# ISOMETRIC DEFORMATIONS OF HYPERSURFACES IN A EUCLIDEAN SPACE PRESERVING MEAN CURVATURE 

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#### Abstract

Under some conditions we classify hypersurfaces in a Euclidean space which admit isometric deformations preserving mean curvature.


1. Introduction. The isometric deformations of surfaces in a 3-dimensional space form preserving mean curvature (which are called $H$-deformations) have been studied by a number of mathematicians. Bonnet [2] showed over a century ago that all the surfaces with constant mean curvature in a Euclidean 3-space except planes or spheres are locally $H$-deformable. Referring to this problem in the case of surfaces with non-constant mean curvature, the work done by Cartan [3] is authoritative and Chern [4] gave an interesting characterization for their existence.

Recently, Colares and Kenmotsu [5] classified $H$-deformable surfaces with constant Gaussian curvature in a Euclidean 3-space. Umehara [9] proved that a compact surface in a 3-dimensional space form is locally $H$-deformable if and only if it has constant mean curvature.

In this paper, we study such a deformation of hypersurfaces in $\boldsymbol{R}^{n}$ (as a direct generalization of that of a surface in $\boldsymbol{R}^{3}$ ).

Definition. Let $f: M^{n} \leftrightarrows \boldsymbol{R}^{n+1}$ be an isometric immersion as a hypersurface of an $n$-dimensional Riemannian manifold and $H$ the mean curvature of $f$.

An $H$-deformation of the immersion $f$ is a continuous mapping $F:(-\varepsilon, \varepsilon) \times M^{n} \rightarrow$ $\boldsymbol{R}^{n+1}(\varepsilon>0)$ such that
(1.1) $f_{t}:=F(t, \cdot)$ for any fixed $t \in(-\varepsilon, \varepsilon)$ is an isometric immersion whose mean curvature is equal to $H$,
(1.2) $f_{0}=f$.

An $H$-deformation is said to be trivial if for each $t \in(-\varepsilon, \varepsilon)$, there exists a motion $T_{t}$ of $\boldsymbol{R}^{n+1}$ such that $f_{t}=T_{t} \circ f . f$ is said to be $H$-deformable if there exists a non-trivial $H$-deformation. $f$ is said to be locally $H$-deformable if for each point of $M^{n}$ there exists a neighborhood $U$ such that $\left.f\right|_{U}$ is $H$-deformable.

First we mention the following well-known theorem:
Theorem A (Beez [1], Killing [7]). A hypersurface $M$ is rigid in $\boldsymbol{R}^{n+1}$ if the type
number is greater than 2 at every point (where the type number is the rank of the shape operator as a linear transformation of the tangent spaces of $M$ ).

Consequently, such a hypersurface as that in Theorem A cannot admit a non-trivial $H$-deformation. From such a point of view, we deal only with hypersurfaces with type number $\leq 2$.

Our main result is the following.
Theorem. Let $f: M^{n} \hookrightarrow \boldsymbol{R}^{n+1}$ be an isometric immersion with type number 2 which has distinct non-zero principal curvatures. If $f$ is locally $\mathbf{H}$-deformable, then $f$ is one of the following:
(a) a minimal immersion
(b) an open piece of a cylinder $M^{2} \times \boldsymbol{R}^{n-2}$, where $M^{2}$ is a locally H-deformable surface in $\boldsymbol{R}^{3}$
(c) an open piece of a cylinder $C N \times \boldsymbol{R}^{n-3}$, where $N$ is a locally $H$-deformable surface in $\boldsymbol{S}^{3}$ and $C N$ is a cone over $N$ in $\boldsymbol{R}^{4}$
(d) a hypersurface mixed with that of (a), (b) and (c).

Moreover, we assume the real analyticity of $f$ and $M^{n}$. Then $f$ is H-deformable if and only if $f$ is (a) or (b) or (c).

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2. Preliminaries. We use the moving frame method.

