# *isl*: An Integer Set Library for the Polyhedral Model

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# Outline

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1. Introduction
2. Internals
3. Operations
   - Set Difference
   - Set Coalescing
   - Parametric Vertex Enumeration
   - Bounds on Quasi-Polynomials
4. Conclusion

# An Integer Set Library

`isl` is an **LGPL thread-safe C** library for manipulating
**sets and relations** of **integer tuples** bounded by **affine constraints**

$\rightsquigarrow$ finite unions of projections of parametric lattice polytopes

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`isl` is an **LGPL thread-safe C** library for manipulating
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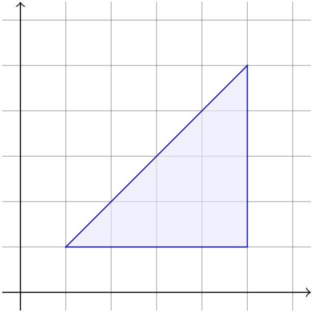
$\rightsquigarrow$ finite unions of projections of parametric lattice polytopes

- very similar to `Omega` and `Omega+` libraries
- similar to `polymake`, but different focus/philosophy
  - integer values instead of rational values
  - designed for the polyhedral model for program analysis and transformation (but also useful for other applications)
  - library ("calculator" interface is available too)
    $\Rightarrow$ embeddable in a compiler
  - works best on sets of small dimensions (up to about 10; some operations also work for higher dimensions)
  - self-contained (apart from GMP)
  - closed representation
  - objects may be sets or relations (or piecewise quasipolynomials)

# Examples of Sets and Relations

$$S = \{\,(x, y) \mid 1 \leq y \leq x \leq 5\,\}$$



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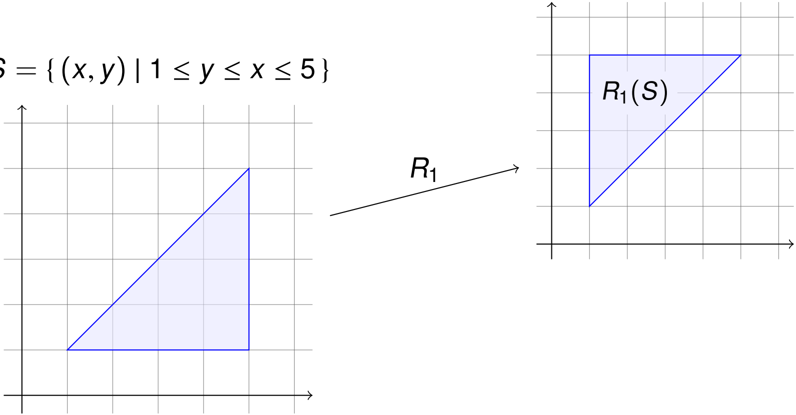


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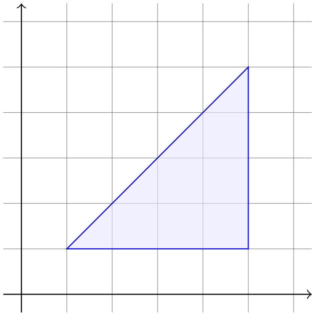

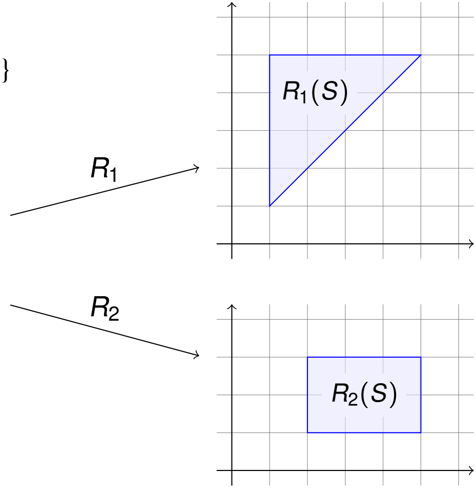


$$R_1 = \{\, (x, y) \to (y, x) \,\} = \{\, (x, y) \to (x', y') \mid x' = y \wedge y' = x \,\}$$
$$R_2 = \{\, (x, y) \to (x, y') \mid x \geq 2 \wedge 1 \leq y' \leq 3 \,\}$$

# Sets and Relations in the Polyhedral Model

```
for (i = 0; i < n; ++i)
    for (j = 0; j < i; ++j)
        f(a[j][i+j][2*i]);
```

Typical sets and relations

- Iteration domain
  ⇒ set of all possible values of the iterators

$$n \to \{\, (i, j) \mid 0 \le i < n \wedge 0 \le j < i \,\}$$

- Access relation
  ⇒ maps iteration vector to array index

$$\{\, (i, j) \to (j, i + j, 2i) \,\}$$

# Comparison to Related Libraries

- Compared to double description based libraries (`PolyLib`, `PPL`)
  - All operations are performed on constraints

    Reason: objects in target application domain usually have few constraints, but may have many vertices
  - Full support for parameters
  - Built-in support for existentially quantified variables
  - Built-in support for relations
  - Focus on integer values

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  - Full support for parameters
  - Built-in support for existentially quantified variables
  - Built-in support for relations
  - Focus on integer values
- Compared to `Omega` and `Omega+`
  - All operations are performed in arbitrary integer arithmetic using GMP
  - Different way of handling existentially quantified variables
  - Named and nested spaces
  - Parametric vertex enumeration

    ⇒ useful for the `barvinok` counting library and for computing bounds
  - Support for piecewise quasipolynomials

    ⇒ results of counting problems

# Interaction with Other Libraries and Tools

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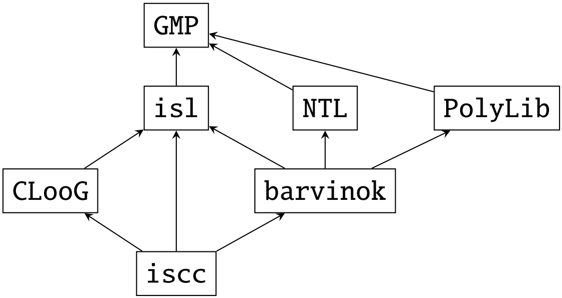
CLooG: generates code to scan elements in parametric affine sets
iscc: interactive isl calculator (included in barvinok distribution)



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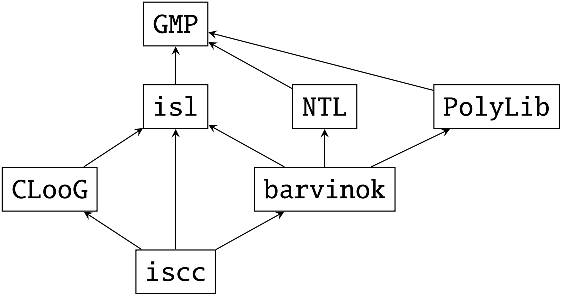


Future work:

- remove dependence on PolyLib and NTL

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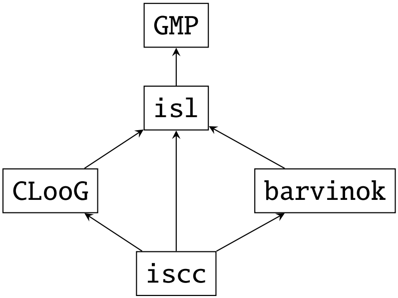


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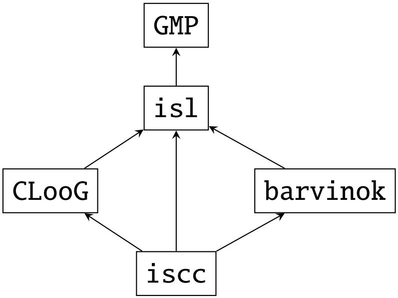
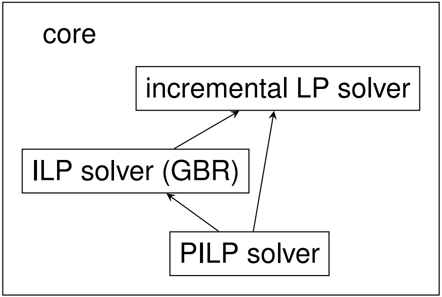



Future work:

- remove dependence on `PolyLib` and `NTL`
- merge `barvinok` into `isl`

# Outline

# Internal Structure



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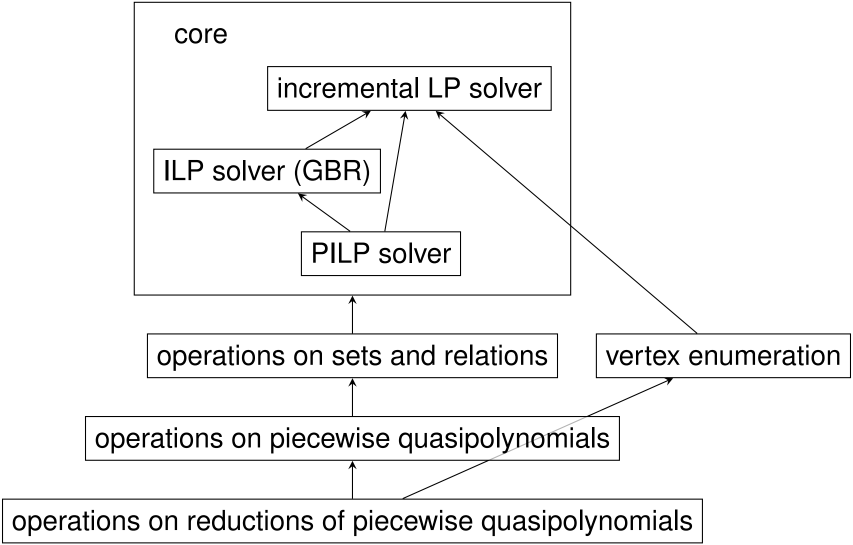

# Internal Representation

$$S(\mathbf{s}) = \{\, \mathbf{x} \in \mathbb{Z}^d \mid \exists \mathbf{z} \in \mathbb{Z}^e : A\mathbf{x} + B\mathbf{s} + D\mathbf{z} \geq \mathbf{c} \,\}$$

