X}$ is a Zariski-closed set.*
2. *If $\mathfrak{X} \simeq \mathrm{Spf}(A)$ is affine, then $\mathbf{P}(\mathfrak{X})$ is equivalent to $\mathbf{P}(A)$.*

Proof. — It suffices to consider $\mathfrak{X} = \mathrm{Spf}(A)$. As we have noted earlier, A is excellent (using [V1], [V2]). Let I be an ideal defining the separated and complete topology of A. Let $\mathfrak{p}$ be an open prime ideal (*i.e.* $I \subseteq \mathfrak{p}$). Using the multiplicative set $S = A - \mathfrak{p}$, we define $A_{\{S\}}$ and $A\{S^{-1}\}$ as in [EGA], 0_{I}, 7.6.15, 7.6.1, and for simplicity we denote these by $A_{\{\mathfrak{p}\}}$ and $A\{\mathfrak{p}^{-1}\}$ respectively. Thus, $A\{\mathfrak{p}^{-1}\}$ is the $I_{\mathfrak{p}}$-adic completion of $A_{\mathfrak{p}}$ and $A_{\{\mathfrak{p}\}} \simeq \mathcal{O}_{\mathfrak{X},\mathfrak{p}}$. By [EGA], 0_{I}, 7.6.17, 7.6.18, there is a natural local map $A_{\{\mathfrak{p}\}} \to A\{\mathfrak{p}^{-1}\}$ of local noetherian rings (with maximal ideals generated by $\mathfrak{p}$) which induces an isomorphism of completions. The local map $A_{\mathfrak{p}} \to A_{\{\mathfrak{p}\}}$ of local noetherian rings also induces an isomorphism on completions. To see this, recall that by definition,

$$A_{\{\mathfrak{p}\}} = \varinjlim_{f \notin \mathfrak{p}} A_{\{f\}},$$

with $A_{\{f\}}$ the I_f-adic completion of A_f. Since $\mathfrak{p}$ is open, it is therefore clear that $A_{\mathfrak{p}}/\mathfrak{p}^n \to A_{\{\mathfrak{p}\}}/\mathfrak{p}^n$ is an isomorphism for all $n \geq 1$. In particular, we deduce that the local map of local noetherian rings $A_{\mathfrak{p}} \to A_{\{\mathfrak{p}\}}$ is flat.

For an open prime $\mathfrak{p}$ of A (such as a maximal ideal), we claim that $\mathbf{P}(A_{\mathfrak{p}})$ is equivalent to $\mathbf{P}(A_{\{\mathfrak{p}\}})$. Since the property of a local noetherian ring being regular, Cohen-Macaulay, Gorenstein, and a complete intersection ring can be checked on completions, we only have to consider the properties R_i and S_i. The map $A_{\mathfrak{p}} \to A_{\{\mathfrak{p}\}}$ induces an isomorphism on completions, so there is a flat local map of local noetherian rings $A_{\{\mathfrak{p}\}} \to \widehat{A_{\mathfrak{p}}}$. Since we also have a flat local map of local noetherian rings $A_{\mathfrak{p}} \to A_{\{\mathfrak{p}\}}$, we can use [Mat2], Thm 23.9 to deduce the implications $\mathbf{P}(\widehat{A_{\mathfrak{p}}}) \Rightarrow \mathbf{P}(A_{\{\mathfrak{p}\}}) \Rightarrow \mathbf{P}(A_{\mathfrak{p}})$. But $\mathbf{P}(A_{\mathfrak{p}}) \Rightarrow \mathbf{P}(\widehat{A_{\mathfrak{p}}})$ since $A_{\mathfrak{p}}$ is excellent [EGA], IV_2, 7.8.3(v), so $\mathbf{P}(A_{\mathfrak{p}})$ and $\mathbf{P}(A_{\{\mathfrak{p}\}})$ are equivalent.

This argument show that the non-$\mathbf{P}$ locus of $\mathrm{Spf}(A)$ is exactly the set of open primes in the non-$\mathbf{P}$ locus of $\mathrm{Spec}(A)$. But A is an excellent noetherian ring which is the quotient of a regular ring, so the non-$\mathbf{P}$ locus of $\mathrm{Spec}(A)$ is the Zariski-closed set cut out by some ideal J. Thus, $\mathrm{Spf}(A/J)$ is the non-$\mathbf{P}$ locus of $\mathrm{Spf}(A)$. In particular, this locus is empty if and only if $J = A$, so $\mathbf{P}(\mathfrak{X})$ is equivalent to $\mathbf{P}(A)$. □

To conclude our commutative algebra discussion, we want to establish the basic algebraic fact which will ensure that we can globally construct normalizations of rigid spaces and formal schemes in FS_R (in case k has discrete valuation), much like in complex analytic and algebraic geometry. If A is an excellent noetherian ring, we denote by $\widetilde{A}$ the normalization of A_{red} (*i.e.* the integral closure of A_{red} in its total ring of fractions). This is a finite A-module. There is no risk of confusion of this with the quotient of the subring of power-bounded elements by topologically nilpotent elements in the affinoid case (also denoted $\widetilde{A}$ in [BGR]). Before we can get a good global theory of normalization, we must check how affine/affinoid normalization behaves under passage to open affines/affinoids.

THEOREM 1.2.2. — *Let A be a reduced Japanese ring, A' a flat noetherian A-algebra, $\widetilde{A}$ the normalization of A (which is finite over A).*

Assume that $A' \otimes_A \widetilde{A}$ is normal and that $A'/\mathfrak{p}$ is reduced for all minimal primes $\mathfrak{p}$ of A. Then the natural map $f : A' \to A' \otimes_A \widetilde{A}$ is a normalization.

Proof. — Since $A \to A'$ is flat, and $A \to \widetilde{A}$ is an finite *injection*, we see that f is a finite injection into a normal ring. In particular, A' is reduced. Let S denote the multiplicative set of elements of A' which are not zero divisors. Since A' is reduced, $S^{-1}A'$ is the product of the finitely many fraction fields of A' at its generic points. If we localize $A' \otimes_A \widetilde{A}$ at S, then we also invert elements of A which aren't zero divisors (since $A \to A'$ is flat), so f becomes an isomorphism after localizing at S.

In order to conclude that f is a normalization map, we must show that it is an isomorphism at generic points, which algebraically amounts to showing that the elements of S are not zero divisors in $A' \otimes_A \widetilde{A}$ (*i.e.* no generic points of $A' \otimes_A \widetilde{A}$ lie over non-generic points of A', so we don't lose any factor rings of the total fraction ring of $A' \otimes_A \widetilde{A}$ by inverting S). Since $A \to \widetilde{A}$ is not flat, some argument is needed. Let $\{\mathfrak{p}_i\}$ be the finite set of minimal primes of A. Suppose $s \in A'$ kills a non-zero element of $A' \otimes_A \widetilde{A}$. We want to conclude that s is a zero-divisor in A', so $s \notin S$. Since $\widetilde{A} = \prod \widetilde{A/\mathfrak{p}_i}$ and $A/\mathfrak{p}_i \hookrightarrow \widetilde{A/\mathfrak{p}_i}$ induces an isomorphism of fraction fields, s must kill an element of some $A' \otimes_A (A/\mathfrak{p}_{i_0}) = A'/\mathfrak{p}_{i_0}$ (note that $A \to A'$ is flat, so we may "kill" denominators in normalizations to detect if an element is a zero divisor).

By hypothesis, $A'/\mathfrak{p}_{i_0}$ is reduced. Since s can be viewed as a zero divisor in the ring $A'/\mathfrak{p}_{i_0}$, s must lie in a minimal prime $\mathfrak{q}$ over $\mathfrak{p}_{i_0}A'$. The going-down theorem holds for flat maps [Mat2], Thm 9.5, so $\mathfrak{q}$ must contract to $\mathfrak{p}_{i_0}$. Since $\mathfrak{p}_{i_0}$ is a minimal prime of A, it therefore follows that $\mathfrak{q}$ is a minimal prime of A'. This implies that $s \in A'$ is a zero divisor, as desired. □

COROLLARY 1.2.3. — *Let* $\mathrm{Sp}(A') \to \mathrm{Sp}(A)$ *be an open immersion of k-affinoids,* $\widetilde{A}$ *the normalization of A. Then* $\mathrm{Sp}(A' \otimes_A \widetilde{A}) \to \mathrm{Sp}(A')$ *is a normalization. If k has discrete absolute value,* $\mathrm{Spf}(A') \to \mathrm{Spf}(A)$ *is an open immersion in* FS_R*, and* $\widetilde{A}$ *is the normalization of A, then* $\mathrm{Spf}(A' \otimes_A \widetilde{A}) \to \mathrm{Spf}(A')$ *is a normalization.*

Proof. — We may replace A by A_{red}, which has the effect of replacing A' by A'_{red}. It remains to check the hypotheses in Theorem 1.2.2. Note that a maximal ideal of A' contracts to a maximal ideal of A. In the rigid case this is clear (since the residue fields at maximal ideals are finite over k) and in the formal case this is because a maximal ideal in a formal affine is a closed point in the formal spectrum, but our formal affines have the property that their underlying topological spaces are Spec of finite type rings over a field (so the property of a point being closed is local). Thus, if $\mathfrak{m}$ is a maximal ideal of A' then its contraction to A is a maximal ideal $\mathfrak{n}$, and the natural map $A_{\mathfrak{n}} \to A'_{\mathfrak{m}}$ is faithfully flat because it induces an isomorphism on completions (the rigid case is [BGR], 7.3.2/3 and in the formal case this arises from the fact that $A_{\{\mathfrak{n}\}} \to A'_{\{\mathfrak{m}\}}$ is an isomorphism). The flatness of $A \to A'$ follows.

We know that A is Japanese (even excellent), and $A' \otimes_A \widetilde{A}$ is normal because $\mathrm{Sp}(A' \otimes_A \widetilde{A})$ is an open in the normal space $\mathrm{Sp}(\widetilde{A})$ in the rigid case and likewise with Spf's in the formal case. Finally, if $\mathfrak{p}$ is a prime of A, then $\mathrm{Sp}(A/\mathfrak{p})$ is a reduced space in the rigid case and $\mathrm{Spf}(A/\mathfrak{p})$ is a reduced space in the formal case, but $\mathrm{Sp}(A'/\mathfrak{p}) \to \mathrm{Sp}(A/\mathfrak{p})$ is an open immersion in the rigid case and likewise with Spf's in the formal case, so $A'/\mathfrak{p}$ is also reduced. □

2. Global normalizations.

2.1. Basic constructions.

Let X be a rigid space over k or (if k has discrete valuation) an object in FS_R. If X is affinoid/affine and $A = \Gamma(X, \mathcal{O}_X)$, we define the *affinoid/affine normalization* of X to be the canonical finite map $\widetilde{X} \overset{\mathrm{def}}{=} \mathrm{Sp}(\widetilde{A}) \to \mathrm{Sp}(A) \simeq X$ in the rigid case and likewise with Spf's in the formal case. If $V \subseteq U$ is an inclusion of affinoid/affine opens in X, then by Corollary 1.2.3, there is a unique isomorphism $\widetilde{V} \simeq V \times_U \widetilde{U}$ over V. Using this as gluing data, we glue the $\widetilde{U}$'s, and call the resulting finite surjective map $\widetilde{X} \to X$ the *normalization* of X (see [EGA], III_1, §4.8 for the theory of finite morphisms of locally noetherian formal schemes). We say that a morphism $Y \to X$ is a *normalization* if $Y \simeq \widetilde{X}$ over X (in which case the isomorphism is unique, so the notion of $Y \to X$ being a normalization is local on X). Of course, the same construction for locally finite type k-schemes gives the usual normalization for such schemes. The normalization of a rigid space "localizes well" in the following sense:

LEMMA 2.1.1. — *Let X be a rigid space, $x \in X$, $p : \widetilde{X} \to X$ a normalization. Then $p_x^\sharp : \mathcal{O}_{X,x} \to (p_*\mathcal{O}_{\widetilde{X}})_x$ is a normalization map.*

Proof. — Without loss of generality, $X = \mathrm{Sp}(A)$ is a reduced affinoid, so $p_x^\sharp$ is the map $\mathcal{O}_{X,x} \to \widetilde{A} \otimes_A \mathcal{O}_{X,x}$. By Theorem 1.2.2, we just have to check that $\widetilde{A} \otimes_A \mathcal{O}_{X,x} \simeq (p_* \mathcal{O}_{\widetilde{X}})_x$ is a normal ring. The argument at the end of the proof of Theorem 1.1.3 shows that the natural map

$$(p_* \mathcal{O}_{\widetilde{X}})_x \to \prod_{y \in p^{-1}(x)} \mathcal{O}_{\widetilde{X},y}$$

is an isomorphism, and the right side is a finite product of normal domains. □

In order to study the behavior of irreducible components on rigid spaces (once they are defined) with respect to the analytification of schemes and formal schemes, we will first need to see how the formation of the normalization morphism behaves with respect to the functors $X \rightsquigarrow X^{\mathrm{an}}$ and $\mathfrak{X} \rightsquigarrow \mathfrak{X}^{\mathrm{rig}}$. For this, an intrinsic characterization of normalization morphisms is useful. Before we give the characterization, we review some terminology. A closed set Z in a topological space X is said to be *nowhere dense* if Z contains no non-empty opens of X. This can be checked locally on X. If X is a locally noetherian scheme, this is equivalent to requiring $\dim \mathcal{O}_{Z,z} < \dim \mathcal{O}_{X,z}$ for all $z \in Z$. For a noetherian ring A and ideal I in A, $\mathrm{Spec}(A/I)$ is nowhere dense in $\mathrm{Spec}(A)$ if and only if I is not contained in any minimal primes of A. An analytic set Z in a rigid space X is said to be *nowhere dense* if $\dim \mathcal{O}_{Z,z} < \dim \mathcal{O}_{X,z}$ for all $z \in Z$ (or equivalently, Z does not contain a non-empty admissible open in X). This can be checked locally on X and when $X = \mathrm{Sp}(A)$ is affinoid and Z is defined by an ideal I, it is easy to see that this is equivalent to $\mathrm{Spec}(A/I)$ being nowhere dense in $\mathrm{Spec}(A)$.

THEOREM 2.1.2. — *Consider the category of locally finite type k-schemes or rigid spaces over k. A map $f : Y \to X$ of reduced objects is a normalization if and only if f is finite, Y is normal, and there is a nowhere dense Zariski-closed set Z in X with $f^{-1}(Z)$ nowhere dense in Y such that f is an isomorphism over the open $X - Z$ complementary to Z.*

When f is a normalization, $\mathcal{O}_X \to f_*(\mathcal{O}_Y)$ is injective and we may take Z to be the support of the coherent cokernel sheaf.

Proof. — We handle both cases simultaneously. First, assume f is a normalization, so certainly f is finite and Y is normal. Since our spaces are reduced, the map of coherent sheaves $\mathcal{O}_X \to f_*(\mathcal{O}_Y)$ is a monomorphism. Let Z be the Zariski-closed support of the coherent cokernel sheaf. By the construction of normalizations, trivially f is an isomorphism over $X - Z$, and any open over which f is an isomorphism lies inside of $X - Z$. In fact, Z is exactly the "non-normal locus" (*i.e.* Z is the set of all $x \in X$ such that $\mathcal{O}_{X,x}$ is not normal), so it remains to check that in the affine/affinoid cases, the non-normal locus on a reduced space is nowhere dense and its preimage in the normalization is nowhere dense. In the case of reduced excellent schemes this is obvious, and we thereby deduce the same result in the rigid case.

Conversely, consider a finite map $f : Y \to X$ with the given properties. Since an isomorphism $Y \simeq \widetilde{X}$ over X is unique if it exists, we may work locally over X and so in the rigid case we can assume $X = \mathrm{Sp}(A)$ is affinoid, so $Y = \mathrm{Sp}(B)$ is affinoid (and likewise we need only consider the affine case in the scheme setting, again denoting the rings by A and B respectively). Clearly $\varphi : A \to B$ is finite. Also, φ is injective. To see the injectivity of φ, we need that in both cases, if Z is a Zariski closed set in a "reduced space" X which is nowhere dense, and U is the complement of Z, then $\Gamma(X, \mathcal{O}_X) \to \Gamma(U, \mathcal{O}_X)$ is injective. If $f \in \Gamma(X, \mathcal{O}_X)$ vanishes on U, then the zero locus $Z(f)$ of f is a Zariski-closed set which contains U, so its complement is a Zariski-open set inside of the nowhere dense Z, which forces the complement to be empty. Thus, $Z(f) = X$ as sets, so $f = 0$ since X is reduced.

In both cases, $\mathrm{Spec}(\varphi)$ induces a bijection on maximal ideals away from Z and $f^{-1}(Z)$. Let I be the radical ideal in A corresponding to the Zariski-closed set Z. Consider a maximal ideal $\mathfrak{m}$ of A not containing I, so B has a unique maximal $\mathfrak{n}$ over $\mathfrak{m}$. Since φ is a finite injection, $B_{\mathfrak{m}} = B_{\mathfrak{n}}$. We claim that $\varphi_{\mathfrak{m}} : A_{\mathfrak{m}} \to B_{\mathfrak{m}} = B_{\mathfrak{n}}$ is an isomorphism. This is clear in the scheme case. In the rigid case, $\varphi_{\mathfrak{m}}$ induces an isomorphism on completions. Since B is finite over A, $\varphi_{\mathfrak{m}}$ becomes an isomorphism after the faithfully flat base change $\otimes_{A_{\mathfrak{m}}} \widehat{A_{\mathfrak{m}}}$, so $\varphi_{\mathfrak{m}}$ is an isomorphism.

We conclude that in both cases $\mathrm{Spec}(\varphi)$ is an isomorphism over the Zariski-open complement of $\mathrm{Spec}(A/I)$ in $\mathrm{Spec}(A)$. Since φ is a finite injection with B normal, it suffices to check that no minimal prime of A contains I and no minimal prime of B contains IB (so φ induces a bijection on generic points and an isomorphism between the resulting residue fields at these points). This is clear, by the hypothesis that Z and $f^{-1}(Z)$ are nowhere dense. □

Note that there seems to be no analogue of the above theorem in the case of formal schemes. Indeed, if k has discrete absolute value and A is the completion of a reduced finite type R-scheme at a non-normal point over the maximal ideal of R, then $\mathrm{Spf}(A)$ is reduced and not normal, but consists of a single point!

THEOREM 2.1.3. — *If X is a locally finite type scheme over* $\mathrm{Spec}(k)$ *and $f : \widetilde{X} \to X$ is a normalization, then f^{an} is a normalization of X^{an}. Likewise, if k has discrete valuation and $\mathfrak{X}$ is a formal scheme in* FS_R*, and $f : \widetilde{\mathfrak{X}} \to \mathfrak{X}$ a normalization, then f^{rig} is a normalization of $\mathfrak{X}^{\mathrm{rig}}$.*

Proof. — The normality of $\widetilde{X}^{\mathrm{an}}$ follows from excellence, as we have already observed in §1.2, and the finiteness of f^{an} in this case is a consequence of the behavior of coherent sheaves with respect to analytification and the description of finite morphisms via coherent sheaves of algebras in both the scheme and rigid case cases (or see Theorem 5.2.1). The normality of $\widetilde{\mathfrak{X}}^{\mathrm{rig}}$ follows from [deJ], 7.2.4(c), and finiteness of f^{rig} in this case follows from [deJ], 7.2.4 (e), or an argument with coherent sheaves of algebras.

Consider the scheme case. By Theorem 2.1.2 and the basic properties of the analytification functor on schemes, it remains to check that if $i : Z \hookrightarrow Y$ is a "nowhere dense" closed immersion of locally finite type k-schemes, then the closed immersion i^{an} is "nowhere dense". For any $z \in Z^{\mathrm{an}}$, viewed as a closed point of Z, the map of local noetherian rings $\mathcal{O}_{Z,z} \to \mathcal{O}_{Z^{\mathrm{an}},z}$ induces an isomorphism on completions, so these two local rings have the same dimension. The same statement holds on Y. Since i^{an} on the level of sets is the map induced by i on closed points, we see that $\dim \mathcal{O}_{Z,z} < \dim \mathcal{O}_{Y,i(z)}$ if and only if $\dim \mathcal{O}_{Z^{\mathrm{an}},z} < \dim \mathcal{O}_{Y^{\mathrm{an}},i^{\mathrm{an}}(z)}$. This settles the scheme case.