Assume that $M^{n}$ is immersed as a hypersurface in $\boldsymbol{R}^{n+1}$ with constant type number $d(0 \leq d \leq n)$ and that its non-zero principal curvatures are distinct. Let $A$ denote the shape operator. From now on we shall use the following convention on the ranges of indices:

$$
\begin{gathered}
1 \leq A, B, C, \cdots \leq n \\
1 \leq i, j, k, \cdots \leq d \\
d+1 \leq p, q, r, \cdots \leq n
\end{gathered}
$$

Choose a local orthonormal frame field $\left\{e_{1}, \ldots, e_{d}, e_{d+1}, \ldots, e_{n}\right\}$ of $M$ in such a way that $e_{1}, \ldots, e_{n}$ are principal vectors and their principal curvatures are $k_{1}, \ldots, k_{d}, 0, \ldots, 0$ respectively ( $k_{1} k_{2} \cdots k_{d} \neq 0, k_{i} \neq k_{j}$ for $i \neq j$ ). Let $\left\{\theta^{1}, \ldots, \theta^{n}\right\}$ be the dual frame field of $\left\{e_{A}\right\}$. The Levi-Civita connection of $M$ is denoted by $\nabla$ and the connection form of $\nabla$ is denoted by

$$
\omega=\left(\omega_{B}^{A}\right) \quad \text { with respect to }\left\{e_{A}\right\} .
$$

Then the shape operator $A$ can be written as

$$
A=\left(\begin{array}{cccccc}
k_{1} & & & & &  \tag{2.1}\\
& \ddots & & & 0 & \\
& & k_{d} & & & \\
& & & 0 & & \\
& 0 & & & \ddots & \\
& & & & 0
\end{array}\right)
$$

Let $\Omega=\left(\Omega_{B}^{A}\right)$ be the curvature form of $M$. Then the Gauss equation implies

$$
\begin{equation*}
\Omega_{j}^{i}=k_{i} k_{j} \theta^{i} \wedge \theta^{j}, \quad \Omega_{q}^{i}=0, \quad \Omega_{q}^{p}=0 \tag{2.2}
\end{equation*}
$$

and the Codazzi equation is given by

$$
\begin{equation*}
\left(\nabla_{e_{A}} A\right) e_{B}=\left(\nabla_{e_{B}} A\right) e_{A} \tag{2.3}
\end{equation*}
$$

Lemma 2.1. Under the above assumption, the Codazzi equation (2.3) is equivalent to the following equations.

$$
\begin{align*}
& e_{i} k_{j}=\left(k_{i}-k_{j}\right) \omega_{i}^{j}\left(e_{j}\right) \quad \text { for } \quad i \neq j \\
& \left(k_{j}-k_{l}\right) \omega_{j}^{l}\left(e_{i}\right)=\left(k_{i}-k_{l}\right) \omega_{i}^{l}\left(e_{j}\right) \quad \text { for } \quad l \neq i, j \\
& k_{i} \omega_{i}^{p}\left(e_{j}\right)=k_{j} \omega_{j}^{p}\left(e_{i}\right) \\
& e_{p} k_{i}=k_{i} \omega_{i}^{p}\left(e_{i}\right)  \tag{2.4}\\
& k_{j} \omega_{j}^{p}\left(e_{i}\right)=\left(k_{i}-k_{j}\right) \omega_{i}^{j}\left(e_{p}\right) \quad \text { for } \quad i \neq j \\
& \omega_{i}^{q}\left(e_{p}\right)=0 .
\end{align*}
$$

From the last three equations of (2.4), we have

$$
\begin{equation*}
\omega_{p}^{i}=-\frac{e_{p} k_{i}}{k_{i}} \theta^{i}+\sum_{j \neq i} \frac{k_{i}-k_{j}}{k_{i}} \omega_{j}^{i}\left(e_{p}\right) \theta^{j} . \tag{2.5}
\end{equation*}
$$

