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Irreducible Virasoro modules from tensor products

Haijun Tan and Kaiming Zhao

Abstract. In this paper, we obtain a class of irreducible Virasoro modules by taking tensor products of the irreducible Virasoro modules $\Omega(\lambda, b)$ with irreducible highest weight modules $V(\theta, h)$ or with irreducible Virasoro modules $\mathrm{Ind}_\theta(N)$ defined in Mazorchuk and Zhao (Selecta Math. (N.S.) 20:839–854, 2014). We determine the necessary and sufficient conditions for two such irreducible tensor products to be isomorphic. Then we prove that the tensor product of $\Omega(\lambda, b)$ with a classical Whittaker module is isomorphic to the module $\mathrm{Ind}_{\theta,\lambda}(\mathbb{C}_{\mathbf{m}})$ defined in Mazorchuk and Weisner (Proc. Amer. Math. Soc. 142:3695–3703, 2014). As a by-product we obtain the necessary and sufficient conditions for the module $\mathrm{Ind}_{\theta,\lambda}(\mathbb{C}_{\mathbf{m}})$ to be irreducible. We also generalize the module $\mathrm{Ind}_{\theta,\lambda}(\mathbb{C}_{\mathbf{m}})$ to $\mathrm{Ind}_{\theta,\lambda}(\mathcal{B}_{\mathbf{s}}^{(n)})$ for any non-negative integer n and use the above results to completely determine when the modules $\mathrm{Ind}_{\theta,\lambda}(\mathcal{B}_{\mathbf{s}}^{(n)})$ are irreducible. The submodules of $\mathrm{Ind}_{\theta,\lambda}(\mathcal{B}_{\mathbf{s}}^{(n)})$ are studied and an open problem in Guo et al. (J. Algebra 387:68–86, 2013) is solved. Feigin–Fuchs' Theorem on singular vectors of Verma modules over the Virasoro algebra is crucial to our proofs in this paper.

1. Introduction

Throughtout this paper, we use $\mathbb{Z}$, $\mathbb{Z}_+$, $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ to denote the sets of all integers, non-negative integers, positive integers, real numbers and complex numbers, respectively.

The Virasoro algebra $\mathfrak{V}{:=}\mathrm{Vir}[\mathbb{Z}]$ (over $\mathbb{C}$) is the Lie algebra with the basis $\{c, d_i | i{\in}\mathbb{Z}\}$ subject to the Lie brackets defined by

$$[c, d_i] = 0 \quad \text{and} \quad [d_i, d_j] = (j-i)d_{i+j} + \delta_{i,-j}\frac{i^3-i}{12}c, \quad \forall i, j \in \mathbb{Z}.$$

The algebra $\mathfrak{V}$ is widely known as one of the most important Lie algebras both in mathematics and in mathematical physics, see for example [KR], [IK] and references therein. The Virasoro algebra theory has been extensively used in physics and other mathematical branches, for example, quantum physics [GO], conformal field theory [FMS], higher-dimensional WZW models [IKUX], [IKU], Kac-Moody algebras [K2], [MoP], vertex algebras [LL], and so on.

The theory of irreducible weight modules over the Virasoro algebra with finite-dimensional weight spaces is fairly well developed. It is well-known that a classification of irreducible weight Virasoro modules with finite-dimensional weight spaces was obtained by Mathieu [M], and a classification of irreducible weight Virasoro modules with at least one finite-dimensional nonzero weight space was completely determined in [MZ1]. There are also some known irreducible weight Virasoro modules with infinite-dimensional weight spaces, see [Zh], [CM], [CGZ] and [LLZ]. We remark that the tensor products of intermediate series modules over the Virasoro algebra are never irreducible [Zk]. For non-weight irreducible Virasoro modules, there are Whittaker modules, see [BM], [LGZ] and [MW], and other non-Whittaker modules, see [MW], [MZ2], [LZ] and [LLZ].

The purpose of the present paper is to construct new irreducible (non-weight) Virasoro modules by taking tensor products of some known irreducible Virasoro modules defined quite recently. Let us first recall some notions and results which will be used later.

For any pair $(\lambda,b)\in\mathbb{C}^*\times\mathbb{C}$, the Virasoro module $\Omega(\lambda,b)$ is defined on the polynomial (associative) algebra $\mathbb{C}[\partial]$ in one indeterminant ∂ over $\mathbb{C}$ with the action of $\mathfrak{V}$ given by

$$c\cdot\partial^j=0 \quad\text{and}\quad d_n\cdot\partial^j=\lambda^n\big(\partial+n(b-1)\big)(\partial-n)^j,\quad \forall j\in\mathbb{Z}_+, n\in\mathbb{Z}.$$

It was proved in [LZ] that $\Omega(\lambda,b)$ is irreducible if and only if $b\neq1$; if $b=1$ then $\Omega(\lambda,1)$ has a codimension one irreducible submodule isomorphic to $\Omega(\lambda,0)$. The modules $\Omega(1,b)$ were also studied in Section 4 of [BMZ].

Now let us recall the highest weight modules over the Virasoro algebra. Let $U:=U(\mathfrak{V})$ be the universal enveloping algebra of the Virasoro algebra $\mathfrak{V}$. For any $\theta,h\in\mathbb{C}$, let $I(\theta,h)$ be the left ideal of U generated by the set

$$\{d_i\mid i>0\}\bigcup\{d_0-h\cdot 1, c-\theta\cdot 1\}.$$

The Verma module with highest weight (θ,h) for $\mathfrak{V}$ is defined as the quotient $\bar{V}(\theta,h):=U/I(\theta,h)$. It is a highest weight module of $\mathfrak{V}$ and has a basis consisting of all vectors of the form

$$d_{-1}^{k_{-1}}d_{-2}^{k_{-2}}\dots d_{-n}^{k_{-n}}v_h \quad\text{and}\quad k_{-1},k_{-2},\dots,k_{-n}\in\mathbb{Z}_+,\ n\in\mathbb{N},$$

where $v_h=1+I(\theta,h)$. Each nonzero scalar multiple of v_h is called a highest weight vector of the Verma module. Then we have the *irreducible highest weight module*

$V(\theta,h)=\overline{V}(\theta,h)/J$, where J is the maximal proper submodule of $\overline{V}(\theta,h)$. For the structure of $V(\theta,h)$, refer to [FF] and [A] (which is a refined version of [FF]).

Denote by $\mathfrak{V}_+$ the Lie subalgebra of $\mathfrak{V}$ spanned by all d_i with $i\geq 0$. For $n\in\mathbb{Z}_+$, denote by $\mathfrak{V}_+^{(n)}$ the Lie subalgebra of $\mathfrak{V}$ generated by all d_i for $i>n$. For any $\mathfrak{V}_+$-module N and $\theta\in\mathbb{C}$, consider the induced module $\mathrm{Ind}(N):=U(\mathfrak{V})\otimes_{U(\mathfrak{V}_+)}N$, and denote by $\mathrm{Ind}_\theta(N)$ the module $\mathrm{Ind}(N)/(c-\theta)\mathrm{Ind}(N)$. From [MZ2] we know that for an irreducible $\mathfrak{V}_+$-module N, if there exists $k\in\mathbb{N}$ such that d_k acts injectively on N and $d_i\cdot N=0$ for all $i>k$, then $\mathrm{Ind}_\theta(N)$ is an irreducible $\mathfrak{V}$-module for any $\theta\in\mathbb{C}$.

The present paper is organized as follows. In Section 2, we obtain a class of irreducible non-weight modules by taking the tensor product of $\Omega(\lambda,b)$ with the highest weight module $V(\theta,h)$ or with the modules $\mathrm{Ind}_\theta(N)$ (see Theorem 1). In Section 3, we determine the necessary and sufficient conditions for two irreducible modules $\Omega(\lambda,b)\otimes V$ and $\Omega(\lambda',b')\otimes V'$ to be isomorphic (Theorem 2). In Section 4, we compare the tensor product modules $\Omega(\lambda,b)\otimes V$ with all other known non-weight irreducible modules in [LZ], [LLZ], [MZ2] and [MW]. In particular, we prove that the tensor product of $\Omega(\lambda,b)$ with the classical Whittaker module (see [OW] and [LGZ]) is isomorphic to the module $\mathrm{Ind}_{\theta,\lambda}(\mathbb{C}_{\mathbf{m}})$ defined in [MW]. As a by-product, we obtain the necessary and sufficient conditions for the modules $\mathrm{Ind}_{\theta,\lambda}(\mathbb{C}_{\mathbf{m}})$ to be irreducible which was not solved in [MW] (Theorem 5). From these we conclude that the modules $\Omega(\lambda,b)\otimes V$ are new when V are not the classical irreducible Whittaker modules. In Section 5, we generalize the modules $\mathrm{Ind}_{\theta,\lambda}(\mathbb{C}_{\mathbf{m}})$ which were defined and studied in [MW] to the modules $\mathrm{Ind}_{\theta,\lambda}(\mathcal{B}_{\mathbf{s}}^{(n)})$ for any $n\in\mathbb{Z}_+$. More precisely, for $n\in\mathbb{Z}_+,\lambda\in\mathbb{C}^*$, first we define the subalgebra of $\mathfrak{V}$ as follows

$$\mathfrak{b}_{\lambda,n}=\mathrm{span}_{\mathbb{C}}\{d_k-\lambda^{k-n+1}d_{n-1}:k\geq n\}.$$

