ISOMETRIC DILATIONS OF NON-COMMUTING FINITE RANK N-TUPLES

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ABSTRACT. A contractive n-tuple $A = (A_1, \dots, A_n)$ has a minimal joint isometric dilation $S = (S_1, \dots, S_n)$ where the S_i's are isometries with pairwise orthogonal ranges. This determines a representation of the Cuntz-Toeplitz algebra. When A acts on a finite dimensional space, the WOT-closed nonself-adjoint algebra $\mathfrak{S}$ generated by S is completely described in terms of the properties of A. This provides complete unitary invariants for the corresponding representations. In addition, we show that the algebra $\mathfrak{S}$ is always hyper-reflexive. In the last section, we describe similarity invariants. In particular, an n-tuple B of $d \times d$ matrices is similar to an irreducible n-tuple A if and only if a certain finite set of polynomials vanish on B.

In **[15, 16]**, the first author and David Pitts studied a class of algebras coined *free semigroup algebras*. These are the WOT–closed (nonself-adjoint) unital operator algebras generated by an n-tuple of isometries with pairwise orthogonal ranges. When these ranges span the whole space, the associated norm-closed self-adjoint algebra is a representation of the Cuntz algebra. This nonself-adjoint algebra can contain detailed information about fine unitary invariants of the corresponding C*-algebra representation. Indeed in **[15]** the set of atomic representations of the Cuntz algebra is completely classified. On the other hand, when the ranges span a proper subspace, the representation contains a multiple of the left regular representation of the free semigroup on n letters. The WOT-closed algebra of the left regular representation is called the *non-commutative analytic Toeplitz algebra.* This nomenclature is justified by a good analogue of Beurling's Theorem **[29, 1, 15]**, hyper-reflexivity **[15]** and the relationship **[16]** between its automorphism group and the group of conformal automorphisms of the ball in $\mathbb{C}^n$.

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The connection with dilation theory derives from a theorem of Frahzo, Bunce and Popescu [**19, 11, 26**]. If $A = (A_1, \dots, A_n)$ is an n-tuple of operators such that $AA^* = \sum_{i=1}^n A_i A_i^* \le I$, then there is a unique minimal isometric dilation to isometries S_i on a larger space with pairwise orthogonal ranges. Popescu [**26**] establishes the analogue of Wold's decomposition which splits this into a direct sum of a multiple of the left regular representation and a representation of the Cuntz algebra. Moreover, Popescu [**28**] obtains the non-commutative analogue of von Neumann's inequality in this context. We mention in passing that there has been recent interest in dilating commuting n-tuples as well [**30, 3, 4**].

On the other hand, representations of the Cuntz algebra correspond to endomorphisms of $\mathcal{B}(\mathcal{H})$ [**31, 25, 8, 9**]. This has created new interest in classifying these representations up to unitary equivalence. The well-known theorem of Glimm [**22**] shows that this classification is non-smooth because $\mathcal{O}_n$ is anti-liminal (or NGCR). Nevertheless, interesting classes of representations do lend themselves to a complete analysis. In [**10**], Bratteli and Jorgensen introduced a class of representations which turned out to be a special case of the atomic representations classified in [**15**] using nonself-adjoint techniques. In [**9**] they introduce a different class associated to finitely correlated states. The reader will see a lot of parallels between their results and ours, though the approach is quite different. In the end, they specialize to the subclass of diagonalizable shifts in order to obtain a classification theorem. In this paper, we obtain good unitary invariants for the class of all of these finitely correlated representations.

The goal of this paper is two-fold. First we wish to understand the structure of the free semigroup algebra generated by the dilation of an n-tuple A in terms of information obtained from the n-tuple itself (and the algebra it generates). In particular, we seek unitary invariants for the associated C*-algebra representation. Secondly, we wish to determine whether these algebras are reflexive and even hyper-reflexive. In this paper, we focus on the case in which the n-tuple A acts on a finite dimensional space. Here we obtain a complete description of the algebra. This enables us to decompose the associated representation as a direct sum of irreducible representations and obtain complete unitary invariants. These algebras all turn out to be hyper-reflexive.

In the last section, we discuss similarity invariants. One of the surprising consequences is a complete invariant for an irreducible n-tuple of $d \times d$ matrices up to similarity. An algorithm for determining if two n-tuples of matrices are similar is provided by Friedland [**21**]. But this method rapidly gets complicated. So it is not clear whether it is superior to ours. We find that there is a finite set of no more than $1 + (n-1)d^2$ polynomials p_j so that another n-tuple B is similar to A if and only if $p_j(B) = 0$ for all j. These polynomials are obtained from a computable set of generators of an ideal of the left regular free semigroup algebra as a right ideal, which amounts to computing an orthonormal basis for a certain subspace. In practice, one only needs generators as a two-sided ideal, and hence the actual number needed is normally smaller.