Furthermore, it follows from (2.5) and the structure equation $d \theta^{A}=-\sum \omega_{B}^{A} \wedge \theta^{B}$ that

$$
\begin{gather*}
d \theta^{i}=-\sum_{j} \omega_{j}^{i} \wedge \theta^{j}+\sum_{p} \frac{e_{p} k_{i}}{k_{i}} \theta^{i} \wedge \theta^{p}-\sum_{p, j \neq i} \frac{k_{i}-k_{j}}{k_{i}} \omega_{j}^{i}\left(e_{p}\right) \theta^{j} \wedge \theta^{p}  \tag{2.6}\\
d \theta^{p}=\sum_{i<j} \frac{\left(k_{i}-k_{j}\right)^{2}}{k_{i} k_{j}} \omega_{j}^{i}\left(e_{p}\right) \theta^{i} \wedge \theta^{j}-\sum_{q} \omega_{q}^{p} \wedge \theta^{q} \tag{2.7}
\end{gather*}
$$

Thus we have proved the following lemma.
Lemma 2.2. (a) The distribution $V^{0}$ defined by $\theta^{1}=\cdots=\theta^{d}=0$ is completely integrable.
(b) The distribution $V^{1}$ defined by $\theta^{d+1}=\cdots=\theta^{n}=0$ is completely integrable if and only if $\left(k_{i}-k_{j}\right) \omega_{j}^{i}\left(e_{p}\right)=0$ holds for all $i, j, p$.

Remark. It is easy to see that the integral manifolds of the distribution $V^{0}$ are totally geodesic submanifolds in $\boldsymbol{R}^{n+1}$. So we can choose $e_{d+1}, \ldots, e_{n}$ in such a way that $\nabla_{e_{p}} e_{q}=0$, i.e., $\omega_{p}^{A}\left(e_{q}\right)=0$ holds. From now on we assume that $e_{p}$ 's are chosen in that way.
3. $H$-deformability of hypersurfaces of constant type number 2 . We shall apply formulas obtained in the previous section to the case $d=2$.

Let $f: M^{n} \hookrightarrow \boldsymbol{R}^{n+1}$ be an immersed hypersurface with type number 2 and $H$ the mean curvature. Assume that the two non-zero principal curvatures are distinct at every point.

Suppose that $f$ is locally $H$-deformable. Then there exists a simply-connected neighborhood $U$ for an arbitrary point $x_{0}$ in $M^{n}$ and an isometric immersion $f^{\prime}: U G \boldsymbol{R}^{n+1}$ with mean curvature $H$ such that $f^{\prime} \neq f$ on $U$. In other words, there exists a symmetric (1, 1)-tensor field $A^{\prime}$ on $U$ satisfying the Gauss equation, the Codazzi equation and $\operatorname{tr} A^{\prime}=n H$, because of the fundamental theorem for hypersurfaces. The type number of $f^{\prime}$ is also equal to 2 . This follows from the fact that the null space of the shape operator does not depend on the immersion in this case, because of the following theorem.

Theorem B ([8, Theorem 6.1]). For an isometric immersion $M^{\boldsymbol{n}} \varsigma \boldsymbol{R}^{\boldsymbol{n + 1}}$, if the type number $\geq 2$ at a point $x$, then $\operatorname{ker} A_{x}=\left\{X \in T_{x} M ; R(X, Y)=0\right.$ for all $\left.Y \in T_{x} M\right\}$, where $R$ denotes the curvature tensor of $M$.

Consider two orthonormal frame fields $\left\{e_{A}\right\}$ and $\left\{e_{A}^{\prime}\right\}$ on $U$ which consist of principal vectors of $f$ and $f^{\prime}$, respectively, and let $k_{1}, k_{2}, 0, \ldots, 0$ (resp. $k_{1}^{\prime}, k_{2}^{\prime}, 0, \ldots, 0$ ) be principal curvatures with respect to $f$ (resp. $f^{\prime}$ ). From the Gauss equations and $\operatorname{tr} A=\operatorname{tr} A^{\prime}$, we have $k_{1} k_{2}=k_{1}^{\prime} k_{2}^{\prime}, k_{1}+k_{2}=k_{1}^{\prime}+k_{2}^{\prime}$, so we may assume $k_{1}=k_{1}^{\prime}, k_{2}=k_{2}^{\prime}$. Then from the above consideration,

$$
\operatorname{ker} A_{x}=\operatorname{ker} A_{x}^{\prime} \quad \text { at every } \quad x \in U
$$

