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Irreducible Unitary Representations of a Diffeomorphisms Group of an Infinite-dimensional Real Manifold

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Summary. - *Groups of diffeomorphisms* $Diff^t_{\beta,\gamma}(M)$ *of infinite-dimensional real Banach manifolds* M *are defined. Their structure is studied. Irreducible unitary representations of a group of diffeomorphisms associated with quasi-invariant measures on a Banach manifold are constructed.*

1. Introduction

For a locally compact (finite-dimensional) manifold M irreducible unitary representations of a group of diffeomorphisms were constructed in [13] with the help of a measure on M induced by the Lebesgue measure on $\mathbb{R}^n$ and the Riemannian metric g on M. Each group of diffeomorphisms is an infinite-dimensional manifold itself. Their structure for locally compact M was investigated in [2,7].

This article is devoted to the definition of a group of diffeomorphisms of a Banach manifold and the construction its irreducible unitary representations. For this are used quasi-invariant Gaussian measures on M.

In Section 2 notations and definitions are given. Section 3 contains results about the structure of a group of diffeomorphisms. Irreducible unitary representations of a group of diffeomorphisms associated with a quasi-invariant measure on a Banach manifold are

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described in Section 4. There is the great difference in investigations between cases of finite-dimensional and infinite-dimensional M. The main results of the present paper are deduced for the first time and given below in Theorems 3.3, 4.1, 4.17, 4.18.

2. Notations and definitions

To avoid misunderstandings, we first present our notations and terminology.

Definition 2.1. Let U and V be open subsets in l_2. We consider a space of all infinitely many times Frechét (strongly) differentiable functions $f, g : U \to V$ fulfilling (i, ii) and with a finite metric $\rho^t_{\beta,\gamma}(f,h) < \infty$, where h is some fixed smooth mapping $h : U \to V$ (that is of class C^∞);

$$(i) \quad \rho^t_{\beta,\gamma}(f,g) := \sup_{x\in U,\ y\ne x,\ y\in U} (\sum_{n=0}^{\infty}[d^t_{n,\beta,\gamma}(f,g)]^2)^{1/2} < \infty;$$

$$d^t_{0,\beta,\gamma}(f,g) := \| < x >^\beta (f(x) - g(x))\|_{l_{2,\gamma}},$$

$$\begin{aligned}(d^t_{n,\beta,\gamma}(f,g))^2 := & \sum_{\substack{\alpha^n\ne 0,\ |\alpha|\le t\\ \alpha=(\alpha^1,\dots,\alpha^n)}} \|\bar{n}^{\alpha\gamma} < x >^{\beta+|\alpha|} D^\alpha_x(f(x)-g(x))\|^2_{l_{2,\gamma}} + \\ & + \sum_{\substack{\alpha=(\alpha^1,\dots,\alpha^n)\\ |\alpha|=[t]}} \|n\bar{n}^{\alpha\gamma} < \tilde{x} >^{\beta+|\alpha|+b} [D^\alpha_x(f(x)-g(x)) \\ & - D^\alpha_y(f(y)-g(y))]\|^2_{l_{2,\gamma}} / |x^n - y^n|^{2b},\end{aligned}$$

for $n \in \mathbb{N} := \{1,2,3,\dots\}$, $d^t_{n,\beta,\gamma}\ (f,g) = d^t_{n,\beta,\gamma}(f,g)(x,y)$, such that

$$(ii) \quad \lim_{R\to\infty} \rho^t_{\beta,\gamma}(f|U^c_R, h|U^c_R) = 0.$$

Here $x = (x^j : j\in\mathbb{N},\ x^j\in\mathbb{R}) \in l_{2,\gamma}$ that is

$$\|x\|_{l_{2,\gamma}} = \left\{ \sum_{j=1}^{\infty}(x^j j^\gamma)^2 \right\}^{1/2} < \infty,$$

$\infty > \gamma \ge 0$, $l_2 = l_{2,0}$ is the standard separable Hilbert space over $\mathbb{R}$ with the orthonormal base $\{e_n : n \in \mathbb{N}\}$, $U^c_R := (x \in U : \|x\|_{l_2} > R)$, $f(x) = (f^j(x) : j \in \mathbb{N}, f^j(x) \in \mathbb{R})$, $t \ge 0$, $[t]$ is the integral part of t (the largest integer such that) $[t] \le t$, $b = \{t\} := t - [t]$, $0 \le b < 1$ (for $b = 0$ the last term in the definition of $d^t_{n,\beta,\gamma}$ is omitted), $D^{e_j}_x := \partial/\partial x^j =: \partial_j$, $D^{\alpha+\gamma}_x f(x) := D^\gamma_x(D^\alpha_x f(x))$, $e_j = (0,\dots,0,1,0,\dots)$ with 1 in the j-th place, $\alpha = (\alpha^1,\dots,\alpha^n)$, $\alpha^j \in \mathbb{N} \cup 0 =: \mathbb{N}_o$, $|\alpha| = \alpha^1 + \dots + \alpha^n$, $\beta \in \mathbb{R}$, $< \tilde{x} >:= \min(< x >, < y >)$, $< x >:= (1+\|x\|^2_{l_2})^{1/2}$, $f(x) - g(x) \in l_2$, $f|A$ denotes a restriction of f on a subset $A \subset U$, $\bar{n}^\alpha := 1^{\alpha^1} 2^{\alpha^2} \dots n^{\alpha^n}$ for $n \in \mathbb{N}$.

We denote by $E^{t,h}_{\beta,\gamma}(U,V)$ the completion of such metric space, $E^\infty_\beta := \bigcap_{j=1}^\infty E^j_\beta(U,V)$ with the topology given by the family $(\rho^j_{\beta,\gamma} : j \in \mathbb{N})$ in the latter case. For $V = l_2$ and $h(u) = 0$ it is the Banach space with $\|f - g\|_{E^{t,h}_{\beta,\gamma}(U,l_2)} := \rho^t_{\beta,\gamma}(f,g) = \rho^t_{\beta,\gamma}(f - g, 0)$ that is, the infinite-dimensional separable analog of the weighted Hölder space $C^t_\beta(U', \mathbb{R}^m)$ (compare with [5]) for open $U' \subset \mathbb{R}^k$, k and $m \in \mathbb{N}$. When $\gamma = 0$ or $h(U) = 0$ we omit γ or h respectively. It is evident that each cylindrical function $g(P_k x)$ is in $E^t_\beta(U, l_2)$ if $g \in C^t_\beta(U', \mathbb{R}^m)$, $P_k : l_2 \to \mathbb{R}^k$ is the orthogonal projection, $U = (P_k)^{-1}(U')$, $g(P_k x) := (g^1(P_k x), \dots, g^m(P_k x), 0, 0, \dots)$. The spaces $E^t_\beta(U,V)$ differ from $E^t_0(U,V) =: E^t(U,V)$ for unbounded U if $\beta > 0$.

Definition 2.2. Let M be a manifold modelled on l_2 and fulfilling conditions (i-vi) below:

(i) an atlas $At(M) = [(U_j, \phi_j) : j = 1, \dots, k]$ is finite, $k \in \mathbb{N}$ (or countable, $k = \infty$), $\phi_j : U_j \to l_2$ are homeomorphisms of U_j onto $\phi_j(U_j) \ni 0$, U_j and $\phi_j(U_j)$ are open in M and l_2 respectively, $(\phi_j \circ \phi_i^{-1} - id) \in E^\infty_{\omega,\delta}(\phi_i(U_i \cap U_j), l_2)$ for each $U_i \cap U_j \ne \emptyset$, where $\omega > 0$, $\gamma \ge 0$, id is the identity mapping $id(x) = x$ for each x;

(ii) TM is a Riemannian vector bundle with a projection $\pi : TM \to M$ and a metric g_x in $T_x M$ induced by $\| * \|_{l_2}$ with a RMZ-structure. This means that a connector K and g are such that $g_c(X,Y)$ is constant for each C^∞-curve $c : I \to M$, $I = [0,1] \subset \mathbb{R}$ and parallel translation along c of X and $Y \in \Xi(M)$,

$\Xi(M) := \Xi_{TM}(M)$ is the algebra of infinitely differentiable vector fields on M (see 3.7 in [10]);

(iii) (M, g) is geodesically complete and supplied with the Levi-Civita connection and the corresponding covariant differentiation ∇ (see 1.1, 2.1 and 5.1 in [10]);

(iv) the charts (U_j, ϕ_j) are natural with the natural (Gaussian) coordinates with locally convex $\phi_j(U_j)$ and the exponential mapping $exp_p : V_p \to M$ corresponding to ∇, where V_p is open in T_pM for each $p \in M$, each restriction $exp_p|V_p$ is the local homeomorphism (see Section III.8 in [15], Section 6, 7 in [10]) such that $r_{inj} := \inf_{x \in M} r_{inj}(x) > 0$, where $r_{inj}(x)$ is a radius of injectivity for exp_x, r_{inj} is for entire M;

(v) M is Hilbertian at infinity, that is, there exists $\tilde{M}_R \subset M$ with $M \setminus \tilde{M}_R =: M^c_R$ equal to finite (or countable) disjoint union of connected open components Ω_a, $a = 1, \dots, p$, such that $\phi_a^{-1}(\Omega_a) = l_2 \setminus B_a$, where B_a are closed balls in l_2, each Ω_a is with a metric $\tilde{e}$ induced by ϕ_a^{-1} and the standard metric in l_2. Let a metric g for M be elliptic, that is, there exists $\lambda > 0$ such that $\lambda \tilde{e}_x(\xi, \xi) \le g_x(\xi, \xi)$ for each $\xi \in T_xM$ and $x \in M$, where $\tilde{M}_R := [x \in M : \ d_M(x, x_0) \le R]$, x_0 is some fixed point in M, d_M is the distance function on M induced by g, $\infty > R > 0$ (see for comparison the finite-dimensional case of M in [5]);