Now consider the formal scheme case, so k has discrete valuation and R is a discrete valuation ring. Let $\mathfrak{X}$ be an object in FS_R, with normalization $\widetilde{\mathfrak{X}}$. We will prove directly that $\widetilde{\mathfrak{X}}^{\mathrm{rig}} \to \mathfrak{X}^{\mathrm{rig}}$ is a normalization. Using [deJ], 7.2.2, 7.2.4(c), formation of the "underlying reduced space" commutes with $(\cdot)^{\mathrm{rig}}$, so we may assume $\mathfrak{X}$ is reduced. We may also work locally over $\mathfrak{X}^{\mathrm{rig}}$, so we may work locally over $\mathfrak{X}$ [deJ], 7.2.3. Thus, it is enough to consider the case where $\mathfrak{X} = \mathrm{Spf}(A)$ is affine, with A reduced. Letting $\widetilde{A}$ denote the normalization of A, we will prove directly that $\mathrm{Spf}(\widetilde{A})^{\mathrm{rig}} \to \mathrm{Spf}(A)^{\mathrm{rig}}$ is a normalization. Let $C_n(A) = B_n(A) \otimes_R k$, with $B_n(A)$ the J-adic completion of the subring $A[J^n/\pi]$ in $A \otimes_R k$ generated by the image of A and elements of the form j/π with $j \in J^n$ and π a fixed uniformizer of R. By [BGR], 7.1.2(a), $C_n(A)$ is flat over A, and by construction the $\mathrm{Sp}(C_n(A))$'s give an "increasing" admissible cover of the reduced space $\mathrm{Spf}(A)^{\mathrm{rig}}$. In particular, all $C_n(A)$ are reduced.

By [deJ], 7.2.2, if A' is any finite A-algebra, the preimage under $\mathrm{Spf}(A')^{\mathrm{rig}} \to \mathrm{Spf}(A)^{\mathrm{rig}}$ of $\mathrm{Sp}(C_n(A))$ is the affinoid $\mathrm{Sp}(C_n(A) \otimes_A A')$. Taking $A' = \widetilde{A}$, $\mathrm{Sp}(C_n(A) \otimes_A \widetilde{A})$ is an admissible open in the normal space $\mathrm{Spf}(\widetilde{A})^{\mathrm{rig}}$, so $C_n(A) \otimes_A \widetilde{A}$ is normal. Taking $A' = A/\mathfrak{p}$ for a prime $\mathfrak{p}$ of A, $\mathrm{Sp}(C_n(A)/\mathfrak{p})$ is an admissible open in the reduced space $\mathrm{Spf}(A/\mathfrak{p})^{\mathrm{rig}}$, so $C_n(A)/\mathfrak{p}$ is reduced. By Theorem 1.2.2, the maps $C_n(A) \to C_n(A) \otimes_A \widetilde{A}$ are normalizations, so $\mathrm{Spf}(\widetilde{A})^{\mathrm{rig}} \to \mathrm{Spf}(A)^{\mathrm{rig}}$ is a normalization. □

We will define irreducible components in terms of connected components of a normalization. Before doing this, we need to recall a few facts concerning connected components and connected normal spaces. Recall that a rigid space Z is said to be *connected* if an admissible open cover $\{U, V\}$ with disjoint U and V must have U or V empty, or equivalently if $\Gamma(Z, \mathcal{O}_Z)$ has no non-trivial idempotents (this is equivalent to the definition of connectedness given in [BGR], 9.1.1). For example, an affinoid $\mathrm{Sp}(A)$ is connected if and only if $\mathrm{Spec}(A)$ is connected.

Choose $z \in Z$. A straightfoward argument shows the admissibility of the set of all $z' \in Z$ such that there is a finite set $Z_1, \ldots, Z_n$ of connected affinoid opens in Z with $z \in Z_1$, $z' \in Z_n$, and $Z_i \cap Z_{i+1}$ non-empty for all $1 \le i < n$. This is called the *connected component of* z. It is not hard to check that the connected components of any two points $z, z' \in Z$ are either disjoint or equal, and that for any set $\{Z_i\}$ of such components, $\cup Z_i$ is an admissible open and $\{Z_i\}$ is an admissible covering of this union. We define the *connected components of* Z to be the connected components of the points of Z (and if Z is empty, it is viewed as its own connected component).

It follows that Z is connected if and only if all points have the same connected component, and any non-empty connected admissible open in an arbitrary rigid space Z lies in a unique maximal such open, which is a connected component. The connected components of any Z are analytic sets and are "locally finite" in the sense that any quasi-compact admissible open meets only finitely many connected components. We noted above that if $\{U_i\}_{i \in \Sigma}$ is a subset of the set of connected components of Z, then the union of these U_i's is an admissible open with the U_i for $i \in \Sigma$ its connected components; the admissible opens of Z obtained in this way are exactly the admissible opens of Z which are also analytic sets ("open and closed"). We call such subsets *clopen sets*.

Before we define the irreducible components of a rigid space, we make a useful observation about the connected components on a normal rigid space. This will later be extended to irreducible spaces, once they are defined.

LEMMA 2.1.4. — *Let X be a connected normal rigid space, U a non-empty admissible open. The only analytic subset Z of X which contains U is X itself, and if $X = Z_1 \cup Z_2$ is a union of two analytic sets, then $X = Z_1$ or $X = Z_2$.*

Proof. — Give Z its reduced structure. To show that $Z = X$, we may replace U by an affinoid subdomain which we may take to be connected, and we may replace Z by its connected component containing U. Since X is normal and connected, it is not hard to see (using Lemma 2.1.5 below) that all $\mathcal{O}_{X,x}$'s have a common dimension n. Let $A = \{z \in Z \mid \dim \mathcal{O}_{Z,z} = n\}$ and let $B = \{z \in Z \mid \dim \mathcal{O}_{Z,z} < n\}$.

Since Z is a reduced analytic set in a purely n-dimensional normal rigid space X, if A meets a connected affinoid subdomain $U \subseteq X$ then A contains U (here we use normality). Thus, $\{A, B\}$ is an admissible cover of Z by disjoint admissible opens. But Z is connected and A is non-empty, so $Z = A$. Thus, Z contains every connected open affinoid of X which it meets, so $\{Z, Z - X\}$ is an admissible cover of X. But X is connected and Z is non-empty, so $Z = X$.

Now suppose that $X = Z_1 \cup Z_2$ for analytic sets Z_i. If $Z_1 \neq X$, then Z_2 contains the non-empty admissible open $X - Z_1$, so $Z_2 = X$. □

Due to lack of a reference, we include a proof of the following lemma, the first part of which was used above.

LEMMA 2.1.5. — *Let A be a k-affinoid domain of dimension d. Then all maximal ideals of A have height d. If k'/k is a complete extension field, then $A\widehat{\otimes}_k k'$ is equidimensional with dimension d.*

Proof. — If $A = T_d(k)$, then all k-rational maximal ideals clearly have height d, by computing the completions at such points. For general maximal ideals, we may make a finite base change on k to reduce to the case of rational points. This settles the case of Tate algebras. By Noether normalization [BGR], 6.1.2/2, there is a finite injection of domains $T_n \hookrightarrow A$. Since T_n is factorial [BGR], 5.2.6/1, hence integrally closed, we may use the going-up and going-down theorems [Mat2], 9.4 to deduce that all maximal ideals of A have the same height n. Thus, $n = d$.

Now let k'/k be a complete extension and let $A' = A\widehat{\otimes}_k k'$. Since $T_d(k) \to T_d(k')$ is flat and $A' = A\widehat{\otimes}_{T_d(k)}T_d(k') = A \otimes_{T_d(k)} T_d(k')$ (by the finiteness of $T_d(k) \to A$ and the first part of Lemma 1.1.5), we see that the injection $T_d(k) \hookrightarrow A$ induces an *injection* $T_d(k') \hookrightarrow A'$ by base change. Let $\mathfrak{p}'$ be a minimal prime of A'. If $\mathfrak{p}'$ contracts to (0) in $T_d(k')$, then $T_d(k') \to A'/\mathfrak{p}'$ is a finite injection, so the k'-affinoid domain $A'/\mathfrak{p}'$ has the same dimension d as $T_d(k')$. Thus, it suffices to show that $\mathrm{Spec}(A') \to \mathrm{Spec}(T_d(k'))$ takes generic points to generic points.

Since $T_d(k) \to A$ is a finite injection of noetherian domains, there exists some non-zero $f \in T_d(k)$ so that $T_d(k)_f \to A_f$ is a (finite) flat map. Base change by $T_d(k) \to T_d(k')$ implies that $T_d(k')_f \to A'_f$ is a finite flat map, so $\mathrm{Spec}(A'_f) \to \mathrm{Spec}(T_d(k')_f)$ sends generic points to generic points. It remains to show that f is a unit at each generic point of $\mathrm{Spec}(A')$ and $\mathrm{Spec}(T_d(k'))$. Since f is not a zero-divisor in either $T_d(k)$ or in A, by the flatness of $A \to A'$ and $T_d(k) \to T_d(k')$ we see that f is not a zero divisor in A' or $T_d(k')$. Thus, f does not lie in any minimal primes of A' or $T_d(k')$, as desired. □

In the terminology of [CM], Lemma 2.1.4 says that the "Zariski closure" of a non-empty admissible open in a connected normal space is the whole space. This will later imply that for reduced rigid spaces, the definition of "irreducible component" in [CM] is equivalent to the one we will give.

2.2. Irreducible components.

Let $p : \widetilde{X} \to X$ be the normalization of a rigid space X and Z an analytic set in $\widetilde{X}$. Since p is a finite map, $p(Z)$ is an analytic set in X, always understood to have its canonical structure of reduced rigid space. A useful preliminary lemma is:

LEMMA 2.2.1. — *Let $p : \widetilde{X} \to X$ be the normalization of a rigid space X.*

1. *If U is a clopen set in $\widetilde{X}$, then the analytic set $p^{-1}(p(U)) - U$ is nowhere dense in $\widetilde{X}$. In particular, if $V \neq U$ are distinct clopen sets, then $p(U) \neq p(V)$.*

2. *If $\widetilde{X}_i$ is a connected component of $\widetilde{X}$, then $p_i : \widetilde{X}_i \to p(\widetilde{X}_i) = X_i$ is a normalization.*

3. *Assume X is reduced and give all $X_i = p(\widetilde{X}_i)$ the reduced structure. Then $X_i \neq X_j$ for $j \neq i$, any quasi-compact admissible open in X meets only finitely many X_i's, and there is an analytic set $Z_i \subsetneq X_i$ such that the admissible (Zariski) open $X_i - Z_i$ in X_i is non-empty, normal, and disjoint from all X_j's for $j \neq i$. In particular, $X_i - Z_i$ is an admissible (Zariski) open in X.*

Proof. — For the first part, we can work locally on X, so without loss of generality X is affinoid. Then we may appeal to the explicit construction of normalizations in the affinoid case and the analogue for Spec of noetherian rings.

For the second part, we may assume X is non-empty. Let $X_i = p(\widetilde{X}_i)$, an analytic set in X. By the first part, $X_i \neq X_j$ for $i \neq j$. It is obvious from the definitions that $\{X_i\}$ is a "locally finite" set on X, in the sense that a quasi-compact admissible open meets only finitely many X_i's. To show that $p_i : \widetilde{X}_i \to X_i$ is a normalization, we may certainly assume that X is reduced. Clearly p_i is finite. Also, by Lemma 2.1.4 and the surjectivity of p_i, formation of preimage under p_i takes nowhere dense analytic sets to nowhere dense analytic sets. In fact, this same reasoning shows that any analytic set in X_i which is not the whole space is nowhere dense. By Theorem 2.1.2, it therefore remains to check that there is a proper analytic set Z_i in X_i satisfying the conditions of the third part of the lemma.

Define

$$W_i = \bigcup_{j \neq i} (X_i \cap X_j).$$

This is an analytic set, by the local finiteness of the set of irreducible components of X. If $W_i = X_i$ then the clopen union $U \subset \widetilde{X}$ of the $\widetilde{X}_j$'s for $j \neq i$ has the property that $p^{-1}(p(U)) = \widetilde{X}$. By the first part of the lemma, the complement $\widetilde{X}_i$ of U in $\widetilde{X}$ must therefore be nowhere dense in $\widetilde{X}$, an absurdity. Thus, $X_i - W_i$ is a non-empty admissible open set in X_i. Moreover, since p is surjective, $X_i - W_i$ is an admissible open (even Zariski open) in X, since it is equal to the complement in X of the analytic set which is the "locally finite" union of the X_j's over $j \neq i$.

The non-normal locus N_i of X_i is a proper analytic set (since X_i is reduced), so the union $Z_i = N_i \cup W_i$ is a proper analytic set in X_i (by considering preimages in $\widetilde{X}_i$ under the surjective p_i and using

Lemma 2.1.4). But N_i meets $X_i - W_i$ in the non-normal locus of $X_i - W_i$ as an admissible open in X (not just in X_i), so $X_i - Z_i$ is a non-empty admissible open in X (even Zariski open) which is normal. Thus, p (and hence p_i) is an isomorphism over this open. □

In complex analytic geometry, one develops the global theory of irreducible decomposition by using the connected components of the normalization [CAS], 9.2.2. However, the proofs typically make use the of the fact that a reduced complex analytic space is manifold (*i.e.* smooth) on the complement of a nowhere dense analytic set. Since this is not true in rigid geometry over a non-perfect field, we need to argue differently in order to get similar results in the general rigid case. We begin with the following fundamental definition.

DEFINITION 2.2.2. — The *irreducible components* of a rigid space X are the reduced analytic sets of the form $X_i = p(\widetilde{X}_i)$, with $\widetilde{X}_i$ the connected components of the normalization $p : \widetilde{X} \to X$. We say that X is *irreducible* if it is non-empty and has a unique irreducible component. An analytic set Z in X is said to be *irreducible* if it is irreducible as a rigid space with its reduced structure.

Clearly the notions of irreducibility and irreducible component are unaffected by passing to the reduced space, and any non-empty rigid space is covered by its irreducible components, only finitely many of which meet any quasi-compact admissible open (so clearly any union of irreducible components of a rigid space is an analytic set). By Lemma 2.1.1, $\mathrm{Spec}(\mathcal{O}_{X,x})$ is irreducible if and only if there is a unique irreducible component of X passing through x. Also, by the second part of Lemma 2.2.1, the irreducible components of a non-empty rigid space *are* irreducible! By abuse of notation, if $Z \hookrightarrow X$ is a closed immersion whose underlying reduced analytic set is an irreducible component of X, we may sometimes say that Z is an irreducible component of X.

For any normalization $p : \widetilde{X} \to X$ and $x \in X$, there is some $x' \in p^{-1}(x)$ such that $\dim \mathcal{O}_{\widetilde{X},x'} = \dim \mathcal{O}_{X,x}$, so it is clear that if X is irreducible, then $\dim \mathcal{O}_{X,x}$ is the same integer for all $x \in X$, called the *dimension* of X (and this is equal to the dimension of $\widetilde{X}$). In particular, a rigid space is equidimensional if and only if all irreducible components have the same dimension. We first want to relate the above definition with the usual notion of irreducibility from algebraic geometry.

LEMMA 2.2.3. — *Let X be a non-empty rigid space. Then X is irreducible if and only if X is not the union of two proper analytic subsets. Also, X is irreducible if and only if the only analytic set Z in X which set-theoretically contains a non-empty admissible open of X is $Z = X$.*

Proof. — Let $p : \widetilde{X} \to X$ be the surjective normalization map. Suppose X is irreducible in our sense and $X = Z_1 \cup Z_2$ for analytic sets Z_1, Z_2 in X. Thus, $\widetilde{X}$ is a connected normal space surjecting onto X and $\widetilde{X} = p^{-1}(Z_1) \cup p^{-1}(Z_2)$. By Lemma 2.1.4 and the surjectivity of $\widetilde{X} \to X$, $X = Z_1$ or $X = Z_2$. Conversely, suppose that X is not the union of two proper analytic sets and $\{U_1, U_2\}$ is an admissible cover of $\widetilde{X}$ with U_1 and U_2 disjoint and non-empty. Consider the images $Z_i = p(U_i)$, where $p : \widetilde{X} \to X$ is the normalization map. These are analytic sets since p is finite, and their union is X. By hypothesis, some $Z_i = X = p(\widetilde{X})$, so by Lemma 2.2.1 some U_i has nowhere dense complement in $\widetilde{X}$, an absurdity.

For the last part, suppose that X is irreducible, U is a non-empty admissible open in X, and Z is an analytic set containing U. Then $p^{-1}(Z)$ is an analytic set in the connected normal space $\widetilde{X}$ containing the non-empty admissible open $p^{-1}(U)$. Thus, $p^{-1}(Z) = \widetilde{X}$ by Lemma 2.1.4, so $Z = X$. Conversely, if X is not irreducible, so X is the union of two proper analytic sets Z_1 and Z_2, then the non-empty admissible open $X - Z_1$ set-theoretically lies inside of the proper analytic set Z_2. □

THEOREM 2.2.4. — *Let X be a rigid space over k.*

1. *The mapping $\widetilde{X}_i \mapsto p(\widetilde{X}_i)$ is a bijection between connected components of $\widetilde{X}$ and irreducible components of X.*

2. *Each irreducible analytic set in X lies in an irreducible component of X.*

3. *If Σ is a set of distinct irreducible components of X, then the union X_Σ of the $X_i \in \Sigma$ is an analytic set in X. Giving X_Σ its reduced structure and defining $\widetilde{X}_\Sigma$ to be the union of the connected components of $\widetilde{X}$ corresponding to the elements of Σ, $p_\Sigma : \widetilde{X}_\Sigma \to X_\Sigma$ is a normalization. In particular, the irreducible components of X_Σ are exactly those $X_i \in \Sigma$.*

Proof. — The first part follows from Lemma 2.2.1. For the second part, let Z be an irreducible analytic set in X, and $z \in Z$ a point. By considering an affinoid open U in X around z and the construction of normalizations, we can certainly find a finite set Σ of irreducible components X_i of X whose union contains $U \cap Z$. Since Z is irreducible, by Lemma 2.2.3 the union of the finitely many $X_i \in \Sigma$ contains Z, so at least one of the $X_i \in \Sigma$ must contain Z.

Now let Σ be as in the third part. By the local finiteness of Σ, X_Σ is an analytic set. Using the reduced structure, we want the finite map $\widetilde{X}_\Sigma \to X_\Sigma$ to be a normalization map. For each $X_i \in \Sigma$, let Z_i be an analytic set in X_i as in the third part of Lemma 2.2.1. Let Z_Σ be the union of Z_i's for $X_i \in \Sigma$. By the local finiteness of Σ, Z_Σ is an analytic set in X and clearly is nowhere dense in X_Σ. Likewise, $p_\Sigma^{-1}(Z_\Sigma)$ is nowhere dense in $\widetilde{X}_\Sigma$. By Theorem 2.1.2, we just have to show that $\widetilde{X}_\Sigma \to X_\Sigma$ is an isomorphism over the complement of Z_Σ. It suffices to check that the union $X_\Sigma - Z_\Sigma$ of the disjoint Zariski opens $X_i - Z_i$ in X (for $X_i \in \Sigma$) is itself an admissible open, with the $X_i - Z_i$'s giving an admissible cover. This follows from the next lemma. □

LEMMA 2.2.5. — *Let W be a rigid space and $\{W_i\}$ a set of mutually disjoint Zariski opens. Assume this set is locally finite in W (i.e. only finitely many W_i's meet any quasi-compact admissible open in W). The union U of the W_i's is an admissible open and the W_i's are an admissible cover of U.*

Proof. — This can be checked in the case of affinoid W, so we reduce to the claim that if $W = \mathrm{Sp}(A)$ is an affinoid and $I_1, \ldots, I_n$ are ideals in A, with $Z_i = \mathrm{Sp}(A/I_i)$, $W_i = W - Z_i$, then the union U of the W_i's is an admissible open and the W_i's are an admissible cover of U. Since U is a finite union of Zariski opens, it is also Zariski open. Thus, the admissibility of U and of any covering of U by Zariski opens follows from [BGR], 9.1.4/7. □

In [CM], a *component part* of a rigid space X is defined to be the minimal closed immersion $Z \hookrightarrow X$ through which a fixed admissible open $U \hookrightarrow X$ factors; it is easy to see by a noetherian argument that such a minimal Z exists. It is reduced if X is reduced. The following theorem relates the terminology in [CM] to our setup.