1. Background

Let $\mathbb{F}_n^+$ denote the unital free semigroup on n letters $\{1, 2, \dots, n\}$, and let $\mathcal{K}_n = \ell^2(\mathbb{F}_n^+)$ denote the Hilbert space with basis $\{\xi_w : w \in \mathbb{F}_n^+\}$, which is known as n-variable Fock space. The left regular representation λ of $\mathbb{F}_n^+$ is given by $\lambda(v)\xi_w := L_v\xi_w = \xi_{vw}$. In particular, the generators of $\mathbb{F}_n^+$ determine isometries L_i for $1 \le i \le n$ with orthogonal ranges such that $\sum_{i=1}^n L_i L_i^* = I - P_e$ where $P_e = \xi_e\xi_e^*$ is the rank one projection onto the basis vector for the empty word e, which is the identity of $\mathbb{F}_n^+$. The algebra $\mathfrak{L}_n$ is the WOT-closed algebra generated by the n-tuple $L = (L_1, \dots, L_n)$. See **[15, 16, 17, 23, 27, 29]** for detailed information about this algebra.

More generally if S_i, $1 \le i \le n$, are isometries with $\sum_{i=1}^n S_i S_i^* \le I$, let $\mathfrak{S}$ denote the unital WOT-closed (nonself-adjoint) algebra generated by them. We denote by S_v the isometry $v(S) := v(S_1, \dots, S_n)$ for each $v \in \mathbb{F}_n^+$. A subspace $\mathcal{W}$ is called *wandering* for the n-tuple $S = (S_1, \dots, S_n)$ provided that the subspaces $S_v\mathcal{W}$ are pairwise orthogonal for all $v \in \mathbb{F}_n^+$. Thus the smallest $\mathfrak{S}$-invariant subspace containing a wandering space $\mathcal{W}$ is $\mathfrak{S}[\mathcal{W}] = \sum_{v\in\mathbb{F}_n^+}^{\oplus} S_v\mathcal{W}$. The restriction of $\mathfrak{S}$ to this subspace is evidently a multiple of the left regular representation algebra $\mathfrak{L}_n$, where the multiplicity is given by $\dim \mathcal{W}$. Popescu's Wold decomposition **[26]** works as follows: the subspace $\mathcal{W} = \operatorname{Ran}(I - \sum_{i=1}^n S_i S_i^*)$ is easily seen to be wandering. Moreover the complement $\mathcal{N} = \mathfrak{S}[\mathcal{W}]^\perp$ is also invariant for $\mathfrak{S}$, and the restriction to $\mathcal{N}$ yields isometries $T_i = S_i|_{\mathcal{N}}$ satisfying $\sum_{i=1}^n T_i T_i^* = I_{\mathcal{N}}$.

Suppose $A = (A_1, \dots, A_n)$ is an n-tuple of operators on a Hilbert space $\mathcal{V}$ such that $AA^* = \sum_{i=1}^n A_i A_i^* \le I$. Frahzo **[19]** (for $n = 2$), Bunce **[11]** (for $n < \infty$) and Popescu **[26]** (for $n = \infty$) show that there is a joint dilation of the A_i to isometries S_i on a Hilbert space $\mathcal{H} = \mathcal{V} \oplus \mathcal{K}$ which have pairwise orthogonal ranges. Popescu observes that if this dilation is minimal in the sense that $\mathcal{H} = \operatorname{span}\{S_v\mathcal{V} : v \in \mathbb{F}_n^+\}$, then the dilation is unique (up to a unitary equivalence which fixes $\mathcal{V}$). We will always work with this minimal isometric dilation.

Popescu also observes **[28]** that the norm-closed nonself-adjoint algebra $\mathcal{A}_n$ spanned by $\{L_w : w \in \mathbb{F}_n^+\}$ is the appropriate non-commutative analogue of the disk algebra for a version of von Neumann's inequality. Namely, if A is a contractive n-tuple as above, then $\|p(A)\| \leq \|p(L)\|$ for every non-commuting polynomial in n variables. This is immediate from the dilation theorem and the fact that there is a contractive homomorphism of $\mathcal{E}_n$ onto $\mathcal{O}_n$, the two possible C*-algebras for the dilation. However, it turns out that this quotient map is completely isometric on $\mathcal{A}_n$. So this norm estimate is an equality for any contractive n-tuple of *isometries*. This shows that $\mathcal{O}_n$ is the C*-envelope of $\mathcal{A}_n$.