For any $\theta\in\mathbb{C}$ and $\mathbf{s}=(s_n,s_{n+1},\dots,s_{2n})\in\mathbb{C}^{n+1}$, we define the 1-dimensional $\mathfrak{b}_{\lambda,n}$-module $\mathcal{B}_{\mathbf{s}}^{(n)}$ on $\mathbb{C}$ by

$$\left(d_k-\lambda^{k-n+1}d_{n-1}\right)\cdot 1=s_k,\quad n\leq k\leq 2n.$$

Then we have our Virasoro module

$$\mathrm{Ind}_{\theta,\lambda}\left(\mathcal{B}_{\mathbf{s}}^{(n)}\right):=\left(\mathrm{Ind}_{\mathfrak{b}_{\lambda,n}}^{\mathfrak{V}}\mathcal{B}_{\mathbf{s}}^{(n)}\right)/(c-\theta)\left(\mathrm{Ind}_{\mathfrak{b}_{\lambda,n}}^{\mathfrak{V}}\mathcal{B}_{\mathbf{s}}^{(n)}\right),$$

where $\mathrm{Ind}_{\mathfrak{b}_{\lambda,n}}^{\mathfrak{V}}\mathcal{B}_{\mathbf{s}}^{(n)}=U(\mathfrak{V})\otimes_{U(\mathfrak{b}_{\lambda,n})}\mathcal{B}_{\mathbf{s}}^{(n)}$. We use the above established results to obtain the necessary and sufficient conditions for the modules $\mathrm{Ind}_{\theta,\lambda}(\mathcal{B}_{\mathbf{s}}^{(n)})$ to be irreducible in Theorems 7, 8 and 9 for different cases of n. We remark that the three cases are totally different. We also study the submodules of $\mathrm{Ind}_{\theta,\lambda}(\mathcal{B}_{\mathbf{s}}^{(n)})$ in Theorem 10. As a by-product, Corollary 11 solves the open problem in [GLZ], i.e.,

$\mathrm{Ind}_{\theta,\lambda}(\mathcal{B}_{\mathbf{s}}^{(0)})$ has a unique maximal submodule. Our main technique used in this paper is Feigin–Fuchs' Theorem in [FF] on singular vectors of Verma modules over the Virasoro algebra.

Using the present paper's results, we further study irreducibility for the tensor product of finitely many modules of the form $\Omega(\lambda, b)$ and $\mathrm{Ind}_\theta(N)$ in the subsequent paper [TZ].

2. Constructing non-weight modules

In this section we will obtain a class of irreducible non-weight modules over $\mathfrak{V}$ by taking the tensor products of $\Omega(\lambda, b)$ with two classes of other modules, which is the following

Theorem 1. *Let $\lambda\in\mathbb{C}^*$ and $b\in\mathbb{C}\setminus\{1\}$. Assume that V is an irreducible module over $\mathfrak{V}$ such that each d_k is locally finite on V for all $k\geq R$ where R is a fixed positive integer. Then $\Omega(\lambda, b)\otimes V$ is an irreducible Virasoro module.*

Proof. From Theorem 2 in [MZ2] we know that V has to be $V(\theta, h)$ for some $\theta, h\in\mathbb{C}$ or $\mathrm{Ind}_\theta(N)$ defined in [MZ2]. Let $W=\Omega(\lambda, b)\otimes V$. It is clear that, for any $v\in V$, there is a positive integer $K(v)$ such that $d_l\cdot v=0$ for all $l\geq K(v)$.

Suppose M is a nonzero submodule of W. It suffices to show that $M=W$. Take a nonzero $w=\sum_{j=0}^{s}\partial^j\otimes v_j\in M$ such that $v_j\in V$, $v_s\neq 0$ and s is minimal.

Claim 1. $s=0$.

Let $K=\max\{K(v_j): j=0,1,\ldots,s\}$. Using $d_l\cdot v_j=0$ for all $l\geq K$ and $j=0,1,\ldots,s$, we deduce that

$$\lambda^{-l}d_l\cdot w=\sum_{j=0}^{s}\big(\partial+l(b-1)\big)(\partial-l)^j\otimes v_j\in M,\quad \forall l\geq K,$$

which right-hand side can be written in the following form

$$\sum_{j=0}^{s+1}l^j w_j\in M,\quad \forall l\geq K, \tag{2.1}$$

where all the $w_j\in W$ are independent of the choice of l $(\geq K)$. Taking $l=K$, $K+1,\ldots,K+s+1$, we see that the coefficient matrix of the w_j is a Vandermonde matrix. So each $w_j\in M$. In particular,

$$0\neq w_{s+1}=(b-1)(-1)^s\otimes v_s\in M.$$

Consequently, $s=0$.

Claim 2. $M=W$.

From Claim 1 we know that $1\otimes v\in M$ for some nonzero $v\in V$. By induction on t and using

$$\begin{aligned} d_l\cdot\partial^t\otimes v &= \big(\lambda^l\big(\partial+l(b-1)\big)(\partial-l)^t\big)\otimes v \\ &= \lambda^l(\partial-l)^{t+1}\otimes v+lb\lambda^l(\partial-l)^t\otimes v, \end{aligned}$$

where $l\geq K(v)$, $t\in\mathbb{Z}_+$, we deduce that $\partial^t\otimes v\in M$ for all $t\in\mathbb{Z}_+$, i.e., $\Omega(\lambda,b)\otimes v\subset M$. Let X be a maximal subspace of V such that $\Omega(\lambda,b)\otimes X\subset M$. We know that $X\neq 0$. Clearly, X is a nonzero submodule of V. Since V is irreducible, we obtain that $X=V$. Therefore, $M=W$.

The theorem follows from Claim 2. $\square$

Example 1. Let $\lambda_1,\lambda_2,\theta\in\mathbb{C}$ and let J be the left ideal of $U(\mathfrak{V}_+)$ generated by $d_1-\lambda_1, d_2-\lambda_2, d_3, d_4, \ldots$. We define $N:=U(\mathfrak{V}_+)/J$. Then $V=\mathrm{Ind}_\theta(N)$ is the classical Whittaker module (See [OW] or [MZ2]). From [LGZ] we know that if $\lambda_1\neq 0$ or $\lambda_2\neq 0$, then V is both an irreducible $\mathfrak{V}$-module and a locally nilpotent $\mathfrak{V}_+^{(2)}$-module. By Theorem 1 we know that $\Omega(\lambda,b)\otimes V$ is an irreducible $\mathfrak{V}$-module for any $\lambda\in\mathbb{C}^*$ and $b\in\mathbb{C}\setminus\{1\}$. These modules will be studied in detail in Section 4.

3. Isomorphisms

In this section we will determine the necessary and sufficient conditions for two irreducible tensor products defined in Theorem 1 to be isomorphic, which is the following

Theorem 2. *Let $\lambda,\lambda'\in\mathbb{C}^*$, $b,b'\in\mathbb{C}\setminus\{1\}$, and let V and V' be two irreducible modules over $\mathfrak{V}$ such that each d_k is locally finite on both V and V' for all $k\geq R$ where R is a fixed positive integer. Then $\Omega(\lambda,b)\otimes V$ and $\Omega(\lambda',b')\otimes V'$ are isomorphic as $\mathfrak{V}$-modules if and only if $(\lambda,b)=(\lambda',b')$ and $V\cong V'$ as $\mathfrak{V}$-modules.*

Proof. The "If" part of the theorem is obvious. We need only to prove the "only if" part. Let φ be an isomorphism from $\Omega(\lambda,b)\otimes V$ to $\Omega(\lambda',b')\otimes V'$.