(vi) M contains a sequence of M_k and N_k. They are supposed to be closed $E^{\infty}_{\omega,\gamma}$-submanifolds with finite dimensions $dim_{\mathbb{R}} M_k = k$ for M_k and codimensions $codim_{\mathbb{R}} N_k = k$ for N_k, $k = k(n) \in \mathbb{N}$, $k(n) < k(n+1)$ for each n, $M_k \subset M_l$ and $N_k \supset N_l$ for each $k < l$, $M = M_k \cup N_k$, $M_k \cap N_k = \partial M_k \cap \partial N_k$ for each k such that $\bigcup_k M_k$ is dense in M, $At(M)$ and M are foliated in accordance with this decompositions. These means that (α) $\phi_{i,j} := \phi_i \circ \phi_j^{-1}|\phi_j(U_i \cap U_j) \to l_2$ are of the form $\phi_{i,j}((x^l : l \in \mathbb{N})) = (\alpha_{i,j,k}(x_1, \dots, x_k), \gamma_{i,j,k}((x^l : l > k)))$ for each $n \in \mathbb{N}$, $k = k(n)$, when M is without boundary, $\partial M = \emptyset$. If $\partial M \ne \emptyset$ there is the following additional condition: (β) for each boundary component M_0 of M and $U_i \cap M_0 \ne \emptyset$ we have $\phi_i : U_i \cap M_0 \to$

H_l, where $H_l = \{(x_j : j \in \mathbb{N}) | \, x^l \ge 0\}$. If $U_i \cap M_0 \ne \emptyset$ and $U_j \cap M_0 \ne \emptyset$ we have both images in H_1 (or in H_l with $l > 1$), then the foliation is called transverse (tangent respectively) to M_0. Then the equivalence relation of $E^{\infty}_{\omega,\gamma}$-atlases that produces foliated M (see also [12] for finite-dimensional C^r-manifolds) is as usually considered.

Definition 2.3. Let M and $\tilde{M}$ be two manifolds as in 2.2 with a smooth mapping (for example, an embedding) $\theta : \tilde{M} \hookrightarrow M$, ω and $\tilde{\omega} \ge max(0,\beta)$, $\beta \in \mathbb{R}$, $t \in \bar{\mathbb{R}}_+ := [0,\infty)$, $\infty > \gamma \ge 0$, δ and $\tilde{\delta} \ge \gamma$. We denote by $\tilde{E}^{t,\theta}_{\beta,\gamma}(\tilde{M},M)$ a space of functions $f : \tilde{M} \to M$ with $f_{i,j} := \phi_i \circ f \circ \tilde{\phi}_j^{-1} | (\tilde{\phi}_j(\tilde{U}_j) \cap \tilde{\phi}_j(f^{-1}(U_i)))$, $(f_{i,j} - \theta_{i,j}) \in E^{t,\theta}_{\beta,\gamma}(\tilde{\phi}_j(\tilde{U}_j) \cap \tilde{\phi}_j(f^{-1}(U_i))), \phi_i(U_i))$ for each i and j. When $At(M)$ is finite it is metrizable by a metric (i) $\tilde{\rho}^t_{\beta,\gamma}(f,\theta) := \sum_{i,j} \rho^t_{\beta,\gamma}(f_{i,j},\theta_{i,j})$ with (ii) $\lim_{R\to\infty} \tilde{\rho}^t_{\beta,\gamma}(f|M^c_R,\theta) = 0$. For infinite countable $At(M)$ we denote by $\tilde{E}^{t,\theta}_{\beta,\gamma}(\tilde{M},M)$ the strict inductive limit $str{-}ind{-}\lim[\tilde{E}^{t,\theta}_{\beta,\gamma}(\tilde{U}^E, M), \Pi^F_E, \Sigma]$, where $E \in \Sigma$, Σ is the family of all finite subsets of $\mathbb{N}$ directed by the inclusion $E < F$ if $E \subset F$, $\tilde{U}^E := \bigcup_{j\in E} \tilde{U}_j$, $(\tilde{U}_j, \tilde{\phi}_j)$ are charts of $At(M)$, $\Pi^F_E : \tilde{E}^{t,\theta}_{\beta,\gamma}(U^E, M) \hookrightarrow \tilde{E}^{t,\theta}_{\beta,\gamma}(U^F, M)$ and $\Pi_E : \tilde{E}^{t,\theta}_{\beta,\gamma}(\tilde{M},M)$ are uniformly continuous embeddings (isometrical for $0 \le t < \infty$). Evidently, $\tilde{E}^{t,\theta}_{\beta,\gamma}(\tilde{M},M)$ is the space of functions f of the class $\tilde{E}^{t,\theta}_{\beta,\gamma}$ with supports $supp(f) := cl\{x \in \tilde{M} : f(x) \ne 0\} \subset U^{E(f)}$, $E(f) \in \Sigma$ and $0 \in W \subset \tilde{E}^{t,\theta}_{\beta,\gamma}(\tilde{M},M)$ is open if and only if $\Pi^{-1}_E(W) \cap \tilde{E}^{t,\theta}_{\beta,\gamma}(U^E, M)$ is open for each $E \in \Sigma$.
Let $Hom(M)$ be a group of homeomorphisms of M and $Diff^t_{\beta,\gamma}(M) := [f \in Hom(M) : f$ and $f^{-1} \in \tilde{E}^t_{\beta,\gamma}(M,M)]$ be a group of homeomorhisms (diffeomorhisms for $t \ge 1$) of class $\tilde{E}^t_{\beta,\gamma}$. When $At(M)$ is finite it is metrizable with the right-invariant metric

$$(iii) \quad d(f,g) := \tilde{\rho}^t_{\beta,\gamma}(g^{-1}f, id),$$

where θ is the identity map for $\tilde{M} = M$, $\theta = id$ (in this case the index θ is omitted), $\beta \ge 0$ (see also [14] for finite-dimensional M, correctness of this definition is proved in Theorem 3.1). Henceforth, we omit tilde in $\tilde{E}$.

Definition 2.4. A Riemannian metric g for M Hilbertian at infinity is called regular Hilbertian asymptotically, if there exist $\delta > 0$, $t' > 1$, $\beta' > 0$, $\infty > \gamma' \ge 0$ such that $(g - \tilde{e})_x(\xi,\xi) \in E^{t'}_{\beta',\gamma'}(M,\mathbb{R})$ by x for each $\xi \in TM$, $\xi = (\xi_x : x \in M)$, $\|\xi_x\|_{l_2} \le 1$ for each $x \in M$, $\sup_{\xi\in TM,\ \|\xi_x\|\le 1} \|(g-\tilde{e})_x(\xi,\xi)\|_{E^{t'}_{\beta',\gamma'}(M,\mathbb{R})} \le \delta$. For spaces $E^t_{\beta,\gamma}(M,N)$ with $M = N$ or N being a Banach space over $\mathbb{R}$ we assume that $\omega \ge \max(0,\beta)$ and $\beta' \ge max(0,\beta)$, $t' > t+1$, $\gamma' \ge \gamma$ in 2.2, 2.4.

Definition 2.5.1. Let X be separable BS over $\mathbb{R}$. Suppose that $F_n \subset F_{n+1} \subset \cdots \subset X$, $dim_{\mathbb{R}} F_n = n$, is a sequence of finite-dimensional subspaces. Let $\{z_n : n \in \mathbb{N}\}$ be a sequence of linearly independent vectors in X with $\|z_n\|_X = 1$, $sp_{\mathbb{R}}\{z_1,\dots,z_n\} = F_n$ for each n. For open U and V in X we consider a space of all infinitely many times Frechèt differentiable functions $f, g : U \to V$ fulfilling (i, ii) in 2.1 and with $\rho^t_{\beta,\gamma}(f,h) < \infty$, where $h : U \to V$ is some fixed smooth (of class C^∞) mapping $h : U \to V$, D^α_x for $\alpha = (\alpha^1,\dots,\alpha^n)$ is the operator of differentiation by $(x^1,\dots,x^n) \in F_n$, but with $U^c_R := \{x \in U : \|x\|_X > R\}$ and $< x >= (1 + \|x\|^2_X)^{1/2}$. We denote by $E^{t,h}_{\beta,\gamma}$ the completion of such metric space and consider $E^\infty_\beta(U,V)$ as in 2.3.