This presents a rather precise picture for the *norm-closed* algebra generated by an n-tuple of isometries with orthogonal ranges. However, the WOT-closed algebras can be quite different. They can reflect the fine unitary invariants of the representation. The case $n = 1$ is familiar, where the WOT-closed algebra depends on the spectral invariants of the unitary part and the multiplicity of the shift (from the Wold decomposition).

When $\sum_{i=1}^n S_i S_i^* = I$, the C*-algebra generated by the isometries S_i is the Cuntz algebra $\mathcal{O}_n$; and when $\sum_{i=1}^n S_i S_i^* < I$, this C*-algebra is $*$-isomorphic to the Cuntz–Toeplitz algebra $\mathcal{E}_n$ generated by the left regular representation λ. This algebra is an extension of the compact operators $\mathfrak{K}$ by $\mathcal{O}_n$. We associate to each n-tuple S_i the representation σ of $\mathcal{E}_n$ given by $\sigma(s_i) = S_i$, where s_i are the canonical generators of $\mathcal{E}_n$. When $\sum_{i=1}^n S_i S_i^* = I$, we may consider this as a representation of $\mathcal{O}_n$ instead. Let $\mathfrak{S}_\sigma$ denote the WOT-closed non-self-adjoint algebra determined by the representation σ. One can view the Wold decomposition as the spatial view of the C*-algebra fact that every representation σ of $\mathcal{E}_n$ splits as a direct sum $\sigma = \lambda^{(\alpha)} \oplus \tau$ of a representation $\lambda^{(\alpha)}$, which is faithful on $\mathfrak{K}$ and thus is a multiple of the identity representation λ, and a representation τ which factors through $\mathcal{O}_n$.

A representation is called *atomic* if there is an orthonormal basis $\{\xi_j\}$ which are permuted up to scalars by the generating isometries S_i. That is, for each i there is an endomorphism $\pi_i : \mathbb{N} \to \mathbb{N}$ and scalars $\lambda_{i,j}$ of modulus 1 such that $S_i \xi_j = \lambda_{i,j} \xi_{\pi_i(j)}$. These representations decompose as a direct integral of irreducible atomic representations **[15]**, and these irreducible atomic representations are of three types. The first is just the left regular representation; which is the only one which does not factor through $\mathcal{O}_n$. The second type is a class of inductive limits of the left regular representation, and are classified by an infinite word (up to shift-tail equivalence) that describes the imbeddings. The third type fits into the context of this paper, and so we describe it in more detail. See **[15]** for a complete description.

The third type is given by a word $u = i_1 i_2 \dots i_d$ in $\mathbb{F}_n^+$ and a scalar λ of modulus 1. A finite dimensional space $\mathcal{V}$ of dimension d is formed with a basis $e_1, \dots, e_d$. Operators A_j, $1 \le j \le n$, are partial isometries given by

$$\begin{aligned} A_j e_k &= \delta_{j i_k} e_{k+1} \quad \text{for} \quad 1 \le k < d \\ A_j e_d &= \lambda \delta_{j i_d} e_1. \end{aligned}$$

The minimal isometric dilation of this n-tuple yields isometries S_j acting on a space $\mathcal{H} = \mathcal{V} \oplus \mathcal{K}$. The isometry S_{i_k} maps e_k to e_{k+1} (or λe_1 when $k = d$) and the other $n-1$ isometries send e_k to pairwise orthogonal vectors which are all wandering vectors for $\mathfrak{S}$. Thus $\mathcal{K} = \mathcal{V}^\perp$ is determined by a wandering space $\mathcal{W}$ of dimension $d(n-1)$, and therefore $\mathcal{K} = \mathfrak{S}[\mathcal{W}] \simeq \mathcal{K}_n^{(d(n-1))}$. The associated representation $\sigma_{u,\lambda}$ is irreducible precisely when the word u is primitive, meaning that it is not a power of a smaller word. In this case, $\mathfrak{S}$ can be completely described as the sum of $\mathcal{B}(\mathcal{H})P_\mathcal{V}$ and a multiple of $\mathfrak{L}_n$ acting on $\mathcal{K}$ via its identification with $\mathcal{K}_n^{(d(n-1))}$. The invariant subspaces of this algebra are readily described, and it turns out to be hyper-reflexive. (See below).