Take a nonzero element $1\otimes v\in W$. Suppose

$$\varphi(1\otimes v)=\sum_{j=0}^{n}\partial^j\otimes w_j,$$

where $w_j \in V'$ with $w_n \neq 0$. There is a positive integer $K = \max\{K(v), K(w_0), K(w_1), ..., K(w_n)\}$ such that $d_l \cdot v = d_l \cdot w_j = 0$ for all integers $l \geq K$ and $0 \leq j \leq n$. For any $l, l' \geq K$, we have

$$\big(\lambda^{-l} d_l - \lambda^{-l'} d_{l'}\big) \cdot (1 \otimes v) = \big(l - l'\big)(b-1)(1 \otimes v).$$

Then

$$\begin{aligned}
&\big(l-l'\big)(b-1)\sum_{j=0}^{n} \partial^j \otimes w_j \\
&\qquad = \big(\lambda^{-l} d_l - \lambda^{-l'} d_{l'}\big) \cdot \sum_{j=0}^{n} \partial^j \otimes w_j \\
&\qquad = \sum_{j=0}^{n} \left(\left(\frac{\lambda'}{\lambda}\right)^{l} \big(\partial + l\big(b'-1\big)\big)(\partial - l)^j - \left(\frac{\lambda'}{\lambda}\right)^{l'} \big(\partial + l'\big(b'-1\big)\big)\big(\partial - l'\big)^j \right) \otimes w_j.
\end{aligned}$$

We deduce that

$$\left(\left(\frac{\lambda'}{\lambda}\right)^{l} - \left(\frac{\lambda'}{\lambda}\right)^{l'} \right) \big(\partial^{n+1} \otimes w_n\big) = 0, \quad \forall l, l' \geq K.$$

So $\lambda' = \lambda$. The previous equation becomes

$$\begin{aligned}
\big(l-l'\big)(b-1)\sum_{j=0}^{n} \partial^j \otimes w_j = {} & \sum_{j=0}^{n} \big(b'-1\big)\big(l(\partial - l)^j - l'\big(\partial - l'\big)^j\big) \otimes w_j \\
& + \sum_{j=0}^{n} \partial\big((\partial - l)^j - \big(\partial - l'\big)^j\big) \otimes w_j,
\end{aligned}$$

where $l, l' \geq K$. If $n > 0$, the coefficient of l^{n+1} is $(-1)^n (b'-1)(1 \otimes w_n)$ which is nonzero, yielding a contradiction. So $n = 0$, hence $b' = b$. Thus there is a one to one and onto linear map $\tau : V \to V'$ such that

$$\varphi(1 \otimes v) = 1 \otimes \tau(v), \quad \forall v \in V. \tag{3.1}$$

Since

$$\varphi\big(d_l \cdot (1 \otimes v)\big) = d_l \cdot \big(\varphi(1 \otimes v)\big), \quad \forall l \geq K,$$

that is,

$$\lambda^l \varphi(\partial \otimes v) + \lambda^l l(b-1)\big(1 \otimes \tau(v)\big) = \lambda^l \big(\partial \otimes \tau(v)\big) + \lambda^l l(b-1)\big(1 \otimes \tau(v)\big),$$

we see that $\varphi(\partial\otimes v)=\partial\otimes\tau(v)$. Hence, $\varphi((d_j\cdot 1)\otimes v)=(d_j\cdot 1)\otimes\tau(v), j\in\mathbb{Z}$. From $\varphi(d_j\cdot(1\otimes v))=d_j\cdot(\varphi(1\otimes v))$, $j\in\mathbb{Z}$ we can deduce that $\varphi(1\otimes(d_j\cdot v))=1\otimes(d_j\cdot\tau(v))$. So

$$\tau(d_j\cdot v)=d_j\cdot\big(\tau(v)\big),\quad \forall j\in\mathbb{Z},\ v\in V.$$

Clearly, $\varphi(c\cdot(1\otimes v))=c\cdot(\varphi(1\otimes v))$ implies that $\tau(c\cdot v)=c\cdot\tau(v)$. Thus $\tau:V\to V'$ is a $\mathfrak{V}$-module isomorphism, i.e., $V\cong V'$. This completes the proof. □

4. New irreducible modules $\Omega(\lambda,b)\otimes V$

In this section we will compare the irreducible tensor products in Theorem 1 with all other known non-weight irreducible Virasoro modules in [LZ], [LLZ], [MZ2] and [MW].

For any $s\in\mathbb{Z}_+$, $l,m\in\mathbb{Z}$, as in [LLZ], we denote by

$$\omega_{l,m}^{(s)}:=\sum_{i=0}^{s}\binom{s}{i}(-1)^{s-i}d_{l-m-i}d_{m+i}\in U(\mathfrak{V}).$$

Proposition 3. *Let $\lambda\in\mathbb{C}^*$, $b\in\mathbb{C}\setminus\{1\}$, and let V be an infinite-dimensional irreducible $\mathfrak{V}$-module such that each d_k is locally finite on V for all $k\geq R$ for a fixed $R\in\mathbb{N}$.*

(i) *For any positive integer n, the action of $\mathfrak{V}_+^{(n)}$ on $\Omega(\lambda,b)\otimes V$ is not locally finite.*

(ii) *For any integer $s>4$, there exists $v\in V, m,l\in\mathbb{Z}$ such that in $\Omega(\lambda,b)\otimes V$ we have*

$$\omega_{l,-m}^{(s)}\cdot(1\otimes v)\neq 0.$$

Proof. As we mentioned before, V has to be $V(\theta,h)$ for some $\theta,h\in\mathbb{C}$ or $\mathrm{Ind}_\theta(N)$ defined in [MZ2]. Part (i) follows from considering $d_R^k\cdot(1\otimes v)$ for any nonzero $v\in V$ and any $k\in\mathbb{N}$.

(ii) Take v to be the highest weight vector of V if $V=V(\theta,h)$ for some $(\theta,h)\in\mathbb{C}^2$, otherwise v can be any nonzero vector of V. Let $S=\{v,d_{-2}v,d_{-3}v,\ldots,d_{-s-2}v\}$. In the case $V=V(\theta,h)$, by Theorem 3.1 in [A] we see that S consists of nonzero elements with pair-wise different weights, which means that the elements in S are linearly independent. If $V=\mathrm{Ind}_\theta(N)$, from the structure of $\mathrm{Ind}_\theta(N)$ described in [MZ2] we can easily see that S consists of linearly independent vectors. Obviously, there exists a positive integer K (depending on v and s) such that $d_l\cdot S=0$ for $l>K$. Letting $l>K$ and $m=s+2$, and noting that $\omega_{l,-m}^{(s)}\cdot 1=0$ in $\Omega(\lambda,b)$, we deduce that

$$\begin{aligned}\omega^{(s)}_{l,-m}\cdot(1\otimes v) &= \sum_{i=0}^{s}\binom{s}{i}(-1)^{s-i}d_{l+m-i}d_{-m+i}\cdot(1\otimes v)\\ &= \sum_{i=0}^{s}\binom{s}{i}(-1)^{s-i}(d_{l+m-i}\cdot 1)\otimes(d_{-m+i}\cdot v)\\ &= \sum_{i=0}^{s}\binom{s}{i}(-1)^{s-i}\lambda^{l+m-i}\big(\partial+(l+m-i)(b-1)\big)\otimes(d_{-m+i}\cdot v),\end{aligned}$$

which is clearly nonzero. $\square$

Let us recall the known non-weight irreducible $\mathfrak{V}$-modules. We first recall the irreducible Virasoro modules $A_{b'}$ defined in [LZ] where $b'\in\mathbb{C}$ and A is an irreducible module over the associative algebra $\mathbb{C}[t,t^{-1},t\frac{d}{dt}]$. The action on $A_{b'}$ is

$$c\cdot w=0 \quad \text{and} \quad d_n\cdot w=\big(t^n\partial+nb't^n\big)w, \quad \forall n\in\mathbb{Z},\ w\in A,$$

where $\partial=t\frac{d}{dt}$ and the left hand side is associative algebra action.

Let $\mathfrak{a}_r:=\mathfrak{V}_+/\mathfrak{V}_+^{(r)}$, $r\in\mathbb{Z}_+$ and let $\bar{d}_i$ be the image of d_i in $\mathfrak{a}_r$ for $0\le i\le r$. Now let us consider the modules $\mathcal{N}(M,\beta)$ described in [LLZ], where M is an irreducible module over $\mathfrak{a}_r$ such that the action of $\overline{d_r}$ on M is injective, $\beta\in\mathbb{C}[t^{\pm1}]\setminus\mathbb{C}$. We know that $\mathcal{N}(M,\beta)=M\otimes\mathbb{C}[t^{\pm1}]$, and the action of $\mathfrak{V}$ on $\mathcal{N}(M,\beta)$ is defined by

$$\begin{aligned}&d_n\circ\big(v\otimes t^k\big)=\left(kv+\sum_{i=0}^{r}\left(\frac{n^{i+1}}{(i+1)!}\bar{d}_i\right)\cdot v\right)\otimes t^{n+k}+v\otimes\big(\beta t^{n+k}\big),\\ &c\cdot\big(v\otimes t^k\big)=0, \quad n,k\in\mathbb{Z},\ v\in M.\end{aligned}$$

Corollary 4. *Let $\lambda\in\mathbb{C}^*$, $b\in\mathbb{C}\setminus\{1\}$, and let V be an infinite-dimensional irreducible $\mathfrak{V}$-module such that each d_k is locally finite on V for all $k\ge R$ for a fixed $R\in\mathbb{N}$. Then $\Omega(\lambda,b)\otimes V$ is not isomorphic to any one of the irreducible modules* $\mathrm{Ind}_\theta(N)$, $A_{b'}$, *and* $\mathcal{N}(M,\beta)$ *for corresponding parameters.*

Proof. Since there is an $n\in\mathbb{N}$ such that the action of $\mathfrak{V}_+^{(n)}$ on the modules $\mathrm{Ind}_\theta(N)$ is locally finite, from Proposition 3(i) we see that $\Omega(\lambda,b)\otimes V$ is not isomorphic to any module described in [MZ2].