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- “basic” types: “convex” sets and maps (relations)
  - equality + inequality constraints
  - parameters **s**
  - (optional) explicit representation of existentially quantified variables as integer divisions
    - ⇒ useful for aligning dimensions when performing set operations (e.g., set difference)
    - ⇒ can be computed using PILP

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    - $\Rightarrow$ can be computed using PILP
- sets and maps
  - $\Rightarrow$ (disjoint) unions of basic sets/maps
- union sets and union maps
  - $\Rightarrow$ unions of sets/maps in different spaces

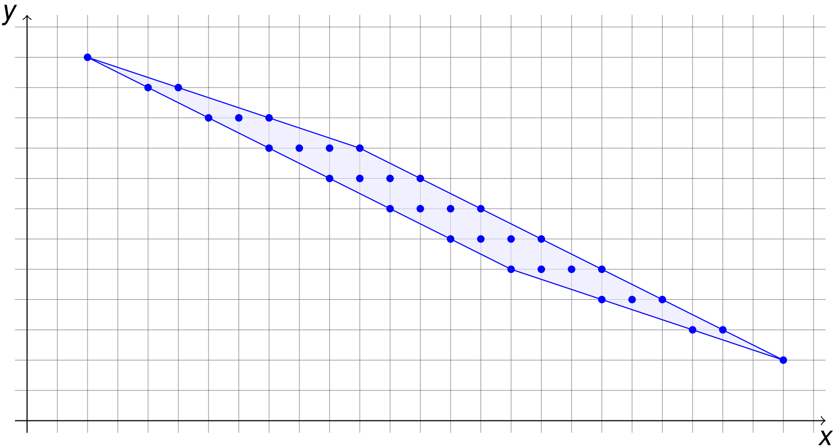
# Parametric Integer Linear Programming

$$R(\mathbf{s}) = \{\, \mathbf{x}_1 \to \mathbf{x}_2 \in \mathbb{Z}^{d_1} \times \mathbb{Z}^{d_2} \mid \exists \mathbf{z} \in \mathbb{Z}^{e} : A_1\mathbf{x}_1 + A_2\mathbf{x}_2 + B\mathbf{s} + D\mathbf{z} \geq \mathbf{c} \,\}$$

Lexicographic minimum of $R$:

$$\operatorname{lexmin} R = \{\, \mathbf{x}_1 \to \mathbf{x}_2 \in R \mid \forall \mathbf{x}_2' \in R(\mathbf{s}, \mathbf{x}_1) : \mathbf{x}_2 \preccurlyeq \mathbf{x}_2' \,\}$$

# Parametric Integer Linear Programming Example

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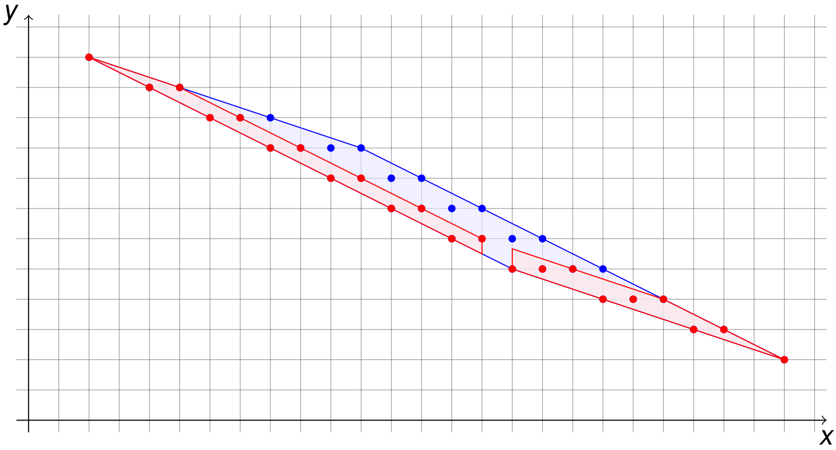


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$\text{lexmin}\, R = \{ x \to y \mid (x \leq 25 \land x \geq 16 \land 3y \geq 31 - x \land 3y \leq 33 - x \land 2y \leq 29 - x) \lor (3y \leq 38 - x \land x \leq 15 \land x \geq 2 \land 2y \geq 26 - x \land 2y \leq 27 - x) \}$

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Parametric integer linear programming computes $\operatorname{lexmin} R$ in the form

$$\begin{aligned} \operatorname{lexmin} R = \bigcup_i \{\, \mathbf{x}_1 \to \mathbf{x}_2 \in \mathbb{Z}^{d_1} \times \mathbb{Z}^{d_2} \mid \exists \mathbf{z}' \in \mathbb{Z}^{e'} : A_i\mathbf{x}_1 + B_i\mathbf{s} \geq \mathbf{c}_i \wedge \\ \mathbf{z}' = \left\lfloor \frac{P_i\mathbf{x}_1 + Q_i\mathbf{s} + \mathbf{r}_i}{m} \right\rfloor \wedge \\ \mathbf{x}_2 = T_i\mathbf{x}_1 + U_i\mathbf{s} + V_i\mathbf{z}' + \mathbf{w}_i \,\} \end{aligned}$$

- explicit representation of existentially quantified variables
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Technique: dual simplex + Gomory cuts

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# PILP Example: Dataflow Analysis

*Given a read from an array element, what was the last write to the same array element before the read?*