For the future, we wish to name the type of algebra which occurs here. Let $\mathfrak{B}_{n,d}$ denote the WOT-closed algebra on a Hilbert space $\mathcal{H} = \mathcal{V} \oplus \mathcal{K}_n^{(d(n-1))}$ where $\dim \mathcal{V} = d$ given by

$$\mathfrak{B}_{n,d} = \mathcal{B}(\mathcal{H})P_\mathcal{V} + \big(0_\mathcal{V} \oplus \mathfrak{L}_n^{(d(n-1))}\big).$$

Another class of representations which have been studied are the *finitely correlated* representations [**9**]. A representation of $\mathcal{O}_n$ is finitely correlated if there is a finite dimensional cyclic subspace $\mathcal{V}$ which is invariant for each S_i^*. Likewise, a *finite correlated state* is a state φ such that in the GNS construction, the invariant subspace for the S_i^*'s generated by the cyclic vector ξ_φ is finite dimensional. It is evident that these representations are exactly those which we will study from the viewpoint of dilation theory. In this paper, we will obtain a complete classification of these representations up to unitary equivalence. We will explain later how our classification relates to the work of Bratteli and Jorgensen.

If $\mathfrak{A}$ is an algebra of operators, $\operatorname{Lat}\mathfrak{A}$ denotes the lattice of all $\mathfrak{A}$-invariant subspaces. And if $\mathcal{L}$ is a lattice of subspaces, $\operatorname{Alg}\mathcal{L}$ denotes the WOT-closed unital algebra of all operators which leave each element of $\mathcal{L}$ invariant. The algebra $\mathfrak{A}$ is reflexive if it equals $\operatorname{Alg}\operatorname{Lat}\mathfrak{A}$. For each reflexive algebra, there is a quantitative measure of the distance to $\mathfrak{A}$ given by

$$\beta_{\mathfrak{A}}(T) = \sup_{L\in\mathcal{L}} \|P_L^\perp T P_L\|.$$

It is easily seen that $\beta_{\mathfrak{A}}(T) \leq \operatorname{dist}(T,\mathfrak{A})$. The algebra is called *hyper-reflexive* if there is a constant C such that $\operatorname{dist}(T,\mathfrak{A}) \leq C\beta_{\mathfrak{A}}(T)$. The optimal C, if it is finite, is called the distance constant for $\mathfrak{A}$.

The list of algebras known to be hyper-reflexive is rather short. Arveson **[2]** showed that nest algebras have distance constant 1, so that equality is achieved. Christensen **[12]** showed that AF von Neumann algebras have distance constant at most 4. Concerning the algebras studied in this paper, the first author **[14]** showed that the analytic Toeplitz algebra has distance constant at most 19; and with Pitts **[15]**, that all atomic free semigroup algebras where shown to have distance constant at most 51. The worst case for these estimates was the algebra $\mathfrak{L}_n$. However a recent general result of Bercovici **[6]** applies to show that $\mathfrak{L}_n$ actually has a distance constant no greater than 3.

2. Main Results

In this paper, we generally take n to be a finite integer with $n \geq 2$. However, Popescu's version of the dilation theorem is valid for $n = \infty$, as are the results of **[15, 16]** on the structure of $\mathfrak{L}_n$ which we shall use. So the results of this paper go through for $n = \infty$ with only a few minor changes in notation, not in substance. For ease of presentation, we will write this paper as though n were finite, and let the interested reader interpolate the $n = \infty$ case.

Consider a contractive n-tuple $A = (A_1, \dots, A_n)$ acting on a finite dimensional space $\mathcal{V}$ of dimension d; i.e. $\sum_{i=1}^n A_iA_i^* \leq I$. The Frahzo–Bunce–Popescu minimal dilation yields isometries S_i acting on a larger space $\mathcal{H}$. We let $\mathfrak{A}$ denote the algebra generated by the A_i's, and let $\mathfrak{S}$ be the WOT-closed algebra generated by the S_i's. We will make important use of an associated completely positive contractive map on $\mathcal{B}(\mathcal{V})$ given by

$$\Phi(X) = \sum_{i=1}^n A_iXA_i^*.$$

The operator $\Phi^\infty(I) := \lim_{k\to\infty} \Phi^k(I)$ will also be useful.