COROLLARY 2.2.6. — *Let X be a reduced rigid space. The component parts of X are exactly the unions of irreducible components of X (with their reduced structure). An irreducible component part is the same thing as an irreducible component.*

Proof. — Let U be an admissible open in X, and let Σ denote the set of irreducible components of X which contain U. If $p : \widetilde{X} \to X$ is the normalization, then Σ is in canonical bijection with the set of connected components of $\widetilde{X}$ which meet $p^{-1}(U)$. Let Z be an analytic set in X which contains U. By Lemma 2.1.4, $p^{-1}(Z)$ is contains all connected components of $\widetilde{X}$ that meet $p^{-1}(U)$, so Z contains all elements of Σ. By the final part of Theorem 2.2.4, we conclude that Z is a union of irreducible components of X. Since X is reduced, every component part of X must be as claimed.

Conversely, for each irreducible component X_i of X, there is a non-empty Zariski open U_i in X disjoint from X_j for all $j \neq i$ (by Lemma 2.2.1). If Σ is a non-empty set of X_i's, then by Lemma 2.2.5, the union U_Σ of the U_i's for $X_i \in \Sigma$ is an admissible open with the U_i's for $X_i \in \Sigma$ giving an admissible cover. Since X is reduced, the smallest closed immersion $Z \hookrightarrow X$ through which $U_\Sigma \to X$ factors is $X_\Sigma \hookrightarrow X$, where the union X_Σ of all $X_i \in \Sigma$ is given its reduced structure. Thus, X_Σ is a component part. □

The following corollary (when combined with the preceding one) generalizes [CM], 1.2.3 in the reduced case.

COROLLARY 2.2.7. — *Let X be a rigid space which is equidimensional of dimension d. The analytic sets in X which are a union of irreducible components of X are exactly those which are equidimensional of dimension d.*

Proof. — We just need to show that if X is irreducible and Z is an irreducible analytic set in X, then Z fills up all of X if Z has the same dimension as X. Let $p : \widetilde{X} \to X$ be the normalization and d the common dimension of X and $\widetilde{X}$. Since p is finite and surjective, if $z \in Z$ has $\dim \mathcal{O}_{Z,z} = d$ then there exists $z' \in p^{-1}(z)$ with $\dim \mathcal{O}_{p^{-1}(Z),z'} = d$. But $\widetilde{X}$ is normal of dimension d, so $p^{-1}(Z)$ must contain an open around z'. By the connectedness of $\widetilde{X}$ and Lemma 2.1.4, $p^{-1}(Z) = \widetilde{X}$. Therefore $Z = X$. □

We can now formulate the uniqueness of global irreducible decomposition.

COROLLARY 2.2.8. — *Let X be a non-empty rigid space, $\Sigma = \{Y_j\}$ a set of distinct irreducible analytic sets in X which is locally finite in the sense that each quasi-compact admissible open U in X meets only finitely many Y_j's. Assume that the union of the Y_j's is X and that Σ is minimal in the sense that no Y_j is contained in the union of the Y_i's for $i \neq j$. Then Σ coincides with the set of irreducible components of X.*

Proof. — Without loss of generality, X is reduced. Let $\{X_i\}$ be the set of (reduced) irreducible components of X, and choose (by Lemma 2.2.1) a non-empty normal Zariski-open U_i in X which is disjoint from all irreducible components of X aside from X_i. By the local finiteness and covering assumptions on Σ, some Y_j contains a non-empty open in U_i, and hence contains all of X_i. By the second part of Theorem 2.2.4 and the third part of Lemma 2.2.1, $Y_j = X_i$. Thus, Σ contains as a subset the set of irreducible components of X. By the minimality assumption, Σ is the set of irreducible components of X. □

We end this section by recording a useful easy fact which generalizes [CM], 1.2.4 in the reduced case.

COROLLARY 2.2.9. — *Let X be a rigid space, with irreducible components $\{X_i\}$. For each non-empty admissible open U in X, each irreducible component U_j of U is contained in a unique X_i. For all i, $X_i \cap U$ is a (possibly empty) union of irreducible components of U.*

Proof. — Since the formation of the normalization is local on the base, $X_i \cap U$ is the image of a clopen under the normalization map for U, so the last part is clear. Lemma 2.2.1 implies that U_j contains a non-empty connected normal admissible open U' which is also an admissible open in U, and hence in X. The existence and uniqueness of the X_i containing U_j therefore follows from Lemma 2.2.3 and the construction of the X_i's. □

2.3. Relation with irreducible components on schemes and formal schemes.

Let X be a non-empty locally finite type k-scheme and, in case k has discrete absolute value, $\mathfrak{X}$ a non-empty object in FS_R which is R-*flat*. Let $\{X_i\}$ and $\{\widetilde{\mathfrak{X}}_i\}$ denote the set of irreducible components of X and the set of connected components of the normalization $\widetilde{\mathfrak{X}}$ respectively, given their canonical reduced structure.

Our goal in this section is to prove the following theorem:

THEOREM 2.3.1. — *The set of irreducible components of X^{an} is $\{X_i^{\mathrm{an}}\}$ and the set of irreducible components of $\mathfrak{X}^{\mathrm{rig}}$ is $\{\mathfrak{X}_i^{\mathrm{rig}}\}$, where $\mathfrak{X}_i^{\mathrm{rig}}$ is the image of $\widetilde{\mathfrak{X}}_i^{\mathrm{rig}}$ in $\mathfrak{X}^{\mathrm{rig}}$.*

Proof. — Note that $(\cdot)^{\text{an}}$ and $(\cdot)^{\text{rig}}$ take finite surjections to finite surjections. Indeed, the case of $(\cdot)^{\text{an}}$ is clear and for $(\cdot)^{\text{rig}}$ use [deJ], 7.2.2, 7.2.4(*e*). Thus, by Theorem 2.1.3, it is enough to consider the case where X and $\mathfrak{X}$ are connected, normal, and non-empty, and we want to show that the normal rigid spaces X^{an} and $\mathfrak{X}^{\text{rig}}$ are connected and non-empty. In the case of formal schemes, $\mathfrak{X}^{\text{rig}}$ is connected by [deJ], 7.4.1 and $\mathfrak{X}^{\text{rig}}$ is non-empty by [deJ], 7.1.9 and our R-flatness assumption. Thus, we now concentrate exclusively on the scheme case, with X a normal connected locally finite type k-scheme. Clearly X^{an} is non-empty and normal. We want X^{an} to be connected.

As a first step, we want to reduce to the case when k is algebraically closed and X is affine. Let $\{U_i\}$ be an open affine covering of X, so the U_i are irreducible opens in X, any two of which can be "linked up" by a finite chain of opens in this covering (since X is connected). Thus, $\{U_i^{\text{an}}\}$ is an admissible covering of X^{an}, any two of which can be "linked up" by a finite chain of opens in this covering, so it suffices to show that the U_i^{an} are connected. Hence, we may assume X is affine, so in particular is quasi-compact. Therefore, by [EGA], IV_2, 4.5.11, there is a finite Galois extension k'/k such that the irreducible (= connected) components of the normal (affine) k'-scheme $X' = k' \times_k X$ remain irreducible after arbitrary change of the base field over k'. Since $X' \to X$ is a Galois covering with Galois group $G = \text{Gal}(k'/k)$, so is $X'^{\text{an}} \to X^{\text{an}}$. By the following lemma, G acts transitively on the connected components of X', so it is enough to handle the connected components of X' (viewed as schemes over k'). This will allow us to assume that X is a normal affine k-scheme which is geometrically irreducible over k.

LEMMA 2.3.2. — *Let $f : Z' \to Z$ be a finite map of rigid spaces over k, with Z connected. Assume that f is Galois with Galois group G. That is, there is a map of groups $G \to \text{Aut}(Z'/Z)$ such that the canonical map $G \times Z' \to Z' \times_Z Z'$ given by $(g, z') \mapsto (z', gz')$ is an isomorphism. Then G acts transitively on the connected components of Z'. In particular, Z' has only finitely many connected components.*

Proof. — If $U' \subseteq Z'$ is the union of the G-orbits of a connected component of Z', then U' is a non-empty G-stable admissible open. Working over an admissible open affinoid covering of Z, we see by Galois descent of rings that $U' = f^{-1}(U)$ for a unique non-empty clopen $U \subseteq Z$. But Z is connected, so $U = Z$. Thus, $U' = Z'$, so G acts transitively on the connected components of Z'. □

Let $k' = \widehat{\overline{k}}$, so $X' = X \times_k k'$ is an *irreducible* affine k'-scheme. In particular, the normalization of X' is affine and connected, so if we can handle the case of a normal connected affine over an algebraically closed complete ground field, then by Theorem 2.1.3 it will follow that $X^{\text{an}} \times_k k' \simeq (X \times_k k')^{\text{an}}$ is connected (this isomorphism of rigid spaces is explained in the appendix). To deduce the connectedness of X^{an} from this, we claim more generally that if Y is a quasi-separated rigid space over k and k'/k is an extension field such that $Y \times_k k'$ is connected, then Y is connected. Let $\{U, V\}$ be an admissible cover of Y by disjoint admissible opens, so we get by base change and Lemma 3.1.1 an analogous such cover $\{U', V'\}$ of the connected space $Y \times_k k'$. Thus, one of U' or V' is empty. By the injectivity of $A \to A\widehat{\otimes}_k k'$ for k-affinoids A, we see that U or V is empty, so Y is connected. This completes the reduction to the case where k is algebraically closed and X is normal, affine, and connected.

Now that we have an algebraically closed ground field, the method of proof in the complex analytic case [SGA1], XII, Prop 2.4 can be carried over. The only extra input we need from rigid geometry is a theorem of Lütkebohmert [L1], Thm 1.6 which asserts that if Y is a normal rigid space over an *algebraically closed* field and Z is a nowhere dense analytic set in Y with complement U, then every bounded element in $\Gamma(U, \mathcal{O}_Y)$ extends uniquely to $\Gamma(Y, \mathcal{O}_Y)$. Applying this to to idempotents, it follows that Y is connected if and only if U is connected. This applies in particular to the analytification of a dominant open immersion $X \hookrightarrow P$ with P a normal projective k-scheme, thereby reducing us to the case in which X is projective. Then one needs rigid GAGA in the case of projective schemes over a (possibly non-discretely-valued) complete field. A proof of this based on Serre's arguments over $\mathbb{C}$ can be found in [Ko], Satz 4.7 (though it should be noted that once the base is not a field, the formulation of GAGA in [Ko] is not an analogue of the complex-analytic version [SGA1], XII, but instead is more in the spirit of GAGA for noetherian formal schemes [EGA], III_1, 5.1.2). □

3. Behavior with respect to base change.

We want to study how formation of irreducible components behaves with respect to base change (always understood to mean extension of the base field, rather than formation of a fiber product with a rigid space over the same base field). For example, if X is a quasi-compact quasi-separated rigid space over k (such as a proper rigid space over k), or more generally has only finitely many irreducible components, does there exist a finite separable extension k'/k such that the irreducible components of $X \times_k k'$ remain the irreducible components after arbitrary base change to a complete extension of k'? In [EGA], IV_2, 4.5ff this type of question is systematically studied in the case of schemes, particularly those locally of finite type over a field. We want to obtain similar results, but must deal with the fact that elements of Tate algebras (in contrast to polynomial rings) typically have infinitely many non-zero coefficients.

3.1. Preliminary remarks on base change.

The basic construction and local properties of the functor $(\cdot) \times_k k'$ on quasi-separated rigid spaces are given in [BGR], 9.3.6. The reason for quasi-separatedness in this construction is that we know how base change should be defined for affinoids (*i.e.* we should define $\mathrm{Sp}(A) \times_k k' = \mathrm{Sp}(A \widehat{\otimes}_k k')$ for any k-affinoid algebra A), and in order to globalize this to more general rigid spaces X, it seems necessary to require that the overlap of any two open affinoids is quasi-compact (*i.e.* covered by finitely many open affinoids). Such spaces are precisely those for which the diagonal map $X \to X \times_k X$ is quasi-compact, which is to say that X is quasi-separated. The essential reason why we need this quasi-compactness condition to define $(\cdot) \times_k k'$ is because of the finiteness conditions built into the definition of the Grothendieck topology on affinoid rigid spaces.

Before investigating geometric questions, it seems worthwhile to make some remarks which clarify the role of the quasi-separatedness condition in our study of base change below. If X is an arbitrary rigid space over k, k'/k is a complete extension, and $\{X_i\}$ is an admissible covering by affinoid opens, then the overlaps $X_{ij} = X_i \cap X_j$ might not be quasi-compact (if X is not quasi-separated), but they are open in an affinoid and so are at least quasi-separated (even separated). Thus, the base change $X'_{ij} = X_{ij} \times_k k'$ makes sense. If the maps $X'_{ij} \to X'_i = X_i \times_k k'$ and $X'_{ij} \to X'_j = X_j \times_k k'$ induced by functoriality are *open immersions*, then we can glue X'_i and X'_j along X'_{ij} and thereby (with a little care) define $X \times_k k'$ without a quasi-separatedness condition. If it were always true that for any open immersion $\iota : U \hookrightarrow X$ into a quasi-separated rigid space, the base change $\iota' = \iota \times_k k'$

is again an open immersion, then we could define a base change functor $(\cdot) \times_k k'$ on arbitrary rigid spaces over k, with all of the usual properties one expects.

It is immediate from the construction of the base change functor that ι' as above is always an injective local isomorphism and that ι' is an open immersion when ι is quasi-compact. This enables us to reduce the general problem of whether ι' is an open immersion to the case where X is affinoid. From the "classical" point of view (as in [BGR]), the case of non-quasi-compact ι appears to be rather difficult to settle, even when X is affinoid. However, the only case of non-quasi-compact ι which we need is when U is the complement of an analytic set Z in X; that is, when U is Zariski open. In this case, we have a more precise statement, given by the following easy lemma:

LEMMA 3.1.1. — *Let $\iota : U \hookrightarrow X$ be a Zariski open set in a quasi-separated rigid space X over k. Let k'/k be a complete extension field, let $\iota' : U' \to X'$ be the base change of ι, and let $Z \hookrightarrow X$ be a closed immersion whose underlying space in X is the complement of U. The analytic set $Z' = Z \times_k k'$ in X' is the complement of $\iota'(U')$ and ι' is an open immersion. In particular, $\iota'(U')$ is a Zariski open set in X'.*

Proof. — Keeping in mind how the base change functor is defined, it is not difficult to reduce to the case where $X = \mathrm{Sp}(A)$ is affinoid, so $Z = \mathrm{Sp}(A/I)$ for some ideal I in A. Choose generators $f_1, \dots, f_n$ of I, so by the Maximum Modulus Theorem, U has an admissible covering by Laurent domains of the form $U_\varepsilon = \mathrm{Sp}(A\langle \varepsilon/f_i \rangle) = \{x \in X \mid |f_i(x)| \geq \varepsilon\}$ for $\varepsilon \in |k^\times|$ (i.e. $\varepsilon = |c|$ for some $c \in k^\times$). By the construction of the base change functor, U' has an admissible covering by the open affinoids $U'_\varepsilon = \mathrm{Sp}(A'\langle \varepsilon/f'_i \rangle)$, with $f'_i = f_i \widehat{\otimes} 1 \in A' = A \widehat{\otimes}_k k'$. By restriction, the map $U' \to X'$ induces a map $U'_\varepsilon \to X'$ which corresponds exactly to the canonical map $A' \to A'\langle \varepsilon/f'_i \rangle$, so $\iota'(U')$ is exactly the complement of the analytic set in X' cut out by the f'_i's. But this analytic set is exactly the one corresponding to Z'. Moreover, the affinoid opens $\iota'(U'_\varepsilon)$ in X' clearly form an admissible covering of $\iota'(U') = X' - Z'$, so $U' \to \iota'(U')$ is an isomorphism. □

Because of the technical problem that we do not know if the base change functor for (quasi-)separated rigid spaces takes open immersions to open immersions, we are forced to impose quasi-separatedness conditions whenever we discuss base change questions (unless $[k' : k]$ is finite, but this is not the case which interests us). However, all proofs in §3.2–§3.4 are written so that the *only* role of the quasi-separatedness hypothesis is in the definition of the functor $(\cdot) \times_k k'$. If the above technical question about open immersions can be resolved in the affirmative, then everything we say about base change in § 3.2–§ 3.4 will carry over verbatim without any quasi-separatedness conditions.

3.2. Geometric connectivity.

We begin by studying connected components, as these are somewhat simpler than irreducible components (and constitute a necessary prerequisite, due to how irreducible components are defined). The following is analogous to a special case of [EGA], IV_2, 4.5.13.

THEOREM 3.2.1. — *Let X be a quasi-separated connected rigid space over k. Assume that X has a k-rational point. For any extension k'/k, $X' = X \times_k k'$ is connected.*

Proof. — As we saw near the end of the proof of Theorem 2.3.1, a quasi-separated rigid space is connected if it is so after an extension of the base field. In particular, to prove the theorem for some k', it suffices to check it after replacing k' by any desired extension. We first treat the case where k'/k is a finite extension, so we may consider just the two cases when the finite extension k'/k is Galois and when it is purely inseparable. The purely inseparable case follows from the more general fact that if k'/k is a finite purely inseparable extension, taking images and preimages under the natural finite bijective map $\pi : X' = X \times_k k' \to X$ sets up a bijection between admissible opens (resp. admissible covers) on X and admissible opens (resp. admissible covers) on X' (so π is a "homeomorphism" of Grothendieck topologies). This assertion can be checked in case $X = \mathrm{Sp}(A)$ is affinoid, and it is enough to check that if U' is an affinoid open in $X' = \mathrm{Sp}(A \otimes_k k')$, then $U' = U \times_k k'$ for a quasi-compact admissible open U in X (necessarily unique). We may assume k has positive characteristic p. By the Gerritzen-Grauert Theorem [BGR], 7.3.5, U' is a finite union of rational subdomains, so since $(A \otimes_k k')^{p^n} \subseteq A$ for some n, it is easy to define a finite union U of rational subdomains in X (necessarily a quasi-compact admissible open in X [BGR], 9.1.4/4) such that $U' = U \times_k k'$ as opens in X'. In fact, extra work shows that U is affinoid since U' is, but we don't give the proof since we don't need this.

Now suppose k'/k is finite Galois with Galois group G, so $X' \to X$ is Galois with Galois group G. By Lemma 2.3.2, G acts transitively on the connected components of X'. Since the fiber of $X' \to X$ over a k-rational point of X consists of a *single* point on X', X' must have only one connected component. This proves the theorem for finite extensions of k. Thus, it is enough to prove the theorem after replacing k by any desired finite extension.

There is an admissible cover of X by connected affinoids $U_i = \mathrm{Sp}(A_i)$, and since X is connected there is a finite chain of these "linking up" any U_i and $U_{i'}$. Some U_{i_0} has a k-rational point, so joining this up to any other U_i we easily reduce to the case where X is quasi-compact. Let $\{U_i\}$ be an admissible covering of X by finitely affinoids. We need the following lemma.

LEMMA 3.2.2. — *Let A be a non-zero k-affinoid. There exists a finite extension k'/k such that the connected components of* $\mathrm{Sp}(A \otimes_k k')$ *have k'-rational points.*

Proof. — Let $B = A\widehat{\otimes}_k\widehat{\overline{k}}$. Since B is noetherian and the natural map $A \otimes_k k' \to B$ is injective for any finite k'/k inside of $\overline{k}$, the number of idempotents of $A \otimes_k k'$ is bounded by the number of idempotents of B (more geometrically, $\mathrm{Spec}(B) \to \mathrm{Spec}(A \otimes_k k')$ is surjective, by Lemma 1.1.5). Thus, we can find a "sufficiently large" k' of finite degree over k such that the connected components of $\mathrm{Sp}(A \otimes_k k')$ remain connected after any further finite base change over k'. Enlarging k' a little more, we may assume that all connected components of $\mathrm{Sp}(A \otimes_k k')$ have rational points. □

By Lemma 3.2.2, we can choose a finite extension k'/k so that each connected component of each $U_i \times_k k'$ has a k'-rational point. Replacing X by $X \times_k k'$ and k by k', we may assume that the connected non-empty X has an admissible covering by finitely many connected affinoids $\{U_i\}$, each of which has a k-rational point. Since X is connected and the covering $\{U_i\}$ is admissible, each U_{i_0} can be "linked up" to each U_{i_1} by some chain of U_j's, so it suffices to treat the U_i's separately. Thus, we reduce to the case where X is affinoid, say $X = \mathrm{Sp}(A)$, so X remains connected after any finite extension of k. The rest of this proof follows a suggestion by Bosch, and it is simpler and more geometric than my original proof (which used Noether normalization and an algebraic analysis of Tate algebras and power series rings).