Simple case: array written through a single access

```
for (i = 0; i < N; ++i)
    for (j = 0; j < N - i; ++j)
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Access relations:

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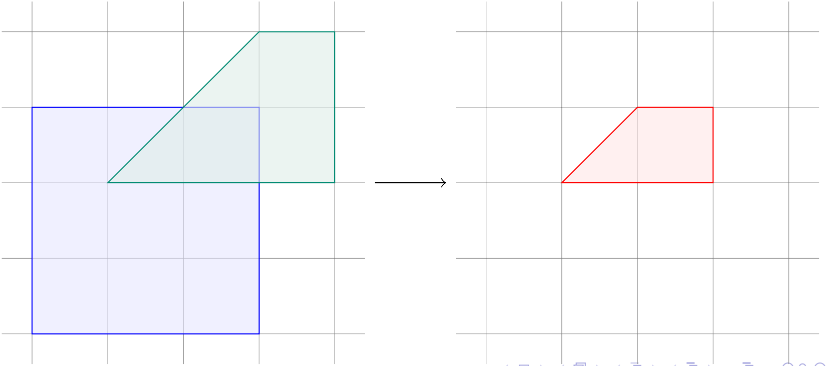
In general: impose lexicographical order on shared iterators

# Outline

# Supported Operations

- Intersection



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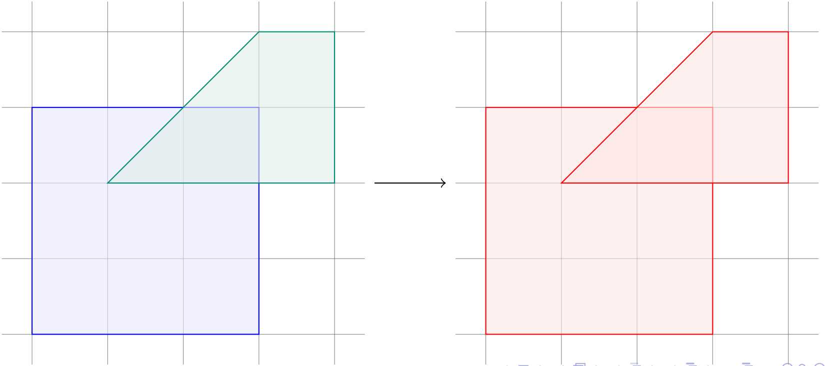
- Intersection
- Union



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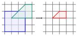
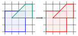
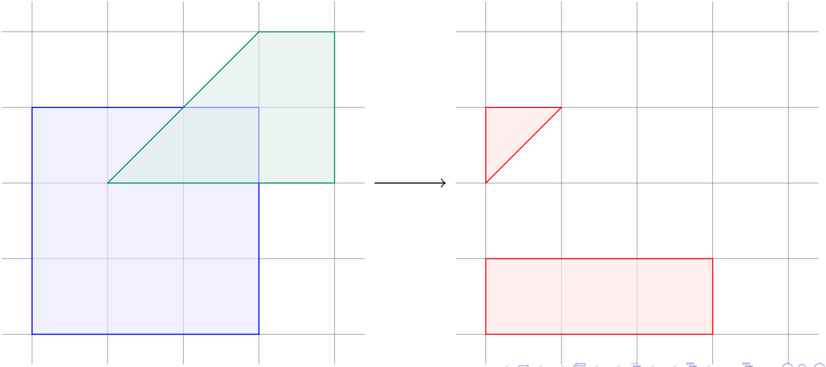
- Intersection 
- Union 
- Set difference



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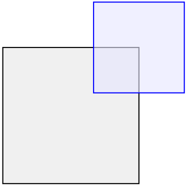
Set difference $S_1 \setminus S_2$

- no existentially quantified variables

$$S_2(\mathbf{s}) = \{\, \mathbf{x} \in \mathbb{Z}^d \mid \bigwedge_i \langle \mathbf{a}_i, \mathbf{x} \rangle + \langle \mathbf{b}_i, \mathbf{s} \rangle \geq c_i \,\}$$

$$S_1 \setminus S_2 = \bigcup_i (S_1 \cap \{\, \mathbf{x} \mid \neg(\langle \mathbf{a}_i, \mathbf{x} \rangle + \langle \mathbf{b}_i, \mathbf{s} \rangle \geq c_i) \,\})$$



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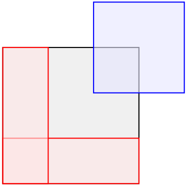
$$= \bigcup_i \left( S_1 \cap \{\, \mathbf{x} \mid \langle \mathbf{a}_i, \mathbf{x} \rangle + \langle \mathbf{b}_i, \mathbf{s} \rangle \le c_i - 1 \,\} \right)$$



# isl Operation: Set Difference

$$S(\mathbf{s}) = \{\, \mathbf{x} \in \mathbb{Z}^d \mid \exists \mathbf{z} \in \mathbb{Z}^e : A\mathbf{x} + B\mathbf{s} + D\mathbf{z} \geq \mathbf{c} \,\}$$

Set difference $S_1 \setminus S_2$

- no existentially quantified variables

$$S_2(\mathbf{s}) = \{\, \mathbf{x} \in \mathbb{Z}^d \mid \bigwedge_i \langle \mathbf{a}_i, \mathbf{x} \rangle + \langle \mathbf{b}_i, \mathbf{s} \rangle \geq c_i \,\}$$

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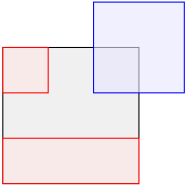

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- with existentially quantified variables
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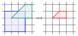
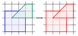
$$S_2(\mathbf{s}) = \{\, \mathbf{x} \in \mathbb{Z}^d \mid \bigwedge_i \langle \mathbf{a}_i, \mathbf{x} \rangle + \langle \mathbf{b}_i, \mathbf{s} \rangle + \left\langle \mathbf{d}_i, \left\lfloor \frac{\langle \mathbf{p}, \mathbf{x} \rangle + \langle \mathbf{q}_i, \mathbf{s} \rangle + r}{m} \right\rfloor \right\rangle \geq c_i \,\}$$

# Supported Operations

- Intersection 
- Union 
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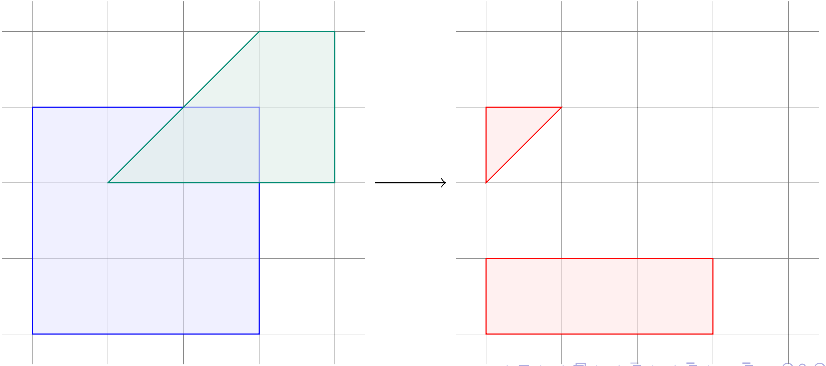

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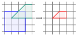
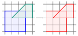
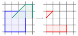
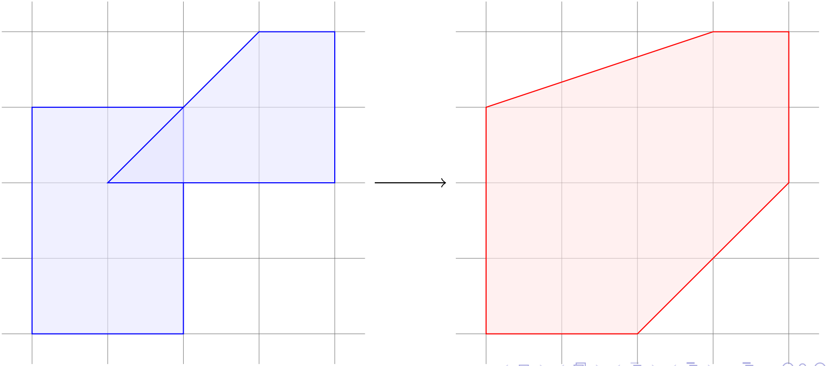
- Intersection 
- Union 
- Set difference 
- Closed convex hull ("wrapping", FLL2000)



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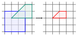
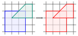
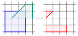
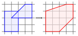
- Intersection 
- Union 
- Set difference 
- Closed convex hull (“wrapping”, FLL2000) 
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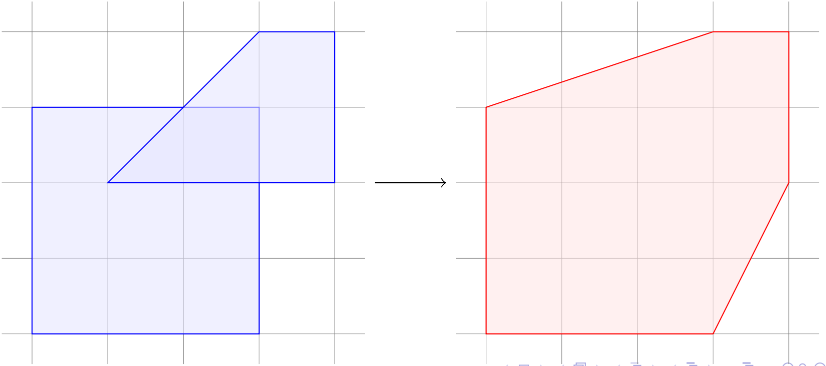