The first fairly easy observation is that the dilation is of Cuntz type ($\sum_{i=1}^n S_iS_i^* = I$) if and only if $\sum_{i=1}^n A_iA_i^* = I$; or equivalently $\Phi(I) = I$. In general, we define the *pure rank* of $\mathfrak{S}$ to be the multiplicity of the left regular representation in the Wold decomposition of $\mathfrak{S}$. This is the dimension of the wandering space $\mathcal{W} = \operatorname{Ran}(I - \sum_{i=1}^n S_iS_i^*)$. Simple examples show that this wandering space need not be contained in $\mathcal{V}$, and that even when this pure rank is one, the pure part may have large intersection with $\mathcal{V}$. Nevertheless, it turns out that this pure rank may be easily computed as

$$\operatorname{pure\,rank}(\mathfrak{S}) = \operatorname{rank}(I - \Phi(I)) = \operatorname{rank}\Big(I - \sum_{i=1}^{n} A_i A_i^*\Big).$$

The irreducible summands of Cuntz type are determined by the minimal $\mathfrak{A}^*$-invariant subspaces $\mathcal{M}$ of $\mathcal{V}$ on which $\sum_{i=1}^n A_i A_i^*|_{\mathcal{M}} = I_{\mathcal{M}}$. Such a subspace generates an invariant subspace $\mathcal{H}_{\mathcal{M}} = \mathfrak{S}[\mathcal{M}]$ for $\mathfrak{S}$ which is necessarily reducing. The restriction $\mathfrak{S}|_{\mathcal{H}_{\mathcal{M}}}$ of $\mathfrak{S}$ to this subspace is isomorphic to the algebra $\mathfrak{B}_{n,m}$, where $m = \dim \mathcal{M}$, described in the Background section. A crucial feature is that the projection $P_{\mathcal{M}}$ belongs to this algebra. This makes it possible to show that the restriction of the n-tuple A to $\mathcal{M}$ is a unitary invariant for the dilation.

The subspace $\widetilde{\mathcal{V}}$ spanned by all the minimal $\mathfrak{A}^*$-invariant subspaces of this type completely determines the Cuntz part of the dilation. The restriction of $\mathfrak{A}^*$ to $\widetilde{\mathcal{V}}$ is a finite dimensional C*-algebra. The well-known invariants for a finite dimensional C*-algebra allow one to compute the multiplicities of each irreducible subrepresentation. In general, this information may be used to completely decompose the representation into a direct sum of finitely many irreducible representations of the types given above. This yields complete unitary invariants: the pure rank and the unitary equivalence class of the restriction of A^* to $\widetilde{\mathcal{V}}$.

For example, one can show that $\mathfrak{S}$ is irreducible if and only if either
(1) $\operatorname{rank}(I - \Phi(I)) = 1$ and $\Phi^\infty(I) = 0$, the pure case, or
(2) $\{X : \Phi(X) = X\} = \mathbb{C}I$, the Cuntz case.

The algebras $\mathfrak{L}_n$ and $\mathfrak{B}_{n,d}$ were shown to be hyper-reflexive in **[15]**. This analysis can be used to show that all of these algebras $\mathfrak{S}$ determined by a finite rank n-tuple are hyper-reflexive. The constant 51 of that paper may be improved to 5 using recent results of Bercovici **[6]** which show that the distance constant for $\mathfrak{L}_n$ is at most 3.

Then we turn our attention to similarity. If two contractive n-tuples are similar, it follows that their Cuntz parts are unitarily equivalent. However, the pure rank can change. Indeed, this rank can be 0 in one case and non-zero in a similar n-tuple.

The major interest lies in the pure case. In this case, the algebra $\mathfrak{S}$ is unitarily equivalent to a multiple of $\mathfrak{L}_n$, and thus completely isometrically isomorphic and weak-$*$ homeomorphic to $\mathfrak{L}_n$. The compression Φ_A of $\mathfrak{S}$ to $\mathcal{V}$ is thus a weak-$*$ continuous representation of $\mathfrak{L}_n$. The study of these representations was initiated in **[16]**. The kernel of such a representation is a WOT-closed ideal. A WOT-closed ideal $\mathfrak{J}$ of $\mathfrak{L}_n$

is determined [**16**, Theorem 2.1] by its range $\mathcal{M} = \overline{\mathfrak{J}\mathcal{K}_n}$, which is a subspace invariant for both $\mathfrak{L}_n$ and its commutant $\mathfrak{R}_n$. Thus we consider the associated representation of $\mathfrak{L}_n$ obtained as restriction to $\mathcal{M}^\perp$. This has the same kernel $\mathfrak{J}$. In the case of an irreducible n-tuple, the minimal $\mathfrak{L}_n^*$-invariant subspaces of $\mathcal{M}^\perp$ yield all of the n-tuples similar to A which have pure rank 1. These are the extreme points of all such representations in the sense that A can be recovered as a C*-convex combination of them.