i.e.,

$$
\begin{aligned}
\left\{\left.e_{1}\right|_{x},\left.e_{2}\right|_{x}\right\} & =\left\{\left.e_{1}^{\prime}\right|_{x},\left.e_{2}^{\prime}\right|_{x}\right\} \quad \text { at every } \quad x \in U \\
\left\{\left.e_{3}\right|_{x}, \ldots,\left.e_{n}\right|_{x}\right\} & =\left\{\left.e_{3}^{\prime}\right|_{x}, \ldots,\left.e_{n}^{\prime}\right|_{x}\right\} \quad \text { at every } \quad x \in U
\end{aligned}
$$

where $\{\cdots\}$ denotes the subspace of $T_{x} M$ spanned by $\cdots$. Thus the frame $\left\{e_{A}\right\}$ corresponds to $\left\{e_{A}^{\prime}\right\}$ by an $S O(2) \times S O(n-2)$-valued function. Therefore $A^{\prime}$ can be written as

$$
A^{\prime}=\left(\begin{array}{cc|ccc}
P^{-1}\left(\begin{array}{cc}
k_{1} & 0 \\
0 & k_{2}
\end{array}\right) P & & & \\
\hline & & 0 & & \\
& & & \ddots & \\
& & & 0
\end{array}\right) \text { with respect to }\left\{e_{A}\right\}
$$

for some $S O(2)$-valued function $P$. Putting

$$
P=\left(\begin{array}{cc}
\cos \xi & \sin \xi \\
-\sin \xi & \cos \xi
\end{array}\right), \quad \xi \in C^{\infty}(U)
$$

we have

$$
A^{\prime}=\frac{1}{2}\left(\begin{array}{cc|ccc}
k_{1}(1+\cos \tau)+k_{2}(1-\cos \tau) & \left(k_{1}-k_{2}\right) \sin \tau & & \\
\left(k_{1}-k_{2}\right) \sin \tau & k_{1}(1-\cos \tau)+k_{2}(1+\cos \tau) & & \\
\hline & & 0 & & \\
& & & \ddots & 0
\end{array}\right)
$$

where $\tau=2 \xi$.
Lemma 3.1. The Codazzi equation $\left(\nabla_{e_{A}} A^{\prime}\right) e_{B}=\left(\nabla_{e_{B}} A^{\prime}\right) e_{A}$ can be written as follows:
(3.1) $\left(e_{1}\left(k_{1}+k_{2}\right)\right) \sin \tau-\left(e_{2}\left(k_{1}+k_{2}\right)\right)(1-\cos \tau)+\left(k_{1}-k_{2}\right)\left\{\left(e_{1} \tau\right) \cos \tau+\left(e_{2} \tau\right) \sin \tau\right\}=0$
(3.2) $\quad\left(e_{1}\left(k_{1}+k_{2}\right)\right)(1-\cos \tau)+\left(e_{2}\left(k_{1}+k_{2}\right)\right) \sin \tau-\left(k_{1}-k_{2}\right)\left\{\left(e_{1} \tau\right) \sin \tau-\left(e_{2} \tau\right) \cos \tau\right\}=0$

$$
\begin{array}{r}
(1-\cos \tau)\left(k_{1}+k_{2}\right)\left(k_{1}-k_{2}\right) \omega_{2}^{1}\left(e_{p}\right)-\sin \tau\left\{\left(e_{p} k_{2}\right) k_{1}-\left(e_{p} k_{1}\right) k_{2}\right\}=0 \\
\frac{(1-\cos \tau)}{k_{1}}\left\{\left(e_{p} k_{2}\right) k_{1}-\left(e_{p} k_{1}\right) k_{2}\right\}+\frac{\sin \tau}{k_{2}}\left(k_{1}+k_{2}\right)\left(k_{1}-k_{2}\right) \omega_{2}^{1}\left(e_{p}\right) \\
-\left(k_{1}-k_{2}\right)\left(e_{p} \tau\right) \sin \tau=0 \\
\frac{\sin \tau}{k_{1}}\left\{\left(e_{p} k_{2}\right) k_{1}-\left(e_{p} k_{1}\right) k_{2}\right\}-\frac{(1-\cos \tau)}{k_{2}}\left(k_{1}+k_{2}\right)\left(k_{1}-k_{2}\right) \omega_{2}^{1}\left(e_{p}\right)  \tag{3.5}\\
-\left(k_{1}-k_{2}\right)\left(e_{p} \tau\right) \cos \tau=0
\end{array}
$$

