

# Isochoric specific heat in the Dual Model of Liquids and comparison with the Phonon theory of Liquid Thermodynamics

Fabio Peluso (✉ [fpeluso65@gmail.com](mailto:fpeluso65@gmail.com))

Leonardo Finmeccanica <https://orcid.org/0000-0003-0890-9772>

---

## Research Article

**Keywords:** Liquid model, Phonons in liquids, Mesoscopic model of liquids, Phonon – particle interaction, Specific heat

**Posted Date:** March 31st, 2021

**DOI:** <https://doi.org/10.21203/rs.3.rs-357792/v1>

**License:** © ⓘ This work is licensed under a Creative Commons Attribution 4.0 International License.

[Read Full License](#)

---

# Isochoric specific heat in the Dual Model of Liquids and comparison with the Phonon theory of Liquid Thermodynamics.

*Fabio Peluso*

*Leonardo SpA – Electronics Division – Defense Systems LoB  
Via Monterusciello 75, 80078 Pozzuoli (NA) – Italy  
Onorary Member of Marscenter Scientific Committee  
mailto: fpeluso65@gmail.com.*

## **Abstract.**

We continue in this paper to illustrate the implications of the Dual Model of Liquids (DML) by deriving the expression for the isochoric specific heat as function of the collective degrees of freedom available at a given temperature and comparing it with the analogous expression obtained in the Phonon Theory of Liquid Thermodynamics.

The Dual Model of Liquids has been recently proposed as a model describing the dynamics of liquids at mesoscopic level. Bringing together the early pictures of Brillouin and Frenkel and the recent experimental outcomes obtained by means of high energy scattering, liquids are considered in the DML as constituted by a population of wave packets, responsible for the propagation of elastic and thermal perturbations, and of dynamic aggregates of molecules, in continuous rearrangement, diving in an ocean of amorphous, disordered liquid. The collective degrees of freedom contribute to the exchange of energy and momentum between the material particles and the lattice particles, which the liquids are supposed to be composed of in the DML.

First, we show that the expression obtained for the specific heat in the DML is in line with the experimental results. Second, its comparison with that of the Phonon Theory of Liquid Thermodynamics allows to get interesting insights about the limiting values of the collective degrees of freedom and on that of the isobaric thermal expansion coefficient, two quantities that appear related each other in this framework.

**Keywords:** Liquid model, Phonons in liquids, Mesoscopic model of liquids; Phonon – particle interaction; Specific heat.

## **1. Introduction**

In a previous paper [1] the mesoscopic model of liquids dubbed Dual Model of Liquids (DML) has been introduced. It assumes a liquid be a system made up of molecules arranged in solid-like dynamic structures in continuous rearrangement, swimming in an ocean of amorphous

liquid. These solid-like structures, that we like to call icebergs, interact with lattice particles, the phonons, present in liquids and responsible for the propagation of elastic and thermal energy. As consequence of the interactions, the dynamic icebergs of molecules and the phonons exchange among them energy and momentum. This interaction is described in Figure 1, where two elementary events are represented. In events of type (a), an energetic wave-packet interact with a solid-like structure, the *liquid particle*, (i.e. a mesoscopic aggregate of molecules as described above) transferring to it energy and momentum. Events of type (b) are just the opposite, a wave-packet interact with a liquid particle and emerges from the interaction with increased energy. The two events are commuted one into the other by time reversal. These interactions are responsible for momentum, energy and mass propagation in a liquid, and in this frame thermal energy is seen as a form of elastic energy. The interaction lasts  $\tau_p$ , at the end of which the liquid particle relaxes the energy stored into internal degrees of freedom (DoF); then it travels by  $\Lambda_R$  during  $\tau_R$ . Figure 2 is a close-up of the first part of the wave-packet  $\leftrightarrow$  particle interaction shown in Figure 1a, that lasting  $\tau_p$ .  $\Lambda_{wp}$  is the extension of the wave-packet and  $d_p$  that of the liquid particle [1].

Starting from this considerations and applying a simple kinetic model, we have calculated, among others, the expressions for several macroscopic quantities in terms of elementary parameters. Examples are the thermal conductivity, the thermal diffusivity, the specific heat, etc.. The evaluation of the order of magnitude of the particle and phonon mean free paths and in particular of the relaxation times involved in their interactions, have also been provided [1].

The internal energy associated to the pool of wave-packets,  $q_T^{wp}$ , is represented in the DML as a fraction of the total one,  $q_T$ , namely:

$$1. \quad q_T^{wp} = mq_T = m \int_0^T \rho C_V d\theta = \mathcal{N}^{wp} \langle \varepsilon^{wp} \rangle$$

where  $\rho$  is the medium density,  $C_V$  the isochoric specific heat per unit mass,  $\mathcal{N}^{wp}$  the number of wave-packets per unit of volume<sup>1</sup>, and  $\langle \varepsilon^{wp} \rangle$  their average energy (here and in the rest of the paper, the two brackets  $\langle \rangle$  indicate the average over a statistical ensemble of the quantity inside them). The dynamics described above occurs at high frequencies and involves only the DoF of the lattice; the parameter  $m$  that we have introduced accounts for the ratio between the number of collective DoF surviving at temperature T, and the total number of available DoF. We will return on the meaning and values of  $m$  in the course of the paper, for now it is relevant to note that  $0 \leq m \leq 1$ .

---

<sup>1</sup>  $\mathcal{N}^{wp}$  is the average density of the wave-packets statistical distribution. As such  $\mathcal{N}^{wp}$  is represented by a Bose-Einstein distribution function

Figure 1a shows that the energy acquired by the liquid particle is given back to the pool of elastic excitation a  $\langle \Lambda \rangle = \langle \Lambda_p \rangle + \langle \Lambda_R \rangle$  step forward and a  $\langle \tau \rangle = \langle \tau_p \rangle + \langle \tau_R \rangle$  time lapse later; this last assumes the right meaning of a relaxation time. One of the immediate consequences of this model is that heat propagation in liquids is naturally described by a Cattaneo equation, as shown in another recent paper [2], the delay time introduced into the propagation equation being physically interpreted as the time taken by icebergs to displace from one site to the next, as in Figure 1.

Few years ago the “Phonon theory of Liquid Thermodynamics” (PLT) has been presented [3-12]. One of the main achievements of the PLT is that, starting from a solid-like Hamiltonian of the liquid, modified to account for the presence in liquids of anharmonic DoF, provides an expression for the isochoric specific heat  $C_V$  covering the solid, glassy, liquid, gas and quantum liquids states of matter. Two expressions are obtained for the specific heat, one in harmonic approximation, the other including the anharmonic contributions to the pool of energy. The theoretical expressions arrived at have been confirmed experimentally in 21 different liquids. Moreover, PLT has also provided a theoretical interpretation of the elastic, visco-elastic and viscous behaviour response of matter to external disturbances as due to the value of the product<sup>2</sup>  $\nu\tau$ , where  $\nu$  is the frequency of the excitation of the DoF, and  $\tau$  the relaxation time as defined above. It depends on whether  $\nu$  is larger, comparable or smaller than  $1/\tau$ , respectively.

Here in this paper we shortly recall the calculations that allow to get in the frame of reference of the DML the expression for the isochoric specific heat as function of the number of collective DoF available at a given temperature, and compare it with the analogous expression arrived at in the Phonon Theory of Liquid Thermodynamics (PLT) [3-12]. This comparison allows to get interesting insights about the limiting values of the collective DoF, and on that of the isobaric thermal expansion coefficient,  $\alpha$ .

The paper is organized as follows. In Section 2 the DML is shortly summarized and compared with the PLT. In Section 3 the expressions for the isochoric specific heat derived along with the two approaches, DML and PLT, are compared among them, and their similarities, insights and the implications are discussed. Finally in the last Section some further consequences and comparisons even between the two theories are discussed.

---

<sup>2</sup> This product is normally reported as  $\omega\tau$  by many authors instead of  $\nu\tau$ . It is clear that there is no ambiguity among the two definitions.