Let $K = \widehat{\overline{k}}$. If we can prove that $\overline{X} = K \times_k X$ is connected, then we may replace k by the algebraically closed field K. Suppose that $e \in \overline{A} = K\widehat{\otimes}_k A$ is an idempotent on $\overline{X}$. Replacing k by a suitably large finite extension inside of $\overline{k}$ and using the definition of the completed tensor product, we can suppose that there is an element $e_0 \in A$ with $|1\widehat{\otimes}e_0 - e|_{\overline{A}} = \varepsilon < 1$. Here, $|\cdot|_{\overline{A}}$ is some norm defining the K-Banach topology on $\overline{A}$. Since $\overline{A}$ might be non-reduced even if A is reduced, we generally cannot use the sup-norm to define the topology on $\overline{A}$. However, by [BGR], 6.2.3/3, we at least have $|e_0(\overline{x}) - e(\overline{x})| \leq \varepsilon < 1$ for all $\overline{x} \in \overline{X}$, and of course for each $\overline{x} \in \overline{X}$, either $e(\overline{x}) = 0$ or $e(\overline{x}) = 1$. Consider the disjoint open affinoids $U = \{x \in X \mid |e_0(x)| = 1\}$ and $V = \{x \in X \mid |e_0(x)| \leq \varepsilon\}$ in X. The open affinoids $U \times_k K$ and $V \times_k K$ cover $\overline{X}$, so by Lemma 1.1.5 we see that U and V set-theoretically cover X. Thus, $\{U, V\}$ is an admissible covering of X by disjoint (affinoid) opens, so by the connectedness of X we see that $X = U$ or $X = V$. By base change to K, $|e_0(\overline{x})| = 1$ for all $\overline{x} \in \overline{X}$ or $|e_0(\overline{x})| \leq \varepsilon$ for all $\overline{x} \in \overline{X}$. Since $|e_0(\overline{x}) - e(\overline{x})| \leq \varepsilon < 1$ for all $\overline{x} \in \overline{X}$, it follows that $|e(\overline{x})| = 1$ for all $\overline{x} \in \overline{X}$ or $|e(\overline{x})| \leq \varepsilon < 1$ for all $\overline{x} \in \overline{X}$. But e is an *idempotent* in $\overline{A}$, so $e = 0$ or $e = 1$, as desired. This completes the reduction to the case where k is algebraically closed.

For any affinoid algebra B over a complete field K, let $\widetilde{B}$ denote the "reduction" of B, which is defined to be the quotient of the subring of power-bounded elements in B by the ideal of topologically nilpotent elements. $\widetilde{B}$ is an algebra over the residue field $\widetilde{K}$ of the valuation ring of K, and there will be no risk of confusion with our notation for normalization. The $\widetilde{K}$-algebra $\widetilde{B}$ is reduced and is functorial in B in an obvious manner. It is less obvious, but true, that $\widetilde{B}$ is a finite type algebra over $\widetilde{K}$ [BGR], 6.3.4/3 (and in particular, $\widetilde{B}$ is a Jacobson ring), so scheme-theoretic results can often be used to study $\widetilde{B}$. This concept of "reduction" is carefully studied in [BGR], §6.3–§6.4. Questions about affinoid spaces can sometimes be reduced to questions about the "reduction". For example, we claim that $\mathrm{Spec}(\widetilde{B})$ is connected if and only if $\mathrm{Sp}(B)$ is connected. There is a canonical surjective "reduction" map of sets $\pi : \mathrm{Sp}(B) = \mathrm{MaxSpec}(B) \to \mathrm{MaxSpec}(\widetilde{B})$ with the property that for any power-bounded element $b \in B$ (e.g., an idempotent) with image $\widetilde{b} \in \widetilde{B}$, the preimage of $\{x \in \mathrm{MaxSpec}(\widetilde{B}) \mid \widetilde{b}(x) \neq 0\}$ under π is the Laurent domain $\mathrm{Sp}(B\langle 1/b\rangle)$ [BGR], 7.1.5/2,4. Also, we note that since $\widetilde{B}$ is of finite type over field,

$$\{x \in \mathrm{MaxSpec}(\widetilde{B}) \mid \widetilde{b}(x) \neq 0\} = \mathrm{MaxSpec}(\widetilde{B}[1/\widetilde{b}])$$

and this is empty if and only if $\widetilde{b}$ is nilpotent. Thus, if $\mathrm{Spec}(\widetilde{B})$ is disconnected and $\widetilde{e} \in \widetilde{B}$ is a *non-trivial* idempotent, then taking $\widetilde{b} = \widetilde{e}$

and $\widetilde{b} = 1 - \widetilde{e}$ gives a set-theoretic covering of $\mathrm{Sp}(B)$ by disjoint *non-empty* Laurent domains. This covering is necessarily admissible, so $\mathrm{Sp}(B)$ is disconnected. Conversely, if $\mathrm{Sp}(B)$ is disconnected and $e \in B$ is a non-trivial idempotent, then the image $\widetilde{e}$ of e in $\widetilde{B}$ is an idempotent which is obviously non-trivial, so $\mathrm{Spec}(\widetilde{B})$ is disconnected.

Back in our above setting with an algebraically closed k and a connected affinoid $\mathrm{Sp}(A)$, let $A' = A\widehat{\otimes}_k k'$. The scheme $\mathrm{Spec}(\widetilde{A})$ is connected and we want to prove that $\mathrm{Spec}(\widetilde{A'})$ is connected. The residue field $\widetilde{k}$ is *algebraically closed* since k is algebraically closed [BGR], 3.4.1/5, so by [EGA], IV_2, 4.4.4, the scheme $\mathrm{Spec}(\widetilde{A}) \times_{\widetilde{k}} \widetilde{k}'$ is connected. Thus, it suffices to show that the canonical map of $\widetilde{k}'$-algebras

$$\widetilde{A} \otimes_{\widetilde{k}} \widetilde{k}' \to \widetilde{A'}$$

is an isomorphism. This map is visibly an isomorphism when A is a Tate algebra over k. In general, $\widetilde{A}$ is reduced and $\widetilde{k}$ is perfect, so the ring $\widetilde{A} \otimes_{\widetilde{k}} \widetilde{k}'$ is always *reduced* [EGA], IV_2, 4.6.1. Therefore, the general case can be deduced from the case of Tate algebras by means of a well-chosen surjection (called a distinguished epimorphism in [BGR], 6.4.3/2) from a Tate algebra onto A. The details are given in [B], §6, Satz 4, and the basic technical facts needed in this argument are also explained in [BGR], 2.7.3/2, 6.4.3. □

As in the case of schemes over a field, we say that a quasi-separated rigid space X over k is *geometrically connected* (relative to k, or over k) if $X \times_k k'$ is connected for all finite extensions k'/k. By Theorem 3.2.1 and its proof, it suffices to consider just finite separable extensions k'/k in this definition, and for any geometrically connected and quasi-separated X over k, $X \times_k k'$ is connected for arbitrary k'/k. We summarize the basic facts concerning geometric connectedness, analogous to what one knows in the case of schemes over a field:

COROLLARY 3.2.3. — *For a connected quasi-separated X and $x \in X$, with k' the separable closure of k in $k(x)$, there is a finite separable extension k''/k such that the connected components of $X \times_k k''$ are geometrically connected over k'' and there are no more than $[k' : k]$ such components.*

Proof. — Using Theorem 3.2.1, one can modify the proofs of [EGA], IV_2, 4.5.14–4.5.17 to apply in the rigid setting. □

3.3. Geometric reducedness and geometric normality.

Our study of "geometric irreducible components" will require some preliminary facts about "geometric reducedness" and "geometric normality". We say a quasi-separated rigid space X over k is *geometrically reduced* (relative to k, or over k) if $X \times_k k'$ is reduced for all k'/k and X is *geometrically normal* over k if $X \times_k k'$ is normal for all k'/k. In this section, when we write $k^{p^{-n}}$ for a field k, we mean the field of p^nth roots of elements of k in case k has characteristic $p > 0$, and we mean k if the characteristic is 0. Note that $k^{p^{-n}}$ is complete, though it might not have finite degree over k. After the results in this section were worked out, I found that [BKKN] considers questions along the lines of Lemma 3.3.1 below, but in a slightly less general context.

The perfect closure k_p of our ground field k is typically not complete if k has positive characteristic, so it is better to consider its completion $\widehat{k_p}$. Note that this is again perfect. Indeed, we may assume k has positive characteristic p and then if $a \in \widehat{k_p}$ is the limit of a sequence $a_i \in k_p$, the sequence of pth roots $a_i^{1/p} \in k_p$ *is* Cauchy, with limit a pth root of a (on a related note, if k_s/k is a separable closure of a complete field k, then by [BGR], 3.4.2/5 the completion $\widehat{k_s}$ is separably closed).

It is instructive to begin with two examples. Let k be a complete field of characteristic $p > 0$ such that the complete field $k^{p^{-1}}$ is of infinite degree over k (e.g., $k = F((t))$, where F is a field of characteristic p for which $[F : F^p]$ is infinite). Let $a_1, a_2, \ldots$ be a countable sequence of elements in k whose pth roots generate an extension of k of infinite degree, and such that $a_i \to 0$. Let $f = \sum a_i x^{ip} \in T_1$, and $A = T_1[Y]/(Y^p - f)$. This is an affinoid algebra over k and for any extension k'/k of finite degree, $A \otimes_k k'$ is reduced. However, $A \widehat{\otimes}_k k^{p^{-1}}$ is non-reduced. For $p \neq 2$, the two-dimensional affinoid $B = k \ll x, y, t \gg /(t^2 - (y^p - \sum a_i x^{ip}))$ remains normal after any finite extension of scalars and $B \widehat{\otimes}_k k^{p^{-1}}$ is reduced (so by Lemma 3.3.1 below, B is even geometrically reduced over k), but $B \widehat{\otimes}_k k^{p^{-1}}$ is *not* normal. Indeed, over $k^{p^{-1}}$ we can define $y' = y - \sum a_i^{1/p} x^i$ and have

$$B \widehat{\otimes}_k k^{p^{-1}} \simeq k^{p^{-1}} \ll x, y', y \gg /(t^2 - y'^p),$$

a domain which is not normal (look at t/y'). Thus, B is not geometrically normal over k. Unlike the case of "geometric" properties such as geometric connectedness or geometric regularity (*i.e.* smoothness), we cannot expect to determine geometric reducedness for rigid spaces by looking only at base change by *finite* extensions of k, and likewise for geometric normality (even if we assume geometric reducedness). The above examples are as bad as things can get, as Lemma 3.3.1 and Theorem 3.3.6 make clear.

LEMMA 3.3.1. — *Let X be a quasi-separated rigid space over k.*

1. *Assume that $X \times_k k^{p^{-1}}$ is reduced. Then $X \times_k k'$ is reduced for any k'/k. In particular, if k is perfect then any reduced rigid space over k is geometrically reduced.*

2. *If X is quasi-compact, then $(X \times_k k^{p^{-n}})_{\mathrm{red}}$ is geometrically reduced over $k^{p^{-n}}$ for large n.*

Proof. — We first consider the case where k is perfect, and we want to show that $X \times_k k'$ is reduced for any k'/k. It is not difficult to see that we may study the normalization of X instead of X, so without loss of generality, X is normal, connected, and even affinoid, say $X = \mathrm{Sp}(A)$ for a (normal) domain A. Let U be the regular locus of X, which coincides with the k-smooth locus since k is perfect [BL3], Lemma 2.8(b) (the point is that if X is a rigid space and $x \in X$ is a regular point with the finite extension $k(x)/k$ separable, then $\widehat{\mathcal{O}_{X,x}}$ is *k-isomorphic* to a formal power series ring over $k(x)$ in $d = \dim\widehat{\mathcal{O}_{X,x}} = \dim\mathcal{O}_{X,x}$ variables, so $\Omega^1_{X/k,x} \otimes_{\mathcal{O}_{X,x}} \widehat{\mathcal{O}_{X,x}} \simeq \widehat{\Omega}^1_{\widehat{\mathcal{O}_{X,x}}/k}$ is a finite free $\widehat{\mathcal{O}_{X,x}}$-module with rank d, which implies that $\Omega^1_{X/k,x}$ is a finite free $\mathcal{O}_{X,x}$-module with rank d). Thus, we have an open immersion $i : U \hookrightarrow X$ with the complement Z of U in X an analytic set given by a proper ideal I in A. Let $g_1, \dots, g_n$ be generators of I. By Lemma 3.1.1, for any k'/k and $A' = A\widehat{\otimes}_k k'$, the base change $i' : U' \hookrightarrow X' = \mathrm{Sp}(A')$ is again an open immersion with U' reduced and the complement of U' is defined by the $g_i' = g_i \widehat{\otimes} 1 \in A'$. If we can show that the map $A' = \Gamma(X', \mathcal{O}_{X'}) \to \Gamma(U', \mathcal{O}_{U'})$ is injective, then A' is reduced, as desired.

Suppose that $f' \in A'$ vanishes on U'. We want to show that $f' = 0$. For every maximal ideal $\mathfrak{m}'$ of A' which does not contain some g_i' (*i.e.* $\mathfrak{m}' \in \mathrm{Sp}(A')$ lies in U'), f' vanishes in $A'_{\mathfrak{m}'}$. Any maximal ideal $\mathfrak{m}'$ of A' containing the annihilator of f' must then contain some g_i', or in other words must contain $g' = g_1' \cdots g_n'$. Since A' is a Jacobson ring [BGR], 6.1.1/3, it follows that the annihilator of f' contains a power of g'. To deduce that $f' = 0$, we just need to check that g' is not a zero divisor in A'. But $g = g_1 \cdots g_n \in A$ is non-zero and A is a domain, so g is not a zero divisor in A. The flatness of $A \to A'$ (see Lemma 1.1.5) then implies that $g' \in A'$ is not a zero divisor, as desired. This settles the case of a perfect base field k.

Now we may assume k is not perfect, or more generally that k has positive characteristic p. Without loss of generality, $X = \mathrm{Sp}(A)$ is affinoid. For a general affinoid A over k with positive characteristic p, let $A_n = A\widehat{\otimes}_k k^{p^{-n}}$ and $k_n = k^{p^{-n}}$. The nilradical N_n of A_n generates an ideal in $A\widehat{\otimes}_k \widehat{k_p}$. This latter ring is noetherian, so N_n and N_{n+1} generate the same ideal in $A\widehat{\otimes}_k \widehat{k_p}$ for large n. Since $A\widehat{\otimes}_k \widehat{k_p}$ is faithfully flat over A_n for all n (by Lemma 1.1.5), N_n generates the nilradical of A_{n+1} for large n. Using [BGR], 6.1.1/12, we conclude that for large n, $(A_n)_{\mathrm{red}}\widehat{\otimes}_{k_n} k_{n+1} \simeq (A_{n+1})_{\mathrm{red}}$, which is reduced. It remains to prove the first part of the lemma.

Assume that $A\widehat{\otimes}_k k^{1/p}$ is reduced and fix an extension k'/k. Define $A' = A\widehat{\otimes}_k k'$. To show that A' is reduced, it suffices to show that if $a' \in A'$ with $a'^p = 0$, then $a' = 0$. Choose a (countable) topological basis $\{e_i\}$ of A over k [BGR], 2.8.2/2, so $A \simeq \widehat{\oplus} k e_i$ as k-Banach spaces and therefore (by [BGR], 2.1.7/8) $A' \simeq \widehat{\oplus} k' e_i$ as k'-Banach spaces, where we identify $e_i \in A$ with $e_i \widehat{\otimes} 1 \in A'$. Writing $a' = \sum a'_i e_i$, with $a'_i \in k'$ tending to 0, we want to show that if $\sum a'^p_i e^p_i = 0$ in A' then $a'_i = 0$ for all i. It suffices to show that the canonical continuous map $\varphi : \widehat{\oplus} k e^p_i \to A$ is injective with a closed image (an assertion that has nothing to do with any k'/k), for then there is a continuous section to φ [BGR], 2.7.1/4 and so the natural map $\varphi' : \widehat{\oplus} k' e^p_i \to A'$ is injective. Since φ' sends $\widehat{\oplus} a'^p_i e^p_i$ to $\sum a'^p_i e^p_i \in A'$, when this sum vanishes we get that all $a'^p_i = 0$, so all $a'_i = 0$, as desired.

To see that φ is injective, choose $a_i \in k$ with $a_i \to 0$ and $\sum a_i e^p_i = 0$. We want $a_i = 0$ for all i. Clearly $\sum a^{1/p}_i e_i \in A\widehat{\otimes}_k k^{1/p}$ has pth power equal to 0. Since $A\widehat{\otimes}_k k^{1/p}$ is assumed to be reduced, $\sum a^{1/p}_i e_i = 0$ in $A\widehat{\otimes}_k k^{1/p}$. This implies that all $a^{1/p}_i = 0$, so all $a_i = 0$. Thus, φ is injective. To see that the image of φ is closed, consider $A' = A\widehat{\otimes}_k k^{1/p}$. The natural map $A \to A'$ is a topological embedding [BGR], 6.1.1/9 and this map sends the image of φ onto the subring $(A')^p$ of pth powers in A'. It therefore suffices to show more generally that in characteristic $p > 0$, the subring of pth powers in a *reduced* affinoid B (such as A') is a closed subset. If $\{b_i\}$ is a sequence in B with $b^p_i \to b$, we just need to show that the sequence $\{b_i\}$ converges. The Banach topology on a reduced affinoid can be defined by the power-multiplicative sup-norm. Since $\{b^p_i\}$ is Cauchy, the power-multiplicative property of the sup-norm implies that $\{b_i\}$ is Cauchy *in characteristic p* and therefore $\{b_i\}$ is convergent. □

As we noted in the proof of Lemma 3.3.1 above, over a perfect (e.g., algebraically closed) base field, the non-smooth locus on a rigid space is the same as the non-regular locus, and on a reduced space this is a nowhere dense analytic set (thanks to the excellence of affinoids). On the other hand, the first part of Lemma 1.1.5 and the criterion for smoothness (over any base field) in term of the flatness and rank of the coherent sheaf $\Omega^1_{X/k}$ show that the formation of the (Zariski open) smooth locus commutes with change of the base field. Combining these facts with the last part of Lemma 1.1.5, we see that the smooth locus on a *geometrically reduced* (quasi-separated) rigid space over any complete field is the complement of a nowhere dense analytic set.

The "topological invariance" of base change by purely inseparable extension of the base field in the scheme setting suggests that base change by $k \to \widehat{k_p}$ should in some sense have no topological effect. Before proving a result in this direction, we establish two preliminary lemmas.

LEMMA 3.3.2. — *Let k be a complete field, F/k a finite separable extension, k_p/k a perfect closure. Give the algebraic extension fields k_p and $F \otimes_k k_p$ their unique absolute values extending the one on k, and give the finite reduced $\widehat{k_p}$-algebra $F \otimes_k \widehat{k_p}$ its unique absolute value extending the one on $\widehat{k_p}$* ([BGR], 3.2.2/3). *Then the natural injection of k_p-algebras*

$$i : F \otimes_k k_p \hookrightarrow F \otimes_k \widehat{k_p}$$

is an isometry with dense image, so it identifies the $\widehat{k_p}$-algebra $F \otimes_k \widehat{k_p}$ with the completion of the field $F \otimes_k k_p$. In particular, $F \otimes_k \widehat{k_p}$ is a field.