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After many applications of projection, set difference, union,
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PolyLib way:

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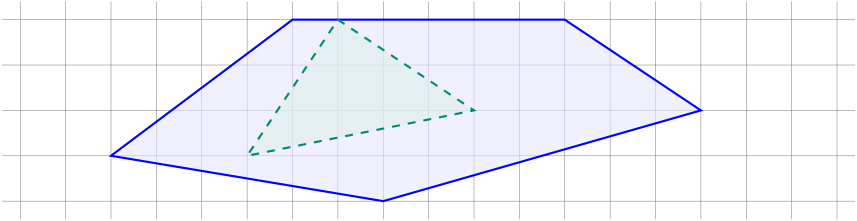

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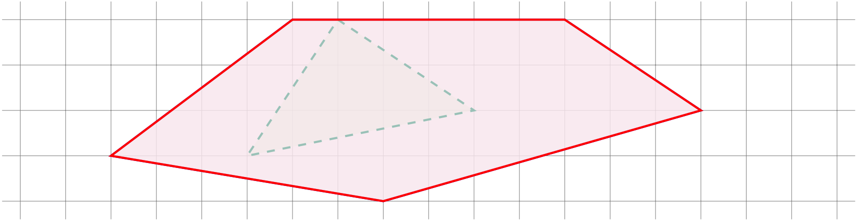

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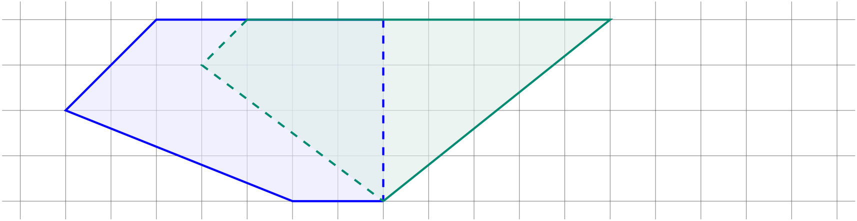

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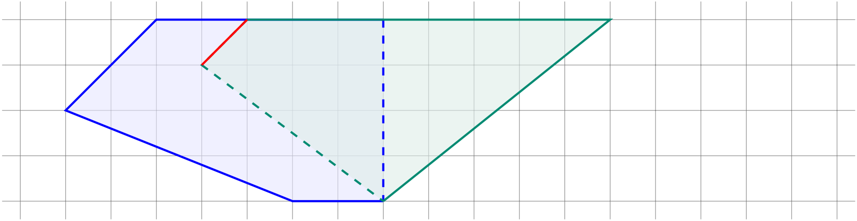

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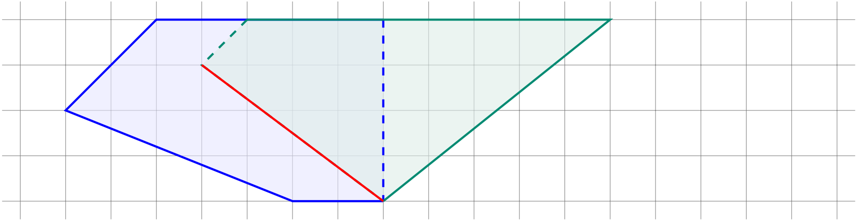

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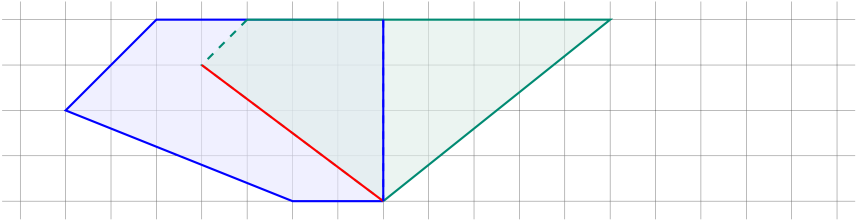

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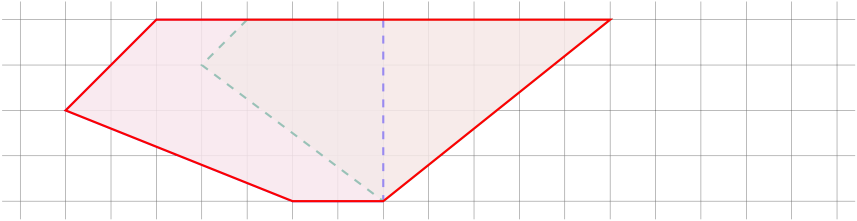

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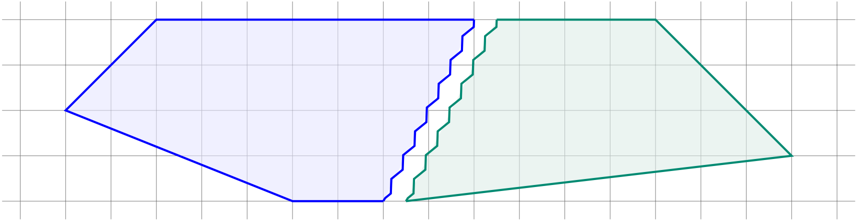

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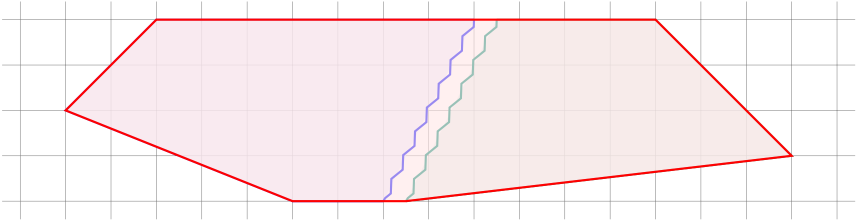

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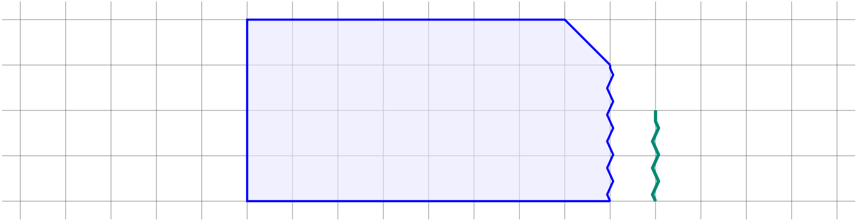

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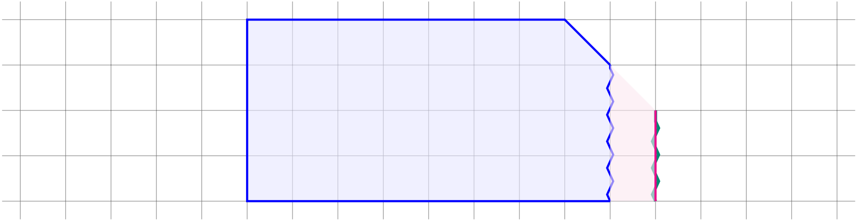

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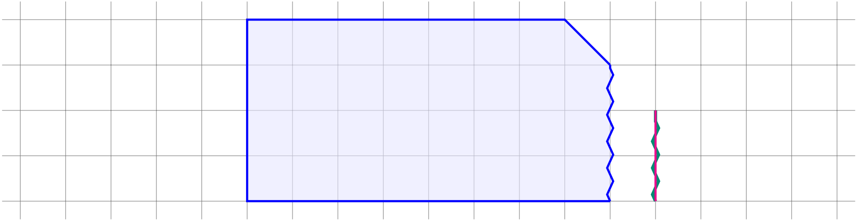

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   3. single pair of adjacent inequalities (other constraints valid)
   4. single adjacent pair of an inequality ($S_1$) and an equality ($S_2$)
      + other constraints of $S_1$ are valid
      + constraints of $S_2$ valid for facet of relaxed inequality
      $\Rightarrow$ drop $S_2$ and relax adjacent inequality of $S_1$

# isl Operation: Set Coalescing

1. Classify constraints
   - redundant: $\min \langle \mathbf{a}_i, \mathbf{x} \rangle > c_i - 1$ over remaining constraints
   - valid: $\min \langle \mathbf{a}_i, \mathbf{x} \rangle > c_i - 1$ over $S_2$
   - separating: $\max \langle \mathbf{a}_i, \mathbf{x} \rangle < c_i$ over $S_2$
     special cases:
     - adjacent to equality: $\langle \mathbf{a}_i, \mathbf{x} \rangle = c_i - 1$ over $S_2$
     - adjacent to inequality: $\left\langle (\mathbf{a}_i + \mathbf{b}_j), \mathbf{x} \right\rangle = (c_i + d_j) - 1$ over $S_2$
   - cut: otherwise
2. Case distinction
   1. non-redundant constraints of $S_1$ are valid for $S_2$, i.e., $S_2 \subseteq S_1$
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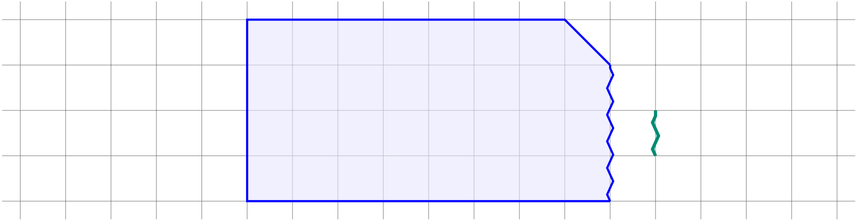
   5. single adjacent pair of an inequality ($S_1$) and an equality ($S_2$)
      + other constraints of $S_1$ are valid
      + inequality and equality can be wrapped to include union
      $\Rightarrow$ replace $S_1$ and $S_2$ by valid and wrapping constraints

# isl Operation: Set Coalescing



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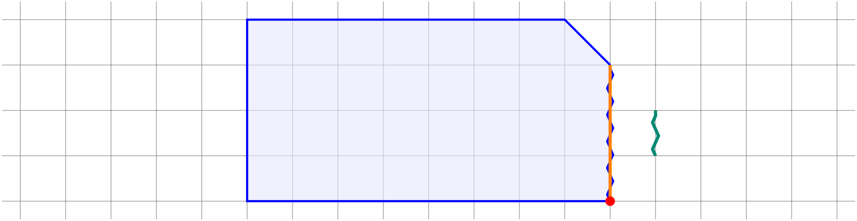

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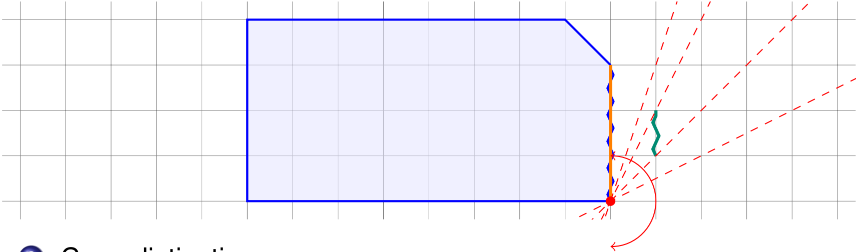

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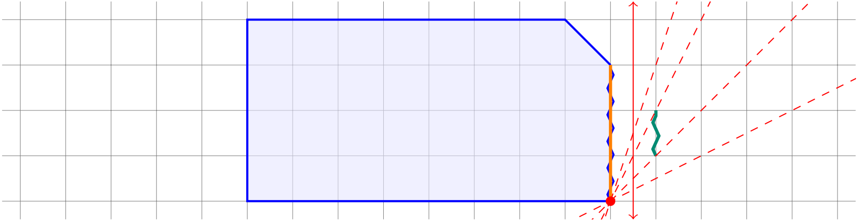

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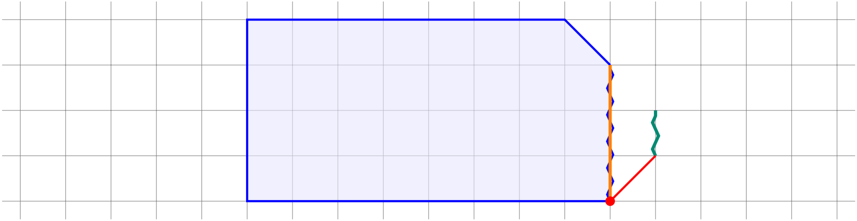

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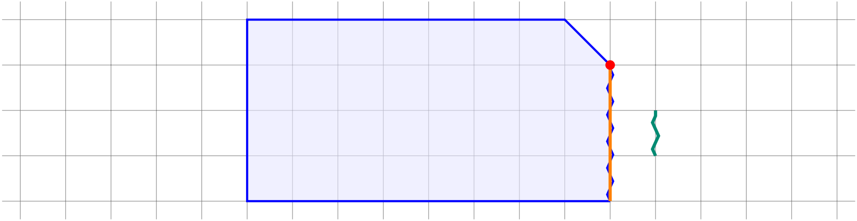

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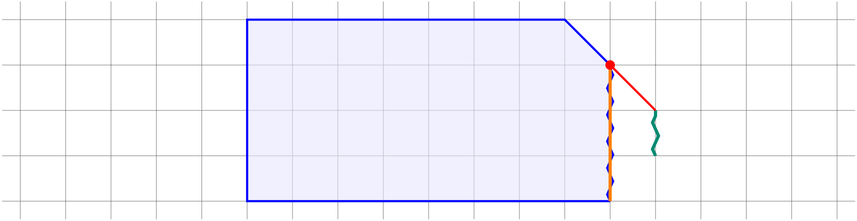