## **2. Dual Model of Liquids and Phonon theory of Liquid Thermodynamics: the two sides of the same coin.**

It is known that historically the liquid state has been initially dealt with as a sort of extension of the gaseous one, this mainly because of its property of flowing like gases, of the lacking of the capability of transmitting shear stresses, and of adapting to the shapes of their containers. However, this approach has never allowed to go very far in the formulation of theories that were able to provide a fairly general picture of the physical quantities characterizing the liquid state. Let us take for instance the specific heat, for which strong theoretical basis exist for gaseous and solid states, but not for the liquid one.

The alternative approach, to consider the liquid state in the same way as that of a solid, has illustrious predecessors, such as Debye [13-14], Brillouin [15-16], and Frenkel [17]. However, they too have never gone so far as to provide a model that went beyond an intuitive description, this because of the lack of experimental evidences that could guide and/or validate it. Nevertheless, recent experimental discoveries have given credit again to the ideas of Frenkel and Brillouin. At low frequencies, i.e. large wavelengths, it is not possible to "see" the mesoscopic structure of liquids. The entire liquid oscillates, compressing and expanding under the effect of pressure waves, which travel at the speed of sound. When one investigates the behaviour of a liquid at very high frequencies, and therefore at small wavelengths, it is discovered that it has a mesoscopic structure similar to that of a solid, the one identified here in a picturesque and imaginative way with the *icebergs*, elsewhere simply called pseudo-crystalline structures (see for instance [13-29]).

The first experimental clear evidence of the presence in liquids of solid-like local structures with a pseudo-periodicity came in 1996 [25], when Ruocco and Sette measured by IXS experiments, therefore at high frequency, in liquid water at ambient conditions, the propagation speed of elastic waves, and found it equal to 3200 m/s, i.e. more than double that known at traditional frequencies, 1500 m/s, and very close to that of solid water, 4000 m/s. Starting from the results of experiments performed on various liquids with INS and IXS techniques, [13-29], the evidence that pseudo-crystalline structures "exist and persist" in liquids has gradually consolidated, their size and number depending on the liquid temperature (and pressure). These structures can obviously be discovered only when high frequency measurements are carried out, typical of INS and IXS techniques, the wavelengths of the radiation involved being small enough to reveal their presence. In the DML these pseudo-crystalline structures (the *icebergs*) organized by means of phonons, interact with the rest of the (amorphous) liquid through anharmonic interactions which arise at their border. Ultimately, a quasi-elastic propagation within *icebergs* is transmitted to the amorphous matrix in the form of anharmonic wave-packets, and vice versa. Unlike in crystalline

solids, the anharmonicity allows a non-infinitesimal duration of the interactions, with consequent transport not only of energy but also of momentum, and furthermore exchange of energy with the internal DoF of the icebergs. Figure 3 shows the picture described above.

How is the liquid state organized in the DML and how does it evolve? At the Triple Point the solid structure is not entirely and abruptly lost. The liquid state resembles an ocean where pieces of solid continue to exist and participate in the propagation of energy and momentum. Being solids, they are also able to transmit shear stresses and not only longitudinal compressions and expansions of the medium. As the temperature rises, the fusion is not instantaneous, the solid phase giving progressively way to solid-like dynamic structures in continuous evolution (the icebergs) swimming in an ocean of amorphous liquid; the number and size of the solid-like structures decrease more and more and the amount of amorphous liquid increases, to the point where it reaches the pure liquid, at the Frenkel line. On the other hand, it would not be surprising if the transition from the solid to the liquid phase took place over a wide range of temperature and pressure rather than over a restricted interval around the triple point. The DML provides an intuitive way of describing liquids how they emerge from the experiments.

This vision of liquids is not entirely new; from the historically point of view, besides the pioneering ideas of Debye and Brillouin, Frenkel proposed that liquids were constituted by a very large number of randomly oriented crystals of submicroscopic size, those he defined *cybotactic groups*<sup>3</sup>. The concept of relaxation time was early introduced by Maxwell in the liquid dynamics as a phenomenological concept related to the visco-elastic nature of liquid [30]. Frenkel, trying to compile a solid-like vision of liquids with their peculiarity of flowing, recovered this concept giving it a microscopic interpretation as “*the average time between particle jumps at one point in space in a liquid*” [17]. Its inverse,  $\nu_F = 1/\tau_F$ , is the frequency of occurrence of particle jumps. In the Frenkel model the way of distinguishing a liquid from a solid or a gas goes around the value of the relaxation time  $\tau_F$ . If, at a given temperature, the characteristic time  $t$  of a perturbation propagating in the liquid is much shorter than  $\tau_F$  (or the frequency  $f$  much larger than  $\nu_F$ ), over the time interval  $t$  the medium is seen by the perturbation as a solid, because the particles have not enough time to rearrange. The dynamic response is that of a free recoil of the liquid particle after the collision with the probe. Incidentally, this figure matches the interdependence between the relaxation time and the excitation frequency shown in the *Introduction*.

---

<sup>3</sup> In his book [17] Frenkel actually affirms that this idea was introduced by Stewart around 30's; Stewart proposed to denote such submicroscopic crystals, consisting of few tens of molecules, at most, by the term “*cybotactic group*” (or regions), and assumed them to be connected with each other by thin layers of the wholly amorphous phase.

The presence of relaxation times is a distinguishing aspect of liquids vs gases. The thermal unrest in liquids in contrast has the same character as in solids, consisting of oscillations of molecules around their equilibrium positions. As in solids, a liquid particle may oscillate along with 3 preferred directions, one longitudinal (L) and two transversal (T). At low temperatures therefore liquids manifest also the capability of reacting to shear stresses [3, 28-29], this behaviour being explained by the presence of the icebergs. Because of the progressive melting of icebergs, the capability of liquids to react to shear stresses by means of transversal DoF, is progressively lost as it approaches the Frenkel line. This is in line with the structural cross-over existing in liquids [5,21]. Consequently, the number  $m$  of lattice collective DoF available in a liquid, introduced in Eq.(1), decreases as the temperature increases,  $dm/dT < 0$ . Indeed, at low temperature energy and momentum in liquids are propagated also by means of transversal DoF, because the propagation mechanism involves a structural relaxation of the pseudo-crystalline structure. As the temperature increases this property is progressively lost and energy and momentum are propagated only by means of longitudinal DoF, because only collisional mechanisms are involved. Therefore, at low temperature  $m$  approaches unity, while it decreases by increasing the temperature because the number of lattice collective DoF decreases with increasing T,  $0 \leq m \leq 1$  (at the Frenkel line only the longitudinal DoF survive). This is a fundamental point of the DML. It is worth noting that, apart the hypothesis of the mesoscopic organizations of liquids by means of dynamic icebergs, this figure matches that of the PLT, representing its mesoscopic counterpart. The above argument related to the variation with temperature of the number of DoF is supported also by measurements [13-29].

The positions of atoms, or better, of liquid particles in fluids are of course not permanent as in solids but temporary. After performing a number of oscillations around a given position, the liquid particle can jump to another equilibrium position, far  $\delta$  from the previous one,  $\delta$  being of the same order of magnitude as the average distance among molecules in that liquid. From a dynamical point of view, this is the manifestation of the fact that the kinetic energy,  $E_p^k$ , of a liquid particle is comparable with the potential energy,  $\Psi_p$ , that kept it to a definite lattice point. Unlike in solids, where obviously  $\Psi_p \gg E_p^k$ , or in gases, where the opposite holds,  $\Psi_p \ll E_p^k$ , in liquids the intermediate situations is true,  $\Psi_p \approx E_p^k$ . This step-by-step wandering, totally lasting a time  $\tau_F$  for each jump, constitutes a sort of self diffusion motion, leading to a gradual mixing up of all the atoms. It must proceed much faster in liquids than in solids, and must have a simpler character because of the absence of definite lattice sites.