Definition 2.5.2. Let M be a paracompact separable metrizable manifold modelled on X [17] and fulfilling (i, ii) below:

(i) an atlas $At(M) = [(U_j,\phi_j) : 1 \le j < k+1]$ is finite, $k \in \mathbb{N}$ (or countable $k = \omega_0$), $\phi_j : U_j \to X$ are homeomorphisms of U_j onto $\phi_j(U_j) \ni 0$, U_j and $\phi_j(U_j)$ are open in M and X respectively, $(\phi_j \circ \phi_i^{-1} - id) \in E^\infty_{\omega,\delta}(\phi_j(U_i \cap U_j), X)$ for each $U_i \cap U_j \ne \emptyset$, where $\omega > 0$, $\gamma \ge 0$, $id(x) = x$ is the identity mapping, ω_0 is the initial number of cardinality $\aleph_0$ [9];

(ii) M contains a sequence of M_k and L_k submanifolds. They are of class $E^\infty_{\omega,\gamma}$ with $dim_{\mathbb{R}} M_k = k$ for M_k and $codim_{\mathbb{R}} L_k = k$ for L_k, $k = k(n) \in \mathbb{N}$, $k(n) < k(n+1)$ for each n, $M_k \subset M_l$ and $L_k \supset L_l$ for each $k < l$, $M = M_k \cup L_k$, $M_k \cap L_k = \partial M_k \cap \partial L_k$ for each k such that $\bigcup_k M_k$ is dense in M. Moreover, M and $At(M)$ are foliated. That is, they fulfil (α,β):

(α) $\phi_{i,j} : \phi_i \circ \phi_j^{-1}|\phi_j(U_i \cap U_j) \to X$ are of the form $\phi_{i,j}((x^l : l \in \mathbb{N})) = (\alpha_{i,j,k}(x^1, \dots, x^k), \gamma_{i,j,k}((x^l : l > k)))$ for each $n \in \mathbb{N}$, $k = k(n)$, when M is without a boundary, $\partial M = \emptyset$. If $\partial M \neq \emptyset$ then:

(β) for each boundary component M_0 of M and $U_i \cap M_0 \neq \emptyset$ we have $\phi_i : U_i \cap M_0 \to H_l$, where $H_l = \{x \in X : x^l \geq 0\}$, $x^l = P_{z_l}(x)$ is the projector of X onto $\mathbb{R}z_l$ along $X \ominus \mathbb{R}z_l$ (see [22]).

Definition 2.5.3. Analogously to Definition 2.3 we consider spaces $E^{t,\theta}_{\beta,\gamma}(\tilde{M}, M)$ and $Diff^t_{\beta,\gamma}(M)$ for M and $\tilde{M}$ as in 2.5.2.
Then $Diff^\infty_{\infty,\gamma}(M)$ is defined as $\bigcap_{l\in\mathbb{N}} Diff^\infty_{l,\gamma}(M)$ and $Diff^\infty_{\beta,\gamma}(M) = \bigcap_{t\in\mathbb{N}} Diff^t_{\beta,\gamma}(M)$ with the corresponding standard topologies of projective limits [9,22].

Definition 2.6. Let G be a topological group. A Radon measure μ on $Af(G,\mu)$ (or ν on $Af(M,\nu)$) is called left-quasi-invariant relative to a dense subgroup G' of G, if $\mu_\phi(*)$ (or $\nu_\phi(*)$) is equivalent to $\mu(*)$ (or $\nu(*)$ respectively) for each $\phi \in G'$. Henceforth, we assume that a quasi-invariance factor $q_\mu(\phi, g) = \mu_\phi(dg)/\mu(dg)$ (or $q_\nu(\phi, x)$) is continuous by $(\phi, g) \in G' \times G$ (or $\in (G' \times M)$), $\mu : Af(G,\mu) \to [0,\infty)$, $\mu(V) > 0$ (or $\nu : Af(M,\nu) \to [0,\infty)$, $\nu(V) > 0$) for some (open) neighbourhood $V \subset G$ (or $\subset M$) of the unit element $e \in G$ (or a point $x \in M$), $\mu(G) < \infty$ (or $\nu(M) \leq \infty$ and is σ-finite respectively), where $\mu_\phi(E) := \mu(\phi^{-1}E)$ for each $E \in Af(G,\mu)$, $Af(G,\mu)$ is the completion of $Bf(G)$ by μ, $Bf(G)$ is the Borel σ-field on G [6].

Let (M, F) be a space M of measures on $(G, Bf(G))$ (or $(M, Bf(M))$) with values in $\mathbb{R}$ and G" be a dense subgroup in G such that a topology T on M is compatible with G", that is, $\mu \to \mu_h$ (or $\nu \to \nu_h$) is the homeomorphism of (M, F) onto itself for each $h \in G$". Let T be the topology of convergence for each $E \in Bf(G)$ (or $\in Bf(M)$) and W be a neighbourhood of the identity $e \in G$ such that J is dense in W, where $J := [h : h \in G" \cap W =: W"$, there exists $b \in (-1,1)$ and $g(b) = h$ with $[g(c) : c \in (-1,1)] \subset W"]$, $g(c_1 + c_2) = g(c_1)g(c_2)$, $g(0) = e$ are one parameter subgroups, $c_1, c_2 \in \mathbb{R}$. We assume also that for each $f \in W"$ there are $g(b_1), \dots, g(b_k) \in J$ such that $f = g(b_1) \dots g(b_k)$. A measure $\mu \in \mathsf{M}$ (or $\nu \in \mathsf{M}$) is called differentiable along $g(b)$ in a point $g(c)$ if $\mu(g(b)^{-1}E)-\mu(E) = (b-c)(\mu'(g(c);E)+\alpha(g(b);E))$ and there exists $\lim_{b\to c}\alpha(g(b);E)=0$ and $\mu'(g(c);E)\in\mathbb{R}$ is continuous by $g(c)$ for each $E\in Bf(G)$, where b and $c\in\mathbb{R}$, $\mu'(g(c);E)$ is called the derivative (by Lagrange) along $g(b)$ in $g(c)$ (analogously for ν on M). Let by induction $\lambda(*) = \mu^{(j-1)}(g(c_1),\dots,g(c_{j-1});*)$ and there exists $\lambda'(g(c_j);E)$, then it is denoted $\mu^{(j)}(g(c_1),\dots,g(c_j);E)$ and is called the j-th derivative (by Lagrange) of μ along $(g(b_1),\dots,g(b_j))$ in $(g(c_1),\dots,g(c_j))$, where $j\in\mathbb{N}$.

Lemma 2.7. *Let M be a $E^\infty_{\omega,\delta}$-domain in X. Then there exists a Hilbert space Y such that $Y\subset X$, Y is dense in X, $\|x\|_Y\ge\|x\|_X$ for each $x\in Y$ and $Diff^{t'}_{\beta',\gamma'}(N)$ is a dense subgroup in $Diff^t_{\beta,\gamma}(M)$, where $N=M\cap Y$, $\infty\ge t\ge 0$, $t'\ge t$, $\infty\ge t'\ge 1$, $\beta'\ge\beta\ge 0$, $\gamma'>\gamma+2$, $\omega\ge\beta'$, $\delta\ge\gamma'$.*

Proof. In view of Theorem I.4.4 [16] for BS X there exists a Hilbert space Y, $Y\subset X$, $\|x\|_Y\ge\|x\|_X$ for each $x\in X$. We take $\{F_n : n\in\mathbb{N}\}$ in X and an orthonormal base $\{e_n : n\in\mathbb{N}\}$ in Y with $e_1=z_1$, $e_i=\sum_{j=1}^i b_{i,j}z_j$ are chosen by induction, $b_{i,i}\ne 0$. Since $\|\sum_{i=1}^n x^i z^i\|_Y\le\sum_{i=1}^n |x^i|\times\|z_i\|_Y$, $\|\sum_{i=m}^n x^i z_i\|_X\le\sum_{i=m}^n|x^i|\le(\sum_{i=m}^n|x^i|^2)^{1/2}(n-m)^{1/2}$, $\sum_{n=1}^\infty(\sum_{m=n}^{2n} m^d)<\infty$ for each $d<-2$, then there is a Hilbert space Y_0 with an injection $T:Y_0\to X$ being a nuclear operator [20,22], $Tx=\sum_{i=1}^\infty(x,y_i)_{Y_0}z_i$, where $x\in Y_0$, $(*,*)_{Y_0}$ is an inner product in Y_0, $\{y_i\}$ is a base in Y_0 such that $\sum_{i=1}^\infty|y_i|_{Y_0}<\infty$. Moreover, we can choose $e_i=b_{i,i}z_i$. Let $Y_0\subset Y\subset X$, $\|x\|_{Y_0}\ge\|x\|_Y\ge\|x\|_X$ for each $x\in Y_0$. Then from Definition 2.1 of $\rho^t_{\beta,\gamma}$ and $l_{2,\gamma}$, also from the consideration of multipliers $\bar{n}^{\alpha\gamma}$, $n\bar{n}^{\alpha\gamma}$, it follows that each $g\in Diff^{t'}_{\beta',\gamma'}(N)$ belongs to $Hom(M)$, since $F_n\subset Y\subset X$, $t'\ge 1$, $<x>_Y\ge<x>_X$ for each $x\in Y$. Therefore, g has the unique continuous extension $\tilde{g}$ on M such that $\tilde{g}\in Diff^t_{\beta,\gamma}(M)$, since N is dense in M and we can choose for each $0<\epsilon$ the space Y_0 with $|y_i|\le i^{-2-\epsilon}$ for each $i\in\mathbb{N}$. □

Definition 2.8. Let M be a $E^\infty_{\omega,\delta}$-manifold as in 2.5 that has a locally finite partition of unity of the same class of smoothness. Henceforward, we suppose that there exists $E^\infty_{\omega,\delta'}$-submanifold N in M; N is modelled on a Hilbert space Y, where Y is as in 2.7 with $Diff^\infty_{\omega,\delta'}(Y)\subset Diff^\infty_{\omega,\delta}(X)$ for the corresponding $\delta'\ge\delta$, where M

and N are separable. Also let N satisfy conditions in 2.2 and 2.4 such that $M_k \subset N$, $N_k \subset N$, N_k is dense in L_k for each $k \in \mathbb{N}$.