Now let us consider an irreducible Virasoro module $A_{b'}$. We may assume that A is $\mathbb{C}[t\frac{d}{dt}]$-torsion-free, otherwise $A_{b'}$ will be a weight Virasoro module. From the proof of Theorem 9 in [LLZ], we know that

$$\omega^{(s)}_{l,m}\cdot A_{b'}=0, \quad \forall l,m\in\mathbb{Z},\ s\ge 3.$$

From Proposition 3(ii) we see that $\Omega(\lambda,b)\otimes V$ is not isomorphic to the modules $A_{b'}$.

Note that if $r=0$ in the definition, the modules $\mathcal{N}(M,\beta)$ with $\beta\in\mathbb{C}[t^{\pm 1}]\setminus\mathbb{C}$ are some modules of the form $A_{b'}$ (see the beginning of Section 6 in [LLZ] and Section 4.1 in [LZ]). From the computation in (6.7) of [LLZ] we see that

$$\omega_{l,m}^{(s)}\cdot\big(\mathcal{N}(M,\beta)\big)=0,\quad \forall l,m\in\mathbb{Z},\ s>2r+2.$$

From Proposition 3(ii) we see that $\Omega(\lambda,b)\otimes V$ is not isomorphic to the modules $\mathcal{N}(M,\beta)$. $\square$

Now we compare our modules $\Omega(\lambda,b)\otimes V$ in Theorem 1 with the modules $\mathrm{Ind}_{\theta,\lambda}(\mathbb{C}_{\mathbf{m}})$ defined in [MW]. Let us first recall the definition for $\mathrm{Ind}_{\theta,\lambda}(\mathbb{C}_{\mathbf{m}})$ from [MW].

Let $\lambda\in\mathbb{C}^*$, denote by $\mathfrak{b}_\lambda$ the subalgebra of $\mathfrak{V}$ generated by $d_k-\lambda^{k-1}d_1$, $k\geq 2$. For a fixed 3-tuple $\mathbf{m}=(m_2,m_3,m_4)\in\mathbb{C}^3$, Mazorchuk and Weisner defined a $\mathfrak{b}_\lambda$-module on $\mathbb{C}$ by

$$\begin{aligned}&\big(d_k-\lambda^{k-1}d_1\big)\cdot 1=m_k,\quad k=2,3,4;\\&\big(d_k-\lambda^{k-1}d_1\big)\cdot 1=(k-3)m_4\lambda^{k-4}-(k-4)m_3\lambda^{k-3},\quad k>4.\end{aligned}\tag{4.1}$$

The $\mathfrak{b}_\lambda$-module is denoted by $\mathbb{C}_{\mathbf{m}}$. Note that the second equation in (4.1) depends on the first one. For a fixed $\theta\in\mathbb{C}$, the module $\mathrm{Ind}_{\theta,\lambda}(\mathbb{C}_{\mathbf{m}})$ is defined as follows

$$\mathrm{Ind}_{\theta,\lambda}(\mathbb{C}_{\mathbf{m}}):=U(\mathfrak{V})\otimes_{U(\mathfrak{b}_\lambda)}\mathbb{C}_{\mathbf{m}}/(c-\theta)U(\mathfrak{V})\otimes_{U(\mathfrak{b}_\lambda)}\mathbb{C}_{\mathbf{m}}.\tag{4.2}$$

From Theorem 1 in [MW] we know that the $\mathfrak{V}$-module $\mathrm{Ind}_{\theta,\lambda}(\mathbb{C}_{\mathbf{m}})$ is irreducible if $(m_2,m_3,m_4)\in\mathbb{C}^3$ and $\lambda\in\mathbb{C}\setminus\{0\}$ satisfy the following conditions

$$\lambda m_3\neq m_4,\quad 2\lambda m_2\neq m_3,\quad 3\lambda m_3\neq 2m_4\quad\text{and}\quad \lambda^2m_2+m_4\neq 2\lambda m_3.\tag{4.3}$$

Now we are ready to prove the following

Theorem 5. *Let $\lambda\in\mathbb{C}^*$, $\theta\in\mathbb{C}$, $\mathbf{m}=(m_1,m_2,m_3)\in\mathbb{C}^3$ and let $\mathrm{Ind}_{\theta,\lambda}(\mathbb{C}_{\mathbf{m}})$ be defined by (4.2).*

(i) *The module $\mathrm{Ind}_{\theta,\lambda}(\mathbb{C}_{\mathbf{m}})$ is isomorphic to $\Omega(\lambda,b)\otimes V$ where V is the classical Whittaker module described in Example 1 and b,λ_1,λ_2 are given by*

$$\begin{aligned}&b=1+\lambda^{-4}(m_4-\lambda m_3),\\&\lambda_1=\lambda^{-3}(2m_4-3\lambda m_3),\\&\lambda_2=\lambda^{-2}\big(m_4-2\lambda m_3+\lambda^2m_2\big);\end{aligned}\tag{4.4}$$

(ii) *The module $\mathrm{Ind}_{\theta,\lambda}(\mathbb{C}_{\mathbf{m}})$ is irreducible if and only if $\lambda m_3\neq m_4$, and $3\lambda m_3\neq 2m_4$ or $\lambda^2m_2+m_4\neq 2\lambda m_3$.*

To prove the theorem, we need the following

Lemma 6. *Suppose V is a cyclic $\mathfrak{V}$-module with a basis*

$$\{d_{j-n}^{k_{j-n}}...d_{j-1}^{k_{j-1}}d_j^{k_j}\cdot v : n\in\mathbb{N},\ k_j,k_{j-1},...,k_{j-n}\in\mathbb{Z}_+\}$$

where $v\in V$ is a nonzero vector, j is a fixed integer and $d_p\cdot v\in\mathbb{C}v$ for all integers $p>j$. Then for any pair $(\lambda,b)\in\mathbb{C}^*\times\mathbb{C}\setminus\{1\}$, the tensor product $\Omega(\lambda,b)\otimes V$ is also a cyclic $\mathfrak{V}$-module with a generator $1\otimes v$ and a basis

$$\{d_{j-n}^{k_{j-n}}...d_{j-1}^{k_{j-1}}d_j^{k_j}d_{j+1}^{k_{j+1}}\cdot 1\otimes v : n\in\mathbb{N},\ k_{j+1},k_j,k_{j-1},...,k_{j-n}\in\mathbb{Z}_+\}. \tag{4.5}$$

Proof. It is easy to see that $\Omega(\lambda,b)\otimes V$ has a basis

$$B=\{\partial^{k_{j+1}}\otimes d_{j-n}^{k_{j-n}}...d_{j-1}^{k_{j-1}}d_j^{k_j}\cdot v : n\in\mathbb{N},\ k_{j+1},k_j,k_{j-1},...,k_{j-n}\in\mathbb{Z}_+\}.$$

Let us define a partial order on B as follows

$$\partial^{k_{j+1}}\otimes d_{j-n}^{k_{j-n}}...d_{j-1}^{k_{j-1}}d_j^{k_j}\cdot v<\partial^{l_{j+1}}\otimes d_{j-m}^{l_{j-m}}...d_{j-1}^{l_{j-1}}d_j^{l_j}\cdot v$$

if and only if

$$(k_j,k_{j-1},...,k_{j-n},0,0,..,0,k_{j+1})<(l_j,l_{j-1},...,l_{j-m},0,0,..,0,l_{j+1})$$

in the lexicographical order where the first zeros are m copies and the second zeros are n copies, i.e.,

$$(a_1,a_2,...,a_{m+n+2})<(b_1,b_2,...,b_{m+n+2})$$
$$\Longleftrightarrow \quad (\exists r>0)(a_i=b_i\forall i<r)(a_r<b_r).$$

When we expand the elements in (4.5) into linear combinations in terms of B:

$$d_{j-n}^{k_{j-n}}...d_{j-1}^{k_{j-1}}d_j^{k_j}d_{j+1}^{k_{j+1}}\cdot(1\otimes v)=\lambda^{k_{j+1}}\partial^{k_{j+1}}\otimes\big(d_{j-n}^{k_{j-n}}...d_{j-1}^{k_{j-1}}d_j^{k_j}v\big)+\text{lower terms},$$

the leading terms are exactly the corresponding basis elements in B. Thus (4.5) is a basis for $\Omega(\lambda,b)\otimes V$, and this implies that $\Omega(\lambda,b)\otimes V$ is a cyclic $\mathfrak{V}$-module with a generator $1\otimes v$. $\square$

Now we can prove Theorem 5.