The arrangement of liquid molecules on mesoscopic scale along with local lattices justifies the experimental value found for the speed of sound in water (and in other liquids) close to that of the corresponding solid form [23-29]. Liquid molecules are bounded to the local lattice, the iceberg; as far as thermal perturbations propagate inside an iceberg, they behave as in solids. When perturbations cross the boundary between two such local lattices, inertial effects develop, giving rise to momentum transport, as in the Frenkel model. The interactions are accompanied by propagation of elastic energy in forms of wave-packets because of the anharmonicity of the potential field [1]. The anharmonic character of the collisions generated at the border between two nearest-neighbour icebergs makes the interaction time non negligible, thus allowing also momentum transport. This gives origin to the displacement of icebergs and consequently to their diffusion [1-4,6,12,16-17]. When wave-packets propagate in the amorphous phase, the speed of sound decreases and the same happens for the other liquid parameters.

It is clear from above that what we usually mean as a “liquid” is not a liquid at all in the DML, but a mixture of solid-like dynamic structures and an amorphous, i.e. liquid phase in its common sense. As consequence, any liquid parameter whose magnitude is experimentally measured at temperatures and pressures where a usual liquid phase exists, is actually a pondered average of a solid/liquid value, in particular at those frequencies where the duality of liquids phase emerges.

Let us to now compare the DML with the PLT. The PLT describes the liquid from a statistical-thermodynamic point of view, therefore it cannot provide a direct answer on “why” the liquid particles oscillate. DML instead faces the problem from the opposite, mesoscopic point of view, starting from the elementary wave packet  $\leftrightarrow$  *liquid particle* collision that gives rise to the harmonic and anharmonic contributions even present in the PLT. DML provides and analyzes the microscopic elementary mechanism by which elastic and thermal energy, as well as momentum, are exchanged between liquid particles (icebergs) and lattice particles (phonons) and show that this scenario is in agreement with that of the PLT. One of the junction points between the two approaches, PLT and DML, is the relaxation times involved in the elementary interactions. Because of the thermodynamic-statistical nature of the theory, PLT from its side is only able to provide an indication of the variation range of the relaxation time through the value of the product  $\nu\tau$ . On the other side, DML is a mesoscopic model and analyzes the intimate mechanism of interaction of liquid particles with lattice modes; this provides both the answer to the question of “why” liquid particles oscillate, and a way for calculating the order of magnitude of  $\tau$  in ordinary liquids [1]. We have shown [1] that the attained value of  $\tau$  and of the product  $\nu\tau$  are in the range foreseen on thermodynamic statistical basis from PLT. The relaxation time is one of the key-points of both models, and because the only common argument of PLT and DML is the assumption that thermal



energy in liquids is transported by means of collective lattice excitations, both harmonic and anharmonic, the mutual agreement shows that both of them look at the model of liquid state in the same way, although from different observation points.

Another similitude of PLT and DML is the presence of harmonic and anharmonic contributions to the pool of energy. From the statistical-thermodynamic point of view, this fact is accounted for in the PLT by considering both contributions into the Hamiltonian of a liquid, giving rise in turn to the expression for the specific heat (the Hamiltonian adopted in the PLT has however many other interesting consequences, which anyway are not discussed here being out of the specific topic of the present paper). On the other side, harmonic and anharmonic oscillations are believed in the DML responsible for the propagation of energy and momentum within a liquid. In fact, energy is supposed to propagate inside the solid-like aggregates by means of harmonic oscillations. Liquid particles communicate among them and with the disordered liquid by means of anharmonic wave-packets, capable as such to exchange not only energy but also momentum. In such a way the macroscopic phenomena of diffusion (and thermal diffusion) may be easily interpreted in the DML.

The duality of liquids, supposed by both DML and PLT, however, does not exclude of course classical mechanisms of intermolecular interactions, as we will see in the Section 3 when discussing the similarities of the isochoric specific heat expressions and their implications. The propagation of energy in a condensed medium through inter-molecular interactions on one side, or interactions between wave packets and pseudo-crystalline structures on the other, may be supposed working simultaneously and uniformly at the mesoscopic level. Which of the two prevails over the other depends on the thermodynamic conditions of the system. In a solid only the phononic part will be present. When the solid melts, the molecules begin to arrange in local solid-like lattices, phononic propagation gradually leaves way for propagation through molecular interactions, until the former completely disappears when the gaseous state is reached, only the molecular one remaining. While in solids there are always three modes of vibration, two transversal and one longitudinal, in liquids the transverse modes will be active only for frequencies higher than  $\omega_F$ ,  $\omega > \omega_F = \frac{2\pi}{\tau_F}$ , where  $\tau_F$  is the relaxation time. In other words, on interaction times lower than the relaxation time, liquids behave as solids, while for times longer than  $\tau_F$ , liquids lose the ability to support transversal ways, as in the classical approach.

### 3. Isochoric specific heat comparison between DML and PLT: similarities and insights.

Scope of this Section is to compare the expression for the isochoric specific heat obtained in the DML with that of the PLT. The same physical quantity is indeed obtained with two different approaches, namely a thermodynamic approach for the PLT and a mesoscopic-statistical approach for the DML. Consequently the comparison between the two expressions could shed light on some parameters adopted in their derivation. This is actually the case for  $m$ , used in the DML approach, and for  $\alpha$ , used in the PLT model. For both these quantities, some limits are deduced from the comparison of the two expressions for  $C_V$ .

To accomplish the task, we begin to recall the calculation of the specific heat in the frame of DML [1]. Let us return then to Eq.(1), which can be re-written as follows:

$$2. \quad q_T^{wp} = mq_T = \mathcal{N}^{wp} \langle \varepsilon^{wp} \rangle = m^* \rho C_V T$$

where the quantity  $m^* = m \frac{\int_0^T \rho C_V d\theta}{\rho C_V T}$  depends on the nature of the liquid, and has been introduced

for the sake of simplifying the expression.  $q_T^{wp}$  in Eqs. (1) and (2) allow for calculating the phonon specific heat contribution  $C_V^{DML}$  to the total one,  $C_V$ :

$$3. \quad \left\{ \begin{aligned} \rho C_V^{DML} &= \frac{\partial q_T^{wp}}{\partial T} = \frac{\partial}{\partial T} (mq_T) = q_T \frac{dm}{dT} + m \frac{\partial q_T}{\partial T} = q_T \frac{dm}{dT} + m \rho C_V = \\ &= m \rho C_V \left[ \frac{q_T}{m \rho C_V} \frac{dm}{dT} + 1 \right] = m \rho C_V \left[ \frac{m^*}{m^2} \frac{dm}{dT} T + 1 \right] \end{aligned} \right.$$

Let's evaluate now the quantity in square brackets at the last member. Because

$$4. \quad \rho C_V^{DML} = \frac{\partial q_T^{wp}}{\partial T} \geq 0,$$

and  $0 \leq m \leq 1$ , this implies that  $\left[ \frac{m^*}{m^2} \frac{dm}{dT} T + 1 \right] \geq 0$ . As for the temperature dependence of  $m$ , the

basis for the reasoning is the experimental evidence of the presence in liquids of transversal modes [18-30] actives for the propagation of elastic energy by means of shear waves working as in the solid phase. These modes however persist in the liquid phase as long as solid-like structures survive. Experiments have shown that the two transversal modes disappear when the system approaches the critical point, where only the longitudinal collective modes survive accounting for the compression and expansion waves responsible for hydrodynamic modes propagation. This topic

has been extensively discussed elsewhere [1], here it is important to have in mind that the number of collective DoF defined by  $m$  decreases with temperature, so that  $\frac{dm}{dT} < 0$ . Consequently,

$\frac{m^*}{m^2} \frac{dm}{dT} T \geq -1$ , and this defines the lower limit for  $\frac{m^*}{m^2} \frac{dm}{dT} T$ . As for the upper limit, because of the fact that  $\frac{dm}{dT} < 0$  and  $\left[ \frac{m^*}{m^2} \frac{dm}{dT} T + 1 \right] \geq 0$ , we get for the maximum value  $\frac{m^*}{m^2} \frac{dm}{dT} T \leq 0$ .