Proof. — Since $F \otimes_k k_p$ has a unique absolute value extending the one on k, the injection i must be an isometry. By the "equivalence of norms" on finite-dimensional vector spaces over complete fields ([BGR], 2.3.3/5), the topology we are considering on $F \otimes_k \widehat{k_p}$ coincides with the topology defined by using the "sup-norm" with respect to the $\widehat{k_p}$-basis $\{e_j \otimes 1\}$, where $\{e_1, \ldots, e_n\}$ is a k-basis of F. Thus, i has dense image since k_p is dense in $\widehat{k_p}$. □

LEMMA 3.3.3. — *Let X be a non-empty connected rigid space over k which admits an admissible covering $\{U_i\}$ by admissible opens U_i which are irreducible. Then X is irreducible.*

Proof. — We may assume X is reduced. Let X_0 be an irreducible component of X, and let Z be the (perhaps empty) union of all other irreducible components of X. By Theorem 2.2.4, Z is an analytic set not containing X_0. Thus, $Z \cap X_0$ is a proper analytic set in X_0, so it does not contain any non-empty admissible open in X_0 (by Lemma 2.2.3). If $z \in Z \cap X_0$, then z lies in at least two distinct irreducible components X_1 and X_0 of X. One of the admissible open U_i's contains z. Then Corollary 2.2.9 implies that U_i is not irreducible, an absurdity. We conclude that $Z \cap X_0$ is empty, so Z and X_0 are disjoint analytic sets which set-theoretically cover X. Each of Z and X_0 must therefore be a Zariski-open in X, so $\{Z, X_0\}$ is an admissible cover of X by disjoint admissible opens. Since X is connected and X_0 is non-empty, we conclude that $X = X_0$ is irreducible. □

THEOREM 3.3.4. — *Let $\{X_i\}$ be the set of reduced irreducible components of a quasi-separated rigid space X over a field k with positive characteristic p. Then $\{X_i \times_k \widehat{k_p}\}$ is the set of irreducible components of $X \times_k \widehat{k_p}$. Also, if X is connected then $X \times_k \widehat{k_p}$ is connected.*

Proof. — We begin by proving the last part. By Corollary 3.2.3, there is a finite Galois extension k'/k such that the connected components X'_i of $X' = X \times_k k'$ are geometrically connected over k'. Since X is connected, the Galois group $G = \mathrm{Gal}(k'/k)$ must act transitively on these components (by Lemma 2.3.2). Similarly, $Y = X \times_k (\widehat{k_p} \otimes_k k')$ is Galois over $X \times_k \widehat{k_p}$ with Galois group G and G acts transitively on those connected components of Y lying over a common connected component of $X \times_k \widehat{k_p}$. But the finite k'-algebra $k' \otimes_k \widehat{k_p}$ is a *field* by Lemma 3.3.2, so the connected components of Y are the $X'_i \times_{k'} (k' \otimes_k \widehat{k_p}) \simeq X'_i \times_k \widehat{k_p}$. The action of G is clearly transitive on these (after one checks a couple of compatibilities), so $X \times_k \widehat{k_p}$ is connected.

For the first part of the theorem, it is easy to see that the analytic sets $X_i \times_k \widehat{k_p}$ in $X \times_k \widehat{k_p}$ satisfy all of the conditions in Corollary 2.2.8 except perhaps for their irreducibility. Thus, it is enough to consider the case where X is irreducible, and even normal, so X has an admissible covering by connected normal affinoids. By Lemma 3.3.3 and the connectedness part we have already shown, we may assume $X = \mathrm{Sp}(A)$ is affinoid and irreducible (even normal and connected). The map $A \to A \widehat{\otimes}_k k^{p^{-n}}$ is injective, integral, and radicial (every element of the larger ring has its p^nth power inside of A), so $\mathrm{Spec}(A \widehat{\otimes}_k k^{p^{-n}}) \to \mathrm{Spec}(A)$ is a homeomorphism. It follows that $X \times_k k^{p^{-n}}$ is irreducible for all n. We may therefore replace k by any $k^{p^{-n}}$ and X by the normalization of $X \times_k k^{p^{-n}}$, so by Theorem 3.3.6 below, we can assume that $X = \mathrm{Sp}(A)$ is an irreducible, geometrically normal affinoid over k. Thus, $X \times_k \widehat{k_p}$ is normal and is also connected (by the last part of the theorem, which we have already proven). This implies that $X \times_k \widehat{k_p}$ is irreducible. □

Recall that if $R \to S$ is a faithfully flat map between noetherian rings and S is normal, then R is normal ([Mat2], Cor to 23.9). Thus, by Lemma 1.1.5, if X is a quasi-separated rigid space over a field k and k'/k is an extension such that $X \times_k k'$ is normal, then X had to be normal in the first place. In order to go in the reverse direction (*i.e.* establish geometric normality), it turns out that things are very similar to geometric reducedness: we only have to verify normality after base change to $k^{1/p}$. Before we prove this, we need the following lemma which is motivated by the condition R_1 in Serre's criterion for normality.

LEMMA 3.3.5. — *Let X be a quasi-separated rigid space and assume that $X \times_k k^{1/p}$ is normal. Then the smooth locus in X has complementary codimension at least 2 everywhere.*

Proof. — Without loss of generality, $X = \mathrm{Sp}(A)$ is affinoid and non-empty. By our hypotheses, we know that X is normal and geometrically reduced, so A is regular in codimension 1 and the smooth locus of X has nowhere dense complement. Taking X to be connected without loss of generality, so A is a normal domain with some dimension $d \geq 0$, we know that the finite A-module $\Omega^1_{A/k}$ of "continuous differentials" is free with rank d precisely over the Zariski open, non-empty k-smooth locus. We want to show that the stalk of $\Omega^1_{A/k}$ at each height 1 prime $\mathfrak{p}$ of $\mathrm{Spec}(A)$ is free over $A_{\mathfrak{p}}$, necessarily with rank d (by comparison with the generic rank), for then the Zariski-closed non-smooth locus would have to have codimension at least 2 everywhere, as desired. In particular, we may assume $d \geq 1$.

If k has characteristic 0, or more generally k is perfect, then the k-smooth locus on $\mathrm{Sp}(A)$ coincides with the regular locus, and this already has codimension at least 2 everywhere by the normality of A. Thus, we now assume that k has positive characteristic p. Recall that the finite A-module $\Omega^1_{A/k}$ and the k-linear derivation

$$\mathrm{d}_{A/k} : A \to \Omega^1_{A/k}$$

is universal for k-linear derivations from A to a finite A-module, and that k-linear derivations from A to a finite A-module are *automatically* continuous [BKKN], Satz 2.1.5. The construction of $\Omega^1_{A/k}$ involves completed tensor products. Our first step is to identify $\Omega^1_{A/k}$ with another module of differentials whose construction involves no topological conditions. Let $A' = A\widehat{\otimes}_k k^{1/p}$. Since A' is reduced by hypothesis, the pth power map sets up an isomorphism between A' and its subring $(A')^p$ of pth powers. By the proof of [BGR], 6.1.1/9, the map $A \to A'$ is a topological embedding with a *closed* image (here we use that k is closed in k'), so we may conclude from a continuity argument that the domain A contains $(A')^p$ as a *closed* subring (see the end of the proof of Lemma 3.3.1 for why $(A')^p$ is closed in A'). The canonical map of k-algebras $k\widehat{\otimes}_{k^p}A^p \to (A')^p$ is therefore easily seen to be surjective, so (by continuity!) any k-linear derivation from A to a finite A-module must kill $(A')^p$. Also, since $(A')^p \subseteq A$ is a closed k-subalgebra of A which contains A^p, it follows from [BGR], 6.3.3/2 that $(A')^p$ is k-affinoid and the inclusion of k-algebras $(A')^p \hookrightarrow A$ is a *finite* map.

At this point, we can conclude that $(A')^p$ is a k-affinoid subalgebra of A over which A is *finite* and on which any k-linear derivation from A to a finite A-module must vanish. Thus, we obtain a canonical identification of finite A-modules

$$\Omega^1_{A/k} = \Omega^1_{A/(A')^p},$$

where the right side is the purely algebraic A-module of Kähler differentials for the finite ring extension $(A')^p \hookrightarrow A$. It remains to prove that the stalk of the finite A-module $\Omega^1_{A/(A')^p}$ is free at each height 1 prime of A.

Note that $(A')^p$ is *normal*, because of the abstract ring isomorphism $A' \simeq (A')^p$ (here is where we use the hypothesis that A' is normal). Since $A^p \subseteq (A')^p$, the finite injective map $(A')^p \hookrightarrow A$ of normal domains is *radicial*, so $\mathrm{Spec}(A') \to \mathrm{Spec}((A')^p)$ is a homeomorphism. Thus, we get a bijection between height 1 primes of $(A')^p$ and A, and localizing the map $(A')^p \to A$ at a height 1 prime of $(A')^p$ automatically causes A to be replaced by its localization at the corresponding height 1 prime of A. We are now reduced to a purely algebraic assertion: if $R \subseteq S$ is a finite injection of discrete valuation rings with characteristic $p > 0$ and $S^p \subseteq R$, then the finite S-module $\Omega^1_{S/R}$ is *free* over S. We may base change throughout by the completion of R, so we can assume that R and S are complete. In particular, R is *Japanese*. Since $Q(S)^p \subseteq Q(R)$, we have $[Q(S) : Q(R)] = p^m$ for some m, and we may assume $m \geq 1$. By field theory, $Q(S) = Q(R)(a_1, \ldots, a_m)$ with $a_i^p \in Q(R) - Q(R)^p$ and the natural map

$$Q(R)(a_1) \otimes_{Q(R)} \cdots \otimes_{Q(R)} Q(R)(a_m) \to Q(S)$$

is an isomorphism. In other words, we have a $Q(R)$-algebra isomorphism

$$Q(S) \simeq Q(R)[X_1, \ldots, X_m]/(X_i^p - a_i^p),$$

so $Q(S) \otimes_S \Omega^1_{S/R} = \Omega^1_{Q(S)/Q(R)}$ is a $Q(S)$-vector space with dimension m (and basis $\mathrm{d}X_1, \ldots, \mathrm{d}X_m$). By the structure theorem for finite modules over a discrete valuation ring, if we can show that $\Omega^1_{S/R}$ has m generators over S, then we obtain the desired freeness of $\Omega^1_{S/R}$ over S.

Granting the case $m = 1$ for a moment, we argue by induction on m, so assume $m > 1$. Choose an intermediate field $K_1 \subseteq Q(S)$ with degree p over $Q(R)$ and let R_1 be the integral closure of R in K_1. By the Japaneseness of the complete discrete valuation ring R, the R-algebra R_1 is *finite* and is a discrete valuation ring. By the inductive hypothesis and the case $m = 1$, $\Omega^1_{R_1/R}$ is free over R_1 with rank 1 and Ω^1_{S/R_1} is free over S with rank $m - 1$. The "first fundamental exact sequence" ([Mat2], 25.1) is a right exact sequence of S-modules

$$\Omega^1_{R_1/R} \otimes_{R_1} S \to \Omega^1_{S/R} \to \Omega^1_{S/R_1} \to 0$$

and from this it is obvious that $\Omega^1_{S/R}$ has m generators as an S-module.

There remains the case $m = 1$, so $Q(S) = Q(R)[X]/(X^p - a)$ where $a \in Q(R)$ is not a pth power. By suitably rescaling a, we may assume that $a \in R$. We want to put a in a good form for computing the integral closure S of R in $Q(R)[X]/(X^p - a)$. We claim that a can be chosen to be either a uniformizer or a unit whose image in the residue field of R is not a pth power. Suppose we have such an a. It is easy to check that the 1-dimensional local noetherian ring $R[X]/(X^p - a)$ has a principal maximal ideal and therefore is regular (or equivalently, is a discrete valuation ring), so $S = R[X]/(X^p - a)$. In fact, the quotient of $R[X]/(X^p - a)$ by (X) is a field if $a \in R$ is a uniformizer and the quotient by the (principal) maximal ideal of R is a field if $a \in R$ is a unit whose image in the residue field of R is not a pth power. When $S = R[X]/(X^p - a)$, one then checks by direct calculation that $\Omega^1_{S/R}$ is free over S on the basis $\mathrm{d}X$. Now we show that a can be chosen in the desired form. Since R is complete, by the Cohen structure theorem ([Mat2], 29.7) we can write $R = K[\![t]\!]$ for some field K with characteristic p. Thus, $a = \sum a_i t^i$ for $a_i \in K$. Since $a \in R$ is not a pth power but we can add any pth power in R to a, without loss of generality any *non-zero* coefficient a_{ip} is not a pth power in K. We argue by induction on the least i_0 for which $a_{i_0} \neq 0$. If $i_0 = 0$, then the image a_{i_0} of a in the residue field of R is not a pth power, so we are done. If $i_0 > 0$ and i_0 is divisible by p, then replace a by $a/t^p \in R$ and induct. If $i_0 > 0$ and i_0 is not divisible by p, then choose $n, n' \in \mathbb{Z}$ for which $ni_0 + n'p = 1$. The element $b = a^n t^{n'p} \in Q(R)$ has order 1, so b is a uniformizer of R and the order p subgroups in $Q(R)^\times/(Q(R)^\times)^p$ generated by a and b are the same. We may replace a with the uniformizer b. ◻

THEOREM 3.3.6. — *Let X be a quasi-separated rigid space over k. If X is quasi-compact, then for sufficiently large n, the normalization of $X \times_k k^{p^{-n}}$ remains normal after base change to $k^{p^{-m}}$ for $m \geq n$. Without any quasi-compactness assumption, if $X \times_k k^{1/p}$ is normal, then $X \times_k k'$ is normal for any complete extension k'/k (i.e. X is geometrically normal over k). In particular, if $X \times_k L$ is normal for some perfect complete extension L/k, then X is geometrically normal over k.*

Proof. — In order to prove the first part of the theorem, it suffices to consider only "sufficiently large" extensions of k, and we may replace k by any $k^{p^{-n}}$ and X by the normalization of $X \times_k k^{p^{-n}}$. In particular, by Lemma 3.3.1, we may assume that X is geometrically reduced. We now find a large n so that that the normalization of $X \times_k k^{p^{-n}}$ remains normal after any further extension of the base field to some $k^{p^{-m}}$ for $m \geq n$. The idea is the same as that in the proof of Lemma 3.2.2: use the noetherian property of affinoids over a very large extension field. We may assume that $X = \mathrm{Sp}(A)$. Let $B = A\widehat{\otimes}_k\widehat{k_p}$, a reduced excellent noetherian ring. Let $\widetilde{B}$ be the normalization of B, so $\widetilde{B}$ is a finite B-module which contains B as a subring. The ring $A_n = A\widehat{\otimes}_k k^{p^{-n}}$ is a reduced Japanese ring, so its normalization $\widetilde{A_n}$ can be viewed as a finite extension ring of A_n. We will embed $\widetilde{A_n}\widehat{\otimes}_{k^{p^{-n}}}\widehat{k_p}$ into $\widetilde{B}$ as an extension ring of B and then use the noetherian property of the finite B-module $\widetilde{B}$ to see that formation of $\widetilde{A_n}$ is compatible with extension of scalars from $k^{p^{-n}}$ to $k^{p^{-m}}$ once n is sufficiently large.

Since A_n is reduced, there exists an element $f \in A_n$ which is not a zero divisor such that $f \cdot \widetilde{A_n} \subseteq A_n$, so $\widetilde{A_n}\widehat{\otimes}_{k^{p^{-n}}}\widehat{k_p}$ can be viewed as a *finite* extension ring of B such that multiplication by f takes $\widetilde{A_n}\widehat{\otimes}_{k_n}\widehat{k_p}$ into B. By Lemma 1.1.5, $f \in B$ is not a zero divisor in $\widetilde{A_n}\widehat{\otimes}_{k^{p^{-n}}}\widehat{k_p}$, so $\widetilde{A_n}\widehat{\otimes}_{k^{p^{-n}}}\widehat{k_p}$ is a subring of $\widetilde{B}$ and these form an increasing chain B-submodules of the *finite* B-module $\widetilde{B}$ as n grows. Such a rising chain must terminate, so for some n and all $m \geq n$, the finite inclusion maps of $k^{p^{-m}}$-affinoids

$$\widetilde{A_n}\widehat{\otimes}_{k^{p^{-n}}} k^{p^{-m}} \hookrightarrow \widetilde{A_m}$$

must become isomorphisms after applying $\widehat{\otimes}_{k^{p^{-m}}}\widehat{k_p}$. Hence, for such a large n, the normalization of $A_n = A\widehat{\otimes}_k k^{p^{-n}}$ remains normal after any further extension to $k^{p^{-m}}$ for $m \geq n$.

Now we assume that X is a quasi-separated rigid space over k such that $X \times_k k^{1/p}$ is normal and we show that $X \times_k k'$ is normal for any complete extension k'/k. Let U be the k-smooth locus on X, so by

Lemma 3.3.5, the complement of U is an analytic set with codimension at least 2 everywhere. Let k'/k be a complete extension field, and we will show that $X' = X \times_k k'$ is normal.

By the last part of Lemma 1.1.5, the smooth locus $U' = U \times_k k'$ of the reduced space X' has complementary codimension at least 2 everywhere. Without loss of generality, $X = \mathrm{Sp}(A)$ is affinoid, so $X' = \mathrm{Sp}(A')$ for $A' = A\widehat{\otimes}_k k'$ a reduced affinoid which is regular in codimension ≤ 1. By Serre's "$R_1 + S_2$" criterion for normality ([Mat2], 23.8), we just have to prove that A' satisfies condition S_2. We know that A satisfies condition S_2 (since A is normal) and $A \to A'$ is flat (by Lemma 1.1.5), so by [Mat2], 23.9 it is enough to prove that the fibers of $\mathrm{Spec}(A') \to \mathrm{Spec}(A)$ satisfy S_2. More generally, for any affinoid algebra B over k, we define $B' = B\widehat{\otimes}_k k'$ and we claim that the fibers of the faithfully flat map $\mathrm{Spec}(B') \to \mathrm{Spec}(B)$ are Cohen-Macaulay (*i.e.* satisfy S_i for all $i \geq 0$). The reason to expect this is that the analogue for locally finite type schemes over a field is true ([EGA], IV_2, 6.7.1). To prove this result in the affinoid setting, choose $\mathfrak{p} \in \mathrm{Spec}(B)$ whose fiber we want to prove is Cohen-Macaulay.

Replacing B by $B/\mathfrak{p}$ and B' by $B'/\mathfrak{p}B' = (B/\mathfrak{p})\widehat{\otimes}_k k'$ [BGR], 6.1.1/12, we can assume that B is a domain and $\mathfrak{p} = (0)$. Let d be the dimension of B. By Noetherian normalization, there is a finite injection of affinoids $T_d(k) \hookrightarrow B$, inducing a finite extension $Q(T_d(k)) \hookrightarrow Q(B)$ of fraction fields. Since $\mathrm{Spec}(B') = \mathrm{Spec}(B) \times_{\mathrm{Spec}(T_d(k))} \mathrm{Spec}(T_d(k'))$ by Lemma 1.1.5, the generic fiber of $\mathrm{Spec}(B') \to \mathrm{Spec}(B)$ is obtained as a base change by $\mathrm{Spec}(Q(B)) \to \mathrm{Spec}(Q(T_d(k)))$ on the generic fiber of $\mathrm{Spec}(T_d(k')) \to \mathrm{Spec}(T_d(k))$. This latter generic fiber is just a localization of the *regular* ring $T_d(k')$, so it is regular and therefore Cohen-Macaulay. For any field K, any finite extension field E/K, and any noetherian K-algebra R, the finite map $\mathrm{Spec}(E \otimes_K R) \to \mathrm{Spec}(R)$ is faithfully flat with 0-dimensional fibers (which are therefore Cohen-Macaulay), so R is Cohen-Macaulay if and only if $E \otimes_K R$ is Cohen-Macaulay ([Mat2], Cor to 23.3). Applying this with the finite field extension $Q(B)/Q(T_d(k))$, we are done. □

We remark that our method carries over to the scheme setting to show that for a locally finite type scheme X over a field k, X is geometrically normal over k if and only if $X \times_k k'$ is normal for all finite extensions k'/k for which $k'^p \subseteq k$ (here, p is the so-called characteristic exponent of k, equal to 1 if k has characteristic 0 and equal to the characteristic of k otherwise). Surprisingly, this result is not in [EGA].