Corollary 2.9. *Let M be a Banach $E^{\infty}_{\omega,\delta}$-manifold and N be a Hilbert $E^{\infty}_{\omega,\delta'}$-manifold such that they satisfy* 2.8. *Then $Diff^{t'}_{\beta,\gamma'}(N)$ is a dense subgroup of $Diff^{t}_{\beta,\gamma}(M)$, if $\delta' \ge \delta \ge \gamma' > \gamma + 2$, $t' \ge 1$, $\infty \ge t' \ge t \ge 0$ and $\omega \ge \beta$.*

Proof. For charts (V_j, ψ_j) of N with $V_j \cap V_i \ne \emptyset$ a mapping $\psi_j \circ \psi_i^{-1}$ is in the class of smoothness $E^{\infty}_{\omega,\delta'}$. In view of Definitions 2.5, 2.8 and Lemma 2.7 $Diff^{t'}_{\beta,\gamma'}(N)$ is a dense subgroup of $Diff^{t}_{\beta,\gamma}(M)$. □

3. Structure of groups of diffeomorphisms

Theorem 3.1. *Let $G = Diff^{t}_{\beta,\gamma}(M)$ be defined as in* 2.5, 2.8. *Then it is a separable topological group. If $At(M)$ is finite, G is metrizable by a left-invariant metric d.*

Proof. Let at first $At(M)$ be finite. If f and $g \in G$ then $f \circ g^{-1} \in G$ due to Theorem 2.5 [1] and Ch. 5 in [21] about differentiation and difference quotients of composite functions and inverse functions, since $\phi_i \circ \phi_j^{-1} \in E^{\infty}_{\omega,\delta}$ for each i and j. At first we have $d(f, id) > 0$ for $f \ne id$ in G, since there are i and j such that $f_{i,j} \ne id_{i,j}$. Then $d(hf, hg) = d(g^{-1}h^{-1}hf, id) = d(g^{-1}f, id) = d(f, g)$, hence d is left-invariant, where $f, g, h \in G$. Therefore, $d(f^{-1}, id) = d(id, f)$, in view of 2.1 and 2.3(i,ii) we have that $d(id, f) = d(f, id)$, hence $d(f, g) = d(g, f)$.

It remains to verify, that the composition map $(f, g) \to f \circ g$ from $G \times G \to G$ and the inversion map $f \to f^{-1}$ are continuous relative to d. Let $W = [f \in G : d^{t}_{\beta,\gamma}(f, id) < 1/2]$ and $f, g \in W$. We have $f_{i,j} \circ g_{j,l} - id_{i,l} = (f_{i,j} \circ g_{j,l} - f_{i,l}) + (f_{i,l} - id_{i,l})$ for corresponding domain as an intersection of domains of $f_{i,j} \circ g_{j,l}$ and $f_{i,l}$. Hence, using induction by $p = 1, 2, \dots, [t] + 1$ and the Cauchy inequality we have that there are constants $\infty > C_1 > 0$, $\infty > C_2 > 0$ such that $d(f \circ g, id) \le C_1(d(f, id) + d(g, id))$ and $d(f^{-1}, id) \le C_2 d(f, id)$, since $\lim_{n\to\infty}[d^{t}_{n,\beta,\gamma}(f_{i,j}, id_{i,j}) + d^{t}_{n,\beta,\gamma}(g_{j,l}, id_{j,l})] = 0$, $[t] + 1$ and $At(M)$ are finite, $r_{inj} > 0$ and g satisfies 2.4 [8].

Indeed, in normal local coordinates x (omitting indices (i,j) for $f_{i,j}$), $M \ni x = (x^j : j \in \mathbb{N})$, $f = (f^j : C \to \mathbb{R}|j \in \mathbb{N})$, C open in X, using the Cauchy inequality we get: $\sum_{i\in\mathbb{N}}(|(f\circ g)^i - x^i|i^\gamma)^2 \le 2(\sum_i[|(f\circ g)^i - g^i|i^\gamma]^2)^{1/2}\times(\sum_i[|g^i - x^i|i^\gamma)^2)^{1/2} + \sum_i[|(f\circ g)^i - g^i|i^\gamma]^2 + \sum_i[|g^i - x^i|i^\gamma]^2$ and $\sum_{i,j}[(\partial_j(f\circ g)^i - \delta^i_j)i^\gamma j^\gamma]^2 \le a+b+ab+2(a^{1/2}b+ab^{1/2})+2a^{1/2}b^{1/2}$, where $a = \sum_{i,j\in\mathbb{N}}[(\partial_j\{(f\circ g)^i - g^i\})j^\gamma i^\gamma]^2$, $b = \sum_{l,j\in\mathbb{N}}[(\partial_j g^l - \delta^l_j)j^\gamma l^\gamma]^2$, $\delta^i_l = 1$ for $i = l$ and $\delta^i_l = 0$ for each $l \ne i$, $f\circ g = f\circ g(x)$, $f, g \in G$.

Then we can proceed by induction for finite products of $D^\alpha_g(f\circ g)^i$ and $D_x g^l$, because $D^\alpha_x id(x) = 0$ for $|\alpha| > 1$. For $f = g^{-1}$ we can express recurrently $(D^\alpha_x f^{-1})$ by $(D^\xi_x f)$ with $\xi^i \le \alpha^i$ for each i, since $|\alpha| \le t$. Analogously, for difference quotients, since $(1+\zeta)^b = 1 + \sum_{m=1}^\infty \binom{b}{m}\zeta^m$ for $0 < b < 1$ and $0 < |\zeta| < 1$, $\zeta \in \mathbb{R}$ and $(1+\zeta^b)^b = 1 + b\zeta^b + z(\zeta)$ with $z : \mathbb{R} \to \mathbb{R}$, $\lim_{\zeta\to 0}(z(\zeta)/\zeta^b) = 0$ [21]. For countable infinite $At(M)$ for each $f, g \in G$ there are $E(f)$, $E(f^{-1})$, $E(g)$ and $E(g^{-1}) \in \Sigma$ such that $supp(f) \subset U^{E(f)}$, etc., consequently, $f(supp(f)) \cup g^{-1}(supp(g^{-1})) \subset U^F$ for some $F \in \Sigma$, whence $g^{-1}\circ f \in G$ and there is $E \in \Sigma$ with $supp(g^{-1}\circ f) \subset U^E$. If $(f_\gamma : \gamma \in \alpha)$ and $(g_\gamma : \gamma \in \alpha)$ are two nets converging in G to f and g respectively, so for each neighbourhood $W \subset G$ there exist $E \in \Sigma$ and $\beta \in \alpha$ such that $g_\gamma^{-1}\circ f_\gamma \in W$ and $supp(g_\gamma^{-1}\circ f_\gamma) \subset U^E$ for each $\gamma \in \beta$, where α is a limit ordinal.

In view of the Stone-Weierstrass Theorem and 2.1(i,ii) in each $E^\infty_{\beta,\gamma}(U,V)$ for open U and V in X are dense cylindrical polynomial functions with rational coefficients, consequently, G is separable, since $E^\infty_{\beta,\gamma}(U,V)$ is dense in $E^t_{\beta,\gamma}(U,V)$. Due to conditions 2.2(i-vi) and 2.5.2 for each open submanifold $V \subset M$ with $V \supset M_k$ and $\epsilon > 0$ every $f \in Diff^t_\beta(M_k)$ has an extension $\tilde{f}$ onto M such that $\tilde{f} \in Diff^t_{\beta,\gamma}(M)$ with $\tilde{\rho}^t_{\beta,\gamma}(\tilde{f}|(M\setminus M_k)\cap U^{E(\tilde{f})}, id) < \epsilon$. □

Lemma 3.2. *Let M be a manifold defined in* 2.2, 2.4 *with submanifolds M_k and N_k, $k = k(n)$, $n \in \mathbb{N}$. Then there exist connections ${}_k\nabla$ induced on M_k by ∇ are the Levi-Civita connections, where ∇ is the Levi-Civita connection on M.*

Proof. For each chart (U_j, ϕ_j) we have $\phi_j(U_j) \subset l_2$ and in l_2 for each sequence of subspaces $\mathbb{R}^n \subset R^{n+1} \subset \cdots \subset l_2$ there are induced embeddings $\phi_j^{-1}(\mathbb{R}^n) \cap U_j \hookrightarrow \phi_j^{-1}(\mathbb{R}^{n+1}) \cap U_j \hookrightarrow U_j$. The Levi-Civita connection and the corresponding covariant differentiation ∇ for the Hilbertian manifold M induces the Levi-Civita connection ∇' for each submanifold M' embedded into M, if M' is a totally geodesic submanifold. That is, for each $x \in M'$ and $X \in T_xM'$ there exists $\epsilon > 0$ such that a geodesic $\tau = x_t \subset M$ defined by the initial condition (x,X) lies in M' for each t with $|t| < \epsilon$ (Section 5 in [10], Section VII.8 in [15]). Then using Theorem 5 in Section 4.2 [17] and geodesic completeness of M we can choose such $M' = M_k$ with dimensions $dim(M_k) = k \in \mathbb{N}$ and $M_k(n) \hookrightarrow M_{k(n+1)} \hookrightarrow \cdots \hookrightarrow M$ with $\bigcup_k M_k$ dense in M. Each manifold $\overset{\circ}{M}_k$ was chosen Euclidean at infinity, since M is Hilbertian at infinity. In view of Section VII.3 in [15] and 5.2, 5.4 in [10] ${}_{k(n+1)}\nabla$ on $M_{k(n+1)}$ induces ${}_{k(n)}\nabla$ on $M_{k(n)}$. The latter coincides with that of induced by ∇ on M. Here each M_k is geodesically complete, but normal coordinates are defined in M_k in general locally as in M also, since may be $r_{inj}(x) < \infty$ for $x \in M$, so that $At(M)$ induces $At(M_k)$ for each $k = k(n)$, $n \in \mathbb{N}$. □