$$
\begin{array}{r}
\frac{(1-\cos \tau)}{k_{2}}\left\{\left(e_{p} k_{2}\right) k_{1}-\left(e_{p} k_{1}\right) k_{2}\right\}+\frac{\sin \tau}{k_{1}}\left(k_{1}+k_{2}\right)\left(k_{1}-k_{2}\right) \omega_{2}^{1}\left(e_{p}\right)  \tag{3.6}\\
-\left(k_{1}-k_{2}\right)\left(e_{p} \tau\right) \sin \tau=0
\end{array}
$$

$$
\begin{align*}
\frac{\sin \tau}{k_{2}}\left\{\left(e_{p} k_{2}\right) k_{1}-\left(e_{p} k_{1}\right) k_{2}\right\}-\frac{(1-\cos \tau)}{k_{1}} & \left(k_{1}+k_{2}\right)\left(k_{1}-k_{2}\right) \omega_{2}^{1}\left(e_{p}\right)  \tag{3.7}\\
& -\left(k_{1}-k_{2}\right)\left(e_{p} \tau\right) \cos \tau=0 .
\end{align*}
$$

Furthermore, the equations (3.4)-(3.7) are equivalent to

$$
\begin{gather*}
\left(e_{p} k_{2}\right) k_{1}-\left(e_{p} k_{1}\right) k_{2}=0  \tag{3.8}\\
\left(k_{1}+k_{2}\right) \omega_{2}^{1}\left(e_{p}\right)=0  \tag{3.9}\\
e_{p} \tau=0 \tag{3.10}
\end{gather*}
$$

Proof. Apply the formulas (2.4), (2.5) to the case $d=2$. Then (3.1), (3.2), (3.3) are obtained from $\left(\nabla_{e_{1}} A^{\prime}\right) e_{2}=\left(\nabla_{e_{2}} A^{\prime}\right) e_{1}$ and (3.4), (3.5), (3.6), (3.7) from ( $\left.\nabla_{e_{i}} A^{\prime}\right) e_{p}=$ $\left(\nabla_{e_{p}} A^{\prime}\right) e_{i}(i=1,2) .\left(\nabla_{e_{p}} A^{\prime}\right) e_{q}=\left(\nabla_{e_{q}} A^{\prime}\right) e_{p}$ holds automatically. The latter part is obvious.

Let (*) stand for the equations (3.1)-(3.3) and (3.8)-(3.10). Therefore we have:
Proposition 3.2. If $\left.f\right|_{U}$ is $H$-deformable, then there exists $\tau \in C^{\infty}(U)$ satisfying the equations (*).

Conversely, if there exists a one-parameter family of functions $\tau_{t}\left(\tau_{0} \equiv 0\right)$ satisfying (*), then $\left.f\right|_{U}$ is $H$-deformable.

Remark. It is remarked that a necessary and sufficient condition for the $H$-deformability of a surface in a 3-dimensional space form is the existence of a one-parameter family of functions $\tau_{t}$ satisfying (3.1) and (3.2).
4. Proof of Theorem. We investigate the hypersurfaces satisfying (*) in this section. From (3.9), we consider the following three cases:
Case 1. $x_{0}$ has a neighborhood such that $k_{1}+k_{2} \equiv 0$ holds, i.e., a piece of minimal hypersurface.
Case 2. $x_{0}$ has a neighborhood such that $\omega_{2}^{1}\left(e_{p}\right) \equiv 0$ holds.
Case 3. Neither $k_{1}+k_{2} \equiv 0$ nor $\omega_{2}^{1}\left(e_{p}\right) \equiv 0$ holds in any neighborhood of $x_{0}$, but $\left(k_{1}+k_{2}\right) \omega_{2}^{1}\left(e_{p}\right) \equiv 0$ holds in some neighborhood.
Case 1. Assume that $k_{1}+k_{2}=0$. Then (3.8) holds, and (3.1) and (3.2) hold if and only if $e_{1} \tau=e_{2} \tau \equiv 0$. Thus a necessary and sufficient condition for ( $*$ ) is $\tau=$ constant. If we put $\tau_{t}=t$ for example, then $\tau_{t}$ defines an $H$-deformation. Therefore minimal hypersurfaces with type number 2 are locally H -deformable.