Proof of Theorem 5. Let $b,\lambda_1,\lambda_2\in\mathbb{C}$ be given by (4.4). Then

$$\begin{aligned} m_2&=\lambda_2-\lambda\lambda_1+\lambda^2(b-1),\\ m_3&=\lambda^2\big(-\lambda_1+2\lambda(b-1)\big),\\ m_4&=\lambda^3\big(-\lambda_1+3\lambda(b-1)\big).\end{aligned}\tag{4.6}$$

Denote by $v{=}1\otimes 1{+}J\in V$ and by $W{=}\Omega(\lambda,b)\otimes V$. Clearly, $\mathfrak{V}_+^{(0)}\cdot v\subset\mathbb{C}v$. Since V has a basis

$$\left\{d_{-n}^{k_{-n}}...d_{-1}^{k_{-1}}d_0^{k_0}\cdot v : n\in\mathbb{Z}_+,\ k_0,k_{-1},...,k_{-n}\in\mathbb{Z}_+\right\},$$

by Lemma 6 and Theorem 1 we see that

(a) W is cyclic with a generator $1\otimes v$ and a basis

$$Q_1=\left\{d_{-n}^{k_{-n}}...d_{-1}^{k_{-1}}d_0^{k_0}d_1^{k_1}\cdot 1\otimes v : k_1,k_0,k_{-1},...,k_{-n}\in\mathbb{Z}_+,\ n\in\mathbb{Z}_+\right\};$$

(b) W is irreducible if and only if $b{\neq}1$, and $\lambda_1{\neq}0$ or $\lambda_2{\neq}0$.

In W, for $k{=}2,3,4$ we compute that

$$\begin{aligned}\left(d_k-\lambda^{k-1}d_1\right)(1\otimes v) &= \left(d_k-\lambda^{k-1}d_1\right)(1)\otimes v+1\otimes\left(d_k-\lambda^{k-1}d_1\right)(v)\\ &= \lambda^k\left(\left(\partial+k(b-1)\right)-\left(\partial+(b-1)\right)\right)\otimes v+1\otimes\left(\delta_{k,2}\lambda_2-\lambda^{k-1}\lambda_1\right)(v)\\ &= \left(\lambda^k(k-1)(b-1)-\lambda^{k-1}\lambda_1+\delta_{k,2}\lambda_2\right)(1\otimes v)\\ &= m_k(1\otimes v).\end{aligned}$$

It follows that

$$\left(d_k-\lambda^{k-1}d_1\right)(1\otimes v)=\left((k-3)m_4\lambda^{k-4}-(k-4)m_3\lambda^{k-3}\right)(1\otimes v),$$

for all $k{>}4$. Because of the universal property of the module $\mathrm{Ind}_{\theta,\lambda}(\mathbb{C}_{\mathbf{m}})$ we have the following surjective (onto) homomorphism of modules

$$\tau : \mathrm{Ind}_{\theta,\lambda}(\mathbb{C}_{\mathbf{m}})\to W,$$

uniquely determined by $\tau(\bar{1}){=}1\otimes v$ where

$$\bar{1}:=1\otimes 1+(c-\theta)U(\mathfrak{V})\otimes_{U(\mathfrak{b}_\lambda)}\mathbb{C}_{\mathbf{m}}\in\mathrm{Ind}_{\theta,\lambda}(\mathbb{C}_{\mathbf{m}}).$$

Clearly, $\mathrm{Ind}_{\theta,\lambda}(\mathbb{C}_{\mathbf{m}})$ has a basis

$$Q_2=\left\{d_{-n}^{k_{-n}}...d_{-1}^{k_{-1}}d_0^{k_0}d_1^{k_1}\cdot\bar{1} : n\in\mathbb{Z}_+,\ k_1,k_0,k_{-1},...,k_{-n}\in\mathbb{Z}_+\right\}.$$

Since $\tau|_{Q_2}{:}Q_2{\to}Q_1$ is a bijection, this means that $\tau{:}\mathrm{Ind}_{\theta,\lambda}(\mathbb{C}_{\mathbf{m}}){\to}W$ is an isomorphism. Hence (i) holds.

From (i) we see that $\mathrm{Ind}_{\theta,\lambda}(\mathbb{C}_{\mathbf{m}})$ is irreducible if and only if W is irreducible; if and only if $b{\neq}1$, and $\lambda_1{\neq}0$ or $\lambda_2{\neq}0$; if and only if $m_4-\lambda m_3{\neq}0$, and $2m_4-3\lambda m_3{\neq}0$ or $m_4-2\lambda m_3+\lambda^2 m_2{\neq}0$. This is (ii) and completes the proof. $\square$

Note that Theorem 5 actually gives the necessary and sufficient conditions for the module $\mathrm{Ind}_{\theta,\lambda}(\mathbb{C}_{\mathbf{m}})$ described in [MW] to be irreducible.

5. Applications

In this section we will generalize the construction of the Virasoro modules $\mathrm{Ind}_{\theta,\lambda}(\mathbb{C}_{\mathbf{m}})$ described in [MW].

Let $n\in\mathbb{Z}_+$ and $s_n, s_{n+1}, \ldots, s_{2n}$, $\lambda,\theta\in\mathbb{C}$ with $\lambda\neq 0$. Let

$$\mathfrak{b}_{\lambda,n} := \mathrm{span}_{\mathbb{C}}\{d_k-\lambda^{k-n+1}d_{n-1} : k\geq n\},$$

which is a subalgebra of $\mathfrak{V}$. Denote $\mathbf{s}=(s_n,s_{n+1},\ldots,s_{2n})\in\mathbb{C}^{n+1}$. We define the action of $\mathfrak{b}_{\lambda,n}$ on $\mathbb{C}$ by the following

$$\begin{aligned}&\big(d_k-\lambda^{k-n+1}d_{n-1}\big)\cdot 1 = s_k, \quad \forall k=n,n+1,\ldots,2n;\\ &\big(d_k-\lambda^{k-n+1}d_{n-1}\big)\cdot 1\\ &\qquad = -(k-2n)s_{2n-1}\lambda^{k-2n+1}+(k-2n+1)s_{2n}\lambda^{k-2n}, \quad \forall k>2n,\end{aligned} \tag{5.1}$$

where we have assigned that $s_{-1}=0$. We denote the corresponding $\mathfrak{b}_{\lambda,n}$-module by $\mathcal{B}_{\mathbf{s}}^{(n)}$. Note that the second equation in (5.1) follows from the first. Let $\mathrm{Ind}_{\mathfrak{b}_{\lambda,n}}^{\mathfrak{V}}\mathcal{B}_{\mathbf{s}}^{(n)}=U(\mathfrak{V})\otimes_{U(\mathfrak{b}_{\lambda,n})}\mathcal{B}_{\mathbf{s}}^{(n)}$. Then the induced $\mathfrak{V}$-module from $\mathcal{B}_{\mathbf{s}}^{(n)}$ is defined as following

$$\mathrm{Ind}_{\theta,\lambda}\big(\mathcal{B}_{\mathbf{s}}^{(n)}\big) := \big(\mathrm{Ind}_{\mathfrak{b}_{\lambda,n}}^{\mathfrak{V}}\mathcal{B}_{\mathbf{s}}^{(n)}\big)/(c-\theta)\big(\mathrm{Ind}_{\mathfrak{b}_{\lambda,n}}^{\mathfrak{V}}\mathcal{B}_{\mathbf{s}}^{(n)}\big). \tag{5.2}$$

We will determine the necessary and sufficient conditions for $\mathrm{Ind}_{\theta,\lambda}(\mathcal{B}_{\mathbf{s}}^{(n)})$ to be irreducible in the next three theorems for different cases of n. It is interesting to remark that the three cases are totally different. Our main technique used here is Feigin–Fuchs' Theorem in [FF], or Theorem A in [A] (which is a refined version of Feigin–Fuchs' Theorem).