Theorem 3.3. *Let M be a manifold fulfilling 2.2, 2.4 and $Diff^t_{\beta,\gamma}$ (M) be as in 2.3 with $t \ge 1$, $\beta \ge 0$, $\gamma \ge 0$. Then*

(i) for each $E^t_{\beta,\gamma}(M,TM)$-vector field V its flow η_t is a one-parameter subgroup of $Diff^t_{\beta,\gamma}(M)$, the curve $t \to \eta_t$ is of class C^1, the mapping $\tilde{E}xp : T_eDiff^t_{\beta,\gamma}(M) \to Diff^t_{\beta,\gamma}(M)$, $V \to \eta_1$ is continuous and defined on a neighbourhood of the zero section in $T_eDiff^t_{\beta,\gamma}(M)$;

(ii) $T_fDiff^t_{\beta,\gamma}(M) = \{V \in E^t_{\beta,\gamma}(M,TM) | \pi \circ V = f\}$;

(iii) $(V,W) = \int_M g_{f(x)}(V_x,W_x)\mu(dx)$ is a weak Riemannian structure on a Banach manifold $Diff^t_{\beta,\gamma}(M)$, where μ is a measure induced on M by ϕ_j and a Gaussian measure with zero mean value on l_2 produced by an injective self-adjoint operator $Q : l_2 \to l_2$ of trace class, $0 < \mu(M) < \infty$;

(iv) the Levi-Civita connection ∇ on M induces the Levi-Civita connection $\hat{\nabla}$ on $Diff^t_{\beta,\gamma}(M)$;

(v) $\tilde{E} : TDiff^t_{\beta,\gamma}(M) \to Diff^t_{\beta,\gamma}(M)$ *is defined by* $\tilde{E}_\eta(V) = exp_{\eta(x)} \circ V_\eta$ *on a neighbourhood* $\bar{V}$ *of the zero section in* $T_\eta Diff^t_{\beta,\gamma}(M)$ *and is a* $E^\infty_{\omega,\delta}$ *mapping by* V *onto a neighbourhood* $W_\eta = W_{id} \circ \eta$ *of* $\eta \in Diff^t_{\beta,\gamma}(M)$; $\tilde{E}$ *is the uniform isomorphism of uniform spaces* $\bar{V}$ *and* W. *Moreover,* (i, ii, v) *is also true for* $Diff^t_{\beta,\gamma}(M)$, *when* M *satisfies* 2.8.

Proof. Let at first $At(M)$ be finite. In view of [12] we have that $T_f E^t_{\beta,\gamma}(M, N') = [g \in E^t_{\beta,\gamma}(M, TN') : \pi'_N \circ g = f]$, where N' fulfils 2.5, 2.8, $\pi'_N : TN' \to N'$ is the canonical projection. Therefore, $TE^t_{\beta,\gamma}(M, N') = E^t_{\beta,\gamma}(M, TN') = \bigcup_f T_f E^t_{\beta,\gamma}(M, N')$ and the following mapping $w_{exp} : T_f E^t_{\beta,\gamma}(M, N') \to E^t_{\beta,\gamma}(M, N')$, $w_{exp}(g) = exp \circ g$ gives charts for $E^t_{\beta,\gamma}(M, N')$, since TN' has an atlas of class $E^\infty_{\nu,\chi}$ with $\nu \ge \beta \ge 0$, $\chi \ge \gamma$. In view of Theorem 5 about differential equations on Banach manifolds in Section 4.2 [17] a vector field V of class $E^t_{\beta,\gamma}$ on M defines a flow η_t of class $E^t_{\beta,\gamma}$, that is $d\eta_t/dt = V \circ \eta_t$ and $\eta_0 = e$. Then lightly modifying proofs of Theorem 3.1 and Lemmas 3.2, 3.3 in [7] we get that η_t is a one-parameter subgroup of $Diff^t_{\beta,\gamma}(M)$, the curve $t \to \eta_t$ is of class C^1, the map $\tilde{E}xp : T_e Diff^t_{\beta,\gamma}(M) \to Diff^t_{\beta,\gamma}(M)$ defined by $V \to \eta_1$ is continuous.
The curves of the form $t \to \tilde{E}(tV)$ are geodesics for $V \in T_\eta Diff^t_{\beta,\gamma}(M)$, $d\tilde{E}(tV)/dt$ is the map $m \to d(exp(tV(m)))/dt = \gamma'_m(t)$, where $\gamma_m(t)$ is the geodesic on M, $\gamma_m(0) = \eta(m)$, $\gamma'_m(0) = V(m)$. Indeed, this follows from the existence of solutions of corresponding differential equations in the Banach space $E^t_{\beta,\gamma}(M, TM)$ and then as in the proof of Theorem 9.1 [7].

From the definition of μ it follows that for each $x \in M$ there exists open neighbourhood $Y \ni x$ such that $\mu(Y) > 0$ [6]. In view of 2.2-4 there is the following inequality $\sup_x g_{f(x)}(V_x, V_x) < \infty$ and also for W. Consequently, $(V, V) > 0$ for each $V \ne 0$, since V and W are continuous vector fields and for some $x \in M$ and $Y \ni x$ with $\mu(Y) > 0$ we have $V_y \ne 0_y$ for each $y \in Y$. On the other hand $\sup_{x \in M} |g_{f(x)}(V_x, W_x)| < \infty$, hence $|(V, W)| < \infty$. From $g_{f(x)}(V_x, W_x) = g_{f(x)}(W_x, V_x)$ and bilinearity of g by (V_x, W_x) it follows that $(V, W) = (W, V)$ and $(aV, W) = (V, aW)$ for each $a \in \mathbb{R}$. Since $t \ge 1$, the scalar product (iii) gives a weaker topology than the initial $E^t_{\beta,\gamma}$. For two Banach spaces A and B we have the following uniform linear isomorhism $E^t_{\beta,\gamma}(M, A \oplus B) = E^t_{\beta,\gamma}(M, A) \oplus E^t_{\beta,\gamma}(M, B)$, where $\oplus$ denotes the direct sum. Therefore, $E^t_{\beta,\gamma}(M, TM)$ is complemented in $E^t_{\beta,\gamma}(M, T(TM))$, since TM and $T(TM) =: TTM$ are the Banach foliated manifolds of class $E^\infty_{\nu,\chi}$ with $\nu \geq \beta$, $\chi \geq \gamma \geq 0$. Then the right multiplication $\alpha_h(f) = f \circ h$, $f \to f \circ h$ is of class C^∞ on $Diff^t_{\beta,\gamma}(M)$ for each $h \in Diff^t_{\beta,\gamma}(M)$. Moreover, $Diff^t_{\beta,\gamma}(M)$ acts on itself freely from the right, hence we have the following principal vector bundle $\tilde{\pi} : TDiff^t_{\beta,\gamma}(M) \to Diff^t_{\beta,\gamma}(M)$ with the canonical projection $\tilde{\pi}$.

Analogously to [2,7,15] we get the connection $\hat{\nabla} = \nabla \circ h$ on $Diff^t_{\beta,\gamma}(M)$. Then $(\hat{\nabla}_{\hat{X}}\hat{Y}, \hat{Z}) + (\hat{Y}, \hat{\nabla}_{\hat{X}}\hat{Z}) = \int_M [< \nabla_{X_e} Y_e, Z_e >_{h(x)} + < Y_e, \nabla_{X_e} Z_e >_{h(x)}]\mu(dx) = \int_M [X_e g(Y_e, Z_e)]_{h(x)}\mu(dx) = \hat{X}(\hat{Y}, \hat{Z})$, since $Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$ (Satz 3.8 in [10]) and for each right-invariant vector field V on $Diff^t_{\beta,\gamma}(M)$ there exists a vector field X on M with $V_h = X \circ h$ for each $h \in Diff^t_{\beta,\gamma}(M)$, where $\hat{X} := X \circ h$ (see also [18,19]). If ∇ is torsion-free then $\hat{\nabla}$ is also torsion-free. From this it follows that the existence of $\tilde{E}$ and $Diff^t_{\beta,\gamma}(M)$ is the Banach manifold of class $E^\infty_{\omega,\delta}$, since exp and M are of class $E^\infty_{\omega,\delta}$, $\alpha_h(f) = f \circ h$, $f \to f \circ h$ is a C^∞ map with the derivative $\alpha_h : E^t_{\beta,\gamma}(M', TN) \to E^t_{\beta,\gamma}(M, TN)$ whilst $h \in E^t_{\beta,\gamma}(M, M')$, $\tilde{E}_h(\hat{V}) := exp_{h(x)}(V(h(x)))$, $\hat{V}_h = V \circ h$, $V \in \Xi(M)$, $\hat{V} \in \Xi(Diff^t_{\beta,\gamma}(M))$.