Definitively we may conclude that

$$5. \quad 0 \leq \left[ \frac{m^*}{m^2} \frac{dm}{dT} T + 1 \right] \leq 1$$

and finally

$$6. \quad C_V^{DM} \leq C_V$$

as one would have expected.

One of the most important results of the PLT is that of having provided for the first time an expression for the isochoric specific heat  $C_V$  for liquids that was shown to be valid for the solid, glassy, liquid, gas and quantum liquids states of matter [3-12]. To get this expression the Authors proposed a solid-like Hamiltonian of the system in which only harmonic terms are included because of the harmonic character of the interactions in a solid lattice. In order to tailoring the Hamiltonian for the liquid structure, it has been modified to account also for inelastic interactions, thus including the anharmonic contributions to the pool of energy. The theoretical expressions arrived at for  $C_V$  have been confirmed experimentally in 21 different liquids. Without entering into the details of the calculations, it is enough for our purposes to quote the expressions for the specific heat as obtained from the PLT accounting for both harmonic  $C_V^H$  and anharmonic  $C_V^A$  contributions, namely:

$$7. \quad C_V^H = \frac{1}{N} \left( \frac{\partial q_T^H}{\partial T} \right)_V = 3 - \left( \frac{\tau_D G_\infty}{\eta} \right)^3 + \frac{3T}{2\eta} \left( \frac{\tau_D G_\infty}{\eta} \right)^3 \frac{d\eta}{dT};$$

$$8. \quad C_V^A = \frac{1}{N} \left( \frac{\partial q_T^A}{\partial T} \right)_V = 3\alpha T - \alpha T \left( \frac{\tau_D G_\infty}{\eta} \right)^3 + \frac{3\alpha T^2}{2\eta} \left( \frac{\tau_D G_\infty}{\eta} \right)^3 \frac{d\eta}{dT} = \alpha T C_V^H$$

from which we get:

$$9. \quad C_V^{PLT} = C_V^H + C_V^A = C_V^H (1 + \alpha T)$$

In Eqs.(7) through (9) indexes 'H' and 'A' stay for harmonic and anharmonic respectively, while  $q_T$  is the total heat content of the liquid defined in Eq.(1). The remaining terms have the

same meanings as those quoted in [3], i.e.  $\alpha$  is the coefficient of isobaric thermal expansion of the liquid,  $\eta$  its viscosity,  $\tau_D$  the Debye vibration period and  $G_\infty$  the instantaneous shear modulus. Scope of the present section is to compare the expression given by Eq.(9) for the total specific heat  $C_V^{PLT}$  due to collective oscillations (phonons) as calculated in the PLT with the corresponding one calculated in the DML frame of reference,  $C_V^{DML}$ , given by Eq. (3) [1]:

$$10. \quad C_V^{DML} = mC_V \left[ \frac{m^*}{m^2} \frac{dm}{dT} T + 1 \right],$$

Here as pointed out before  $C_V$  is the total specific heat of the liquid. This last definition should not surprise the reader; indeed, notwithstanding in the DML the exchange of energy and momentum is due to the presence of collective oscillations, this picture nevertheless modifies gradually as temperature increases. If at the triple point quite all the energy is propagated by means of phonons and wave-packets, as the temperature increases the solid phase gives progressively way to icebergs swimming in an ocean of amorphous liquid; the number and size of icebergs decrease and the amount of amorphous liquid increases, to the point where it reaches the pure liquid, at the Frenkel line. Thus at the triple point one has  $m \cong 1$ , while  $m$  decreases progressively to zero at the Frenkel line, so that  $\frac{dm}{dT} < 0$ . The result at the mesoscopic scale is that the energy propagation by means of wave-packets becomes less and less important as the temperature increases. This is in line with Eq.(10), from which one easily has that  $C_V^{DML} \leq C_V$ , as per Eq.(6). Ultimately, we may assume that in a liquid the total specific heat  $C_V$  is contributed as:

$$11. \quad C_V = C_V^M + C_V^H + C_V^A = C_V^M + C_V^{DML} = C_V^M + C_V^H (1 + \alpha T),$$

where with  $C_V^M$  we have indicated the classical contribution due to molecular interactions, and the last two members have been obtained using Eq.(9).

We are now ready to accomplish the task by comparing Eqs.(9) and (10) among them:

$$12. \quad C_V^{DML} \equiv C_V^{PLT} \Rightarrow mC_V \left[ \frac{m^*}{m^2} \frac{dm}{dT} T + 1 \right] = C_V^H (1 + \alpha T)$$

We start by deriving the expression for  $\frac{dm}{dT}$ :

$$13. \quad \frac{dm}{dT} = \frac{C_V^H (1 + \alpha T - m) - m(C_V - C_V^H)}{C_V \frac{m^*}{m} T} < 0$$

Because the quantities at the denominator are all positive defined, the above relation reduces to discussing the algebraic sign of the numerator. Two possibilities may be considered, namely:

$$14. \quad \begin{array}{l} A) C_V^H(1 + \alpha T - m) < 0 \\ B) \left\{ \begin{array}{l} a) C_V^H(1 + \alpha T - m) > 0 \\ b) C_V^H(1 + \alpha T - m) < m(C_V - C_V^H) \end{array} \right. \end{array}$$

The case A) is however not acceptable because by definition  $0 \leq m \leq 1$ ; therefore we will discuss case B) only. Because  $C_V^H$  is a positive defined quantity, we have

$$15. \quad \left\{ \begin{array}{l} a) 1 + \alpha T - m > 0 \\ b) 1 + \alpha T - m < m \left( \frac{C_V}{C_V^H} - 1 \right) \end{array} \right.$$

from which we easily get

$$16. \quad \left\{ \begin{array}{l} a) 1 + \alpha T > m \\ b) 1 + \alpha T < m \frac{C_V}{C_V^H} \end{array} \right.$$

Eq.(16) allows us to identify specific limits for the parameter  $m$  :

$$17. \quad \frac{C_V^H}{C_V}(1 + \alpha T) < m < (1 + \alpha T)$$

We then see that the parameter  $m$  is strictly linked to  $\alpha$  . In the general case in which  $\alpha > 0$  , Eq.(17) reduces to:

$$18. \quad \frac{C_V^H}{C_V}(1 + \alpha T) < m < 1, \quad \alpha > 0$$

while the lower limit goes to zero as  $C_V^H$  . This conclusion is in line with the hypothesis of the DML. In fact, at high temperature icebergs melt and disappear from the liquid, and with them also the collective DoF contributing to  $C_V^H$  .

There are however very few cases in nature where  $\alpha < 0$  , as for instance the water in the temperature range  $(0 < T < 4)^\circ\text{C}$  at atmospheric pressure. In these cases the assumption introduced in Eq.(18) cannot be adopted.

There are other interesting insights that can be deduced from Eq.(16-b). Remembering that it is valid for  $\frac{dm}{dT} < 0$  , an upper limit can be deduced for  $\alpha$  , namely:

$$19. \quad \alpha < \frac{1}{T} \left( m \frac{C_V}{C_V^H} - 1 \right); \quad \frac{dm}{dT} < 0$$

Because usually it is also  $\alpha > 0$  , we get the following relation holding for the specific heat:

$$20. \quad C_V^H < m C_V$$

which represents in turn an upper limit for  $C_V^H$ , providing the maximum thermal energy that can be stored into harmonic DoF of icebergs. Incidentally, this is in line also with the limits imposed to the total specific heat  $C_V^{DML}$  [see Eqs.(10) and (12) and the related validity range]. Compiling Eqs.(12) and (20), we get the following expression for the lower limit of  $\alpha$

$$21. \quad \alpha > \frac{m^*}{m^2} \frac{dm}{dT}$$

where the second member is a negative quantity. Thus, also in the very few cases in which  $\alpha < 0$ , there is a minimum value for  $\alpha$  as function of the number of collective excitations present in the liquid. The complete range holding for  $\alpha$  is obtained compiling Eqs.(19) and (21):

$$22. \quad \frac{m^*}{m^2} \frac{dm}{dT} < \alpha < \frac{1}{T} \left( m \frac{C_V}{C_V^H} - 1 \right).$$