In particular, it follows that two irreducible n-tuples of matrices are similar if and only if the induced representations of $\mathfrak{L}_n$ have the same kernel. The range space $\mathcal{M}$ of $\mathfrak{J} = \ker \Phi_A$ has a wandering space of dimension $1+(n-1)d^2$. A basis for this wandering space yields a finite set of generators for $\mathfrak{J}$ as a WOT-closed right ideal. They determine a corresponding finite set of isometries X_j in $\mathfrak{L}_n$ with the property that another contractive n-tuple of $d \times d$ matrices B is similar to A if and only if $\Phi_B(X_j) = 0$ for $1 \le j \le 1+(n-1)d^2$. Since one merely requires generators for $\mathfrak{J}$ as a two-sided ideal, normally this number of tests can be reduced. The set of isometries are canonical, but they are generally not polynomials. A set of polynomial invariants can be obtained by an approximation argument.

3. Wandering Subspaces

Let $\mathcal{V}$ be a d-dimensional space (possibly infinite), and let $A_1, \dots, A_n$ be an n-tuple of operators in $\mathcal{B}(\mathcal{V})$ such that $\sum_{i=1}^n A_i A_i^* \le I$. The Frahzo–Bunce–Popescu minimal dilation yields isometries S_i on a larger space $\mathcal{H}$. Let $P_\mathcal{V}$ denote the projection of $\mathcal{H}$ onto $\mathcal{V}$. We let $\mathfrak{A}$ denote the algebra generated by the A_i's and $\mathfrak{S}$ be the WOT-closed algebra generated by the S_i's. We first identify $\mathcal{V}^\perp$.

Lemma 3.1. *The subspace* $\mathcal{W} = (\mathcal{V} + \sum_{i=1}^n S_i\mathcal{V}) \ominus \mathcal{V}$ *is a wandering subspace for* S*, and* $\sum_{v\in\mathbb{F}_n^+}^{\oplus} S_v\mathcal{W} = \mathcal{V}^\perp$.

Proof. $\mathcal{W}$ is contained in $\mathcal{V}^\perp$, which is invariant for S. Thus $S_u\mathcal{W}$ is orthogonal to $\mathcal{V}$ for every word $u \in \mathbb{F}_n^+$. Consequently, when $|u| \ge 1$, $S_u\mathcal{W}$ is also orthogonal to $S_j\mathcal{V}$, $1 \le j \le n$. It follows that $S_u\mathcal{W}$ is orthogonal to $\mathcal{V} + \sum_{i=1}^n S_i\mathcal{V}$, which contains $\mathcal{W}$. Therefore $\mathcal{W}$ is wandering. Minimality ensures that

$$\mathcal{H} = \operatorname{span}\{S_u\mathcal{V} : u \in \mathbb{F}_n^+\} = \operatorname{span}\{\mathcal{V}, S_u\mathcal{W} : u \in \mathbb{F}_n^+\}.$$

Since $\mathcal{W}$ lies in the invariant subspace $\mathcal{V}^\perp$, this can only occur because $\sum_{v\in\mathbb{F}_n^+}^{\oplus} S_v\mathcal{W} = \mathcal{V}^\perp$. ■

Thus $\mathcal{K} = \mathcal{V}^\perp$ is unitarily equivalent to a multiple $\mathcal{K}_n^{(\alpha)}$ of Fock space, where $\alpha = \dim \mathcal{W}$, and $S_i|_\mathcal{K} \simeq L_i^{(\alpha)}$. Hence decomposing $\mathcal{H} = \mathcal{V} \oplus \mathcal{K}$, we may write each S_i as a matrix $S_i = \begin{bmatrix} A_i & 0 \\ X_i & L_i^{(\alpha)} \end{bmatrix}$.