The case of infinite $At(M)$ may be treated using the strict inductive limit topology. □

Note 3.4. For a manifold $N = \oplus\{M_j : j \in \mathsf{J}\}$, $M_j = M$ for each j, $\mathsf{J} \subset \mathbb{N}$, we have that $Diff^t_{\beta,\gamma}(N)$ is isomorphic to $S \otimes Diff^t_{\beta,\gamma}(M)$, where S is a discrete symmetric group.

Henceforward, we assume that M and M_k are connected for each $k > n$ and some fixed $n \in \mathbb{N}$. For a finite-dimensional manifold M a space $E^t_{\beta,\gamma}(M, \mathbb{R})$ (or $Diff^t_{\beta,\gamma}(M)$) is isomorphic with the usual weighted Hölder space $C^t_\beta(M, \mathbb{R})$ (or $Diff^t_\beta(M)$ correspondingly).

4. Irreducible unitary representations of a group of diffeomorphisms of a Banach manifold

Theorem 4.1. *Let M be a Banach manifold fulfilling 2.5, $G = Diff^t_{\beta,\gamma}(M)$ be a group of diffeomorphisms as in 2.8 with $t \ge 1$, $\beta \ge \omega + \xi$ and $\gamma > 2(1+\delta)+\xi$, where $\xi > 2$ for a Banach manifold, $\xi = 0$ for a Hilbert manifold. Then (for each $1 \le l \le \infty$) there exists a quasi-invariant (and l times differentiable) measure ν on M relative to G.*

Proof. The exponential mapping exp is defined on a neighbourhood of the zero section of the tangent bundle TM and exp is of class $E^{\infty}_{\omega,\delta}$ due to 2.5 (see also [17]). For each $x \in N$ we have $T_xN \tilde{=} l_2$. Suppose F is a nuclear (of trace class) operator on l_2 such that $Fe_i = F_ie_i$, where $i^b \le F_i \le i^c$ for each i, $\{e_i : i\}$ is the standard base in l_2, $1-\gamma+2\delta < b \le c < -1$. Then there exists a σ-additive Gaussian measure λ on l_2 with zero mean and a correlation operator equal F. Then a Gaussian measure on T_xN induces a Gaussian measure on T_xM for $x \in N$ [16]. Therefore, exp_x induces a σ-additive measure ν on $W \ni x$, where $W = exp_x(V)$, $0 \in V$ is open in T_xM, $0 < \mu(V) < \infty$, $\nu(C) = \mu(exp_x^{-1}(C))$ for each $C \in Bf(W)$. The manifold M is paracompact and Lindelöf [9], $GW = M$, hence there is a countable family $\{g_j : j \in \mathbb{N}\} \subset G$, $g_1 = e$, $W_1 = W$ and open $W_j \subset W$ such that $\{g_jW_j : j\}$ is a locally finite covering of M with $W_1 = W$, $g_1 = id$. For $C \in Bf(M)$ let $\nu(C) := \sum_{j\in\mathbb{N}} \nu((g_j^{-1}C)\cap W_j)2^{-j}$ (without multipliers 2^{-j} the measure ν will be σ-finite, but not necessarily finite).

The following mapping $Y_g := (exp \circ g \circ exp_x^{-1})$ on TM for each $g \in G$ satisfies conditions of Theorems 1,2 in Section 26 [23]. Indeed, $(\partial g^i/\partial x^j)_{i,j\in\mathbb{N}}$ in local natural coordinates (x^j) is in the class $E^{t'-1}_{\beta'+1,\gamma'}$ (see 2.4, 2.8). In view of these theorems and [3,6,11] the measure ν is quasi-invariant and l times differentiable, since the continuous extension of the operator $((Y_g)'-I)F^{-1/2}Q$ from T_xN onto T_xM is of trace class on the Banach space T_xM and $dg^t/dt = V \circ g^t$ (see the proof of Theorem 3.3 above and [20,22]), where $g^t = \eta_t$, $Qx = \sum_j x^j j^\delta e_j$, $x = \sum_j x_je_j \in l_2$, $x^i \in \mathbb{R}$. □

Definition 4.2.1. Let M satisfy conditions in 2.5. For a given atlas $At(M)$ we consider its refinement $At'(M) = \{(U'_j, \psi_j) : j \in \mathbb{N}\}$ of the same class $E^{\infty}_{\omega,\delta}$ such that $\{U'_j\}$ is a locally finite covering of M, for each U'_j there is $i(j)$ with $U_{i(j)} \supset U'_j$, exp_x^{-1} is injective on U'_j for some $x \in U'_j$, $exp_x^{-1}(U'_j)$ is bounded in T_xM. Henceforward, M will be supplied by such $At'(M)$ and $Diff^t_{\beta,\gamma}(M)$ will be given relative to such atlas.

Definition 4.2.2. Let μ be a non-negative measure on M quasi-invariant relative to $G = Diff^t_{\beta,\gamma}(M)$ (see Theorem 4.1) such that $\mu(M) = \infty$, μ is σ-finite and $\mu(U'_j) < \infty$ for each j. Then μ is considered on $Af(M,\mu)$. We consider $X = \prod_{i\in\mathbb{N}} M_i$, where $M_i = M$ for each i. Take $E_i \in Af(M_i,\mu)$, put $E = \prod_{i\in\mathbb{N}} E_i$, which is called a unital product subset of X if it satisfies the following conditions:

$$(UPS1)\ \sum_{i\in\mathbb{N}} |\mu(E_i) - 1| < \infty \text{ and } \mu(E_i) > 0 \text{ for each } i;$$

$$(UPS2)\ E_i \text{ are mutually disjoint .}$$

Note 4.3. In view of 4.2 the above definitions 4.2.1,2 and Lemmas 1.1, 1.2 [13] are valuable for the case considered here (G, M, μ) for infinite-dimensional M. Henceforward, we denote by G the connected component of $id \in Diff^t_{\beta,\gamma}(M)$ from 4.2.2. Further, the construction of irreducible unitary representations follows schemes of [13] for finite-dimensional M and [18] for non-Archimedean Banach manifolds, so proofs are given briefly with emphasis on features of the case of the real Banach manifold M.

4.4. Let E be cofinal with E' (ERE') if and only if

$$(CF)\ \sum_{i\in\mathbb{N}} \mu(E_i \triangle E'_i) < \infty,$$

E be strongly cofinal with E' ($E \tilde{=} E'$) if and only if

$$(SCF) \text{ there is } n \in \mathbb{N} \text{ such that } \mu(E_i \triangle E'_i) = 0 \text{ for each } i > n,$$

where $E_i \triangle E'_i = (E_i \setminus E'_i) \cup (E'_i \setminus E_i)$, $\Sigma(E) := \{E' : E'RE\}$.

Put $\nu_E(E') = \prod_{i\in\mathbb{N}} \mu(E'_i)$ for each $E' \in \Sigma(E)$. In view of the Kolmogorov's Theorem [6] ν_E has the σ-additive extension onto the minimal σ-algebra $\mathsf{M}(E)$ generated by $\Sigma(E)$.

The symmetric group of $\mathbb{N}$ is denoted by $\tilde{\Sigma}_\infty$, its subgroup of finite permutations of $\mathbb{N}$ is denoted by Σ_∞. For $g \in G$ there is $gx = (gx_i : i \in \mathbb{N})$, where $x = (x_i : i \in \mathbb{N}) \in X$, for $\sigma \in \tilde{\Sigma}_\infty$ let $x\sigma = (x'_i : i \in \mathbb{N})$, $x'_i = x_{\sigma(i)}$ for each i. Quite analogously to Lemma 1.3 [13] we have the following Lemma 4.5 due to $supp(g) \subset U^{E(g)}$ for some $E(g) \in \Sigma$ and $\mu(U^{E(g)}) < \infty$, where $U^E = \bigcup_{j \in E} U_j$, (U_j, ψ_j) are charts of $At'(M)$.

Lemma 4.5. *Let E be a unital product subset of X. Then*

(i) $(gE)RE$ for each $g \in G$,

(ii) $\Sigma(E)$ is invariant under G and Σ_∞.

4.6. In view of 2.6, 2.8, 4.2.1 and the proof of 4.1 we may choose μ such that for each $g \in G$ there is its neighbourhood W_g and there are constants $0 < C_1 < C_2 < \infty$ such that

$$(i)\ C_1 \le q_\mu(f, z) \le C_2$$

for each $x \in m$ and $f \in W_g$ with $supp(f) \subset U^{E(g)}$. Indeed, for each U_j there exists $y \in U_j$ such that $exp_y^{-1} U_j$ is bounded in $T_y M$. Hence for each fixed R, $\infty > R > 0$, for operators $Y_f = U$ of non-linear transformations the term $|det((Y_f)'(x))|^{-1} exp\{\sum_{l=1}^\infty [2(x - Y_f^{-1}(x), e_l)(x, e_l) - (x - Y_f^{-1}(x), e_l)^2]/F_l\}$ is bounded (see f after (i)) for each $x \in l_2$ with $\|x\| < R$. For $z \in M \setminus U^{E(g)}$ we have $q_\mu(f, z) = 1$. Therefore, we suppose further that μ satisfies (i).