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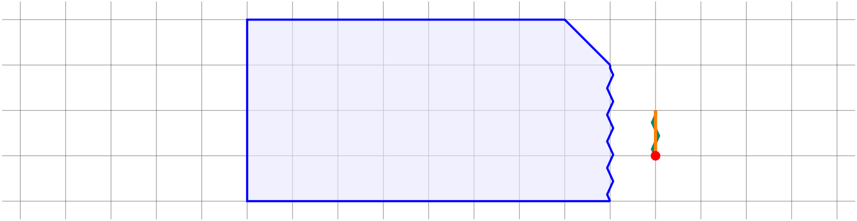

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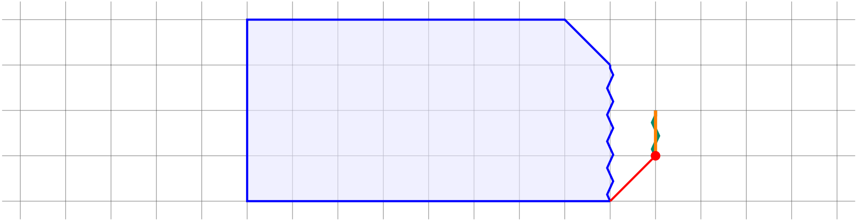

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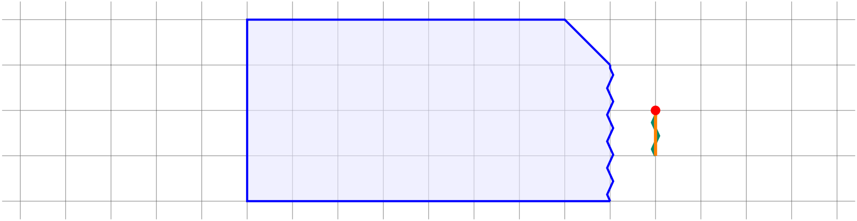

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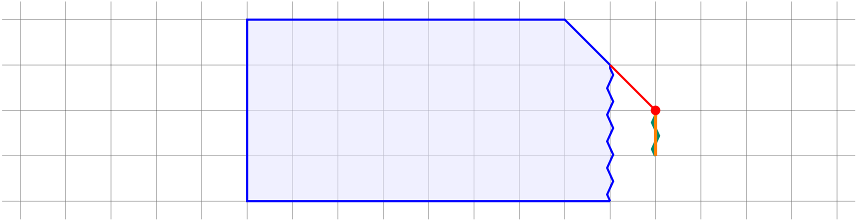

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# isl Operation: Set Coalescing

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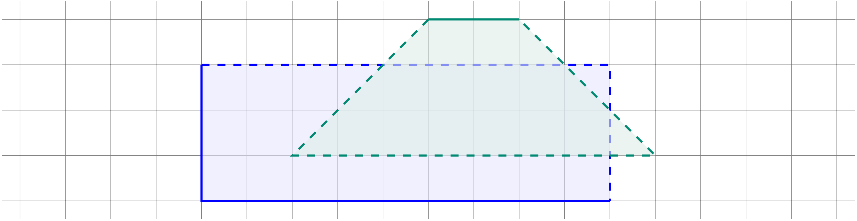

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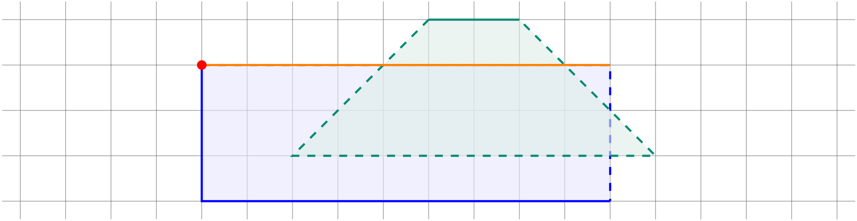

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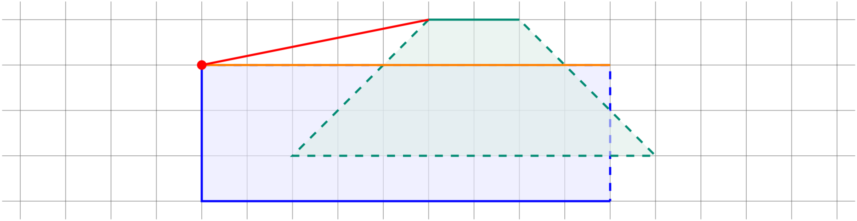

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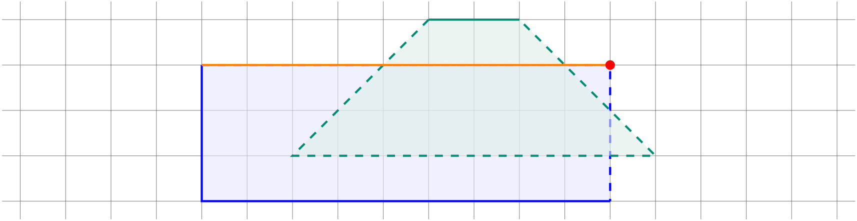

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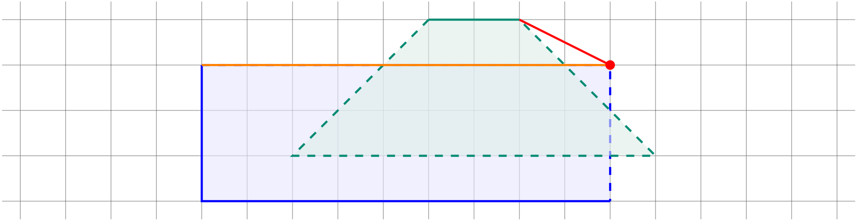

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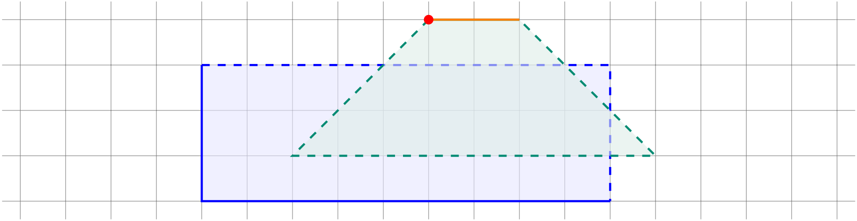

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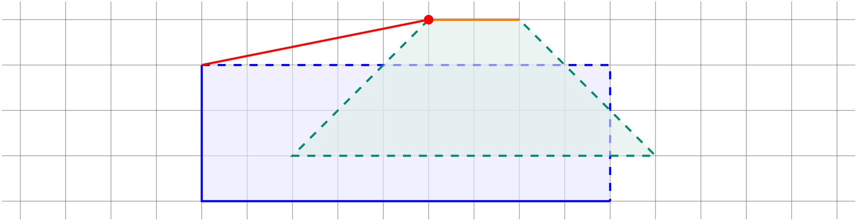

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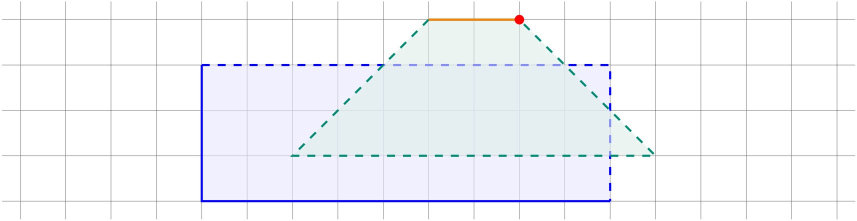

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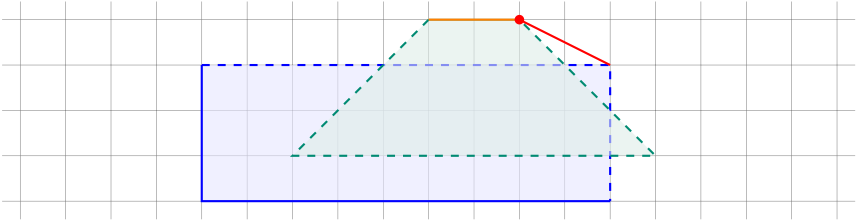

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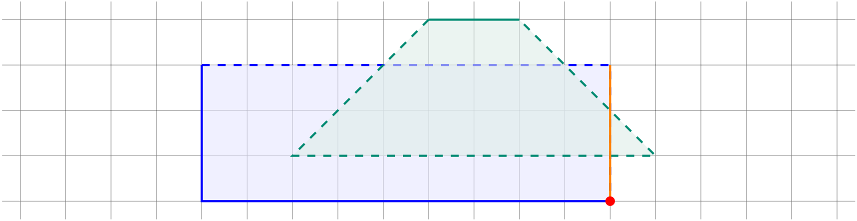

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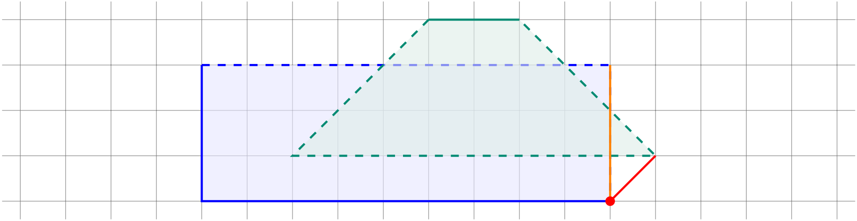
# isl Operation: Set Coalescing



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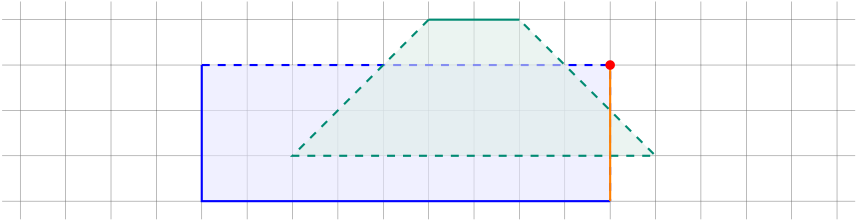

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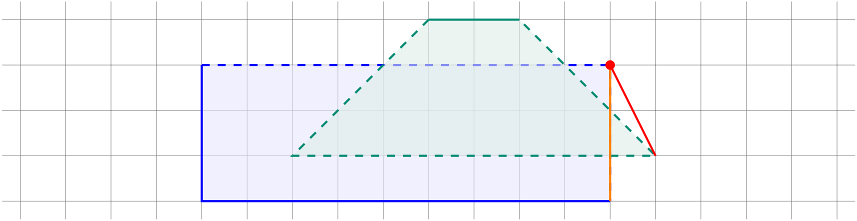

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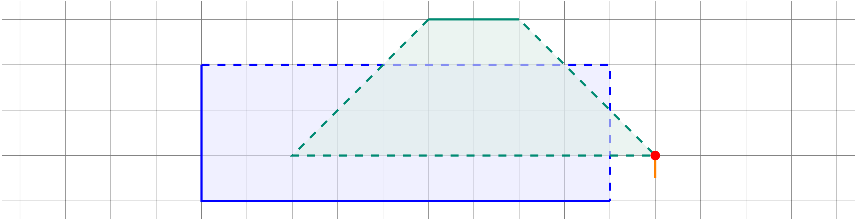

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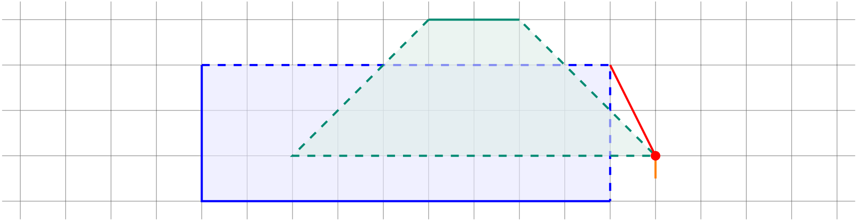

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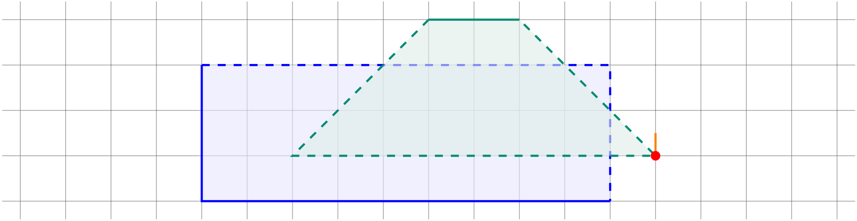

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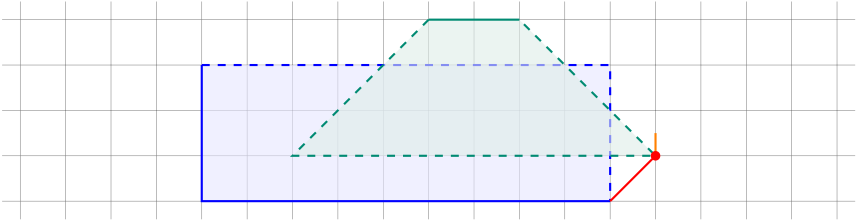

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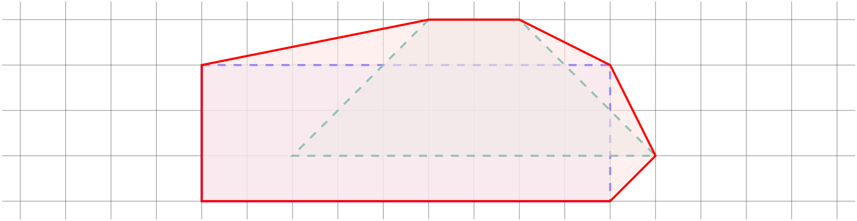

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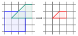
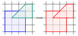
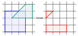
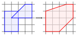
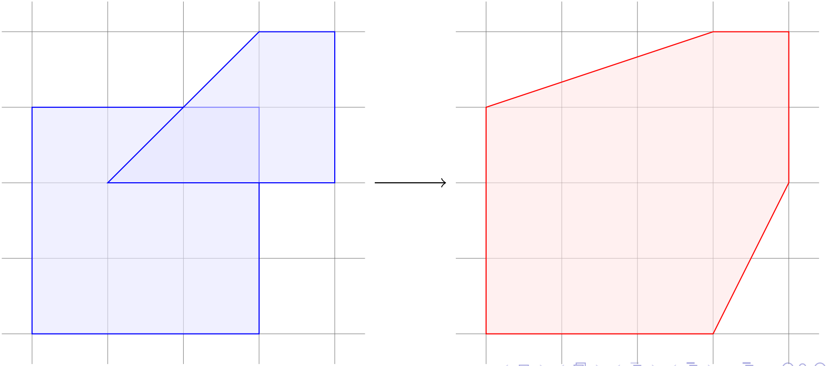
# Supported Operations

- Intersection 
- Union 
- Set difference 
- Closed convex hull (“wrapping”, FLL2000) 
- Coalescing



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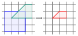
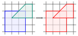
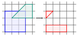
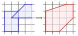
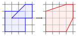
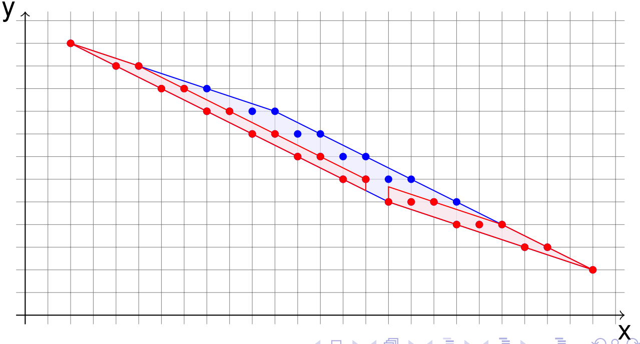
- Intersection 
- Union 
- Set difference 
- Closed convex hull (“wrapping”, FLL2000) 
- Coalescing 
- Lexicographic minimization



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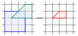
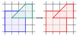
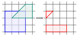
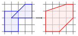
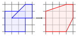
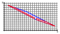
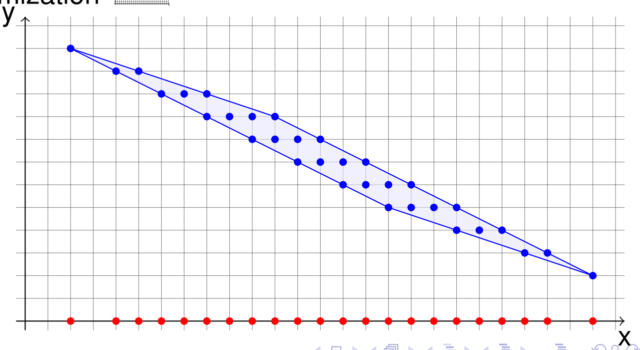
- Intersection 
- Union 
- Set difference 
- Closed convex hull (“wrapping”, FLL2000) 
- Coalescing 