Remark 3.2. The range of $\sum_{i=1}^n S_i S_i^*$ includes $\sum_{i=1}^n S_i \mathcal{V}^\perp = (\mathcal{V}+\mathcal{W})^\perp$ as well as $\sum_{i=1}^n S_i \mathcal{V}$. Hence $\sum_{i=1}^n S_i S_i^* = I$ if and only if $\sum_{i=1}^n S_i \mathcal{V}$ contains $\mathcal{V}$. Since $\mathcal{V}$ is invariant for S_i^* and $S_i^*|_\mathcal{V} = A_i^*$,

$$\sum_{i=1}^n A_i A_i^* = \sum_{i=1}^n P_\mathcal{V} S_i P_\mathcal{V} S_i^*|_\mathcal{V} = P_\mathcal{V} \sum_{i=1}^n S_i S_i^*|_\mathcal{V}.$$

Therefore $\sum_{i=1}^n S_i S_i^* = I$ if and only if its range contains $\mathcal{V}$ if and only if $\sum_{i=1}^n A_i A_i^* = I_\mathcal{V}$.

Let $d = \dim \mathcal{V}$ be finite, and let $\alpha = \dim \mathcal{W}$. Then α can be as large as nd and as small as $(n-1)d$. This is easily seen since $\sum_{i=1}^n S_i \mathcal{V}$ is an orthogonal direct sum and thus has dimension nd, so that $\mathcal{W} = \big(\mathcal{V} + \sum_{i=1}^n S_i \mathcal{V}\big) \ominus \mathcal{V}$ can have no larger dimension than nd, and is at least $(n-1)d$.

When $\sum_{i=1}^n A_i A_i^* = I_\mathcal{V}$, we showed above that $\sum_{i=1}^n S_i S_i^* = I$. Then

$$\mathcal{V} = \sum_{i=1}^n S_i S_i^* \mathcal{V} = \sum_{i=1}^n S_i A_i^* \mathcal{V} \subset \sum_{i=1}^n S_i \mathcal{V}.$$

Hence $\mathcal{W} = \big(\sum_{i=1}^n S_i \mathcal{V}\big) \ominus \mathcal{V}$ has dimension $(n-1)d$.

The case $\dim \mathcal{W} = nd$ occurs, for example, if $A_i = 0$ for $1 \le i \le n$. The minimal dilation is just $L_i^{(d)}$. Indeed, if $x, y \in \mathcal{V}$, then

$$(S_i x, y) = (x, S_i^* y) = (x, A_i^* y) = 0 \quad \text{for} \quad 1 \le i \le n.$$

Thus $\mathcal{V}$ is orthogonal to $\sum_{i=1}^n S_i \mathcal{V}$. Therefore $\mathcal{W} = \sum_{i=1}^n S_i \mathcal{V}$ has dimension nd.

It is easy to combine these examples to obtain any integer in between.

The n-tuple of isometries S is called *pure* if it is unitarily equivalent to a multiple of the left regular representation. Bunce [**11**] shows that whenever $\|A\| < 1$, the dilation S is pure. Popescu [**26**] shows that the dilation is pure if and only if $\text{WOT–}\lim\limits_{k\to\infty} \sum\limits_{|v|=k} A_v A_v^* = 0$.

Lemma 3.1 shows that beginning with an n-tuple, we will always obtain wandering vectors except when the A_i's already are isometries and $\sum_{i=1}^n A_i A_i^* = I$, in which case the dilation is just the A_i's themselves. When there are wandering vectors, each generates a subspace $\mathcal{M}$ on which the isometries S_i are unitarily equivalent to the left regular representation. In particular, the non-$*$ algebra $\mathfrak{S}$ that they generate is very non-self-adjoint. In fact, a strong converse to this exists. Recall that an algebra is *reductive* if all of its invariant subspaces have invariant (orthogonal) complements; and it is *transitive* if it has no proper invariant subspaces at all. It is an open problem equivalent to the transitive algebra variant of the invariant subspace problem [**18**] whether every WOT-closed reductive algebra is self-adjoint.

Lemma 3.3. *Let $\mathfrak{S}$ be a free semigroup algebra. Then either $\mathfrak{S}$ has a wandering vector or $\mathfrak{S}$ is reductive. If the latter is possible, then transitive free semigroup algebras exist.*

Proof. Suppose that $\mathfrak{S}$ has no wandering vectors. Let $\mathcal{M}$ be an invariant subspace for $\mathfrak{S}$. Then $\sum_{i=1}^n S_i\mathcal{M}$ must equal $\mathcal{M}$; for otherwise $\mathcal{M} \ominus \sum_{i=1}^n S_i\mathcal{M}$ is wandering. Thus $\sum_{i=1}^n S_i\mathcal{M}^\perp = \mathcal{M}^\perp$, so that $\mathcal{M}^\perp$ is also invariant. Whence $\mathfrak{S}$ is reductive.