If $S \in Af(M, \mu)$ and $\mu(S) < \infty$ we may consider measures $\mu_k = \mu$ on E'_k, $\nu_k = \mu_k$ on $E'_k \setminus S$ and $\nu_k = 0$ on S, suppose $L_n = \prod_{i=1}^n M_i$, $\mu_{L_n} = \bigotimes_{i=1}^n \mu_i$, $P_n : X \to L_n$ are projections, $\rho_k(x) = \nu_k(dx)/\mu(dx)$. Then $\rho_k(x) = 0$ for each $x \in S$. Using the analog of Lemma 16.1 [23] for our case we obtain the analog of Lemmas 1.4, 1.6, 1.7 and Theorem 1.5 [13], since M has a countable open base $\{\tilde{U}_j : j \in \mathbb{N}$ there is $E \in \Sigma$ such that $\tilde{U}_j \subset U^E\}$.

4.7. The manifold M is Polish, hence M is the Radon space [6] and for each unital product subset E for each i there is a compact $\tilde{E}_i \subset M$ such that $\mu(E_i \triangle \tilde{E}_i) < 2^{-i-1}$ and $\tilde{E}_i \subset U^{h(i)}$ for corresponding $h(i) \in \Sigma$. Since each open covering of $\tilde{E}_i$ has a finite subcovering we may choose $E'_i \in At(M,\mu)$ with finite number of connected components. As in Section 1.8 [13] we can construct $E"RE$ such that $E"_i$ are mutually disjoint.

Proposition 4.8. *Each unital product subset E is cofinal with E^0 satisfying the following conditions:*

(UP3) *the closure $cl(E^0_i)$ and $cl(\bigcup_{j\ne i} E^0_j)$ are mutually disjoint and E^0_i is open for each i and $\inf_i \inf_{x\in E^0_i, y\in \bigcup_{j\ne i} E^0_j} d_M(x,y) > 0$, $E^0_i \subset U^{h(i)}$, $h(i)\in\Sigma$;*

(UP4) *E^0_i and $E^0_{i,k}$ are connected and simply connected, there is $n\in\mathbb{N}$ such that for each $k>n$ and $i\in\mathbb{N}$ there exists $g\in G$ with $g(E^0_{i,k}) = B_{i,k}$ being an open ball in a coordinate neighbourhood of M_k with $g|(M\setminus M_k) = id$ and $\inf_{x\in\partial M_k, y\in E^0_{i,k}} d_M(x,y) > 0$, $g(\bar{E}^0_{i,k}) = \bar{B}_{i,k}$, where $\bar{B} := cl(B)$, $E^0_{i,k} := E^0_i \cap M_k$. For $i\ne j$, E^0_i and E^0_j can be connected by an open path $P_{i,j}$ such that $\bar{P}_{i,j}\cap cl(\bigcup_{k\ne i,j} E^0_k) = \emptyset$.*

Proof. In view of 3.4, M and M_k are connected for each $k>n$ and some fixed $n\in\mathbb{N}$. Then using 3.1, locally finite coverings of M and M_k [9] and shrinking slightly E^0_i such that ∂E^0_i are of class $E^\infty_{\omega,\delta}$ analogously to steps 1-4 [13] and using properties of μ we prove this proposition. Indeed, μ is approximable from beneath by the class of compact subsets [6]. □

4.9. Henceforth, $\Pi : \Sigma_\infty \to U(V(\Pi))$ denotes a unitary representation on a Hilbert space $V(\Pi)$ over $\mathbb{C}$, $H(\sum)$ denotes a Hilbert space that is the completion of $\bigcup_{E'\in\Sigma(E)} H^\Pi_{|E'}$ with the scalar product

$$<\phi_1,\phi_2>= \sum_{\sigma\in\Sigma_\infty}\int_{E^1\cap E^2\sigma} <\phi_1(x),\Pi(\sigma)^{-1}\phi_2(x\sigma^{-1})>_{V(\Pi)} \nu_E(dx),$$

where $H^\Pi_{|E'} := L^2(E';\mathsf{M}(E);\nu_E|E';V(\Pi))$ is a Hilbert space of functions on E' with values in $V(\Pi)$, $\sum := (\Pi;\mu,E)$; $E'RE$, E is a unital product subset of X. Then we define a representation

$$(i)\quad T_{\sum}(g)\phi(x) := \rho_E(g^{-1}|x)^{1/2}\phi(g^{-1}x),$$

where $\rho_E(g^{-1}|x) := (\nu_E)_g(dx)/\nu_E(dx)$, $(\nu_E)_g(C) := \nu_E(g^{-1}C)$ and $\rho_E(g|x) = \prod_{i\in\mathbb{N}}\rho_M(g;x_i)$, $\rho_M(g;x_i) := q_\mu(g^{-1};x_i)$ (see Section 2 [13] and 5.9 [18]).

Proposition 4.10. *The formula* 4.9(i) *determines a strongly continuous unitary representation of* G *(given by* 4.2 *and* 4.3*) on the Hilbert space* $H(\sum)$*.*

Proof. The space $H(\sum)$ is isomorphic with the completion $H'(\sum)$ of $\bigcup_{E'\in\Sigma(E)}H'^{\Pi}_{|E'}$ with the scalar product $< f_1,f_2 >_{H'} := \int_F < f_1(x),f_2(x) >_{V(\Pi)} \nu_E(dx)$, where $f_i \in H'^{\Pi}_{|E^{(i)}}$, $E^{(i)} \in \Sigma(E)$, $F \in \mathsf{M}(E)$, $F\sigma$ for $\sigma\in\Sigma_\infty$ are disjoint and $supp(f_1(x)f_2(x)) \subset \bigcup_{\sigma\in\Sigma_\infty}F\sigma$. Here $H'^{\Pi}_{|E'}$ is a space of functions $f = Q_\Pi\phi$, where $\phi\in H^{\Pi}_{|E'}$ and

(i) $Q_\Pi\phi := \sum_{\sigma\in\Sigma}(R(\sigma)\Pi(\sigma))\phi$, $(Q_\Pi(\phi))(x\sigma) = \Pi(\sigma)^{-1}\phi(x)$;

(ii) $R(\sigma)\phi(x) := \phi(x\sigma)$;

(iii) $\Pi(\sigma)\phi(x) := \Pi(\sigma)(\phi(x))$, $\|f\|^2 = \int_{E'}\|f(x)\|^2_{V(\Pi)}\nu_E(dx) < \infty$,

since $E'\sigma$ for $\sigma\in\Sigma_\infty$ are disjoint for different σ. Therefore, as in 2.1 [13] we get

$$< T_{\sum}(g)f_1,f_2 > \\ = < v_1,v_2 >_{V(\Pi)} \times \prod_{i\in\mathbb{N}}\int_{(gB_i^{(1)})\cap B_i^{(2)}}\rho_M(g^{-1};x_i)^{1/2}\mu(dx_i),$$

for $f_j = Q_\Pi\phi_j$, $\phi_j = \chi_{B^{(j)}}\otimes v_j$, where χ_C is the characteristic function of C (see also 4.6(i)).

Let us fix $J\in\Sigma$ and take $U^J = \bigcup_{j\in J}U_j \subset M$. As in the proof of Theorem 5.6(a) [19] (see 4.6(i)) we can find a neighbourhood $W\ni id$ in G and $0 < c_1 < c_2 < \infty$ such that $c_1 \le \rho_M(g^{-1};y) \le c_2$ for each $y\in U^J$ and $\rho_M(g^{-1};y) = 1$ for each $y\notin U^J$ for each $g\in W$ with $supp(g)\subset U^J$. Hence for each $\epsilon > 0$ there exists $W\ni id$ such that $| < T_{\sum}(g)f_1,f_2 > - < f_1,f_2 > | < \epsilon$, consequently, due to the Banach-Steinhaus Theorem [36] there exists a neighbourhood $V\ni id$ such that $\|(T_{\sum}(g)-I)f_1\| < \epsilon$ and $T_{\sum}$ is strongly continuous.

It is interesting to note that 4.10 may be proved from the inequality:

$$\|T_{\sum}(g)f_1 - f_1\|_{H'(\sum)} \le |v|^2 \int_F |f_1(x) - f_1(g^{-1},x)\rho_E(g^{-1}|x)^{1/2}|^2 \nu_E(dx).$$

Then we consider restrictions $g|M_k$ and properties of $(Y_g)'$ (or g on $M \setminus M_k$) such that $card\{i : supp(g) \cap F_{i,k}\} < \aleph_0$ for each $k \in \mathbb{N}$. In view of Theorems 26.1,2 [23] for each sequence g_n with $\lim_n g_n = e$ and for each $\epsilon > 0$ there is m such that

$$\int_F |f_1(x) - f_1(g_n^{-1}x)\rho_E(g_n^{-1}|x)^{1/2}|^2 \nu_E(dx) < \epsilon,$$

for all $n > m$, since there is $E \in \Sigma$ with $supp(g_n) \subset U^E$ for every $n > m$. $\square$

4.11. Let $E_1, \dots, E_r$ be mutually disjoint open subsets of M, $H_1 := \bigotimes_{i=1}^r L^2(E_i)$, $L^2(E_i) := L^2(E_i; \mu|E_i)$, $G_1 := \prod_{i=1}^r G_{|E_i}$, $G_{|E_i} := \{g \in G : supp(g) \subset E_i\}$, denote by $G(E_i)$ the connected component of $id \in Diff^t_{\beta,\gamma}(E_i)$, also let $\{E_{i,j} : j \in J_i\}$ be the connected components of E_i. Then $G_{|E_{i,j}} = G(E_{i,j})$, since for each continuous mapping $F : [0,1] \to G$ we have by continuity that

(i) $F(\epsilon)(E_{i,j}) \subset E_{i,j}$ for each $\epsilon \in [0,1] \subset \mathbb{R}$ and each $j \in J_i$.