For convenience to study the irreducibility of the $\mathfrak{V}$-module $\mathrm{Ind}_{\theta,\lambda}(\mathcal{B}_{\mathbf{s}}^{(n)})$, we will denote by

$$\bar{1} := 1\otimes 1+(c-\theta)\big(\mathrm{Ind}_{\mathfrak{b}_{\lambda,n}}^{\mathfrak{V}}\mathcal{B}_{\mathbf{s}}^{(n)}\big)\in\mathrm{Ind}_{\theta,\lambda}\big(\mathcal{B}_{\mathbf{s}}^{(n)}\big)$$

in the rest part of the section.

We first consider the case $n=0$. Obviously, the action (5.1) of $\mathfrak{b}_{\lambda,0}$ on $\mathcal{B}_{\mathbf{s}}^{(0)}$, where $\mathbf{s}=(s_0)\in\mathbb{C}$, is equivalent to the following action

$$\big(d_k-\lambda^k d_0\big)\cdot 1 = k\lambda^k s_0, \quad k\geq -1. \tag{5.3}$$

For convenience, we shall use (5.3) to deal with the irreducibility for $\mathrm{Ind}_{\theta,\lambda}(\mathcal{B}_{\mathbf{s}}^{(n)})$ in the case $n=0$. Recall that the Verma module $\overline{V}(\theta,0)$ over the Virasoro algebra was defined in the introduction. Now we have the following

Theorem 7. *Let* $\mathbf{s}=(s_0)\in\mathbb{C}$, $\theta\in\mathbb{C}$, $\lambda\in\mathbb{C}^*$ *and denote*

$$M(\theta,0)=\overline{V}(\theta,0)/U(\mathfrak{V})\big(d_{-1}\big(1+I(\theta,0)\big)\big).$$

(i) *The module* $\mathrm{Ind}_{\theta,\lambda}(\mathcal{B}_{\mathbf{s}}^{(0)})$ *is isomorphic to* $\Omega(\lambda,b)\otimes M(\theta,0)$, *where* $b=s_0+1$.

(ii) *The module* $\mathrm{Ind}_{\theta,\lambda}(\mathcal{B}_{\mathbf{s}}^{(0)})$ *is irreducible if and only if* $s_0\neq 0$ *and* $\theta\neq 1-6\frac{(p-q)^2}{pq}$ *for any coprime integers* $p,q\geq 2$.

Proof. Let $b=s_0+1$. Denote by

$$v=1+U(\mathfrak{V})\big(d_{-1}\big(1+I(\theta,0)\big)\big)\in M(\theta,0).$$

Since $d_j\cdot v\in\mathbb{C}v, j\geq -1$ and $M(\theta,0)$ has a basis

$$\big\{d_{-n}^{k_{-n}}...d_{-2}^{k_{-2}}\cdot v: n\geq 2,\ k_{-2},...,k_{-n}\in\mathbb{Z}_+\big\},$$

by Lemma 6 we see that $\Omega(\lambda,b)\otimes M(\theta,0)$ is cyclic with a generator $1\otimes v$ and a basis

$$Q_1=\big\{d_{-n}^{k_{-n}}...d_{-2}^{k_{-2}}d_{-1}^{k_{-1}}\cdot 1\otimes v: n\geq 2,\ k_{-1},k_{-2},...,k_{-n}\in\mathbb{Z}_+\big\},$$

and by Theorem 1 we deduce that $\Omega(\lambda,b)\otimes M(\theta,0)$ is irreducible if and only if $b\neq 1$, and $M(\theta,0)$ is irreducible.

Now let us consider the irreducibility of $M(\theta,0)$. From Theorem A in [A] we know that $M(\theta,0)$ is not irreducible if and only if the maximal proper submodule $I(\theta,0)$ of the Verma module $\overline{V}(\theta,0)$ cannot be generated by only one singular vector; if and only if Conditions III_- and III_+ in [A] are satisfied; if and only if $\theta=\frac{(3p+2q)(3q+2p)}{pq}\in\mathbb{C}$, where the parameters $p,q\in\mathbb{C}^*$ are such that the straight line $l_{\theta,0}{:}pk+ql-p-q=0$ in the plane $\mathbb{C}^2(k,l)$ contains infinitely many integral points (k,l) with $kl>0$; if and only if $\theta=1-6\frac{(p-q)^2}{pq}$ for any integers p,q with $p,q\geq 2$ and $\gcd(p,q)=1$. Thus $\Omega(\lambda,b)\otimes M(\theta,0)$ is irreducible if and only if $s_0\neq 0$ and $\theta\neq 1-6\frac{(p-q)^2}{pq}$ for any integers p,q with $p,q\geq 2$ and $\gcd(p,q)=1$.

By simple computation we can obtain that

$$\big(d_k-\lambda^k d_0\big)(1\otimes v)=k\lambda^k(b-1)(1\otimes v)=k\lambda^k s_0(1\otimes v),\quad k\geq -1. \tag{5.4}$$

Comparing (5.3) with (5.4) we deduce that there exists a $\mathfrak{V}$-module homomorphism and hence epimorphism $\rho{:}\mathrm{Ind}_{\theta,\lambda}(\mathcal{B}_{\mathbf{s}}^{(0)})\to\Omega(\lambda,b)\otimes M(\theta,0)$ uniquely determined by $\rho(\bar{1})=1\otimes v$. Since $\mathrm{Ind}_{\theta,\lambda}(\mathcal{B}_{\mathbf{s}}^{(0)})$ has a basis

$$Q_2=\big\{d_{-n}^{k_{-n}}...d_{-1}^{k_{-1}}\cdot\bar{1}: k_{-1},...,k_{-n}\in\mathbb{Z}_+,\ n\in\mathbb{N}\big\},$$

noting that $\rho|_{Q_2}{:}Q_2\to Q_1$ is a bijection, we see that ρ is an isomorphism and (i) holds.

Therefore, $\mathrm{Ind}_{\theta,\lambda}(\mathcal{B}_{\mathbf{s}}^{(0)})$ is irreducible if and only if $\Omega(\lambda,b)\otimes M(\theta,0)$ is irreducible. By $s_0=b-1$ and the irreducible conditions for $\Omega(\lambda,b)\otimes M(\theta,0)$ we can deduce (ii). This competes the proof. □

We now handle the case $\mathrm{Ind}_{\theta,\lambda}(\mathcal{B}_{\mathbf{s}}^{(1)})$ for $\mathbf{s}=(s_1,s_2)$.

Theorem 8. *Let* $\lambda\in\mathbb{C}^*$, $\theta\in\mathbb{C}$ *and* $\mathbf{s}=(s_1,s_2)\in\mathbb{C}^2$.

(i) *The module* $\mathrm{Ind}_{\theta,\lambda}(\mathcal{B}_{\mathbf{s}}^{(1)})$ *is isomorphic to* $\Omega(\lambda,b)\otimes\overline{V}(\theta,h)$, *where*

$$\begin{aligned} b&=1+\lambda^{-2}(s_2-\lambda s_1),\\ h&=\lambda^{-2}(s_2-2\lambda s_1). \end{aligned} \tag{5.5}$$

(ii) *The module* $\mathrm{Ind}_{\theta,\lambda}(\mathcal{B}_{\mathbf{s}}^{(1)})$ *is irreducible if and only if* $s_2-\lambda s_1\neq 0$ *and*

$$\left(\frac{s_2-2\lambda s_1}{\lambda^2}+\varphi(k)+\frac{kl-1}{2}\right)\left(\frac{s_2-2\lambda s_1}{\lambda^2}+\varphi(l)+\frac{kl-1}{2}\right)+\frac{(k^2-l^2)^2}{16}\neq 0,\quad \forall k,l\in\mathbb{N}, \tag{5.6}$$

where $\varphi(j)=\frac{(j^2-1)(\theta-13)}{24}$, $j\in\mathbb{N}$.

Proof. Let $b,h\in\mathbb{C}$ be given by (5.5). Denote by $W=\Omega(\lambda,b)\otimes\overline{V}(\theta,h)$. From the structure of $\overline{V}(\theta,h)$ and Lemma 6 we see that W is a cyclic module with a generator $1\otimes v_0$ and a basis

$$Q_1=\left\{d_{-n}^{k_{-n}}\dots d_{-1}^{k_{-1}}d_0^{k_0}\cdot 1\otimes v_0 : n\geq 1,\ k_0,k_{-1},\dots,k_{-n}\in\mathbb{Z}_+\right\}.$$

And by Theorem 1 and the well-known Kac' determinant formula (see, for example [K1]) we can deduce that W is irreducible if and only if $b\neq 1$ and θ,h satisfy the following condition:

$$\left(h+\varphi(k)+\frac{(kl-1)}{2}\right)\left(h+\varphi(l)+\frac{(kl-1)}{2}\right)+\frac{(k^2-l^2)^2}{16}\neq 0, \tag{5.7}$$

where $\varphi(j)=\frac{(j^2-1)(\theta-13)}{24}$, $j\in\mathbb{N}$.