Case 2. It is remarked that the distribution $V^{1}$ generated by $e_{1}$ and $e_{2}$ is integrable in this case. It follows from (2.4) and (2.5) that $\omega_{2}^{1}$ is generated by $\theta^{1}$ and $\theta^{2}$ and

$$
\begin{equation*}
\omega_{p}^{i}=-\frac{e_{p} k_{i}}{k_{i}} \theta^{i} \quad(i=1,2) \tag{4.1}
\end{equation*}
$$

On the other hand, from (3.8) we may put

$$
\varphi_{p}:=\frac{e_{p} k_{1}}{k_{1}}=\frac{e_{p} k_{2}}{k_{2}}
$$

and

$$
\begin{equation*}
\omega_{p}^{i}=-\varphi_{p} \theta^{i} . \tag{4.2}
\end{equation*}
$$

Lemma 4.1. In Case $2, f(U)$ is of the form $M^{2} \times \boldsymbol{R}^{n-2}$ or $M^{3} \times \boldsymbol{R}^{n-3}$, or it is mixed with $M^{2} \times \boldsymbol{R}^{n-2}$ and $M^{3} \times \boldsymbol{R}^{n-3}$, where $M^{2}$ and $M^{3}$ are immersed in $\boldsymbol{R}^{3}$ and $\boldsymbol{R}^{4}$, respectively.

Proof. Assume that $\varphi_{p} \equiv 0$ for all $p$. It is easy to see that $\nabla_{e_{i}} e_{j} \in V^{1}$ so the integral manifold of the distribution $\left\{e_{1}, e_{2}\right\}$ is totally geodesic in $M^{n}$. Let $M^{2}$ be an integral manifold of the distribution $\left\{e_{1}, e_{2}\right\}$. On the other hand, if the Levi-Civita connection of the ambient space $R^{n+1}$ is denoted by $D$, then $D_{e_{i}} e_{p} \in V^{0}$ holds so that $V^{0}$ is parallel in $\boldsymbol{R}^{n+1}$. Therefore the integral manifolds of $V^{0}$ are prallel Euclidean subspaces, and

$$
T_{x} M^{2}=V_{x}^{1} \perp V_{x}^{0} \cong R^{n-2}
$$

at every point $x \in M^{n}$. So $M^{2}$ is contained in $\boldsymbol{R}^{3}$ and it is obvious that $f(U)=M^{2} \times \boldsymbol{R}^{n-2}$.
Otherwise, $x_{0}$ is not an interior point of $B:=\left\{x \in M^{n} \mid \varphi_{p}(x)=0\right.$ for all $\left.p\right\}$. That is,

$$
x_{0} \in B^{c} \quad \text { or } \quad x_{0} \in \partial B
$$

where $B^{c}$ and $\partial B$ denote the complementary set and the boundary set of $B$, respectively.
Whenever $x_{0}$ is a point of $B^{c}$, we can take a neighborhood where $\varphi_{p} \neq 0$ holds. We also write it as $U$. We can see that $\operatorname{dim} \Lambda=1$ holds at each point of $U$ if we set

$$
\Lambda=\left\{\text { the } V^{0} \text {-component of } \nabla_{X} Y ; X, Y \in V^{1}\right\}
$$