Indeed, suppose J is the connected subset of $[0,1]$ such that $0 \in J$ and for each $\epsilon \in J$ is satisfied (i). If $v = \sup(J) < 1$ then by continuity there is $w > v$ for which $[0,w]$ have the same properties as J. Hence the maximal such J coincides with $[0,1]$.

We define and consider $\tilde{G}(E') := \prod{}''_{i\in\mathbb{N}} G(E'_i) := \{g = (g_i : i) : g_i \in G(E'_i), supp(g_i) \subset U^{E(g_i)}, (\bigcup_{i\in\mathbb{N}} E(g_i)) \in \Sigma$ for each $i\}$. Therefore, $\prod{}''_{j\in J_i} G(E_{i,j}) = G_{|E_i}$. Then quite analogously to Lemma 3 [13] and Lemma 5.12 II [18] we get that the following representation L_1 of G_1 is irreducible: $(L_1(g)f)(y) = \prod_{i=1}^r \rho_M(g_i^{-1}; y_i)^{1/2} f(g^{-1}y)$ for $f \in H_1$, $g = (g_i : i) \in G_1$ and $y = (y_i : i) \in \prod_{i=1}^r E_i$, since $G_{|E_i}$ is dense in $G_i := G \cap \prod_{j\in J_i} G(E_{i,j})$ and L_1 is strongly continuous, $G_{|E_i} \subset \prod_{j\in J_i} G(E_{i,j})$. Indeed, in view of Proposition 4.8 $G_{|E_i}$ is connected, since G is connected.

Then L_1 on G_i is decomposable into irreducible components, since L_1 of $G(E_{i,j})$ on $L^2(E_{i,j})$ is irreducible. In view of strong continuity of L_1 on the dense subgroup $G_{|E_i}$ it follows that its strongly continuous extension on G_i is also unitary. Then the rest of Section 3.1 [13] may be transferred onto the case considered here.
Let $L_{E'}(g)f(x) = \rho_E(g^{-1}|x)^{1/2}f(g^{-1}x)$ for $g \in \tilde{G}(E')$, $f \in H_{|E'} := L^2(E', \mathsf{M}(E)|E', \nu_E|E')$, $x \in E'$. Then we get the following.

Lemma 4.12. *Let $E' \in \Sigma(E)$ and E'_i be open and connected. Then the unitary representation $L_{E'}$ of $\tilde{G}(E')$ on $H_{E'}$ is irreducible.*

4.13. Let us consider

(i) $G((E')) := \{g \in G|$ there is $k = k(n), n \in \mathbb{N}$ and $\sigma \in \Sigma_\infty$, such that $g(E'_{i,k}) = E'_{\sigma(i),k}$ for each $i \in \mathbb{N}$ and $g|M \setminus M_k = id\}$, where $E' = \prod_{i\in\mathbb{N}} E'_i$ $(E'_i \subset M)$ satisfies $(UP3-4)$ and $E' \in \Sigma(E)$, $E_{i,k} = E_i \cap M_k$. In view of the foliated structure in M this group is dense in

(ii) $\{g \in G : supp(g) \subset \bigcup_{i\in\mathbb{N}} E'_i\}$.

Lemma 4.14. *Let $E' \in \Sigma(E)$ satisfy $(UP3-4)$. Then for any $\sigma \in \Sigma_\infty$ there is n such that for each $k > n$ there exists $g \in G((E'))$ with $g(E'_{i,k}) = E'_{\sigma(i),k}$ for each i, moreover, $g|E'_i = id|E'_i$ if $\sigma(i) = i$.*

Proof. It is quite analogous to that of Lemma 3.4 [13], since each M_k is locally compact and connected, also due to properties of μ induced as the image of the Gaussian σ-additive measure. On the other hand, the latter is fully characterised by its weak distribution and is with the Radon property (see Lemma 2 and Theorem 1 in Section 2 [23]). □

4.15. Let E' be as in 4.12, $H^{\Pi}_{|E'} = L^2(E', \mathsf{M}(E)|E', \nu_E|E'; V(\Pi))$, $H'^{\Pi}_{|E'} = Q_\Pi H^{\Pi}_{|E'}$ (see the proof of 4.10). For each $g \in G((E'))$ there are $\sigma \in \Sigma_\infty$ and $k = k(n)$, $n \in \mathbb{N}$ such that $g(E'_{i,k}) = E'_{\sigma(i),k}$ for each $i \in \mathbb{N}$ and $g|(M \setminus M_k) = id$. Suppose $f = Q_\Pi\phi$, $\phi \in H^{\Pi}_{|E'}$. If (α) ϕ depends only on $\{x = (x_i : i)|x_i \in E'_{i,k}\}$ then $(T_{\sum}(g)f)(x) = \rho_E(g^{-1}|x)^{1/2}\Pi(\sigma)\phi(g^{-1}x\sigma)$. If (β) ϕ depends only on $\{x = (x_i : i)|x_i \in E'_i \setminus M_k\}$ then $(T_{\sum}(g)f)(x) = f(x)$. Then if $\phi(x) = \phi_1(x) \times \phi_2(x)$, where $\phi_2(x)$ is of type (α) or (β) and ϕ_1 :

$E' \to \mathbb{C}$ is also of type analogous to (α) or (β) then $T_{\sum}(g)f \in H'^{\Pi}_{|E'}$. Let $G_k((E')) = \{g \in G((E')) : g|(M \setminus M_k) = id\}$, then $\bigcup_k G_k((E'))$ is dense in $G((E'))$. Denote $H_k := \{\phi \in H^{\Pi}_{|E'} | \phi(x)$ is constant on $M \setminus M_k\}$, $H'_k := Q_{\Pi} H_k$. In view of Proposition 4.8 we have that the following representation $T_{E'}(g)\phi(x) = \rho_E(g^{-1}|x)^{1/2}\Pi(\sigma)\phi(g^{-1}x\sigma)$ is irreducible, where $\phi \in H_k$, $g \in G_k((E'))$, $x \in E'$, $\sigma \in \Sigma_\infty$ is such that $g(E'_{i,k}) = E'_{\sigma(i),k}$ for each i (see also Lemma 3.5 [13]). Then we obtain analogously to Lemma 4.2 [13] the following lemma.

Lemma 4.16. *Let $F = \prod_{i \in \mathbb{N}} F_i$ satisfies $(UP3-4)$. Then there exists $F' \in \Sigma(F)$ satisfying $(UP3-4)$ and*

$$(UPS5) \quad M \setminus cl(\bigcup_{i>N} F_i) \text{ is connected for every } N > 0.$$

Proof. Consider $F_{i,k} = F_i \cap M_k$ and measures μ_k on M_k induced by μ on M and the projection $P_k : l_2 \to \mathbb{R}^k$ and choose F' such that

$$|\mu_{k(n+1)}(F'_{i,k(n+1)} \triangle F_{i,k(n+1)} - \mu_k(F'_{i,k(n)} \triangle F_{i,k(n)})| < 3^{-i-2(k(n)+1)}\mu(F_i),$$

for each $k = k(n)$ and $i, n \in \mathbb{N}$. Then use Theorem 3.1 [13]. □

Theorem 4.17. *The unitary representation $T_{\sum}$ of G (defined in* 4.2*) on $H(\sum)$ is irreducible.*

Proof. Considering the sequences $\{M_k : k\}$, $\{G_k((E')) : k\}$ and $\{H_k : k\}$, using 4.2-4.16 and strong continuity of $T_{\sum}$ we get from the proof of Theorem 4.1 [13] that $T_{\sum}$ is irreducible. Indeed, we may consider $\Delta := \{E' : E' \tilde{=} E^0, E'$ satisfies $(UP3-4)\}$ instead of Δ in Section 4.3 [13]. □

Theorem 4.18. *Suppose $T_{\sum_i}$ are unitary representations of G with parameters $\sum_i = (\Pi_i; \mu, E')$. Then, $(T_{\sum_i}, H(\sum_i))$, $i = 1, 2$ are mutually equivalent if and only if there exists $a \in \tilde{\Sigma}_\infty$ such that $\Pi_1 \tilde{=} {}^a\Pi_2$ and $E_1 \in \Sigma(E_2 a^{-1})$, where $({}^a\Pi)(\sigma) := \Pi(a^{-1}\sigma a)$.*

Proof. In view of 4.8 and 4.9 we may assume without loss of generality that E^i satisfies $(UP3-4, UPS5)$ for $i = 1$ and 2. Then we consider $G^{(1)} := G((E^{(1)})) \cap G((E^{(2)})) \subset G$ and $G^{(2)} := \prod"_{k \in \mathbb{N}} G(C_k)$, where C_k are all connected components of $E^{(1)}_{i,j} = E^{(2)}_{j,i}$ (with $E^{(2)}$ here instead of $F^{(2)}$ in [13]). Instead of equations (5.7) [13] we have corresponding expressions as intersections with M_k in both sides for some $k = k(n)$, $n \in \mathbb{N}$. Using the sequences $\{M_k\}$, $\{G_k((E'))\}$ and strong continuity of $T_{\sum_i}$ we get the statement of Theorem 4.18 analogously to Section 5 [13]. □

Note 4.19. The construction presented above of irreducible unitary representations is valid as well for each dense subgroup G' of $Diff^t_{\beta,\gamma}(M)$ such that the corresponding non-negative measure λ on M is left-quasi-invariant relative to G' and satisfies 4.2 and 4.6.

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