Isolation of the Weyl conformal tensor for Einstein manifolds

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(Communicated by Heisuke HIRONAKA, M. J. A., Sept. 12, 2002)

Abstract: An isolation theorem of Weyl conformal tensor of positive Einstein manifolds is given, when its $L^{n/2}$-norm is small.

Key words: Einstein manifold; Weyl conformal tensor; Yamabe metric; Sobolev inequality.

1. Introduction. The aim of this note is to show that the Weyl conformal tensor of an oriented positive Ricci Einstein n-manifold, $n \geq 4$ obeys the following isolation theorem.

Theorem. *Let (M, g) be a compact, connected oriented Einstein n-manifold, $n \geq 4$, with positive scalar curvature s and of* $\mathrm{Vol}(g) = 1$. *Then, there exists a constant $C(n)$, depending only on n such that if $L^{n/2}$-norm $\|W\|_{L^{n/2}} < C(n)\, s$, then $W = 0$ so that (M, g) is a finite isometric quotient of the standard n-sphere of unit volume.*

This theorem generalizes the isolation theorem given by M. Singer [8].

Theorem [8]. *Let (M, g) be a compact, connected oriented Einstein n-manifold ($n = 2m \geq 4$) with non-vanishing Euler characteristic $\chi(M)$ and of positive scalar curvature. Then there is a constant $\varepsilon > 0$, depending on n and $\chi(M)$, such that if the $L^{n/2}$-norm $\|W\|_{L^{n/2}} < \varepsilon$, then $W = 0$ and so (M, g) is isometric to a quotient of S^n with the canonical metric up to constant.*

Our theorem is valid even in odd dimension. For related results see [2].

The idea of the proof of our theorem is based on the Weitzenböck formula on the operator $d_L : C^\infty(\Omega^1 \otimes \Omega^2) \longrightarrow C^\infty(\Omega^2 \otimes \Omega^2)$ exploited by the first author in [5] and also on a crucial use of the Sobolev inequality relating to Yamabe metrics.

2. Sobolev inequality. The Sobolev inequality of a compact Riemannian n-manifold (M, g), $n \geq 3$, can be described in terms of Yamabe metrics. Refer also to [1, 4, 7].

Consider the Yamabe functional $Q_g : C^\infty_+(M) \longrightarrow \mathbf{R}$,

$$(1) \quad f \mapsto Q_g(f) = \Big\{4\frac{n-1}{n-2}\int_M |\nabla f|^2 dv_g + \int_M s f^2 dv_g\Big\} \Big/ \Big\{\int_M f^p dv_g\Big\}^{(2/p)},$$

$p = (2n/n-2)$. The infimum of $Q_g(f)$ is called the Yamabe constant $\mu([g])$, a conformal invariant of the conformal class $[g]$;

$$(2) \qquad \mu([g]) = \inf\{Q_g(f) \mid f \in C^\infty_+(M)\}.$$

By the completely solved Yamabe problem there exists for any metric g an $f \in C^\infty_+(M)$ satisfying $Q_g(f) = \mu([g])$. So, the metric $\tilde{g} = f^{(4/n-2)} g$, called the Yamabe metric, a conformal change of g has constant scalar curvature given by $\tilde{s} = \mu([g])\,\mathrm{Vol}(\tilde{g})^{-(2/n)}$. Namely

$$(3) \qquad \mu([g]) = \tilde{s}\,\mathrm{Vol}(\tilde{g})^{(2/n)}.$$

We take a Yamabe metric in the conformal class $[g]$, denoted by g and then obtain the Sobolev inequaility

$$(4) \quad 4\frac{n-1}{n-2}\|\nabla f\|^2_{L^2} \geq s\,\mathrm{Vol}(g)^{(2/n)}\big\{\|f\|^2_{L^p} - \mathrm{Vol}(g)^{-(2/n)}\|f\|^2_{L^2}\big\}, \quad f \in H^2_1(M)$$

where $p = (2n/n-2)$.

Normalizing volume, we have

Proposition 0.1. *Let g be a Yamabe metric on M in a conformal class and with unit volume. Then*

$$(5) \qquad 4\frac{n-1}{n-2}\|\nabla f\|^2_{L^2} \geq s\big\{\|f\|^2_{L^p} - \|f\|^2_{L^2}\big\}, \quad f \in H^2_1(M).$$

2000 Mathematics Subject Classification. 53C25.

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Remark 0.2. The inequality (5) holds when f is replaced by any tensor T because of the Kato's inequality

$$|\nabla|T|| \leq |\nabla T|. \tag{6}$$

Remark 0.3. Theorem 1.3 in [4] (see also Proposition 3.1, [7]) indicates that any Einstein metric must be Yamabe, provided it is not conformally flat.