$W e$ choose a new frame field $\left\{e_{3}, \ldots, e_{n}\right\}$ such that $e_{3} \in \Lambda$. (The observations so far are applicable with respect to this frame.) Then $\varphi_{3} \neq 0, \varphi_{4}=\cdots=\varphi_{n}=0$ holds. Putting $\varphi_{3}=: \varphi$, we see that the connection form $\omega$ can be written as

$$
\left(\begin{array}{cccccc}
0 & \omega_{2}^{1} & -\varphi \theta^{1} & 0 & \cdots & 0  \tag{4.3}\\
-\omega_{2}^{1} & 0 & -\varphi \theta^{2} & 0 & \cdots & 0 \\
\varphi \theta^{1} & \varphi \theta^{2} & 0 & \omega_{4}^{3} & \cdots & \omega_{n}^{3} \\
0 & 0 & -\omega_{4}^{3} & & & \\
\vdots & \vdots & \vdots & & \omega_{p}^{q} & \\
0 & 0 & -\omega_{n}^{3} & & &
\end{array}\right)
$$

We compute $R\left(e_{1}, e_{2}\right) e_{p}$ for $p \geq 4$ using (4.3) directly:

$$
R\left(e_{1}, e_{2}\right) e_{p}=\nabla_{e_{1}} \nabla_{e_{2}} e_{p}-\nabla_{e_{2}} \nabla_{e_{1}} e_{p}-\nabla_{\left[e_{1}, e_{2}\right]} e_{p}=-\varphi \omega_{p}^{3}\left(e_{2}\right) e_{1}+\varphi \omega_{p}^{3}\left(e_{1}\right) e_{2}+\cdots
$$

However, $R\left(e_{1}, e_{2}\right) e_{p}=0$ holds by the Gauss equation. Thus $\omega_{p}^{3} \equiv 0$ for $\omega_{p}^{3}\left(e_{1}\right)=$ $\omega_{p}^{3}\left(e_{2}\right)=0$.

Then it is easy to see that the distribution spanned by $\left\{e_{1}, e_{2}, e_{3}\right\}$ is integrable and $f(U)$ is of the form $M^{3} \times \boldsymbol{R}^{n-3}$ by the argument analogous to that in the case $\varphi_{p} \equiv 0$.

Assume that $x_{0}$ is a boundary point of $B$. If $x_{0}$ has a neighborhood $U$ such that $U \cap B$ is nowhere dense in $U$, then $\varphi \neq 0$ holds and $f(U)=M^{3} \times \boldsymbol{R}^{n-3}$. Otherwise, $x_{0}$ is a boundary point of an open portion of $M^{2} \times \boldsymbol{R}^{n-2}$ and an open portion of $M^{3} \times$ $\boldsymbol{R}^{n-3}$.

We examine the case $\varphi \neq 0$ in the above lemma. We only consider the case of 3-dimensional hypersurfaces $M^{3}$. We give necessary formulas. The structure equations are

$$
\begin{align*}
& d \theta^{1}=-\omega_{2}^{1} \wedge \theta^{2}+\varphi \theta^{1} \wedge \theta^{3} \\
& d \theta^{2}=-\omega_{1}^{2} \wedge \theta^{1}+\varphi \theta^{2} \wedge \theta^{3}  \tag{4.4}\\
& d \theta^{3}=0 .
\end{align*}
$$

From the Gauss equations, we have

$$
\begin{align*}
d \omega_{2}^{1} & =\left(k_{1} k_{2}+(\varphi)^{2}\right) \theta^{1} \wedge \theta^{2} \\
0 & =\left(-d \varphi+(\varphi)^{2} \theta^{3}\right) \wedge \theta^{1}  \tag{4.5}\\
0 & =\left(-d \varphi+(\varphi)^{2} \theta^{3}\right) \wedge \theta^{2} .
\end{align*}
$$

It is immediately seen that

$$
\begin{equation*}
d \varphi=(\varphi)^{2} \theta^{3} \tag{4.6}
\end{equation*}
$$

i.e.,

$$
e_{1} \varphi=e_{2} \varphi=0, \quad e_{3} \varphi=(\varphi)^{2} .
$$

Let $l$ be the integral curve of $e_{3}$ through a point $x$ and $t$ the arc length parameter of $l$ initiated at $x$, i.e., $e_{3}=\partial / \partial t, l(0)=x$. The integral manifold of the distribution $\left\{e_{1}, e_{2}\right\}$ through $l(c)$ is denoted by $M_{c}$.