#### 4. Further discussion and conclusions.

The hypotheses behind the DML, duly described in [1] and to which the reader is referred to for an in-depth study, are actually two, and both have an experimental background. The first is that experiments performed with the IXS and INS techniques [13-29] have made it possible to highlight that the mesoscopic structure of liquids is characterized by the presence of solid-like, pseudo-crystalline structures, whose size being of a few molecular diameters and mass of few molecules, within which the elastic waves propagate as in the corresponding solid phase. The number and size of these structures varies with the temperature and pressure of the liquid. The second hypothesis is a direct consequence of the first. It consists in the assumption that elastic energy and momentum in liquids propagate by means of collective oscillations, or wave-packets, similar to phonons in crystalline solids. This scheme and the DML may be obtained by gathering together the early pictures of liquids of Debye [13-14] Brillouin [15-16] and Frenkel [17] in the light even of what is known today from the high frequency scattering experiments. In fact, as result of the interaction described in Figure 1, being the icebergs in continuous rearrangement, molecules continuously move from an iceberg to the nearest neighbour and icebergs too jump in the liquid from one site to the nearest-neighbour one, as hypothesized by Frenkel for the cybotactic groups introduced in [17].

In Section 2 and also elsewhere [1] we have largely discussed and commented the intimate similitude and differences of the two models, in particular about the presence of harmonic and anharmonic oscillations, the central role of relaxation times, etc.. All these arguments bring to the obvious consequence that DML and PLT show an evident visual identity: in the liquid state both harmonic modes, typical of the solid state, and anharmonic modes, typical of non-crystalline

structures, are simultaneously present. The comparison of the two expressions for the specific heats provides interesting information about the degree of excitation of vibratory modes in liquids. Besides, we have also pointed out the central role of relaxation times in both models; in particular, their role has the consequence, among others, that heat propagation in liquids is described by a Cattaneo equation [2].

As outlined above, and duly discussed in [1], key point of the DML is that the liquid is composed of two systems, the liquid particles and the lattice particles; they interact among them through a harmonic potential inside the iceberg, and through an anharmonic potential outside. As far as the system is in thermodynamic equilibrium, the same holds for the two subsystems; when however a non-equilibrium state is imposed, even the two populations reach two distinct equilibria. The anharmonic character of the interaction allows the two populations to exchange among them not only energy but also momentum. This is another key point of the model, because the interaction represents a mechanism through which energy is dissipated. A possible application of the DML is indeed that of providing an elementary mechanism to explain the viscosity, that is a dissipative phenomenon present in liquids (and not in solids). If the elementary wave-packet  $\leftrightarrow$  *liquid-particle* interactions would be perfectly elastic, all the energy (and momentum) lost by the wave-packet in an event such that of Figure 1a would be acquired by the *particle* and converted into kinetic energy, only the translational DoF being involved in the scattering process. Nevertheless, being the interaction anharmonic, not only it is not instantaneous, the momentum exchanged between wave-packet and *particle* is proportional to the duration of the interaction,  $\langle \tau_p \rangle$  (i.e. to the time-length of the wave-packet) but also the energy lost by a wave-packet in an event of type a) of Figure 1 is only partially converted into kinetic energy of the *particle*. The remaining part is converted into internal molecular energy, exciting the corresponding DoF. This is an important point of the model, which deals with for the first time the exchange of energy with the internal vibratory DoF of the molecular cluster in a phononic model of liquid. It is possible to define an “*effective heat capacity*”  $C_{eff}$  for the total heat absorbed by (or given to) a *liquid particle*, taking into account the two branches of DoF.  $C_{eff}$  results to be a complex quantity whose real part is the static part pertaining to external translational DoF, while the imaginary part depends on that pertaining to the internal vibratory (quantized) DoF, on the frequency and on the relaxation time [34]. The pool of energy stored into the cluster will be released after a time lag  $\langle \tau_r \rangle$ . Depending on the OoM of such time interval, the events of Figure 1 can be considered elastic, quasi-elastic or inelastic [1,3,12,21].

In one of his papers [35], Eckart pointed out that in his model the distinction between solids and liquids becomes a quantitative rather than a qualitative one. He found that the propagation and

absorption of isentropic longitudinal waves exhibit some interesting phenomena. In particular, several physical quantities become complex numbers, as it is the case for instance for the velocity of wave propagation, which is a function of frequency, revealing the dispersion character of this physical parameter [1,21]. Besides, its values determine two relaxation times, one of which related to the viscosity of the medium.

Some final remarks are in order about the comparison of the expression for the isochoric specific heats deduced from PLT and DML. This comparison has provided very interesting information about the physical limits of the parameter  $m$  and the coefficient of isobaric thermal expansion  $\alpha$ . An additional relation is also deduced for the ratio between the harmonic contribution to the specific heat,  $C_V^H$ , with respect to the total specific heat of the liquid. Besides the intrinsic information, unfortunately presently not supported by experimental confirmation, all the attained expressions and limits are internally consistent with the model, and with other limits provided on different basis. Having found limiting values for these quantities stimulates our reasoning on the fact that other Authors [36-40] have identified limiting values albeit for different physical quantities. However an answer to such comparison is still premature, although not absurd in principle. Yet still today there are many questions on the nature of liquids that are still unanswered, as well a deficit of experiments. Performing experiments at the onset of a temperature gradient may allow to investigate how liquid parameters evolve from equilibrium to non equilibrium conditions. It could be possible for instance to investigate the evolution of correlations lengths, sound velocity, thermal conductivity, etc.. Also very interesting could be the investigation of whether and how a temperature gradient affects the viscous coupling between two liquids. Such experiments could be performed, for instance, by applying a temperature gradient to a stabilised isothermal liquid, provided that the average temperature of the system remains unchanged and preventing the convection instability, and performing light scattering experiments until a stationary temperature gradient is reached, i.e. during the transient, exactly the situation that is normally avoided in all the experiments. An alternative way to heat a small volume of liquid could be that on hitting the liquid with a focalised high power laser beam.

A second type of experiments could be aimed at investigating the glassy and liquid-to-solid transitions. Starting from a stationary temperature gradient, both temperatures could be lowered, thus lowering also the average temperature until the liquid solidifies. Light scattering experiments performed during the non stationary phase should allow to investigate the dynamics of the system when the glassy and liquid-to-solid transitions are crossed. In particular, the local domains should be oriented following the external temperature gradient, thus allowing the increase of the correlation lengths, of sound velocity, of thermal conductivity, etc., along the preferential direction of the



external temperature gradient. A difference between the same parameters when measured along the direction of the temperature gradient with respect to those measured along a different direction should be evident from the experimental data collected. This technique could in some way introduce a sort of anisotropy into the liquid matrix, becoming the solid phase anisotropic.

In all the above experimental setups one could fruitfully exploit the characteristics of a special environment such a microgravity laboratory on orbiting platforms, where the above measurements could be pursued without the disturbing, and often overwhelming effects due to the presence of the Earth's gravity.