By simple computation we can obtain the following equalities

$$\big(d_k-\lambda^k d_0\big)(1\otimes v_0)=\lambda^k\big(k(b-1)-h\big)(1\otimes v_0),\quad k\in\mathbb{N}. \tag{5.8}$$

For $k=1,2$ we have

$$s_1=\lambda(b-1-h)\quad\text{and}\quad s_2=\lambda^2\big(2(b-1)-h\big). \tag{5.9}$$

Equations (5.8) and (5.9) imply that

$$\big(d_k-\lambda^k d_0\big)(1\otimes v_0)=\big(-(k-2)s_1\lambda^{k-1}+(k-1)s_2\lambda^{k-2}\big)(1\otimes v_0),\quad k>2.$$

Comparing (5.1) in the case $n=1$ and (5.8) we can deduce that there exists a $\mathfrak{V}$-module homomorphism and hence epimorphism $\sigma:\mathrm{Ind}_{\theta,\lambda}(\mathcal{B}_{\mathbf{s}}^{(1)})\to W$ uniquely determined by $\sigma(\bar{1})=1\otimes v_0$. Since $\mathrm{Ind}_{\theta,\lambda}(\mathcal{B}_{\mathbf{s}}^{(1)})$ has a basis

$$Q_2=\left\{d_{-n}^{k_{-n}}...d_{-1}^{k_{-1}}d_0^{k_0}\cdot\bar{1}:k_0,k_{-1},...,k_{-n}\in\mathbb{Z}_+,\ n\in\mathbb{N}\right\}$$

and since $\sigma|_{Q_2}:Q_2\to Q_1$ is a bijection, this means that σ is an isomorphism and (i) follows.

Therefore, $\mathrm{Ind}_{\theta,\lambda}(\mathcal{B}_{\mathbf{s}}^{(1)})$ is irreducible if and only if W is irreducible; if and only if $b\neq1$ and (5.7) holds; if and only if $s_2-\lambda s_1\neq0$ and

(5.10)
$$\left(\frac{s_2-2\lambda s_1}{\lambda^2}+\varphi(k)+\frac{kl-1}{2}\right)\left(\frac{s_2-2\lambda s_1}{\lambda^2}+\varphi(l)+\frac{kl-1}{2}\right)+\frac{(k^2-l^2)^2}{16}\neq0,\quad\forall k,l\in\mathbb{N},$$

where $\varphi(j)=\frac{(j^2-1)(\theta-13)}{24}$, $j\in\mathbb{N}$. This is (ii) and completes the proof. □

Before treating the case $\mathrm{Ind}_{\theta,\lambda}(\mathcal{B}_{\mathbf{s}}^{(n)})$, $n>1$, let us first recall the Whittaker modules $L_{\psi_n,\theta}$ defined in [LGZ].

Let $n\in\mathbb{N}$ and $(\alpha_n,\alpha_{n+1},...,\alpha_{2n})\in\mathbb{C}^{n+1}$. Define $\psi_n:\mathfrak{V}_+^{(n-1)}\to\mathbb{C}$ by the following

$$\psi_n(d_j)=\alpha_j,\quad j=n,n+1,...,2n;$$
$$\psi_n(d_j)=0,\quad j>2n. \tag{5.11}$$

This can actually define a $\mathfrak{V}_+^{(n-1)}$-module action on $\mathbb{C}$ by $d_j\cdot1=\psi_n(d_j)$, $j\geq n$. Denote the $\mathfrak{V}_+^{(n-1)}$-module by $\mathbb{C}_{\psi_n}$. Then

$$L_{\psi_n,\theta}=U(\mathfrak{V})\otimes_{U(\mathfrak{V}_+^{(n-1)})}\mathbb{C}_{\psi_n}/(c-\theta)U(\mathfrak{V})\otimes_{U(\mathfrak{V}_+^{(n-1)})}\mathbb{C}_{\psi_n}. \tag{5.12}$$

From Theorem 7 in [LGZ] we know that $L_{\psi_n,\theta}$ is an irreducible $\mathfrak{V}$-module if and only if $\alpha_{2n-1}\neq0$ or $\alpha_{2n}\neq0$. Moreover, it is easy to see that $L_{\psi_n,\theta}$ is a locally finite $\mathfrak{V}_+^{(2n)}$-module.

For the modules $\mathrm{Ind}_{\theta,\lambda}(\mathcal{B}_{\mathbf{s}}^{(n+1)})$, where $\mathbf{s}=(s_{n+1},s_{n+2},...,s_{2n+2})\in\mathbb{C}^{n+2}$ with $n\geq1$, we have the following

Theorem 9. *Let* $n\in\mathbb{N}$, $\mathbf{s}=(s_{n+1},s_{n+2},...,s_{2n+2})\in\mathbb{C}^{n+2}$, $\theta\in\mathbb{C}$ *and* $\lambda\in\mathbb{C}^*$.

(i) *The module* $\mathrm{Ind}_{\theta,\lambda}(\mathcal{B}_{\mathbf{s}}^{(n+1)})$ *is isomorphic to* $\Omega(\lambda,b)\otimes L_{\psi_n,\theta}$, *where*

(5.13)
$$b=1+\lambda^{-2n-2}(s_{2n+2}-\lambda s_{2n+1}),$$
$$\alpha_n=\lambda^{-n-2}\big((n+1)s_{2n+2}-(n+2)\lambda s_{2n+1}\big),$$
$$\alpha_k=s_k-\lambda^{k-2n-2}\big(-(k-2n-2)\lambda s_{2n+1}+(k-2n-1)s_{2n+2}\big),\quad n+1\leq k\leq2n.$$

(ii) *The module* $\mathrm{Ind}_{\theta,\lambda}(\mathcal{B}_{\mathbf{s}}^{(n+1)})$ *is irreducible if and only if*

$$s_{2n+2}-\lambda s_{2n+1}\neq 0, \quad \text{and} \tag{5.14}$$

$$s_{2n-1}\neq\lambda^{-3}(3\lambda s_{2n+1}-2s_{2n+2}) \quad \text{or} \quad s_{2n}\neq\lambda^{-2}(2\lambda s_{2n+1}-s_{2n+2}), \tag{5.15}$$

where we have assumed that $s_1=0$ *if* $n=1$.

Proof. Take $b,\alpha_n,\alpha_{n+1},\dots,\alpha_{2n}\in\mathbb{C}$ to be determined by (5.13). Then

$$s_k=\lambda^k(k-n)(b-1)+\big(\alpha_k-\lambda^{k-n}\alpha_n\big), \quad n+1\leq k\leq 2n+2. \tag{5.16}$$

Let $\alpha_k=0$, $k>2n$, and let $L_{\psi_n,\theta}$ be defined by (5.11) and (5.12). Denote

$$v=1+(c-\theta)U(\mathfrak{V})\otimes_{U(\mathfrak{V}_+^{(n-1)})}\mathbb{C}_{\psi_n}\in L_{\psi_n,\theta}$$

and $W=\Omega(\lambda,b)\otimes L_{\psi_n,\theta}$. Clearly, $\mathfrak{V}_+^{(n-1)}\cdot v\subset\mathbb{C}v$ and $L_{\psi_n,\theta}$ has a basis

$$\big\{d_{n-m}^{k_{n-m}}\dots d_{n-1}^{k_{n-1}}\cdot v: m\geq 1,\ k_{n-1},\dots,k_{n-m}\in\mathbb{Z}_+\big\}.$$

Then by Lemma 6 we know that W is a cyclic module with a generator $1\otimes v$ and a basis

$$Q_1=\big\{d_{n-m}^{k_{n-m}}\dots d_{n-1}^{k_{n-1}}d_n^{k_n}\cdot 1\otimes v_0: m\geq 1,\ k_{-m},\dots,k_{n-1},k_n\in\mathbb{Z}_+\big\}.$$

By Theorem 1 and the irreducible conditions for the Whittaker module $L_{\psi_n,\theta}$ we can deduce that W is irreducible if and only if $b\neq 1$, and $\alpha_{2n-1}\neq 0$ or $\alpha_{2n}\neq 0$.