3.4. Geometric irreducibility.

We are now in position to answer the original question of how irreducible components behave under change of the base field. A quasi-separated rigid space Z over k is said to be *geometrically irreducible* (relative to k, or over k) if $Z \times_k k'$ is irreducible for all extensions k'/k.

LEMMA 3.4.1. — *Let X be a rigid space over k with finitely many irreducible components $\{X_i\}$. Let k'/k be a finite extension. Then the $X_i \times_k k'$'s have finitely many irreducible components and these are the irreducible components of $X \times_k k'$. When k'/k is purely inseparable, the $X_i \times_k k'$ are irreducible.*

Proof. — We can assume X is non-empty. It is enough to treat separately the cases when the finite extension k'/k is purely inseparable and when it is separable. First consider the purely inseparable case, so without loss of generality k has positive characteristic p. We claim that the analytic sets $X_i \times_k k'$ are irreducible and are the irreducible components of $X \times_k k'$. By Corollary 2.2.8 and the fact that X_i contains a non-empty admissible open disjoint from all $X_j \neq X_i$, it suffices to consider an irreducible X and to show that the non-empty space $X \times_k k'$ is irreducible. Choose a non-empty admissible open U' in $X \times_k k'$ and a closed immersion $Z' \hookrightarrow X \times_k k'$ which set-theoretically contains U'. By Lemma 2.2.3, it suffices to show that Z' must set-theoretically fill up $X \times_k k'$.

If we replace the coherent ideal sheaf of Z' by its p^nth power for suitably large n (depending on k'), we can find a closed immersion $Z \hookrightarrow X$ whose base change is Z'. Thus, the base change of the Zariski-open complement of Z is the Zariski-open complement of Z'. As was noted in the beginning of the proof of Theorem 3.2.1, $X \times_k k' \to X$ is a "homeomorphism" with respect to admissible opens and admissible covers, so there is a non-empty admissible open U in X whose base change is U'. Since U' is disjoint from the $X' - Z'$, it follows U is disjoint from $X - Z$. The irreducibility of X forces Z to fill up all of X, so the coherent ideal sheaf of Z is locally nilpotent. Thus, the same holds for Z', so X' is irreducible.

Now assume that k'/k is a finite separable extension. Since $k \to k'$ is finite étale, the formation of the normalization of X commutes with base change by k'/k (by Theorem 1.2.2). Thus, we may assume that X is a connected normal rigid space and need to check that $X \times_k k'$ has only finitely many connected components. This follows from Corollary 3.2.3. □

THEOREM 3.4.2. — *Let X be a quasi-separated rigid space over k.*

1. *If X has finitely many irreducible components, then there exists a finite separable extension k'/k such that the finitely many irreducible components of $X \times_k k'$ are geometrically irreducible over k'.*

2. *If $\{X_i\}$ are the irreducible components of X and K/k is an extension, then the irreducible components of the $X_i \times_k K$'s are irreducible components of $X \times_k K$ and every irreducible component of $X \times_k K$ is contained in a unique $X_i \times_k K$, in which it is an irreducible component.*

Proof. — We begin with the first part. Suppose we can construct a finite extension E/k such that the irreducible components of $X \times_k E$ are geometrically irreducible over E. By Lemma 3.4.1, it is easy to see that we may take k' to be the separable closure of k in E. Thus, we may now search for k' finite over k without requiring separability over k. In particular, we may replace k by any finite extension (note that this does not affect the hypothesis on X, by Lemma 3.4.1).

By Theorem 3.3.4, $X \times_k \widehat{k_p}$ has finitely many irreducible components. Suppose we have established the first part of the theorem in the case of a perfect ground field, so there exists a finite extension $E/\widehat{k_p}$ such that the irreducible components of $X \times_k E \simeq (X \times_k \widehat{k_p}) \times_{\widehat{k_p}} E$ are geometrically irreducible over E. Since E has a primitive element over $\widehat{k_p}$, by an approximation argument [BGR], 3.4.2/4,5 we may assume (after replacing k by a finite purely inseparable extension) that there is a finite separable extension k'/k such that $k' \otimes_k \widehat{k_p} \simeq E$. By Lemma 3.3.2, this identifies E with the completion of the perfect closure of k'. Replacing k by k', we may assume that each irreducible component X_i of X has the property that $X_i \times_k \widehat{k_p}$ is geometrically irreducible over $\widehat{k_p}$. If k'/k is any complete extension, then all

$$(X_i \times_k k') \times_{k'} \widehat{k'_p} \simeq (X_i \times_k \widehat{k_p}) \times_{\widehat{k_p}} \widehat{k'_p}$$

are irreducible. Thus, by Corollary 2.2.8 we are reduced to showing that if Z is a rigid space over a complete field k' and $Z \times_{k'} \widehat{k'_p}$ is irreducible, then Z is irreducible. This follows from Lemma 2.2.3 and Lemma 1.1.5.

Now we may assume that k is perfect. We may assume (by hypothesis on X) that X is irreducible, and even connected and normal (so it is normal after any extension of scalars on k, by Theorem 3.3.6). By Corollary 3.2.3, we can replace k by a finite extension and suppose that X is geometrically connected. Thus, for any k'/k, $X \times_k k'$ is connected and normal, hence irreducible.

Finally, we prove the second part of the theorem. Let K/k be an arbitrary complete extension, $\{X_i\}$ the irreducible components of X. Since $\{X_i \times_k K\}$ is a locally finite set of *distinct* non-empty analytic sets in $X \times_k K$ whose union is $X \times_k K$, it is clear that any irreducible analytic set in $X \times_k K$ lies inside of one of the $X_i \times_k K$'s. Since the irreducible components in a non-empty rigid space are the maximal irreducible analytic sets, all that is left to show is that an irreducible component of $X \times_k K$ cannot lie inside of $X_i \times_k K$ and $X_j \times_k K$ for $i \neq j$. Since $(X_i \times_k K) \cap (X_j \times_k K) = (X_i \cap X_j) \times_k K$ is nowhere dense in $X \times_k K$ (by Lemma 1.1.5, because $X_i \cap X_j$ is nowhere dense in X), yet irreducible components of non-empty rigid spaces are never nowhere dense, we are done. □

4. Applications to Fredholm series.

We want to apply the theory of global irreducible decomposition to deduce results of Coleman-Mazur on Fredholm series ([CM], 1.3). We work over general complete ground fields and give more conceptual/geometric proofs. Also, by abuse of notation we let $\mathbf{A}^m = \mathbf{A}^m_k$ denote $\operatorname{Spec}(k[T_1, \ldots, T_m])^{\mathrm{an}}$.

4.1. Basic definitions.

Let X be a rigid space over k. We call $\Gamma(X \times \mathbf{A}^m, \mathcal{O}_{X \times \mathbf{A}^m})$ the ring of *entire functions in m variables over X*. Using pullback along the zero section of $X \times \mathbf{A}^m = X \times_k \mathbf{A}^m_k \to X$, any such f induces an element denoted $f(0) \in \Gamma(X, \mathcal{O}_X)$. Moreover, $f(0) \in \Gamma(X, \mathcal{O}_X)^\times$ if and only if the zero locus of f on $X \times_k \mathbf{A}^m_k$ does not meet the zero section, in which case we can scale f by a unit so that $f(0) = 1$. A *Fredholm series over X in m variables* is an entire function f over X in m variables for which $f(0) = 1$. When m is understood from context, we omit mention of it. For any $f \in \Gamma(X \times \mathbf{A}^m, \mathcal{O}_{X \times \mathbf{A}^m})$, we denote by $Z(f) \hookrightarrow X \times \mathbf{A}^m$ the closed immersion cut out by the coherent ideal sheaf on $X \times \mathbf{A}^m$ generated by f.

For a k-affinoid A, we have an isomorphism

$$A \widehat{\otimes}_k (k \ll T_1, \ldots, T_m \gg) \simeq A \ll T_1, \ldots, T_m \gg$$

([BGR], 6.1.1/7), so there is an injection

$$A \widehat{\otimes}_k (k \ll T_1, \ldots, T_m \gg) \hookrightarrow A[\![T_1, \ldots, T_m]\!]$$

functorial in A. Thus, there is an injection of $\mathcal{O}_X(X)$-algebras $\Gamma(X \times \mathbf{A}^m, \mathcal{O}_{X\times\mathbf{A}^m}) \hookrightarrow \mathcal{O}_X(X)[\![T_1,\ldots,T_m]\!]$ which sends Fredholm series to units and is functorial in X. In particular, we see that a Fredholm series F is nowhere a zero divisor in $\mathcal{O}_{X\times\mathbf{A}^m}$.

When m is fixed, we will sometimes pick $\pi \in k^\times$ with $0 < |\pi| < 1$ and define

$$X_n = X_{n,m} = X \times \mathbf{B}^m(0, |\pi|^{-n}),$$

with $\mathbf{B}^m(0,|\pi|^{-n})$ the "closed" ball in $\mathbf{A}^m$ around the origin of radius $|\pi|^{-n}$. The choice of π is fairly unimportant; the X_n's give an "increasing" admissible open covering of $X \times \mathbf{A}^m$.

The basic facts we need are summarized by the following lemma:

LEMMA 4.1.1. — *Let X be a rigid space over k.*

1. *If $\{U_i\}$ is an admissible covering of $X \times \mathbf{A}^m$ and F, G are two Fredholm series over X such that $G|_{U_i}$ divides $F|_{U_i}$ in $\Gamma(U_i, \mathcal{O}_{X\times\mathbf{A}^m})$ for all i, then $F = GH$ for a unique Fredholm series H over X.*

2. *If X is reduced and H is a Fredholm series with no zeros on $X \times \mathbf{A}^m$, then $H = 1$. In particular, if $H \neq 1$, then H must vanish somewhere.*

3. *If X is reduced and equidimensional and F, G are Fredholm series with $Z(G) \subseteq Z(F)$ as analytic sets in $X \times \mathbf{A}^m$, then $Z(G)_{\mathrm{red}}$ is a union of irreducible components of $Z(F)_{\mathrm{red}}$.*

If X is the product of an affinoid and an affine space and $\mathcal{L}$ is an invertible sheaf on $X \times \mathbf{A}^m$ such that $\mathcal{L}|_{X_n}$ is trivial for all $n \geq 0$, then $\mathcal{L}$ is trivial.

Proof. — The first two parts are essentially ([CM], Lemma 1.3.2). To be precise for the second part, we may reduce to showing that any $f \in \Gamma(\mathbf{A}^m_{k'}, \mathcal{O}_{\mathbf{A}^m_{k'}})$ with $f(0) = 1$ and $f \neq 1$ must vanish somewhere. By suitable scaling of the affine variables, we may assume $f = 1 + \sum_{I\neq 0} a_I T^I$ with all $|a_I| \leq 1$ and some $|a_I| = 1$. Then by [BGR], 5.3.1/1, f has a zero inside of the unit ball $\mathbf{B}^m$.

The third part follows from Corollary 2.2.7, once we observe that if A is a k-affinoid domain of dimension d, then all maximals of A have height d by Lemma 2.1.5 and the k-affinoid domain $A \ll T \gg$ has dimension $d+1$. To prove the latter point, note that the map of domains $A \to A \ll T \gg \simeq A\widehat{\otimes}_k(k \ll T \gg)$ is flat (use the method of proof of flatness in Lemma 1.1.5), so by the dimension formula [Mat2], 15.1 and Lemma 2.1.5, $A \ll T \gg$ has dimension $d+1$.

For the last part, we note that one can carry over to rigid geometry many basic facts from the theory of Stein spaces [TSS], often by much simpler arguments (due to the non-archimedean inequality and the more "algebraic" nature of coherent sheaves in the rigid analytic setting). In particular, the cohomology of a coherent sheaf vanishes on the product of a k-affinoid and the analytification of an affine k-scheme (see [K1], Satz 2.4 for a proof of a more precise general theorem). Thus, to see that $\mathcal{L}$ is trivial we may assume that X is reduced, since if $\mathcal{N}$ is the coherent ideal sheaf of nilpotents on X and s is a generator of $\mathcal{L}/\mathcal{N}\mathcal{L}$ (viewed as a coherent sheaf on X_{red}), any lift of s to a global section of $\mathcal{L}$ is a global generator of $\mathcal{L}$. Since $X \times \mathbf{A}^m \simeq (X \times \mathbf{A}^{m-1}) \times \mathbf{A}^1$, we can also use induction to reduce to the case $m = 1$.

Let $\mathcal{L}_n = \mathcal{L}|_{X_n}$ and pick a generator s_n of $\mathcal{L}_n$. For each $n \geq 1$, we can rescale s_n by an element of $\mathcal{O}_X(X)^\times$ so that the pullback of s_n to the zero section agrees with the pullback of s_0 to the zero section. If we write (for $n \geq 0$) $s_{n+1}|_{X_n} = s_n u_n$ for a unit u_n on $X_n \subseteq X \times \mathbf{A}^1$, we must have $u_n(0) = 1$. Since X is reduced, the convergence argument in the proof of [CM], 1.3.3 (which applies over open affinoids of X when $m = 1$) shows that $v_n = u_n \cdot u_{n+1} \cdot \ldots$ makes sense as a unit on $X_n \subseteq X \times \mathbf{A}^1$. Thus, the sections $s_n v_n$ of $\mathcal{L}_n$ glue to a global generator of $\mathcal{L}$. □

4.2. Relatively factorial rings.

In this section, we fix an integer $m \geq 1$. Following [CM], a k-affinoid $X = \mathrm{Sp}(A)$ is said to be *relatively factorial* (or perhaps *relatively factorial in m variables* to be more precise) if X is reduced and irreducible (so A is a domain) and for every $f \in A \ll T_1, \ldots, T_m \gg$ with $f(0) = 1$, (f) is a product of principal prime ideals, in which case such a factorization is unique and the generators of the prime ideals can be taken to have constant term 1. Tate algebras are relatively factorial [BGR], 5.2.6/1.

LEMMA 4.2.1. — *Let A be relatively factorial over k, $f \in A \ll T_1, \ldots, T_m \gg$ with $f(0) = 1$, f not a unit. Let $(f) = \prod (p_j)^{e_j}$ be the prime factorization. Then $(f) = \cap (p_j)^{e_j}$ and* $\mathrm{rad}((f)) = \cap (p_j) = \prod (p_j)$.

Proof. — Let $B = A \ll T_1, \ldots, T_m \gg$. The local ring $B_{(p_j)}$ is a noetherian local domain with non-zero principal maximal ideal, so by [Mat2], 11.2, $B_{(p_j)}$ is a discrete valuation ring. Thus, $g \in B$ is divisible by p_j^e if and only if the image of g in $B_{(p_j)}$ lies in the e-th power of the maximal ideal. Since p_i maps to a unit in $B_{(p_j)}$ for $i \neq j$, the assertions in the lemma are clear. □

The following result is obtained in [CM], 1.3.4, 1.3.6 by a delicate analysis of limiting arguments over larger and larger X_n's. From our point of view, all such limiting arguments are embedded in the proof of the last part of Lemma 4.1.1, and the global theory of irreducible decomposition thereby enables us to argue more "geometrically".

THEOREM 4.2.2. — *Let X be a relatively factorial affinoid over k, and $F \neq 1$ be a Fredholm series over X in m variables, so $Z(F)$ is non-empty.*

1. *There is a unique Fredholm G over X with $Z(F)_{\text{red}} = Z(G)$ and moreover $G|F$ in the ring of entire functions over X in m variables.*

2. *$Z(F)$ is irreducible if and only if $F = P^e$ for some $e \geq 1$ and some irreducible Fredholm series P (i.e. $P \neq 1$ and P does not factor non-trivially in the ring of entire functions over X). When this occurs, $P = G$.*

3. *The irreducible components of $Z(F)$ are in bijection with irreducible Fredholm series $P|F$, via $P \mapsto Z(P)$.*

Proof. — The uniqueness of G follows from Lemma 4.1.1. For existence, let $\mathcal{J} = \text{rad}((F))$. If $\mathcal{J}$ is principal, with some generator H, then H does not vanish along the zero section, so by scaling by a unit we can assume $H(0) = 1$. Then it is easy to see that we can take this H to be G. By the last part of Lemma 4.1.1, we just have to show that $\mathcal{J}$ is free of rank 1 on each X_n. But X is relatively factorial and $(F)|_{X_n}$ has a generator with constant term 1, so this is clear by Lemma 4.2.1.

Before handling the second part, we check a special case: if $F \neq 1$ is Fredholm over X and $Z = Z(F)$ is *reduced*, then Z is irreducible if and only if F is an irreducible entire function over X. Since $F(0) = 1$, any factorization of F into a product of two entire functions can be scaled by units so that both entire functions have constant term 1. Thus, by Lemma 2.2.3, the second part of Lemma 4.1.1, and the hypothesis that $Z(F)$ is reduced, it is enough to prove that any (reduced) union W of irreducible components of $Z(F)$ is of the form $Z(G)$ for a (necessarily unique) Fredholm G, and moreover $G|F$. If the ideal sheaf of W is globally free with a generator G, then since W does not meet the zero section we can assume $G(0) = 1$ and then since $Z(F)$ is reduced we may use the first part of Lemma 4.1.1 to deduce that $G|F$. Thus, we just have to check that the coherent ideal sheaf $\mathcal{I}_W$ of W is globally free of rank 1. By the last part of Lemma 4.1.1, we only need $\mathcal{I}_W|_{X_n}$ to be globally free of rank 1 for all $n \geq 0$. By Corollary 2.2.9, this follows from the relatively factorial condition and Lemma 4.2.1.

We conclude that for general Fredholm $F \neq 1$, $Z = Z(F)$ is irreducible if and only if $Z_{\text{red}} = Z(P)$ for an irreducible Fredholm P, in which case $(P) = \text{rad}((F))$ as coherent ideal sheaves on $X \times \mathbf{A}^m$ and $F = PH_1$ for some Fredholm H_1. Assuming this to be the case, we want to conclude that $F = P^e$ for some $e \geq 1$. Suppose $H_1 \neq 1$, so $Z(H_1)$ is a non-empty analytic set in $Z(F)$. By the third part of Lemma 4.1.1, $Z(H_1)_{\text{red}} = Z(P)$, so $H_1 = PH_2$ and $F = P^2H_2$ (note in particular that H_1 vanishes everywhere F does). To see that this process must stop, choose some n such that $F|_{X_n}$ has a zero. We will find an upper bound on the power of $P|_{X_n}$ which divides $F|_{X_n}$. Since X is relatively factorial and $F|_{X_n}$ is a non-unit with constant term 1, by Lemma 4.2.1 we see that $P|_{X_n}$ is just the product without multiplicity of the distinct prime factors of $F|_{X_n}$ ("distinct" meaning "taken up to unit multiple"). Thus, the highest exponent occuring in the prime factorization of $F|_{X_n}$ gives the desired upper bound. □

Note that as a consequence of this theorem, if P is an irreducible entire function in m variables over a relatively factorial affinoid X, then P generates a prime ideal in the ring of entire functions over X in m variables. Indeed, if $P|FG$, then the irreducible reduced rigid space $Z(P)$ is contained in the union of the analytic sets $Z(F)$ and $Z(G)$, so the reduced space $Z(P)$ lies inside of $Z(F)$ or $Z(G)$. Thus, $P|F$ or $P|G$.