## 5. List of References

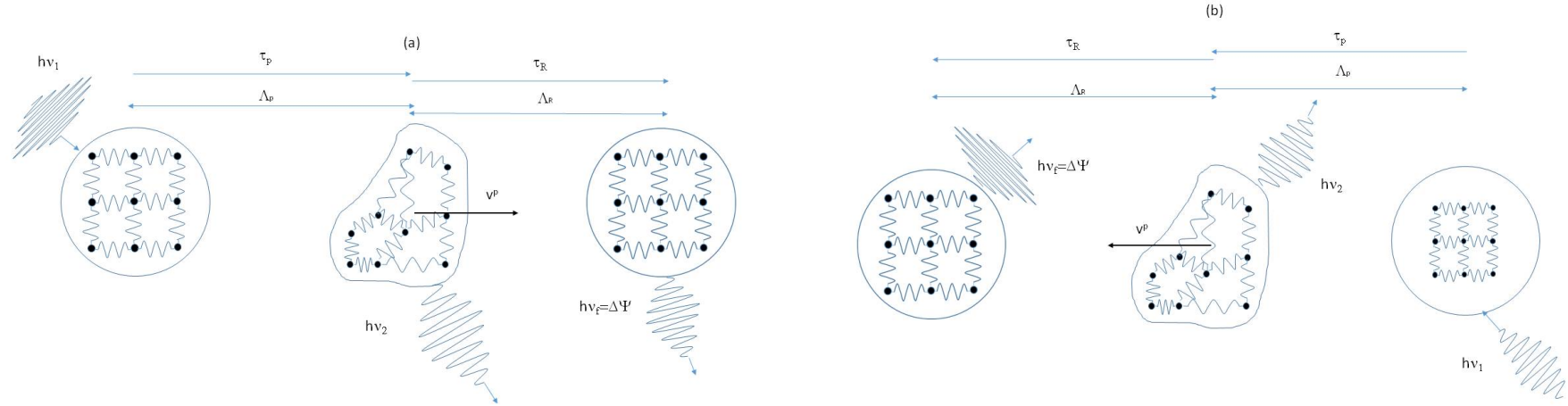
1. Peluso, F., 2021, Mesoscopic dynamics of liquids and the Dual Model, DOI: 10.21203/rs.3.rs-205220/v1, submitted for publication.
2. Peluso, F., 2021, How does heat propagate in liquids?, in preparation.
3. D. Bolmatov, D., Brazhkin, V.V., Trachenko, K., 2012, The phonon theory of liquid thermodynamics, *Scientific Reports*, **2**, 431-427 (DOI:10.1038/srep00421.)
4. Brazhkin, V.V., Trachenko, K., 2012, What separates a liquid from a gas?, *Phys. Today*, **65**(11), 68-69; doi:10.1063/PT.3.1796.
5. Bolmatov, D., Brazhkin, V. V., Fomin, Yu. D., Ryzhov, V. N., Trachenko, K., 2013, Evidence for structural crossover in the supercritical state, *Journal of Chemical Physics*, **139**, 234501.
6. Trachenko, K., Brazhkin, V.V., 2013, Duality of liquids, *Scientific Reports*, **3**, 2188-2193 (DOI: 10.1038/srep02188).
7. Bolmatov, D., Musaev, E. T., Trachenko, K., 2013, Symmetry breaking gives rise to energy spectra of three states of matter, *Scientific Reports*, **3**, 2794.
8. Bolmatov, D., Zhernenkov, M., Zav'yalov, D., Tkachev, S. N., Cunsolo, A., Cai, Y. Q., 2015, The Frenkel Line: a direct experimental evidence for the new thermodynamic boundary, *Scientific Reports*, **5**, 15850.
9. Bolmatov, D., Zav'yalov, D., Zhernenkov, M., Musaev, E.T., Cai, Y.Q., 2015, Unified phonon-based approach to the thermodynamics of solid, liquid and gas states, *Ann. Phys.*, **363**, 221-242.
10. Bolmatov, D., Zhernenkov, M., Zav'yalov, D., Stoupin, S., Cai, Y.Q., Cunsolo, A., 2015, Revealing the mechanism of the viscous-to-elastic crossover in liquids, *J. Phys. Chem. Lett.*, **6**, 3048-3053 (doi: 10.1021/acs.jpcclett.5b01338).
11. Bolmatov, D., Zhernenkov, M., Zav'yalov, D., Stoupin, S., Cunsolo, A., Cai, Y. Q., 2016, Thermally triggered phononic gaps in liquids at THz scale, *Scientific Reports*, **6**, 19469.

12. Trachenko, K., Brazhkin, V.V., 2016, Collective modes and thermodynamics of the liquid state, *Rep. Prog. Phys.* **79**, 016502-016538
13. Debye, P., 1912, Zur Theorie des spezifische Wärmer, *Ann. der Physik*, **344** Nr.14, 798-839.
14. Debye, P., 1914, *Vorträge über die Kinetische Gastheorie*, pp 46-60, Leipzig, B.G. Teubner.
15. Brillouin, L., 1922, *Ann. de Phys.*, **17**, 88.
16. Brillouin, L., 1936, La chaleur spécifique des liquides et leur constitution, *J. Phys. Rad, Serie VII, Tome VII, N. 4*, 153-157
17. Frenkel, J., 1946, *Kinetic theory of liquids*, Oxford, Oxford University Press.
18. Monaco G., Cunsolo A., Ruocco G., Sette F., 1999, Viscoelastic behaviour of water in the THz frequency range: an inelastic X-ray study., *Phys. Rev.E*, **60-5**, 5505-5521.
19. Cunsolo A., Ruocco G., Sette F., Masciovecchio, C., Mermet, A., Monaco, G., Sampoli, M., Verbeni, R., 1999, Experimental determination of the structural relaxation in liquid water, *Phys. Rev. Lett.*, **82-4**, 775-778.
20. Cunsolo, A., 2015, The terahertz spectrum of density fluctuations of water: the viscoelastic regime, *Adv Cond. Matt. Phys.*, **2015**, 137435-137459.
21. Cunsolo, A., 2017, The terahertz dynamics of simplest fluids probed by X-ray scattering, *International Review in Physical Chemistry*, **36-3**, 433-539.
22. Cunsolo, A., 2017, Inelastic X-ray scattering as a probe of the transition between the hydrodynamic and the single particle regimes in simple fluids., <http://dx.doi.org/10.5772/66126>.
23. Sette, F., Ruocco, G., Krisch, M., Bergmann, U., Masciovecchio, C., Mazzacurati, V., Signorelli, G., Verbeni, R., 1995, Collective dynamics in water by high-energy resolution inelastic X-ray scattering, *Phys. Rev. Lett.*, **75**, 850-854.
24. Balucani, U., Brodholt, J.P., Vallauri, R., 1996, Dynamical properties of liquid water, *J. Phys. Cond. Matt.*, (**8**)**47**, 9269-9274.
25. Ruocco, G., Sette, F., Bergmann, U., Krisch, M., Masciovecchio, C., Mazzacurati, V., Signorelli, G., Verbeni, R., 1996, Equivalence of the sound velocity in water and ice at mesoscopic lengths, *Nature*, **379**, 521-523.
26. Sette, F., Ruocco, G., Krisch, M., Masciovecchio, C., Verbeni, R., 1996, Collective dynamics in water by inelastic X-ray scattering, *Physica Scripta*, **T66**, 48-56.
27. Ruocco, G., Sette, F., Krisch, M., Bergmann, U., Masciovecchio, C., Verbeni, R., 1996, Line broadening in the collective dynamics of liquid and solid water, *Phys. Rev. B*, **54**, 14892-14895.
28. Sampoli, M., Ruocco, G., Sette F., 1997, Mixing of longitudinal and transverse dynamics in liquid water, *Phys. Rev. Lett.*, **79**, 1678.

29. Ruocco G., Sette, F., The high-frequency dynamics of liquid water, 1999, *J. Phys. Cond. Matt*, **11**, R259-R293.
30. Maxwell, J.C., 1867, On the dynamical theory of gases, Ph. Trans. Royal Soc. London, **157**, 49-88.
31. see for instance De Groot, S.R., Mazur, P., 1984, *Non-equilibrium thermodynamics*, New York, Dover pub. Inc..
32. Onsager, L., 1931, Reciprocal relations in irreversible processes-I Phys. Rev. **37**, 405-426
33. Onsager, L., 1931, Reciprocal relations in irreversible processes-II, Phys. Rev, **38**, 2265-2279.
34. Herzfeld, K.F., Litovitz, T.A., 1959, *Absorption and dispersion of ultrasonic waves*, New York, Academic Press.
35. Eckart, C., 1948, The theory of anelastic fluid, Rev. Mod. Phys., **20-1**, 232-235
36. Trachencko, K., Brazhkin, V., 2020, Minimal quantum viscosity from fundamental physical constants, Sci. Adv. **6**, eaba3747.
37. Trachencko, K., Brazhkin, V., Baggioli, M., 2020 Similarity between the kinematic viscosity of quark-gluon plasma and liquids at the viscosity minimum., arXiv: 2003.135060.v1.
38. Trachencko, K., Baggioli, M., Behnia, K., Brazhkin, V., 2021, Universal lower bounds on energy and momentum diffusion in liquids, Phys. Rev. B, **103**, 014311.
39. Trachencko, K., Monserrat, B., Pickard, C.J., Brazhkin, V., 2021, Speed of sound from fundamental physical constant, Sci. Adv. **6**, eabc8662.
40. Trachencko, K., Brazhkin, V., 2020, The Purcel question: Why do all viscosities stop at the same place?, arXiv: 2012.00376v1.