- Lexicographic minimization 
- Integer projection



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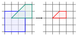
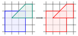
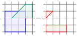
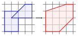
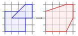
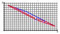
- Intersection 
- Union 
- Set difference 
- Closed convex hull ("wrapping", FLL2000) 
- Coalescing 
- Lexicographic minimization 
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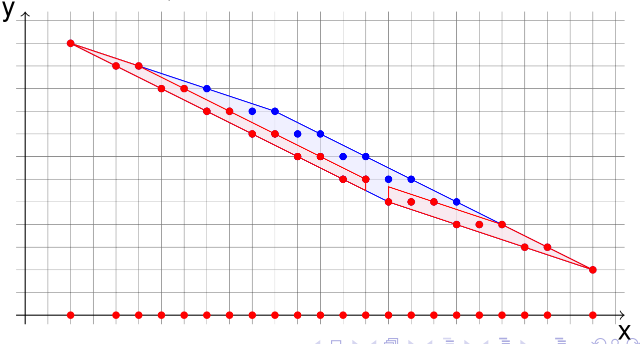

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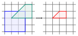
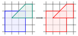
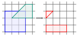
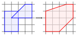
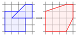
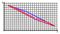
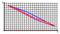
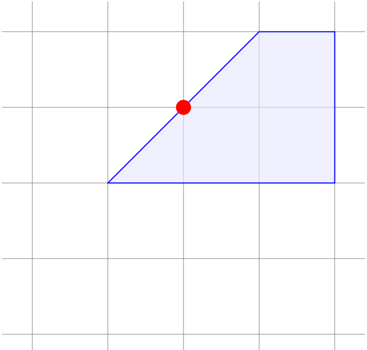
- Intersection 
- Union 
- Set difference 
- Closed convex hull (“wrapping”, FLL2000) 
- Coalescing 
- Lexicographic minimization 
- Integer projection 
- Sampling (GBR)



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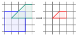
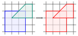
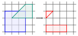
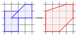
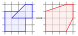
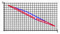
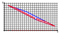
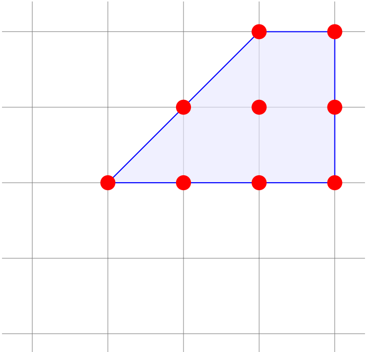
- Intersection 
- Union 
- Set difference 
- Closed convex hull (“wrapping”, FLL2000) 
- Coalescing 
- Lexicographic minimization 
- Integer projection 
- Sampling (GBR) 
- Scanning (GBR)



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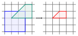
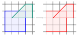
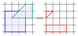
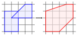
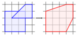
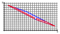
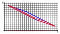
- Intersection 
- Union 
- Set difference 
- Closed convex hull (“wrapping”, FLL2000) 
- Coalescing 
- Lexicographic minimization 
- Integer projection 
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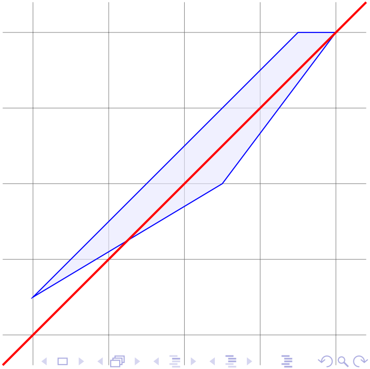

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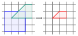
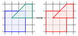
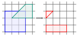
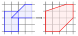
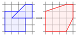
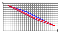
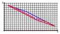
- Intersection 
- Union 
- Set difference 
- Closed convex hull (“wrapping”, FLL2000) 
- Coalescing 
- Lexicographic minimization 
- Integer projection 
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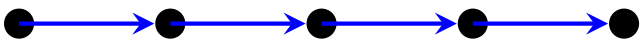
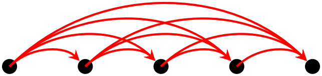


$\{ x \to x + 1 \mid 0 \le x \le 3 \}$



$\{ x \to y \mid 0 \le x < y \le 4 \}$

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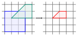
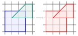
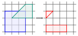
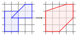
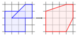
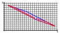
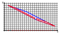
- Intersection 
- Union 
- Set difference 
- Closed convex hull (“wrapping”, FLL2000) 
- Coalescing 