Consider a set $\{P_i\}$ of distinct irreducible Fredholm series over X in m variables which is locally finite in the sense that only finitely many of the (distinct!) analytic sets $Z(P_i)$ meet any quasi-compact admissible open in $X \times \mathbf{A}^m$. It makes sense to form the product $\prod P_i^{n_i}$ for any choice of integers $n_i \geq 1$ (this is entire, hence Fredholm). We can then ask if there is a unique factorization theorem. This was obtained by Coleman-Mazur using limit arguments ([CM], 1.3.7), but we can deduce this in a slightly more precise form from the above by thinking geometrically in terms of global irreducible decompositions:

COROLLARY 4.2.3. — *Let X be a relatively factorial affinoid, $F \neq 1$ Fredholm over X in m variables. Then there is a unique non-empty set of positive integers $\{n_i\}$ and non-empty locally finite set of distinct irreducible Fredholms $\{P_i\}$ such that $F = \prod P_i^{n_i}$. Moreover, the irreducible components of $Z(F)$ endowed with their reduced structures are the $Z(P_i)$'s.*

Proof. — If P is an irreducible Fredholm series with $P|F$, so $Z(P) \subsetneq Z(F)$, by Lemma 4.1.1 we have that $Z(P)$ is an irreducible component of $Z(F)$. Moreover, by the last part of Theorem 4.2.2, every irreducible component of $Z(F)$ has the form $Z(P)$ for a unique irreducible Fredholm series P over X, and $P|F$. By the existence of global irreducible decomposition, it follows that the only possible set $\{P_i\}$ is the set of irreducible Fredholm series which cut out the irreducible components of $Z(F)$. Taking this to be the set $\{P_i\}$, let us show that this works. This set is locally finite. From the proof of Theorem 4.2.2, we see that for fixed i, there is an upper bound on the exponents n for which $P_i^n|F$. Let $n_i \geq 1$ be the largest such n and $G = \prod P_i^{n_i}$. Once we show that $F = GH$ for some Fredholm H, then it is clear that $Z(H)$ cannot have any non-empty irreducible components, so $Z(H)$ is empty and $H = 1$.

By Lemma 4.1.1, it suffices to show $G|_{X_n}$ divides $F|_{X_n}$ for all $n \geq 0$. But $G|_{X_n}$ is just a finite product and the restriction of an irreducible Fredholm to X_n is a product of prime elements which generate distinct prime ideals and have constant term 1. By Lemma 4.2.1 we are done. □

4.3. Consequences of global factorization.

The following result is a more precise version of [CM], 1.3.8.

THEOREM 4.3.1. — *Let X be a relatively factorial affinoid, F a Fredholm series over X, U a non-empty admissible open in $Z(F)$. If Σ is the non-empty set of irreducible components $Z(P_i)$ of $Z(F)$ which meet U, then the "smallest" closed immersion in X through which $U \to X$ factors is $Z(F_\Sigma) \hookrightarrow X$, where F_Σ is the product of $P_i^{n_i}$ for the $Z(P_i) \in \Sigma$.*

Proof. — Without loss of generality, $F = F_\Sigma$. By Corollary 2.2.9, for each irreducible Fredholm $P_i|F$, $U \cap Z(P_i)$ contains a non-empty admissible open U_i which does not meet any $Z(P_j)$ for $j \neq i$. By Lemma 4.2.1 and Corollary 2.2.9, the coherent ideal sheaf (F) is the (locally finite) intersection of the coherent ideal sheaves $(P_i^{n_i})$, so it is enough to replace U by U_i and F by $P_i^{n_i}$. Thus, we may consider an irreducible Fredholm series P, an integer $e \geq 1$, and a non-empty affinoid open U in $Z(P^e)$. If a coherent ideal sheaf $\mathcal{I}$ vanishes under pullback via $U \to X$, we want $\mathcal{I} \subseteq (P^e)$. Since the open immersion $U \hookrightarrow X \times \mathbf{A}^m$ factors through the closed immersion $Z(\mathcal{I}) \hookrightarrow X \times \mathbf{A}^m$ cut out by $\mathcal{I}$, the analytic set $Z(\mathcal{I}) \cap Z(P)$ contains a non-empty admissible open in the irreducible reduced space $Z(P)$. By Lemma 2.2.3, $Z(P) \subseteq Z(\mathcal{I})$, so $\mathcal{I} \subseteq (P)$. Thus, $\mathcal{I} = (P)\mathcal{J}$ for some coherent ideal sheaf $\mathcal{J}$. If $e = 1$, we are done, so suppose $e > 1$. By induction, it is enough to show that $\mathcal{J}$ vanishes on a non-empty admissible open of the rigid space $Z(P^{e-1})$.

Choose large n so that $Z(P) \cap X_n$ meets U. Let $A_n = \mathcal{O}_{X \times \mathbf{A}^m}(X_n)$, $I_n = \Gamma(X_n, \mathcal{I})$. Then $\mathrm{Sp}(A_n/(P|_{X_n}))$ is a reduced non-empty rigid space, so we may write $P|_{X_n}$ as a product of primes $p_1, \ldots, p_{r_n}$ with constant term 1 and all (p_j) distinct prime ideals in A_n. Some $Z(p_{j_0})$ must meet U. The ring $(A_n)_{(p_{j_0})}$ is a discrete valuation ring by Lemma 4.2.1. Since U is an admissible open in the rigid space $Z(P^e)$ and it meets $Z(p_{j_0})$, by Lemma 2.2.3 we see that U contains a point in $Z(p_{j_0})$ not in any other $Z(p_j)$'s. Thus, U contains a non-empty admissible open in the rigid space $Z(p_{j_0}^e)$, so all elements of I_n vanish in some local ring of $(A_n)/(p_j)^e$. Localizing to the generic point of this ring, we see I_n lies in $(p_{j_0})^e$ (by Lemma 4.2.1). Thus, $\mathcal{I}$ vanishes on the non-empty admissible open complement of $\bigcup_{j \neq j_0} Z(p_j)$ in $Z(p_{j_0}^e)$, which is even an admissible open in $Z(P^e)$. It follows that $\mathcal{J}$ vanishes on a non-empty admissible open of $Z(P^{e-1})$, as desired. □

For the last result, we somewhat globalize the above considerations and generalize [CM], 1.3.11 with a proof that uses global irreducible decomposition in place of intricate limit arguments (combining the following with [CM], 1.3.10 recovers [CM], 1.3.11).

THEOREM 4.3.2. — *Let X be a rigid space over k which has an admissible covering by relatively factorial affinoid opens $\{U_j\}$. Let $F \neq 1$ be a Fredholm series over X. The irreducible components of $Z(F)$ have the form $Z(P_i)$ for unique irreducible Fredholm series P_i over X, and there exist positive integers n_i such that $F = \prod P_i^{n_i}$. Up to ordering, this is the unique factorization of F into a product of irreducible Fredholm series over X whose zero loci are locally finite on $X \times \mathbf{A}^m$.*

In particular, an irreducible Fredholm series P over such an X

generates a prime ideal in the ring of entire functions over X in m variables.

Proof. — Let $\{Z_i\}$ be the set of irreducible components of $Z(F)$, given their reduced structure. Working over each U_j and using Corollary 2.2.9, the "radical" coherent ideal sheaf $\mathcal{I}_i$ of Z_i is locally free of rank 1 and $\mathcal{I}_i|_{U_j}$ is generated by a Fredholm series over U_j (this Fredholm series is 1 or a product of irreducible Fredholm series over U_j). By the second part of Lemma 4.1.1, these generators glue, so $\mathcal{I}_i = (P_i)$ for a unique Fredholm series P_i over X. But $Z(P_i) = Z_i$ is a reduced irreducible space, so P_i generates a prime ideal in the ring of entire functions over X in m variables. Since Fredholm series are not zero divisors, P_i is clearly irreducible in this ring as well. By construction and the first part of Lemma 4.1.1, $P_i|F$. In particular, if $F \neq 1$ is a Fredholm series, then F is irreducible if and only if F generates a prime ideal in the ring of entire functions over X in m variables. Thus, irreducibility and primality are the same concept for Fredholm series over X.

If P is any irreducible Fredholm series dividing F, then $Z(P)$ lies inside of $Z(F)$, so by Lemma 4.1.1 it follows that $Z(P)$ must be an irreducible component of $Z(F)$. Thus, P must be one of the P_i's. It is equally clear from the global theory of irreducible decomposition that if $e_i \geq 0$ are integers such that $F = \prod P_i^{e_i}$, then $e_i \geq 1$. For each i, choose a U_j so that Z_i meets $U_j \times \mathbf{A}^m$. The restriction of P_i to $U_j \times \mathbf{A}^m$ is a product of distinct irreducible Fredholm series, so by Corollary 4.2.3, there is a maximal exponent $n_i \geq 1$ such that $P_i^{n_i}|F$. It follows that there is at most one possible factorization of F of the desired type.

Defining $G = \prod P_i^{n_i}$, we are reduced to showing that $G|F$. This can be checked locally over X. Working over each U_j and noting that each $P_i|_{U_j}$ is either equal to 1 or else factors as a locally finite product of distinct irreducible Fredholm series over U_j with order of multiplicity in $F|_{U_j \times \mathbf{A}^m}$ at least n_i, we may appeal to the factorization theory already treated over a relatively factorial affinoid base. □

5. Analytification of schemes.

In this appendix, we want to discuss some fundamental facts concerning the analytification functor from locally finite type k-schemes to rigid spaces over k. The analogous questions in the complex analytic case are treated extensively in [SGA1], XII. Thanks to excellence results in the rigid setting, many of the proofs in the complex analytic case carry over, but extra care is needed for others. We want to indicate here where difficulties arise. We will need to use Theorem 2.3.1 at one point, for example.

5.1. Preliminary facts.

Let X be a locally finite type k-scheme and Y a rigid analytic space over k. A *morphism* $f : Y \to X$ is a continuous map of Grothendieck topological spaces (*i.e.* a map of sets $f : Y \to X$ which pulls Zariski opens and Zariski covers of Zariski opens back to admissible opens and admissible covers of admissible opens) and a map $f^\sharp : \mathcal{O}_X \to f_*(\mathcal{O}_Y)$ of sheaves of k-algebras. For any $y \in Y$, there is an induced map of k-algebras $f_y^\sharp : \mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$. Since the residue field at y is a finite extension of k, it is easy to check that $f_y^\sharp$ is automatically local and $f(y)$ is a closed point on X. Thus, morphisms $Y \to X$ are automatically maps of locally ringed spaces. We denote by $\mathrm{Hom}_k(Y, X)$ the set of such morphisms. The following lemma is essentially a special case of [EGA], II, Errata, I.1.8.1. The only difference is that rigid spaces have a mild Grothendieck topology instead of an ordinary topology, but the same proof carries over.

LEMMA 5.1.1. — *Let Y be a rigid space over k and X an affine k-scheme of finite type. The natural map of sets*

$$\mathrm{Hom}_k(Y, X) \to \mathrm{Hom}_k(\Gamma(X, \mathcal{O}_X), \Gamma(Y, \mathcal{O}_Y))$$

is an isomorphism.

For a fixed locally finite type k-scheme X, $Y \rightsquigarrow \mathrm{Hom}_k(Y, X)$ is a functor from rigid analytic spaces to sets. The basic fact is that this functor is representable. We define an *analytification* of X to be a morphism $i : X^{\mathrm{an}} \to X$ which represents the functor $\mathrm{Hom}_k(\cdot, X)$. Lemma 5.1.1 implies that rigid analytic affine n-space, constructed as in [BGR], 9.3.4/1 by gluing balls, admits a unique morphism to the k-scheme $\mathbf{A}_k^n$ which pulls back coordinate functions to coordinate functions (with the same ordering). This morphism is readily checked to be an analytification because rigid affine n-space has the expected universal mapping property in the rigid analytic category (thanks to the maximum modulus theorem for affinoids). Combining this with Lemma 5.1.1, the method in [SGA1], XII, 1.1 carries over to construct analytifications in general. If $\mathcal{F}$ is an $\mathcal{O}_X$-module, we define $\mathcal{F}^{\mathrm{an}} = i^*(\mathcal{F})$.

LEMMA 5.1.2. — *Let $i : X^{\mathrm{an}} \to X$ be the analytification of a locally finite type k-scheme X.*

1. *The map i is a bijection onto the set of closed points of X.*

2. *For $x \in X^{\mathrm{an}}$, the natural local map of local noetherian rings $\mathcal{O}_{X,i(x)} \to \mathcal{O}_{X^{\mathrm{an}},x}$ induces an isomorphism on completions, so i is a flat map.*

3. *For morphisms $X \to Z$, $Y \to Z$ of locally finite type k-schemes, the canonical map*

$$(X \times_Z Y)^{\mathrm{an}} \to X^{\mathrm{an}} \times_{Z^{\mathrm{an}}} Y^{\mathrm{an}}$$

is an isomorphism

The functor $\mathcal{F} \rightsquigarrow \mathcal{F}^{\mathrm{an}}$ from $\mathcal{O}_X$-modules to $\mathcal{O}_{X^{\mathrm{an}}}$-modules is exact, faithful, takes coherent sheaves to coherent sheaves, and $\mathcal{F}^{\mathrm{an}} = 0$ if and only if $\mathcal{F} = 0$.

Proof. — Since i has image inside of the closed points of X, as indicated above, the first part is a trivial consequence of the fact that $\mathrm{Spec}(k')^{\mathrm{an}} = \mathrm{Sp}(k')$ for finite extension fields k'/k and the characterization of closed points on X as the images of k-morphisms $\mathrm{Spec}(k') \to X$ for variable finite extension fields k'/k. For the second part one reduces to the case $X = \mathbf{A}_k^n$ and by scaling the variables we can assume x lies in the closed unit ball. Then the comparison of completions follows from the fact [BGR], 7.1.1/3 that every maximal ideal in $T_n(k)$ is generated by polynomials in $k[x_1, \dots, x_n]$.

The comparison of fiber products is reduced to the case where all three schemes are affine, and then we can use universal mapping properties and Lemma 5.1.1. The properties of $\mathcal{F} \rightsquigarrow \mathcal{F}^{\mathrm{an}}$ can be proven as in [SGA1], XII, 1.3.1, once we verify the exactness of the functor i^* on arbitrary $\mathcal{O}_X$-modules. This is a delicate point because in the rigid setting one cannot use stalks as nicely as in usual topological settings (there are problems with admissibility of coverings). It should be noted that such general exactness is useful if one wants to prove a relative (rigid) GAGA theorem along the lines of [SGA1], XII, §4, since the use of spectral sequences requires the "analytification" of injective sheaves, which are very non-coherent. Since the sheafication functor from presheaves to sheaves preserves exact sequences, it suffices (due to how i^* is defined) to show that if $X = \mathrm{Spec}(B)$ is affine and $W = \mathrm{Sp}(A)$ is an open affinoid inside of X^{an}, then the map of rings $B = \mathcal{O}_X(X) \to \mathcal{O}_{X^{\mathrm{an}}}(W) = A$ is flat. Pick a maximal ideal $\mathfrak{m}$

of A (corresponding to a point $x \in \mathrm{Sp}(A) \subseteq X^{\mathrm{an}}$), so this contracts back to a maximal ideal $\mathfrak{n}$ of B. It suffices to show that the map $B_{\mathfrak{n}} \to A_{\mathfrak{m}}$ is flat for all $\mathfrak{m}$. But $A_{\mathfrak{m}} \to \mathcal{O}_{X^{\mathrm{an}},x}$ is faithfully flat, so it is enough to study $B_{\mathfrak{n}} \to \mathcal{O}_{X^{\mathrm{an}},x}$. This is exactly the map $i_x^\sharp$, which we have seen is flat. □

Beware that even though all locally finite type k-schemes are quasi-separated, it is not true that the analytification of such a scheme is a quasi-separated rigid space. The problem is that quasi-compactness on the analytic side is a much stronger condition that on the algebraic side (consider an infinite sequence of copies of $\mathrm{Sp}(k)$ mapping to a "discrete" subset of the rigid affine line). As an example, the analytification of the affine line with a "doubled" origin is not quasi-separated. However, since schemes are covered by affine Zariski opens and extension of the base field on a quasi-separated rigid space takes Zariski opens to Zariski opens (by Lemma 3.1.1), we *can* define $X^{\mathrm{an}} \times_k k'$ for a locally finite type k-scheme X and a complete extension k'/k. However, this is somewhat ad hoc and seems to be a functor of X rather than of X^{an} if X^{an} is not quasi-separated, so we will only consider this construction when X^{an} is quasi-separated (the separatedness of X seems to be the only reasonable general condition which ensures that X^{an} is quasi-separated, though it even yields separatedness of X^{an} by Theorem 5.2.1 below). By studying the case of affine spaces and unit balls, it is easy to construct a functorial isomorphism

$$X^{\mathrm{an}} \times_k k' \simeq (X \times_k k')^{\mathrm{an}} \tag{3}$$

for any complete extension k'/k and any locally finite type k-scheme X with X^{an} quasi-separated, and this isomorphism respects formation of fiber products and composite change of the base field. If Z is any quasi-separated rigid space over k, by working over affinoids one can define a "pullback" functor from coherent sheaves on Z to coherent sheaves on $Z \times_k k'$ with nice properties with respect to standard sheaf-theoretic constructions on coherent sheaves, and it is easy to check that this "pullback" is compatible with the $(\cdot)^{\mathrm{an}}$ functor on coherent sheaves via the isomorphism (3).

THEOREM 5.1.3. — *Let X be a locally finite type k-scheme.*

1. *Each of the following properties holds for X if and only if it holds for X^{an}: non-empty, discrete, Cohen-Macaulay, S_i, regular, R_i, normal, reduced, of dimension n, connected, irreducible.*

2. *Let T be a locally constructible set in X, $i : X^{\mathrm{an}} \to X$ the analytification of X. Then $i^{-1}(\overline{T}) = \overline{i^{-1}(T)}$, where the closure on the right is taken with respect to the "naive" (not the rigid analytic) topology on*

X^{an}; that is, taking opens to be arbitrary unions of admissible opens.

For T as above, T is closed (resp. open, resp. dense) if and only if the same holds for $i^{-1}(T)$, again using the naive topology on X^{an}.

Proof. — Thanks to the excellence results in §1, the proofs in [SGA1], XII, §2 carry over for everything except for the last two properties in the first part. In both the scheme and rigid cases, a space X is connected if and only if for any two irreducible components Z and Z', there exists finitely many irreducible components $X_1, \ldots, X_n$ of X such that $Z = X_1$, $Z' = X_n$ and $X_i \cap X_{i+1}$ is non-empty for $1 \leq i < n$. Thus, we can use Theorem 2.3.1. □

5.2. Properties of morphisms.

If f is a morphism of locally finite type k-schemes, one can ask if f has a certain property (e.g., open immersion, separated, etc.) if and only if f^{an} does. There is a long list of such properties in [SGA1], XII, 3.1, 3.2, and most of the proofs there carry over verbatim. However, due to the special nature of the rigid topology, some of these require extra care. In particular, the deduction of properness of f^{an} from properness of f is a very delicate issue.

THEOREM 5.2.1. — *Let $f : X \to Y$ be a morphism of locally finite type k-schemes.*

1. *The following properties are equivalent for both f and f^{an}: flat, unramified, étale, smooth, flat with geometrically normal fibers, flat with geometrically reduced fibers, injective, separated, open immersion, isomorphism, monomorphism.*

2. *Suppose f has finite type (i.e. f is quasi-compact). Then the following properties are equivalent for both f and f^{an}: surjective, having dense image (with respect to the naive topology on the rigid side), closed immersion, immersion (i.e. an open immersion followed by a closed immersion), quasi-finite (i.e. finite fibers), finite. Moreover, if f^{an} is proper then so is f.*

Proof. — Using [EGA], IV_3, 12.1.1(*vii*), (*viii*) and the discussion in §1.2, for the first part, the methods in the complex-analytic case work for all but the last four properties, and the deduction of the last two cases in the first part from the "separated" and "open immersion" cases is also done as in the complex analytic case. Thus, we want to check that f is an open immersion (resp. separated) if and only if f^{an} is an open immersion (resp. separated). The "only if" directions are proven easily as in the complex-analytic case. Conversely, assuming that f^{an} is separated or an open immersion, we want to deduce that f has the same property. Working locally on Y, we can assume that Y is affine, so Y^{an} and X^{an} are trivially separated (and hence quasi-separated). Thus, isomorphisms such as (3) make sense. Since we do not know if extension of the base field on the rigid side takes open immersions to open immersions, we will instead show that if f^{an} is separated (resp. an injective local isomorphism between separated rigid spaces), then f is separated (resp. an injective local isomorphism). This alternative implication is certainly sufficient for the first part of the theorem, since injective local isomorphisms are open immersions in the context of schemes. Moreover, by using [BGR], 7.3.3, 8.2.1/4 and the construction of the base change functor, it is not difficult to check that if a map between separated rigid spaces is an injective local isomorphism, then the same property holds after extension of the base field.