## Proposition 4.2.

(a) $\varphi$ is constant on each leaf $M_{c}$.
(b) $\quad M_{c}$ is contained in an embedded 3-sphere with curvature $|\varphi(c)|$ in $\boldsymbol{R}^{4}$.
(c) The spheres defined in (b) are concentric for all $t$.
(d) $M^{3}$ is a cone.

Proof. (a) is obvious from (4.6). Let $y_{c}$ be the position vector of $M_{c}$ in $\boldsymbol{R}^{4}$ and $D$ the covariant derivative in $\boldsymbol{R}^{4}$. Then

$$
D_{e_{i}}\left(y_{c}+\frac{1}{\varphi(c)} e_{3}\right)=e_{i}+\frac{1}{\varphi(c)} D_{e_{i}} e_{3}=e_{i}+\frac{1}{\varphi(c)}\left(-\varphi(c) e_{i}\right)=0 .
$$

This means that the vector $y_{c}+\varphi(c)^{-1} e_{3}$ is parallel along $M_{c}$ in $\boldsymbol{R}^{4}$. Let $w$ be the vector in $\boldsymbol{R}^{4}$ which is obtained by parallel translation of $y_{c}+\varphi(c)^{-1} e_{3}$ to the origin in $\boldsymbol{R}^{4}$. Then

$$
\left\langle y_{c}-w, y_{c}-w\right\rangle=\frac{1}{\varphi(c)^{2}}
$$

holds. This proves (b). For (c) it is sufficient to show that $y_{t}+\varphi(t)^{-1} e_{3}$ is parallel along $l$ in $\boldsymbol{R}^{4}$ :

$$
D_{e_{3}}\left(y_{t}+\frac{1}{\varphi(t)} e_{3}\right)=e_{3}+e_{3}\left(\frac{1}{\varphi}\right) e_{3}+\frac{1}{\varphi} D_{e_{3}} e_{3}=e_{3}-\frac{e_{3} \varphi}{\varphi^{2}} e_{3}=0 .
$$

If $\varphi$ is restricted to $l$, then $\varphi$ satisfies an ordinary differential equation $d \varphi / d t=\varphi^{2}$ from (4.6). The solution of this equation is

$$
\varphi(t)=-\frac{1}{t-\varphi(0)^{-1}} .
$$

Put $s=t-\varphi(0)^{-1}$. Then $\varphi(s)=-s^{-1}$. We may assume that the centers of the spheres in (c) are the origin in $\boldsymbol{R}^{4}$, that is, $y_{s}+\varphi(s)^{-1} e_{3}=0$. Thus $y_{s}=s e_{3}$. On the other hand, it can be seen that $e_{3}$ is parallel along $l$. So $y_{s}=s y_{1}$. Therefore $y_{s}$ 's make up a cone over $y_{1} G \boldsymbol{S}^{3}(1)$ in $\boldsymbol{R}^{4}$.

Case 3. In this case, even if we take any neighborhood for the $x_{0}$, it contains an open portion where $k_{1}+k_{2}=0$ holds and an open portion where $\omega_{2}^{1}\left(e_{p}\right)=0$ holds. Thus the neighborhood of $x_{0}$ is of the form mixed with that of Case 1 and that of Case 2.

Proof of Theorem. From the remaining condition of (*), we can conclude that the surfaces appearing in Case 2 are $H$-deformable surfaces in $\boldsymbol{R}^{3}$ and $\boldsymbol{S}^{3}$, respectively.

Remarks. 1. It can be seen that an $H$-deformation is a principal curvature preserving deformation.
2. Concerning minimal hypersurfaces with type number 2, Dajczer and Gromoll [6] proved that one can construct them by making use of a minimal surface $S$ in $S^{n}$ and an eigenfunction of the Laplace-Bertrami operator with eigenvalue 2 on $S$. In particular, if we take a fully immersed one as a minimal surface in $S^{n}(n \geq 4)$, then a minimal hypersurface with type number 2 which is not of the form $M^{2} \times \boldsymbol{R}^{n-2}$ or $C N \times R^{n-3}$ is obtained.

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