- Lexicographic minimization 
- Integer projection 
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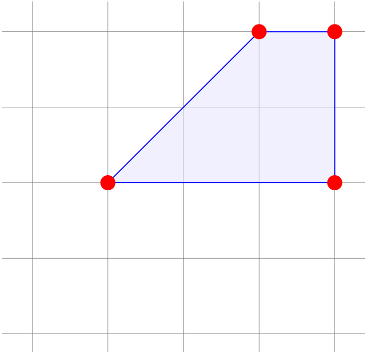

# $\mathcal{H}$-Parametric Polytopes and their Vertices

Polytopes described by hyperplanes that depend linearly on parameters

$$P(\mathbf{s}) = \{\, \mathbf{x} \in \mathbb{Q}^d \mid A\mathbf{x} + B\mathbf{s} \geq \mathbf{c} \,\}$$

Example:

$$P(N) = \{\, (i,j) \mid i \geq 1 \wedge i \leq N \wedge j \geq 1 \wedge j \leq i \,\}$$

Parametric vertices:

$$P = \text{conv.hull} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} N \\ 1 \end{bmatrix}, \begin{bmatrix} N \\ N \end{bmatrix} \right\}$$

In general: different (active) vertices on different parts of the parameter space (chamber decomposition)

# Chamber Decomposition

$$\left\{ \mathbf{t} \in \mathbb{Q}^2 \mid -s_1 + 2s_2 + t_1 - 2t_2 \geq 0 \wedge s_1 - s_2 - t_1 + t_2 \geq 0 \wedge t_1 \geq 0 \wedge t_2 \geq 0 \right\}$$

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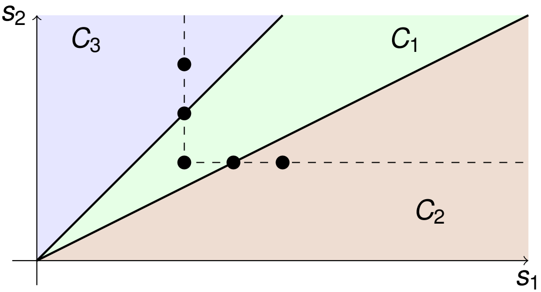

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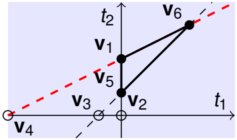


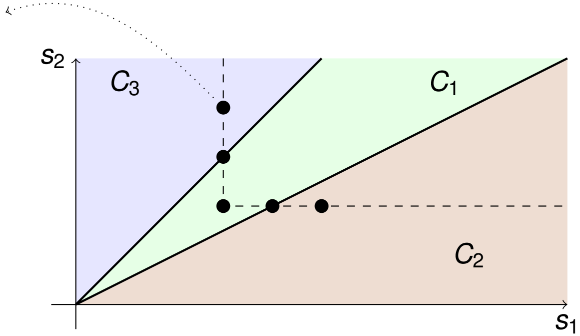

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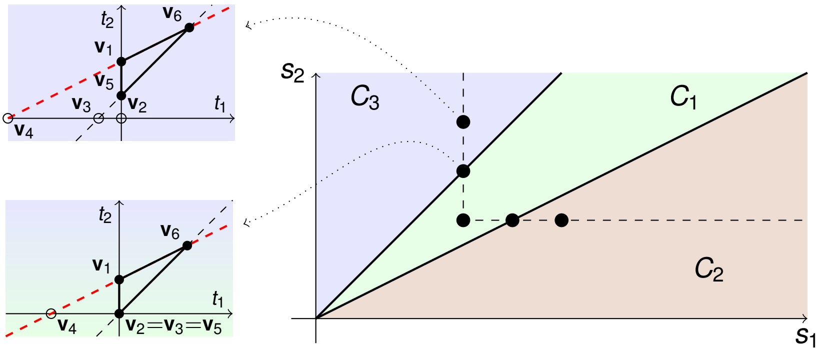

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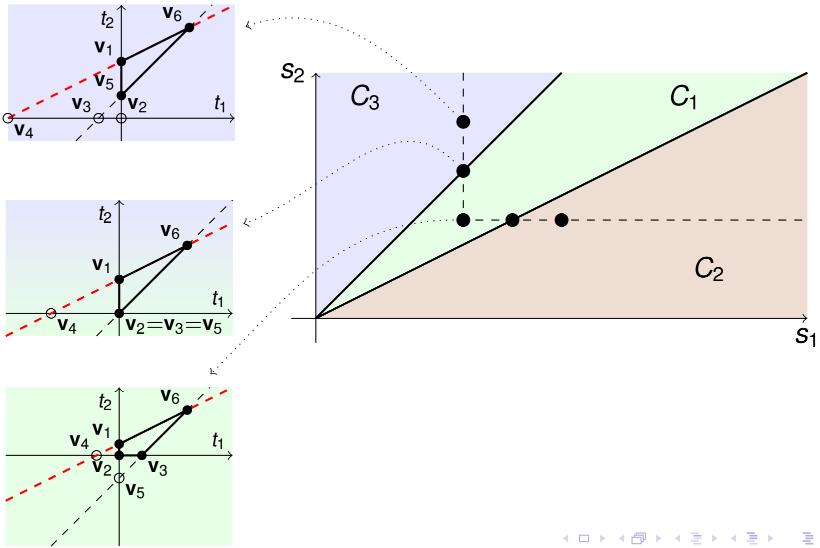

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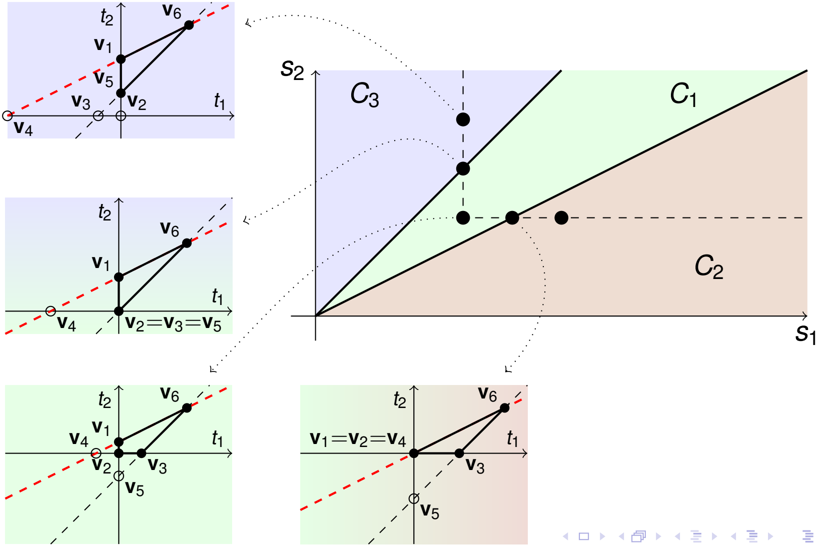

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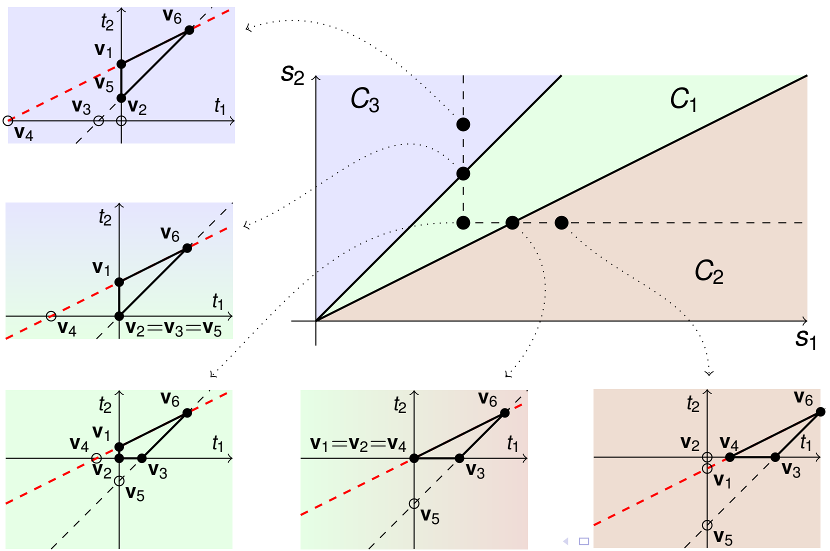

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  - Consider all combinations of $d$ inequalities
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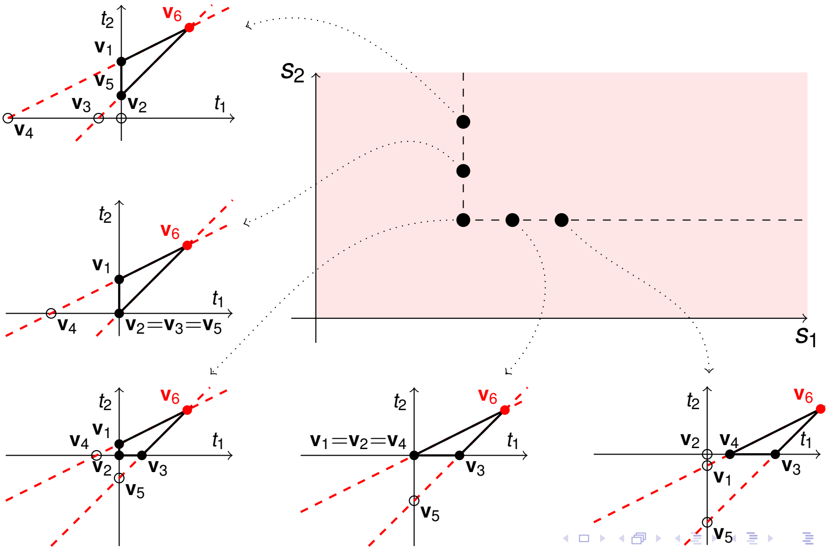

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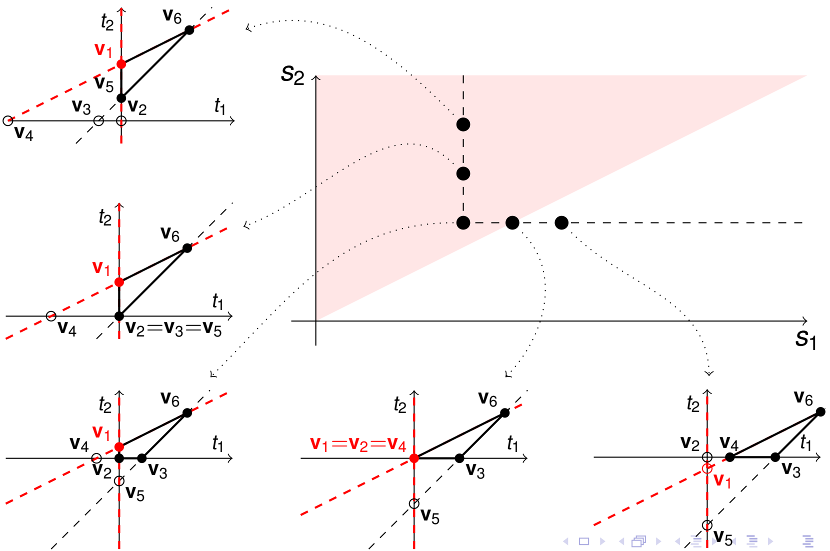

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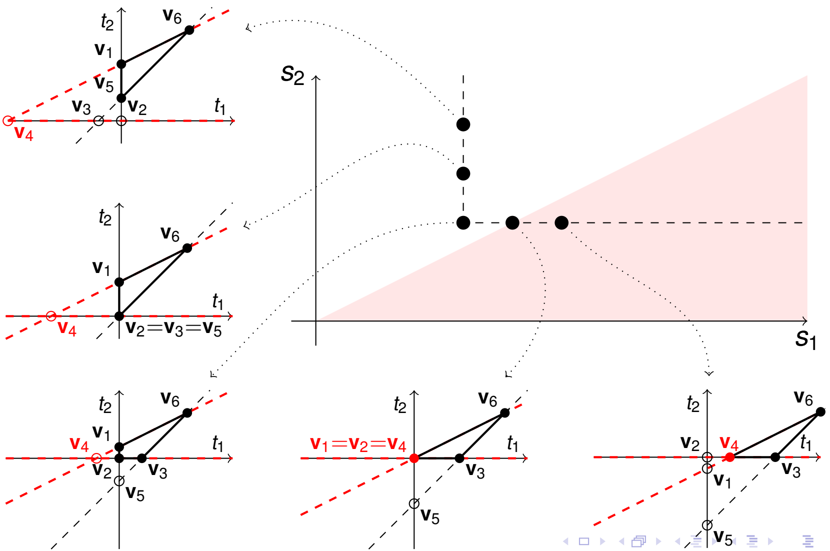

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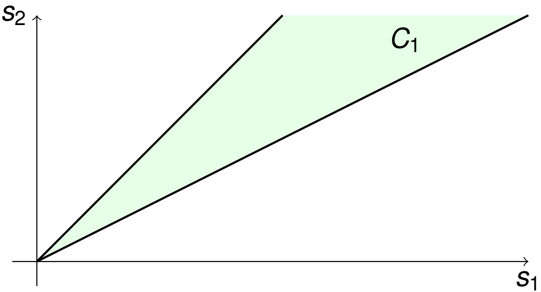

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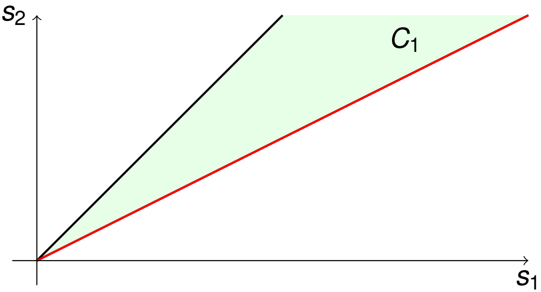

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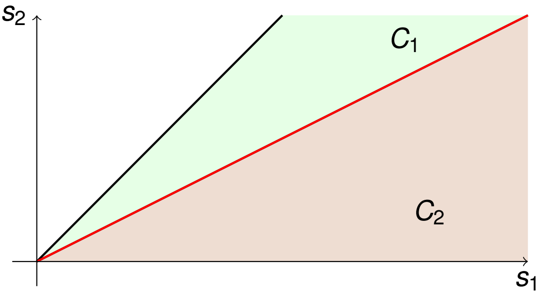

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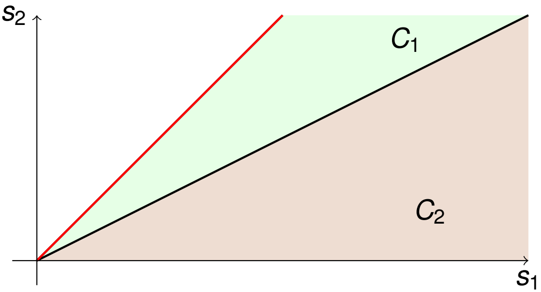

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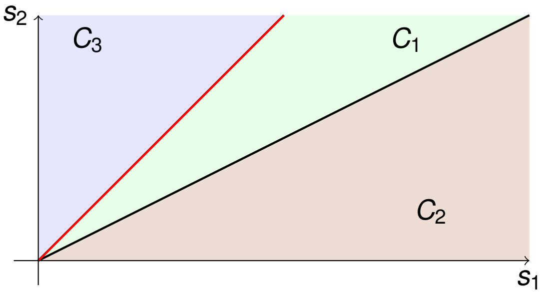

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  $\Rightarrow$ much faster than PolyLib; similar to TOPCOM 0.16.2

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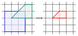
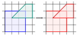
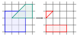
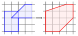
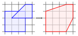
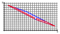
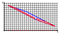
- Intersection 
- Union 
- Set difference 
- Closed convex hull (“wrapping”, FLL2000) 
- Coalescing 
- Lexicographic minimization 
- Integer projection 
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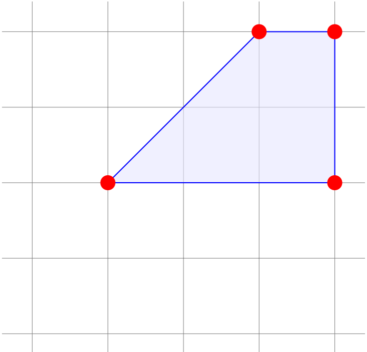

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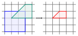
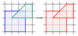
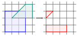
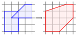
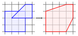
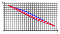