3. Proof of Theorem. We assume that W does not vanish identically. So the Einstein metric g is, from the above remark, a Yamabe metric in the conformal class $[g]$.

We normalize g by constant rescaling so that $\mathrm{Vol}(g) = 1$.

As observed in [8] by M. Singer the Weyl conformal tensor W satisfies

(7)
$$0 = (d_L\delta_L + \delta_L d_L)W = \nabla^*\nabla W + \frac{2s}{n}W + \{W, W\}.$$

Here $\{\cdot,\cdot\}$ denotes a certain quadratic combination of W.

So we get

$$\begin{aligned}\|\nabla W\|_{L^2}^2 + \frac{2s}{n}\|W\|_{L^2}^2 &\leq C_n^{-1}\|W\|_{L^3}^3 \\ &\leq C_n^{-1}\|W\|_{L^{n/2}}\|W\|_{L^p}^2.\end{aligned} \tag{8}$$

Here we used the Hölder inequality together with the pointwise inequality

$$|\langle\{W, W\}, W\rangle| \leq C_n^{-1}|W|^3 \tag{9}$$

for a constant $C_n > 0$, depending only on n. Applying the Sobolev inequality in Proposition 0.1 yields

$$\begin{aligned}&C_n^{-1}\|W\|_{L^{n/2}}\|W\|_{L^p}^2 \\ &\quad\geq \frac{2s}{n}\|W\|_{L^2}^2 + \frac{n-2}{4(n-1)}s(\|W\|_{L^p}^2 - \|W\|_{L^2}^2).\end{aligned} \tag{10}$$

Assume $4 \leq n \leq 9$. Then
$(2/n) - ((n-2)/4(n-1)) > 0$ so that

$$C_n^{-1}\|W\|_{L^{n/2}} \geq \frac{n-2}{4(n-1)}s. \tag{11}$$

If, contrarily, $n \geq 10$, it holds $(2/n) - (n-2/4(n-1)) < 0$. However $\|W\|_{L^2}^2 \leq \|W\|_{L^p}^2$, since $p > 2$. So,

$$\begin{aligned}&\Big(\frac{2}{n} - \frac{n-2}{4(n-1)}\Big)s\|W\|_{L^2}^2 \\ &\quad\geq \Big(\frac{2}{n} - \frac{n-2}{4(n-1)}\Big)s\|W\|_{L^p}^2.\end{aligned} \tag{12}$$

We have thus

$$\begin{aligned}&C_n^{-1}\|W\|_{L^{n/2}}\|W\|_{L^p}^2 \\ &\geq \Big\{\Big(\frac{2}{n} - \frac{n-2}{4(n-1)}\Big)s + \frac{n-2}{4(n-1)}s\Big\}\|W\|_{L^p}^2 \\ &= \frac{2}{n}s\|W\|_{L^p}^2\end{aligned} \tag{13}$$

giving rise to $\|W\|_{L^{n/2}} \geq (2/n)C_n s$.

Therefore, if we put $C(n)$ as

$$C(n) = \frac{n-2}{4(n-1)}C_n, \qquad 4 \leq n \leq 9, \tag{14}$$

$$= \frac{2}{n}C_n, \qquad 10 \leq n, \tag{15}$$

then we get a contradiction giving the complete proof.

4. Remarks. A pointwise isolation theorem is similarly obtained as

Theorem 0.4. Let (M, g) be a compact connected oriented Einstein n-manifold ($n \geq 4$) of scalar curvature $s > 0$. If $|W| \leq (2/n)C_n s$ holds everywhere and strict inequality holds at a point, then $W = 0$, that is, (M, g) is, up to a constant scale, a finite isometric quotient of the standard n-sphere.

In four dimension we have a similar pointwise isolation theorem for the (anti-)self-dual Weyl tensor $W^\pm$ for a wide class of 4-manifolds satisfying $\delta^\pm W^\pm = 0$ in [6]. Remark also that M. Gursky showed the following in [3]:

Let M be a compact, connected, oriented 4-manifold and g a metric on M which satisfies $\delta^+ W^+ = 0$ and the Yamabe constant $\mu([g]) > 0$. If

$$\int_M |W^+|^2\, dv_g \leq \frac{4}{3}\pi^2(2\chi(M) + 3\tau(M)),$$

then, (i) g is anti-self-dual, i.e., $W^+ = 0$, or (ii) g is a positive Einstein metric which is either Kähler or the quotient of a Kähler metric by an isometric anti-holomorphic involution.

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