## 6. Figures

**Figure 1**



$$\Delta\varepsilon^{wp} = h\langle\nu_1\rangle - h\langle\nu_2\rangle = \Delta E_k^p + \Delta\Psi^p = f^{th} \cdot \langle\Lambda_p\rangle$$

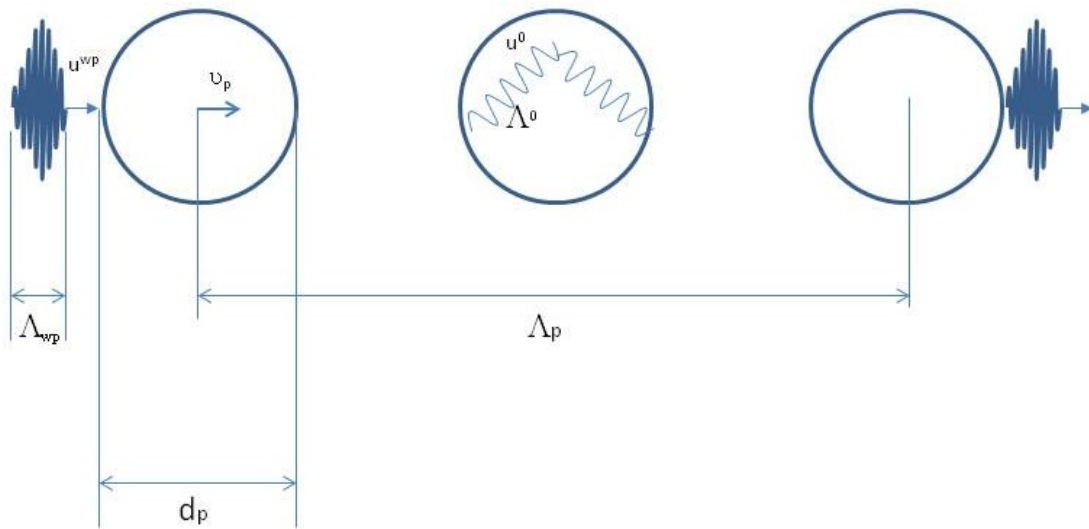
$$\Delta p^{wp} = f^{th} \cdot \langle\Lambda_p\rangle$$

**Figure 1.** Schematic representation of inelastic collisions between wave-packets and liquid particles. The event represented in (a), in which an energetic wave-packet transfers energy and momentum to a liquid particle, is commuted upon time reversal into the one represented in (b), where a liquid particle transfers energy and momentum to a wave-packet. The particle changes velocity and the frequency of wave-packet is shifted by the amount  $(\nu_2 - \nu_1)$ . Due to its time symmetry, we assume this mechanism is the equivalent of Onsager's reciprocity law at microscopic level [31-33].

In a pure isothermal liquid energy and momentum exchanged among the icebergs are statistically equivalent, and no net effects are produced. Events of type a) will alternate with events of type b), to keep the balance of the two energy pools unaltered. Besides, the macroscopic equilibrium will ensure also the mesoscopic equilibrium; events (a) and (b) will be equally probable along any direction, to have a zero average over time and space. On the contrary, if a symmetry breaking is introduced, as for instance a temperature or a concentration gradient, one type of event will prevail over the other along a preferential direction.

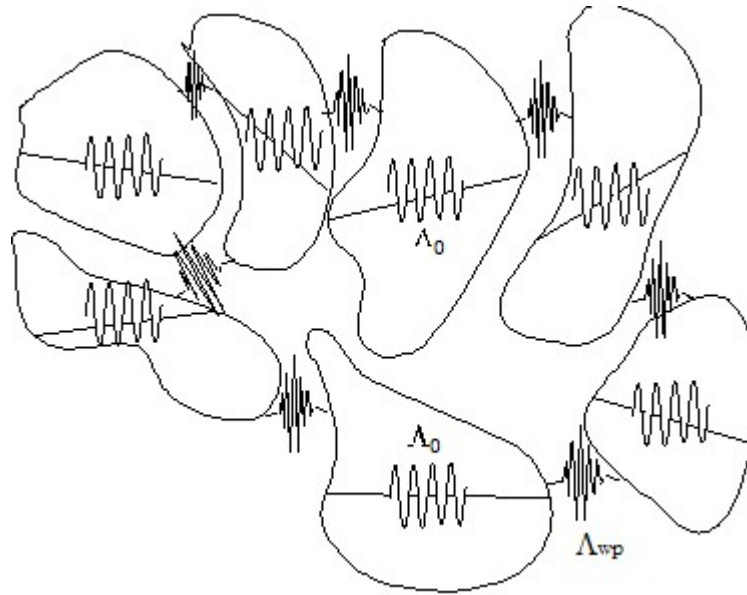
(Re-drawn after Peluso, F., [1])

**Figure 2**



**Figure 2.** Close-up of the first part of the wave-packet – particle interaction shown in Figure 1a, during which the phonon transfers energy and momentum to the liquid particle.  $\Lambda_{wp}$  is the extension of the wave-packet, and  $d_p$  that of the liquid particle. Once  $\tau_p$  has elapsed and the liquid particle has travelled by  $\Lambda_p$ , the particle relaxes the energy stored into internal DoF; then it travels by  $\Lambda_R$  during  $\tau_R$  (not shown in the figure above).

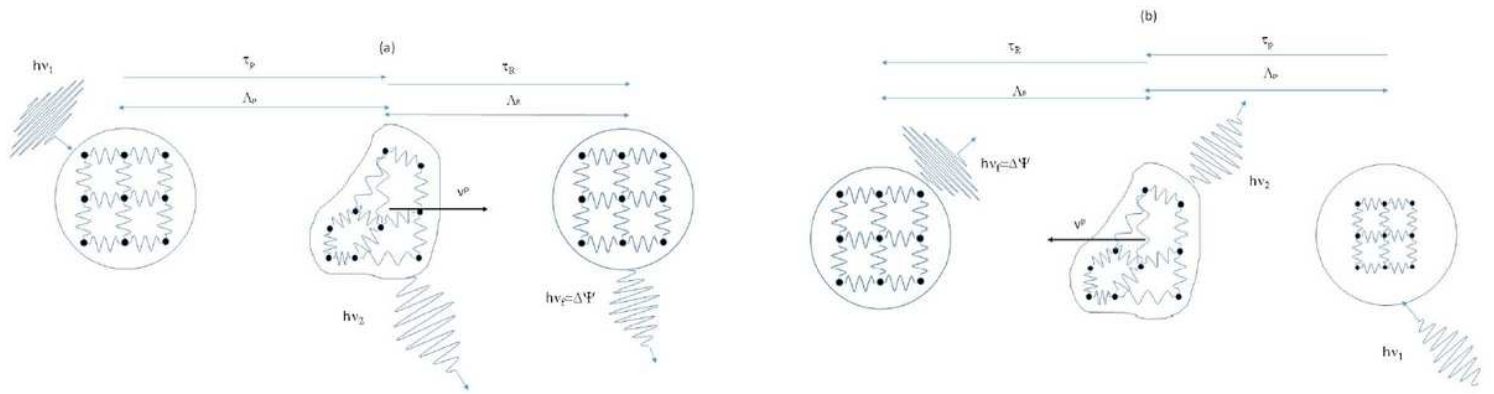
**Figure 3**



**Figure 3.** Icebergs of solid lattice fluctuating and interacting within the liquid global system at equilibrium. As far as elastic (thermal) perturbations propagate within an iceberg, they behave as in solids. Propagation velocity has then the values typical as those of the solid lattice, as found by Ruocco et al. [25] of about 3200 m/s for the case of water. Average sizes of icebergs  $\langle \Lambda_0 \rangle$  have been found of some nanometers. When perturbations cross the boundary between two icebergs,  $f^{th}$  develops and energy and momentum are transmitted from one to the nearest-neighbour iceberg. This pictorial model of liquids at mesoscopic scale, on which the DML is based, reflects also what may be deduced from experiments performed with IXS techniques, able to observe liquids at such scale-lengths.