For $k>n$ we compute

$$\begin{aligned}\big(d_k-\lambda^{k-n}d_n\big)(1\otimes v)&=\big(d_k-\lambda^{k-n}d_n\big)(1)\otimes v+1\otimes\big(d_k-\lambda^{k-n}d_n\big)(v)\\&=\big(\lambda^k\big(\partial+k(b-1)\big)-\lambda^k\big(\partial+n(b-1)\big)\big)\otimes v+1\otimes\big(\alpha_k-\lambda^{k-n}\alpha_n\big)v\\&=\big(\lambda^k(k-n)(b-1)+\big(\alpha_k-\lambda^{k-n}\alpha_n\big)\big)(1\otimes v).\end{aligned}$$

Noting (5.16), we obtain that

$$\big(d_k-\lambda^{k-n}d_n\big)(1\otimes v)=s_k(1\otimes v), \quad n+1\leq k\leq 2n+2.$$

Then

$$\begin{aligned}&\big(d_k-\lambda^{k-n}d_n\big)(1\otimes v)\\&\qquad=\big(-(k-2n-2)s_{2n+1}\lambda^{k-2n-1}+(k-2n-1)s_{2n+2}\lambda^{k-2n-2}\big)(1\otimes v),\end{aligned}$$

where $k>2n+2$. Noting that $\mathrm{Ind}_{\theta,\lambda}(\mathcal{B}_{\mathbf{s}}^{(n+1)})$ satisfies (5.1) in the case $n+1$ and has a basis

$$Q_2=\big\{d_{n-m}^{k_{n-m}}\dots d_{n-1}^{k_{n-1}}d_n^{k_n}\cdot\bar{1}: m\geq 1,\ k_{n-m},\dots,k_{n-1},k_n\in\mathbb{Z}_+\big\},$$

we see that there exists a $\mathfrak{V}$-module homomorphism from $\mathrm{Ind}_{\theta,\lambda}(\mathcal{B}_{\mathbf{s}}^{(n+1)})$ to W, whose restriction to Q_2 is a bijection from Q_2 onto Q_1. Thus this $\mathfrak{V}$-module homomorphism is an isomorphism. Hence (i) holds.

Therefore, $\mathrm{Ind}_{\theta,\lambda}(\mathcal{B}_{\mathbf{s}}^{(n+1)})$ is irreducible if and only if $\Omega(\lambda,b)\otimes L_{\psi_n,\theta}$ is irreducible; if and only if $b\neq 1$, and $\alpha_{2n-1}\neq 0$ or $\alpha_{2n}\neq 0$; and if and only if (5.14) and (5.15) hold. This implies (ii) and completes the proof. $\square$

Note that the modules $\mathrm{Ind}_{\theta,\lambda}(\mathbb{C}_{\mathbf{m}})$ defined in [MW] are just the modules $\mathrm{Ind}_{\theta,\lambda}(\mathcal{B}_{\mathbf{s}}^{(2)})$ we defined here for $\mathbf{s}=(m_2,m_3,m_4)$.

Now we will characterize the submodules of $\mathrm{Ind}_{\theta,\lambda}(\mathcal{B}_{\mathbf{s}}^{(n)})$. Because of Theorems 7, 8 and 9, it is enough to consider submodules of $\Omega(\lambda,b)\otimes V$ where V is determined by Theorems 7, 8, 9, which is the following

Theorem 10. *Let $n\in\mathbb{N}$, $\lambda\in\mathbb{C}^*$, $b,\theta\in\mathbb{C}$, and let V be a highest weight module or $L_{\psi_n,\theta}$ over $\mathfrak{V}$.*

(i) *If $b\neq 1$, then each submodule M of $\Omega(\lambda,b)\otimes V$ is of the form $\Omega(\lambda,b)\otimes X$ for some submodule X of V.*

(ii) *If $b=1$, then each submodule M of $\Omega(\lambda,b)\otimes V$ is of the form $\partial\Omega(\lambda,1)\otimes X_1+\Omega(\lambda,1)\otimes X_2$ where X_1 and X_2 are submodules of V.*

Proof. (i) $b\neq 1$.

Then $\Omega(\lambda,b)$ is an irreducible $\mathfrak{V}$-module. Let Y be a nonzero submodule of $\Omega(\lambda,b)\otimes V$.

Claim 1. *If $\sum_{j=0}^{s}\partial^j\otimes v_j\in Y$, where $v_j\in V$, then $\Omega(\lambda,b)\otimes v_j\subset Y$ for all $j=0,1,\dots,s$.*

Using the same arguments as in the proof of Theorem 1 we can deduce that $\Omega(\lambda,b)\otimes v_s\subset Y$ and hence $\Omega(\lambda,b)\otimes v_j\subset Y$, $j=0,1,\dots,s$, by induction on j.

Claim 2. *$Y=\Omega(\lambda,b)\otimes X$, where X is a submodule of V.*

Let X be the maximal subspace of V satisfying $\Omega(\lambda,b)\otimes X\subset Y$. The maximality of X forces that X is a submodule of V. Using Claim 1 we see that $\Omega(\lambda,b)\otimes X=Y$.

Thus (i) follows.

(ii) Now consider the case $b=1$.

Let Z be a submodule of $\Omega(\lambda,b)\otimes V$. Take a nonzero $w=\sum_{j=0}^{s}\partial^j\otimes v_j\in Z$ where $v_j\in V$.

Claim 3. *$\Omega(\lambda,1)\otimes v_0\subseteq Z$ and $\partial\Omega(\lambda,1)\otimes v_j\subset Z$ for all $j\geq 1$.*

We will prove this by induction on s. This is true for $s=0$ by simple computations. Now suppose $s>0$. As in the proof of Theorem 1, we can take $K=\max\{K(v_j):j=0,1,...,s\}$ such that $d_l\cdot v_j=0$ for all $l\geq K$ and $j=0,1,...,s$. Then

$$d_l\cdot w=\sum_{j=0}^{s}\lambda^l\partial(\partial-l)^j\otimes v_j\in Z,\quad \forall l\geq K.$$

Since the coefficient of l^s is $(-1)^l\lambda^l\partial\otimes v_s$ which has to be in Z, by simple computations we deduce that $\partial\Omega(\lambda,1)\otimes v_s\subset Z$. Claim 3 follows.

Let S_Z be the set consisting of the submodules of Z in the form $\Omega(\lambda,1)\otimes H_1+\partial\Omega(\lambda,1)\otimes H_2$ where H_1, H_2 are subspaces of V. Note that the $U(\mathfrak{V})v_j$ are subspaces (actually, submodules) of V. By Claim 3 we can easily deduce that $\Omega(\lambda,1)\otimes U(\mathfrak{V})v_0$ and $\partial\Omega(\lambda,1)\otimes U(\mathfrak{V})v_j$, $1\leq j\leq s$ are submodules of Z. Then for each $1\leq j\leq s$, we have $\Omega(\lambda,1)\otimes U(\mathfrak{V})v_0+\partial\Omega(\lambda,1)\otimes U(\mathfrak{V})v_j\in S_Z\neq\varnothing$. Let X_1, X_2 be two subspaces of V such that $\Omega(\lambda,1)\otimes X_1+\partial\Omega(\lambda,1)\otimes X_2$ is a maximal element of S_Z. It is easy to see that X_1 and X_2 are submodules of V. Using again Claim 3 it is not hard to deduce that $Z=\Omega(\lambda,1)\otimes X_1+\partial\Omega(\lambda,1)\otimes X_2$. This is (ii) and completes the proof. □

Note that when $\theta=0$, the modules $\mathrm{Ind}_{\theta,\lambda}(\mathcal{B}_{\mathbf{s}}^{(0)})$ are exactly the highest weight-like modules defined in [GLZ]. Now we can answer the open problem in [GLZ]: if $\mathbf{s}=s_0\in\mathbb{C}^*$, whether or not $\mathrm{Ind}_{0,\lambda}(\mathcal{B}_{\mathbf{s}}^{(0)})$ has a unique maximal submodule. From (i) of Theorem 10 we know that the maximal submodules of $\mathrm{Ind}_{\theta,\lambda}(\mathcal{B}_{\mathbf{s}}^{(0)})$ correspond to the maximal submodules of $M(\theta,0)$. Since $M(\theta,0)$ is a highest weight module, it has a unique maximal submodule, so does $\mathrm{Ind}_{\theta,\lambda}(\mathcal{B}_{\mathbf{s}}^{(0)})$. In the special case $\theta=0$, the conclusion is certainly true. Therefore, we have the following

Corollary 11. *Let $\lambda,\mathbf{s}=s_0\in\mathbb{C}^*$. Then $\mathrm{Ind}_{0,\lambda}(\mathcal{B}_{\mathbf{s}}^{(0)})$ has a unique maximal submodule.*

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Haijun Tan
College of Mathematics and
Information Science
Hebei Normal (Teachers) University
Shijiazhuang
Hebei CN-050016
P.R. China
and
Department of Applied Mathematics
Changchun University of Science
and Technology
Changchun
Jilin CN-130022
P.R. China
hjtan9999@yahoo.com

Kaiming Zhao
Department of Mathematics
Wilfrid Laurier University
Waterloo
ON N2L 3C5
Canada
and
Department of Mathematics
Xinyang Normal University
Xinyang
Henan CN-464000
P.R. China
kzhao@wlu.ca

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