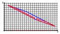
- Intersection 
- Union 
- Set difference 
- Closed convex hull (“wrapping”, FLL2000) 
- Coalescing 
- Lexicographic minimization 
- Integer projection 
- Sampling (GBR) 
- Scanning (GBR) 
- Integer affine hull (GBR) 
- Transitive closure (approx.) 
- Parametric vertex enumeration 
- Bounds on quasipolynomials (approx.)

# $\mathcal{V}$-Parametric Polytopes and Bounds on Polynomials

- $\mathcal{V}$-parametric polytopes

$$P : D \to \mathbb{Q}^n :$$
$$\mathbf{q} \mapsto P(\mathbf{q}) = \{\, \mathbf{x} \mid \exists \alpha_i \in \mathbb{Q} : \mathbf{x} = \textstyle\sum_i \alpha_i \mathbf{v}_i(\mathbf{q}), \alpha_i \geq 0, \sum_i \alpha_i = 1 \,\}$$

$D \subset \mathbb{Q}^r$: parameter domain
$\mathbf{v}_i(\mathbf{q}) \in \mathbb{Q}[\mathbf{q}]$ arbitrary polynomials in parameters
$\mathbf{v}_i(\mathbf{q})$ are generators of the polytope

Note: $\mathcal{V}$-parametric polytope can be computed from $\mathcal{H}$-parametric polytope through parameter vertex enumeration + chamber decomposition

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- Bounds on quasipolynomials (CFGV2009)

Input: Parametric polytope $P$ and quasipolynomial $p(\mathbf{q}, \mathbf{x})$
Output: Bound $B(\mathbf{q})$ on quasipolynomial over polytope

$$B(\mathbf{q}) \geq \max_{\mathbf{x} \in P(\mathbf{q})} p(\mathbf{q}, \mathbf{x})$$

Note: $\mathcal{V}$-parametric polytope can be computed from $\mathcal{H}$-parametric polytope through parameter vertex enumeration + chamber decomposition

# Bounds on Quasipolynomials: Example

$$p(x_1, x_2) = \frac{1}{2}x_1^2 + \frac{1}{2}x_1 + x_2 \qquad P = \text{conv.hull}\{(0,0), (N,0), (N,N)\}$$

To compute:

$$M(N) = \max_{(x_1,x_2)\in P} p(x_1, x_2)$$

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How? $\Rightarrow$ Bernstein expansion

- Express $\mathbf{x} \in P$ as convex combination of vertices

$$(x_1, x_2) = \alpha_1(0,0) + \alpha_2(N,0) + \alpha_3(N,N), \quad \alpha_i \geq 0, \quad \sum_i \alpha_i = 1$$

$$p(\alpha_1, \alpha_2, \alpha_3) = \frac{1}{2}N^2\alpha_2^2 + N^2\alpha_2\alpha_3 + \frac{1}{2}N^2\alpha_3^2 + \frac{1}{2}N\alpha_2 + \frac{3}{2}N\alpha_3$$

- Express $p(\mathbf{x})$ as convex combination of polynomials in parameters

# Bounds on Quasipolynomials: Example

$$p(\alpha) = \frac{N^2}{2}\alpha_2^2 + N^2\alpha_2\alpha_3 + \frac{N^2}{2}\alpha_3^2 + \frac{N}{2}\alpha_2 + \frac{3N}{2}\alpha_3 \quad \alpha_i \geq 0, \quad \sum_i \alpha_i = 1$$

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$$p(\alpha_1, \alpha_2, \alpha_3) = \alpha_1^2 0 + \alpha_2^2\left(\frac{N^2 + N}{2}\right) + \alpha_3^2\left(\frac{N^2 + 3N}{2}\right) + (2\alpha_1\alpha_2)\,\frac{N}{4} + (2\alpha_1\alpha_3)\,\frac{3N}{2} + (2\alpha_2\alpha_3)\,\frac{N^2 + 2N}{2}$$

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# Outline

# Conclusion

- `isl`: a relatively new integer set library
- currently used in
  - equivalence checking tool
  - `barvinok`
  - `CLooG`
- explicit support for parameters and existentially quantified variables
- all computations in exact integer arithmetic using GMP
- built-in incremental LP solver
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Future work: port `barvinok` to `isl`; now uses

- `PolyLib`: GPL, ...
  - $\Rightarrow$ `isl` already supports operations provided by PolyLib, but a lot of code still needs to be ported
- `NTL`: not thread-safe, C++
  - $\Rightarrow$ `isl` needs LLL