In a solution solute particles may be considered as icebergs having elastic impedance different from that of the solvent. Energy and momentum exchanged between the two types of icebergs produce a net effect resulting in the diffusion of the solute along the concentration gradient. If a temperature gradient is imposed externally, the net effect will depend on the prevailing flux of wave packets, which will give rise to thermal diffusion of one species with respect to the other.

# Figures

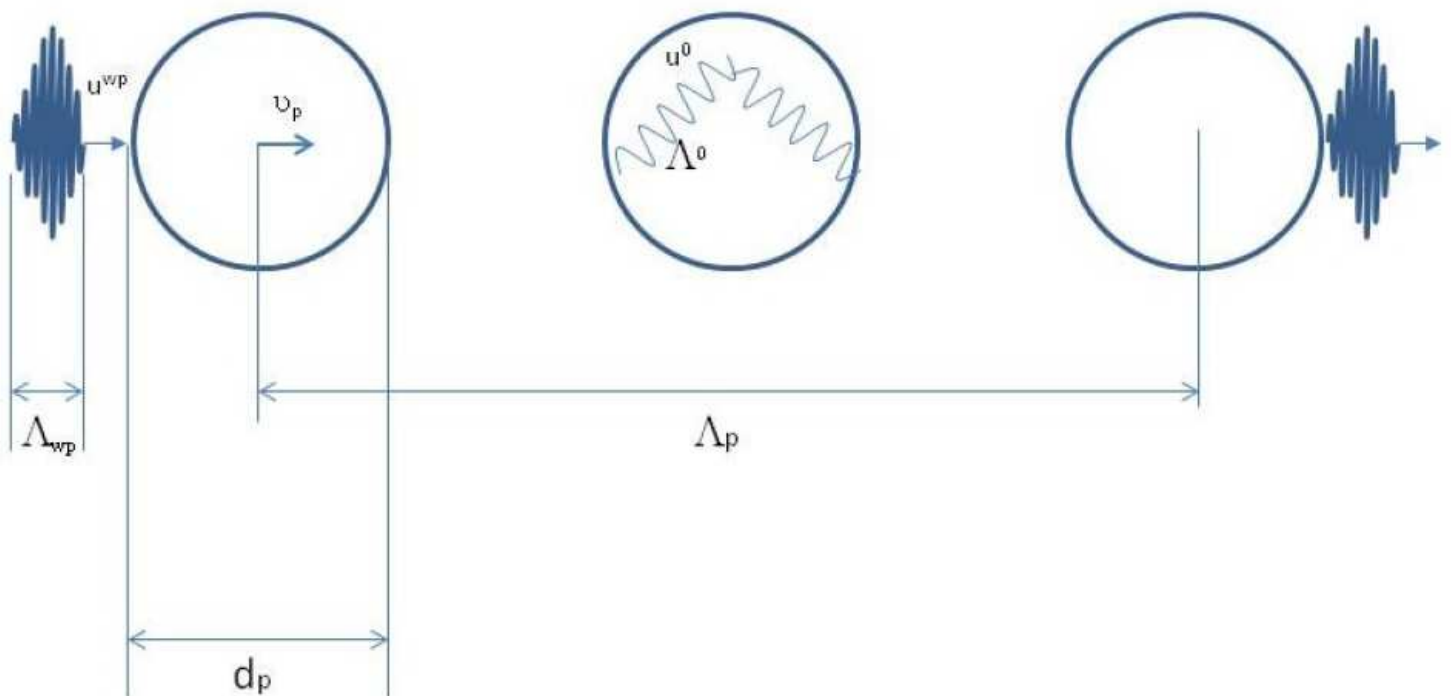


$$\Delta \varepsilon^{wp} = h\langle \nu_1 \rangle - h\langle \nu_2 \rangle = \Delta E_k^p + \Delta \Psi^p = f^{th} \cdot \langle \Lambda_p \rangle$$

$$\Delta p^{wp} = f^{th} \cdot \langle \Lambda_p \rangle$$

**Figure 1**

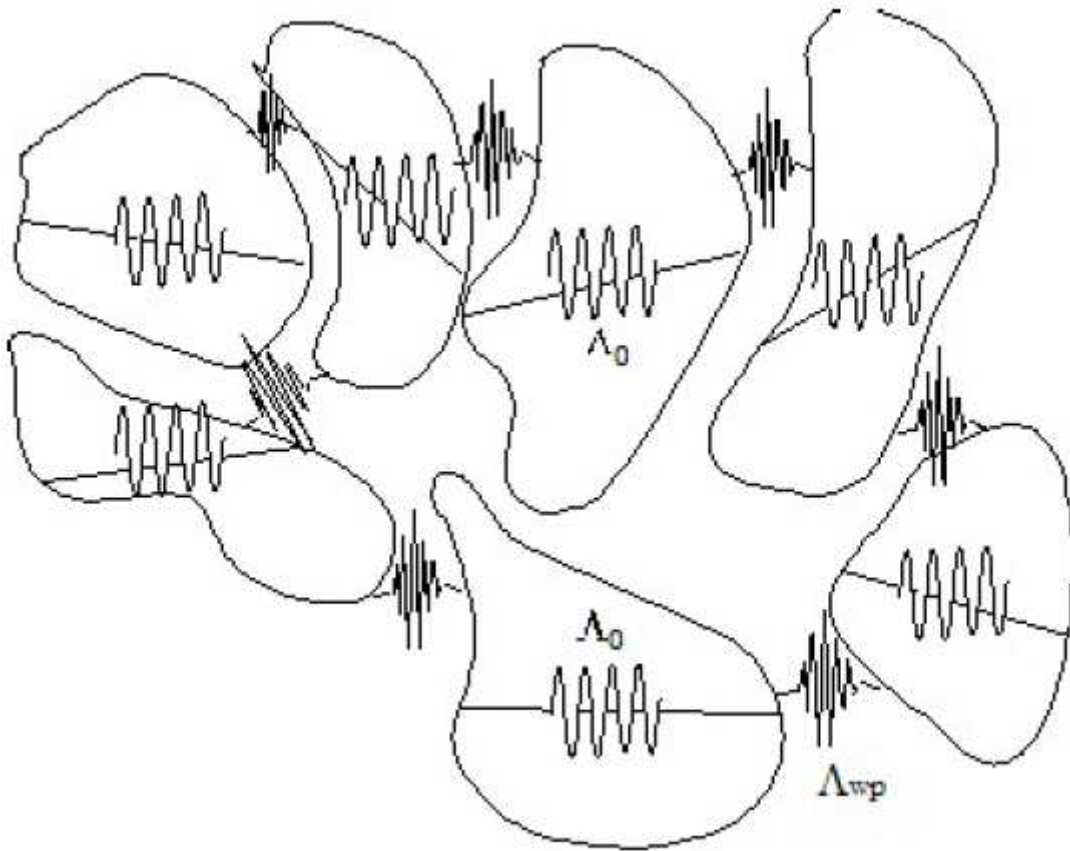
Schematic representation of inelastic collisions between wave-packets and liquid particles. Please see manuscript .pdf for full caption.



**Figure 2**

Close-up of the first part of the wave-packet – particle interaction shown in Figure 1a, during which the phonon transfers energy and momentum to the liquid particle. Please see manuscript .pdf for full

caption.



**Figure 3**

Icebergs of solid lattice fluctuating and interacting within the liquid global system at equilibrium. Please see manuscript .pdf for full caption.