Using [EGA], IV_2, 2.7.1(*i*),(*x*) and the isomorphism (3), we may base change to $\widehat{\overline{k}}$ and thereby reduce to the case in which k is algebraically closed. Once the base field k is algebraically closed, étale injective maps between locally finite type k-schemes are radicial (hence open immersions) and fiber products of rigid spaces over k are fiber products on the level of underlying sets. This allows the complex-analytic proofs to carry over for the "if" direction in the case of open immersions (or injective local isomorphisms) and separated maps.

Now we treat the second part of the theorem. The "surjective" and "dominant" cases are handled as in the complex-analytic case, and likewise for the deduction of the "immersion" case from the "closed immersion" case. The quasi-finite case is seen by using [EGA], IV_3, 9.6.1(*viii*), which implies the local constructibility of $\{y \in Y \mid \dim f^{-1}(y) = 0\}$. Finally, we can carry over the complex-analytic proof that f is proper when f^{an} is proper, using [EGA], II, 5.6.3 and Kiehl's theorem that the image of an analytic set under a proper rigid morphism is again an analytic set [BGR], 9.6.3/3.

It remains to show that a finite type map f is a closed immersion (resp. finite) if and only if f^{an} is a closed immersion (resp. finite). The "only if" direction for closed immersions is trivial, and for finite maps it follows from the observation that a finite f can (locally over Y) be factored as a closed immersion followed by a map $Y[T_1, \ldots, T_m]/(g_1, \ldots, g_m) \to Y$ with each g_i a monic in T_i (the analytification of which is trivially finite over Y^{an} by using the fiber product compatibility of $(\cdot)^{\mathrm{an}}$). One now proves just as in the complex-analytic case that f is a closed immersion (resp. finite) when f^{an} is. □

On both the scheme and rigid sides, one can "classify" finite morphisms to a space X in terms of coherent sheaves of $\mathcal{O}_X$-algebras. The functor $(\cdot)^{\mathrm{an}}$ and the isomorphism (3) are compatible with these constructions. The details are fairly straightfoward and are left to the reader.

We end this section with a closer look at the properness issue. Let $f : X \to Y$ be a finite type map between locally finite type k-schemes. It is a non-trivial matter to determine whether f^{an} is proper when f is proper (note that this fact is *not* needed to prove a relative rigid GAGA theorem using the method in [SGA1], XII, §4). To carry over the complex-analytic proof, it suffices to show the following facts in the rigid setting:

- a composite of proper maps is proper,
- a separated quasi-compact f is proper if $f \circ \pi$ is proper for some surjective morphism π

(actually, one only needs the second claim). One can immediately reduce this to the case in which all of the rigid spaces involved are quasi-compact and separated. Since formal admissible blow-ups of "admissible" formal schemes over $\mathrm{Spf}(R)$ in the sense of [BL1], §2 are proper, a proof of *both* assertions above would follow if one could show that for a map $f : X \to Y$ of quasi-compact quasi-separated rigid spaces over k with a formal model $\mathfrak{f} : \mathfrak{X} \to \mathfrak{Y}$, f is proper if and only if $\mathfrak{f}$ is proper and $\mathfrak{f}$ is surjective if f is.

Using rig-points as in [BL1], Prop 3.5, the "surjective" part is clear. Although Lütkebohmert's proof that properness of f implies properness of $\mathfrak{f}$ [L2], 2.5, 2.6 is given in the case when the valuation on k is a discrete valuation, the method of proof works for arbitrary k by using the formal model setup in [BL1] and the non-trivial fact that an open immersion between quasi-compact quasi-separated rigid spaces has a formal model which is an open immersion [L2], 5.4(b). Conversely, the deduction of properness of f from properness of $\mathfrak{f}$ is given in [L2], Thm 3.1 under the hypothesis that the absolute value on k is a discrete valuation. The proof of this converse implication is much more difficult and it is unclear to what extent one can modify the methods there to work without the noetherian conditions.

For the purpose of proving that the analytification of a proper scheme map is proper, one can use the Nagata Compactification Theorem for finitely presented R-schemes (in conjunction with Theorem 5.3.1 below and direct limit arguments descending to the noetherian case [EGA], IV_3, 8.9.1, 8.10.5) to bypass the complex-analytic method and reduce to the purely formal scheme problem of proving that $\mathfrak{X}^{\mathrm{rig}}$ is a proper rigid space when $\mathfrak{X}$ is a proper admissible formal scheme over $\mathrm{Spf}(R)$. However, there seems to be no reason why this should be any easier to prove for general R than Lütkebohmert's [L2], Thm 3.1 for general R.

5.3. Analytification and rigidification.

Let X be a locally finitely presented scheme over R. It makes sense to form the formal completion $\mathfrak{X}$ of X with respect to an ideal of definition I of R (the result is a locally topologically finitely presented formal scheme over $\mathrm{Spf}(R)$ and is independent of the choice of I). To work with this type of construction, we need to make essential use of the fact that large parts of the "noetherian" theory in [EGA], I, §10 carry over to locally finitely presented R-schemes and locally topologically finitely presented formal schemes over $\mathrm{Spf}(R)$ (and in particular, the process of taking formal completions of locally finitely presented R-schemes transforms open immersions and closed immersions into open immersions and closed immersions respectively). This generalization of [EGA], I, §10 is already needed in the foundations of the theory of formal models of rigid spaces over arbitrary complete fields, and one can work out the lengthy but mostly straightfoward details by using the results in [BL1], §1 and some cohomological facts in [EGA], 0_{III}, III_1. We omit the details.

If X is not R-flat, then the formal scheme $\mathfrak{X}$ over $\mathrm{Spf}(R)$ might not be R-flat (and so might not be "admissible" in the sense of [BL1], §1). Of course, replacing X by the R-flat closed subscheme cut out by the I-power torsion would in general probably lose the locally finitely presented property for X over R which is needed in order to have a good theory of formal completion. For any locally topologically finitely presented formal scheme $\mathfrak{Z}$ over $\mathrm{Spf}(R)$ which is R-flat, we define the associated rigid space $\mathfrak{Z}^{\mathrm{rig}}$ as in [BL1], §4. Without assuming $\mathfrak{Z}$ to be R-flat, by [BL1], 1.4 the I-power torsion ideal sheaf on the formal scheme $\mathfrak{Z}$ is coherent, so we can define the locally topologically finitely presented formal R-scheme $\mathfrak{Z}'$ cut out by this ideal sheaf. This formal scheme *is* R-flat and we define $\mathfrak{Z}^{\text{rig}} = (\mathfrak{Z}')^{\text{rig}}$ in general. If $\mathfrak{Z} = \text{Spf}(A)$, then $\mathfrak{Z}^{\text{rig}} = \text{Sp}(A \otimes_R k)$. This is functorial in $\mathfrak{Z}$ in the obvious manner, and by reduction to the formal affine case we see that $(\cdot)^{\text{rig}}$ commutes with the formation of fiber products.

For X as above, we can now construct two rigid spaces over k: $\mathfrak{X}^{\text{rig}}$ and $(X \times_R k)^{\text{an}}$. We want to explain the relationship between these. This is probably rediscovered by everyone who studies rigid geometry, and one such construction is given in [Ber], 0.3.5 (which seems to say "finite type" where it should say "finite presentation"); we want to give a more "pure thought" construction which we found before becoming aware of [Ber]. In general there will be a morphism $i_X : \mathfrak{X}^{\text{rig}} \to (X \times_R k)^{\text{an}}$ which is an isomorphism for proper X over R and an open immersion when X is separated and admits a locally finite affine covering. As an example, consider $X = \mathbf{A}^n_R = \text{Spec}(A)$ with $A = R[T_1, \ldots, T_n]$. In this case, $\mathfrak{X}^{\text{rig}} = \mathbf{B}^n$ and $(X \times_R k)^{\text{an}} = (\mathbf{A}^n_k)^{\text{an}}$, and i_X will be the canonical open immersion of the unit ball around the origin into rigid affine space.

Before we look at the general case, we first consider the affine case. Let I be an ideal of definition of R. Let A be a finitely presented R-algebra, $\widehat{A}$ its I-adic completion, and $A_k = A \otimes_R k$. We want to construct a morphism

$$i_A : \text{Sp}(\widehat{A} \otimes_R k) \to \text{Spec}(A_k)^{\text{an}}$$

which is functorial in A.

Using Lemma 5.1.1, the k-algebra map $\varphi_A : A_k \to \widehat{A} \otimes_R k$ arises from a unique morphism $\text{Sp}(\widehat{A} \otimes_R k) \to \text{Spec}(A_k)$. By the universal mapping properties of analytifications of schemes, this induces a canonical morphism $i_A : \text{Sp}(\widehat{A} \otimes_R k) \to \text{Spec}(A_k)^{\text{an}}$. To check that this is functorial in A, we can use universal mapping properties with rings of global functions to reduce to the obvious fact that for a map $A \to B$ of finitely presented R-algebras, the diagram of rings

$$\begin{array}{ccc} A_k & \longrightarrow & \widehat{A} \otimes_R k \\ \downarrow & & \downarrow \\ B_k & \longrightarrow & \widehat{B} \otimes_R k \end{array}$$

commutes. By using the universal mapping property of rigid affine space (as well as the construction of rigid affine space), it is clear that when $A = R[T_1, \ldots, T_n]$, i_A is just the canonical open immersion from the unit ball $\mathbf{B}^n$ around the origin into rigid affine n-space.

THEOREM 5.3.1. — *Let X be a locally finitely presented R-scheme with generic fiber X_k and formal completion $\mathfrak{X}$ with respect to an ideal of* *definition of R. There is a unique way to define a morphism of rigid spaces*

$$i_X : \mathfrak{X}^{\mathrm{rig}} \to X_k^{\mathrm{an}}$$

which is functorial in X, respects the formation of fiber products, and is the canonical map $\mathbf{B}^1 \hookrightarrow (\mathbf{A}_k^1)^{\mathrm{an}}$ when $X = \mathbf{A}_R^1$. The following additional properties hold:

1. *For finitely presented R-algebras A, $i_{\mathrm{Spec}(A)} = i_A$.*

2. *If $Z \hookrightarrow X$ is a closed immersion of finite presentation, with formal completion $\mathfrak{Z} \hookrightarrow \mathfrak{X}$, then $i_X^{-1}(Z_k^{\mathrm{an}}) = \mathfrak{Z}^{\mathrm{rig}}$ as closed analytic subspaces of $\mathfrak{X}^{\mathrm{rig}}$.*

3. *When X is separated and admits a locally finite affine covering, i_X is a quasi-compact open immersion; in particular, i_X is a local isomorphism for general X.*

4. *When X is proper over R, i_X is an isomorphism.*

Proof. — The existence of a functorial i_A yields a functorial morphism $i_X : \mathfrak{X}^{\mathrm{rig}} \to X_k^{\mathrm{an}}$ satisfying $i_{\mathrm{Spec}(A)} = i_A$ for finitely presented R-algebras A. Since every affine X of finite presentation over R admits a finitely presented closed immersion into some $\mathbf{A}_R^n \simeq \mathbf{A}_R^1 \times \ldots \times \mathbf{A}_R^1$, uniqueness is clear. Beware that for an open affine $U = \mathrm{Spec}(A)$ in X, $i_X^{-1}(U_k^{\mathrm{an}})$ is typically much larger than the open affinoid $\mathrm{Sp}(\widehat{A} \otimes_R k)$ in $\mathfrak{X}^{\mathrm{rig}}$ (consider $X = \mathbf{A}_R^1$ and U the complement of the origin). To check that i_X respects formation of fiber products, it suffices to consider the affine case. Since analytification commutes with fiber products, we are reduced to the obvious fact that for maps $C \to A$, $C \to B$ of finitely presented R-algebras, the diagram of rings

$$\begin{array}{ccc} A_k \otimes_{C_k} B_k & \longrightarrow & (\widehat{A} \otimes_R k) \widehat{\otimes}_{\widehat{C} \otimes_R k} (\widehat{B} \otimes_R k) \\ \uparrow & & \uparrow \\ (A \otimes_C B)_k & \longrightarrow & (A \widehat{\otimes}_C B) \otimes_R k \end{array}$$

commutes (the columns are isomorphisms).

Compatibility with respect to a finitely presented closed immersion $Z \hookrightarrow X$ is easily reduced to the case of affine X. We have to show that if $B \twoheadrightarrow A$ is a surjection of finitely presented R-algebras, then the diagram

$$\begin{array}{ccc} \mathrm{Sp}(\widehat{A} \otimes_R k) & \xrightarrow{i_A} & \mathrm{Spec}(A_k)^{\mathrm{an}} \\ \downarrow & & \downarrow \\ \mathrm{Sp}(\widehat{B} \otimes_R k) & \xrightarrow{i_B} & \mathrm{Spec}(B_k)^{\mathrm{an}} \end{array}$$

(with columns closed immersions) is cartesian. This is seen by using the universal mapping properties of analytifications and the universal mapping properties of closed immersions for both affinoid rigid spaces and affine schemes.

Every affine X admits a closed immersion into some $\mathbf{A}^n_R$, so by the compatibility with respect to closed immersions and fiber products, i_X is a quasi-compact open immersion for affine X provided $i_{\mathbf{A}^1_R}$ is a quasi-compact open immersion. Since $i_{\mathbf{A}^1_R}$ is the canonical open immersion of the unit ball around the origin into rigid affine space, we see that i_X is a quasi-compact open immersion for affine X.

Now assume that X is separated and admits a locally finite covering by affines. We want to prove that i_X is a quasi-compact open immersion. For now, do not assume that X admits a locally finite affine covering. The first step is to use the valuative criterion for separatedness to show that i_X is injective on the underlying sets. Suppose that $x_1, x_2 \in \mathfrak{X}^{\mathrm{rig}}$ map to the same point $x \in X_k^{\mathrm{an}}$ under i_X. Pick open affines $U_j = \mathrm{Spec}(A_j)$ in X with $x_j \in \mathrm{Sp}(\widehat{A_j} \otimes_R k)$. Letting k' be a sufficiently large finite extension of k with valuation ring R', x_j corresponds to a k-algebra map $\widehat{A_j} \otimes_R k \to k'$, which (by the continuity of maps between affinoids) is equivalent to an R-algebra map $A_j \to R'$. We claim that the corresponding elements in $X(R')$ are equal. Since X is separated over R, it suffices to check that the two resulting maps $A_j \otimes_R k \to k'$ give rise to the same element in $X_k(k')$. Using $X_k(k') = X_k^{\mathrm{an}}(k')$, we get x in both cases. Thus, $x_1 = x_2$, so i_X is injective.

Now we check that i_X is quasi-compact when X is separated and has a locally finite affine open covering $\{\mathrm{Spec}(A_j)\}$. Since each $\mathrm{Spec}(A_j)$ meets only finitely many other $\mathrm{Spec}(A_{j'})$'s, it follows that $\{\mathrm{Sp}(\widehat{A_j} \otimes_R k)\}$ is a locally finite admissible affinoid covering of $\mathfrak{X}^{\mathrm{rig}}$. Letting U be an open affinoid in X_k^{an}, it suffices to check that the overlap of U with each $\mathrm{Sp}(\widehat{A_j} \otimes_R k) \subseteq \mathrm{Spec}(A_j)_k^{\mathrm{an}} \subseteq X_k^{\mathrm{an}}$ is quasi-compact (note this overlap is empty for all but finitely many j). Since X_k^{an} is separated (because X_k is separated), the overlap of any two open affinoids is an open affinoid, hence quasi-compact. Thus, i_X is quasi-compact.

To see that i_X is an open immersion as well, choose an open affinoid U in X_k^{an}. Using the locally finite affine covering $\{\mathrm{Spec}(A_j)\}$ of X, we get a locally finite admissible covering of X_k^{an} by the $\mathrm{Spec}(A_j)_k^{\mathrm{an}}$'s. Thus, there are only finitely many j for which the quasi-compact U meets $\mathrm{Spec}(A_j)_k^{\mathrm{an}}$, and so only finitely many j for which U meets the admissible open $\mathrm{Sp}(\widehat{A_j} \otimes_R k) \hookrightarrow \mathrm{Spec}(A_j)_k^{\mathrm{an}}$. Let $V_j = U \cap \mathrm{Sp}(\widehat{A_j} \otimes_R k)$ when this is non-empty. This is an intersection of open affinoids in the separated space X_k^{an}, so V_j is affinoid, with the canonical maps $V_j \to \mathfrak{X}^{\mathrm{rig}}$ and $V_j \to U \subseteq X_k^{\mathrm{an}}$

both open immersions. Since $X_k^{\rm an}$ and $\mathfrak{X}^{\rm rig}$ are separated, any finite union W of affinoid opens W_i in either of these spaces is an admissible open and the finitely many such affinoids W_i give an admissible cover of W. Thus, $i_X^{-1}(U) = \cup V_j$ is an admissible open in $\mathfrak{X}^{\rm rig}$ with the V_j's giving an admissible cover, and $i_X(i_X^{-1}(U)) = \cup i_X(V_j)$ is an admissible open in $X_k^{\rm an}$ and the $i_X(V_j)$'s give an admissible cover. Since $V_j = i_X^{-1}(i_X(V_j)) \to i_X(V_j)$ is an isomorphism, we conclude that $i_X^{-1}(U) \to U$ has an admissible open image and $i_X^{-1}(U)$ maps isomorphically onto this image. Therefore, i_X is an open immersion. This completes the proof that i_X is a quasi-compact open immersion when X is separated and has a locally finite affine covering.

Finally, suppose that X is proper over R. Since i_X is an open immersion, it is an isomorphism as long as it is surjective on the underlying sets. To prove this surjectivity, we will use the valuative criterion for properness. A point $x \in X_k^{\rm an}$ is the image of a k-morphism $x : {\rm Sp}(k') \to X_k^{\rm an}$ for a finite extension field k'/k. The integral closure R' of R in k' is the valuation ring of the complete field k', so it is I-adically complete. By the universal mapping property of analytifications, such a morphism x is equivalent to giving a map ${\rm Sp}(k') \to X_k$ as locally ringed Grothendieck topological spaces of k-algebras, which is equivalent to giving a map of schemes ${\rm Spec}(k') \to X_k$ over k. By the valuative criterion for properness, there is a unique R-morphism $\psi : {\rm Spec}(R') \to X$ such that $\psi_k^{\rm an} : {\rm Sp}(k') \to X_k^{\rm an}$ is our original map x.

Let $\widehat{\psi} : {\rm Spf}(R') \to \mathfrak{X}$ be the induced map on formal I-adic completions. We want to apply $(\cdot)^{\rm rig}$ and appeal to the functoriality of the $i_{(\cdot)}$ construction with respect to the map ψ. The only problem is that R' might not be finite over R (hence not finitely presented). However, this is easy to circumvent. Choose an open affine $U = {\rm Spec}(A)$ in X around the image of the closed point under ψ, so for topological reasons ψ factors through U. Thus, we get a map $h : A \to R'$. The map h induces a k-algebra map $A_k \to k'$ whose kernel corresponds to the point x (under the identification of closed points on X_k with points of $X_k^{\rm an}$). But h also induces a map $\widehat{A} \to R'$ on I-adic completions, and therefore a map $\widehat{A} \otimes_R k \to k'$. This gives rise to a point x' on ${\rm Sp}(\widehat{A} \otimes_R k) \subseteq \mathfrak{X}^{\rm rig}$ and it is obvious that $i_A(x') = x$. By the functoriality of $i_{(\cdot)}$ with respect to the open immersion ${\rm Spec}(A) = U \hookrightarrow X$, $i_X(x') = x$ must hold. □

If we do not assume that X is separated, then it can happen that i_X is not injective (and so is not an open immersion), even when X_k is separated. This arises from the failure of the valuative criterion. A typical example is the following (essentially [Ber], 0.3.6). Let X be the finitely presented flat R-scheme obtained by gluing two copies of $\mathbf{A}^1_R$ via the identity map along the complement of the origin in the closed fiber. In this case, $\mathfrak{X}^{\mathrm{rig}}$ is obtained from gluing two copies of $\mathbf{B}^1_k$ via the identity map along the "boundary" $\{|t| = 1\}$ and X_k^{an} is the rigid affine line. The map i_X is the obvious map, and this collapses the two "open unit balls" in $\mathfrak{X}^{\mathrm{rig}}$ to the single "open unit ball" in $X_k^{\mathrm{an}} = (\mathbf{A}^1_k)^{\mathrm{an}}$.

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