Now we invoke the direct integral theory for non-self-adjoint operator algebras due to Azoff, Fong and Gilfeather [**5**, Theorem 4.1]. Let $\mathfrak{M}$ be any masa in the commutant of $\mathfrak{S}$. They show that $\mathfrak{S}$ may be decomposed with respect to $\mathfrak{M}$ as an integral of algebras which are transitive almost everywhere. The isometries S_i decompose as an integral of operators which are isometries almost everywhere as well. In particular, transitive algebras generated by isometries with orthogonal ranges would exist. ∎

At this point, we do not know if there are transitive free semigroup algebras. The motivation for suspecting that there may be comes from the case $n = 1$. A unitary operator is reductive unless Lebesgue measure on the whole circle is absolutely continuous with respect to the spectral measure of the unitary [**35**]. This is the case for 'most' unitaries.

We will frequently construct *reducing* subspaces of $\mathfrak{S}$ from $\mathfrak{A}^*$-invariant subspaces. This procedure preserves orthogonality as well.

Lemma 3.4. *Suppose that $\mathcal{V}$ contains an $\mathfrak{A}^*$-invariant subspace $\mathcal{V}_1$. Then $\mathcal{H}_1 = \mathfrak{S}[\mathcal{V}_1]$ reduces $\mathfrak{S}$.*

If $\mathcal{V}$ contains a pair of orthogonal $\mathfrak{A}^*$-invariant subspaces $\mathcal{V}_1$ and $\mathcal{V}_2$, then $\mathcal{H}_j = \mathfrak{S}[\mathcal{V}_j]$ for $j = 1, 2$ are mutually orthogonal.

If in addition $\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_2$, then $\mathcal{H}$ decomposes as $\mathcal{H}_1 \oplus \mathcal{H}_2$ and $\mathcal{H}_j \cap \mathcal{V} = \mathcal{V}_j$ for $j = 1, 2$.

Proof. Since $\mathcal{V}_1$ is invariant for A_i^*, it is also invariant for S_i^*. The $\mathfrak{S}$-invariant subspace $\mathcal{H}_1 = \mathfrak{S}[V_1]$ is spanned by vectors of the form $S_w x$ where $x \in \mathcal{V}_1$ and $w \in \mathbb{F}_n^+$. Notice that $S_i^* S_w x$ equals $S_{w'}x$ if $w = iw'$, 0 if $w = i'w'$ for some $i' \neq i$, and $S_i^* x$ if $w = e$. Since $\mathcal{V}_1$ is invariant for $\mathfrak{S}^*$, each of these possibilities belongs to $\mathcal{H}_1$. Thus $\mathcal{H}_1$ reduces $\mathfrak{S}$.

Likewise, if $\mathcal{V}_1$ and $\mathcal{V}_2$ are orthogonal $\mathfrak{A}^*$-invariant subspaces, it follows that $\mathcal{H}_1$ and $\mathcal{H}_2$ are orthogonal. For if $v_j \in \mathcal{V}_j$, the inner product $(S_u v_1, S_w v_2)$ can be reduced by cancellation of isometries until either u or w is the identity element. Then, for example when $w = e$,

$$(S_u v_1, v_2) = (v_1, S_u^* v_2) = 0$$

by the $\mathfrak{A}^*$-invariance of $\mathcal{V}_2$ and orthogonality.

Now suppose that $\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_2$. Since $\mathcal{H}_1$ contains $\mathcal{V}_1$ and is orthogonal to $\mathcal{V}_2$, it follows that $\mathcal{H}_1 \cap \mathcal{V} = \mathcal{V}_1$. Finally, $\mathcal{H}_1 \oplus \mathcal{H}_2$ is an $\mathfrak{S}$-reducing subspace containing $\mathcal{V}$, so it is all of $\mathcal{H}$ by the minimality of the dilation. ■

4. Finite Dimensional n-tuples

Now let us specialize to the case when $\mathcal{V}$ is finite-dimensional. In general, we can decompose the S_i into a pure part and Cuntz part. Let $\mathcal{X}$ be the range of $I - \sum_{i=1}^n S_i S_i^*$, which is the wandering space for the reducing subspace $\mathcal{H}_p = \sum_{v \in \mathbb{F}_n^+}^{\oplus} S_v \mathcal{X}$. The restriction of the S_i to this space yields a multiple of the left regular representation, where the multiplicity is $\dim \mathcal{X}$. We call this quantity the *pure rank* of the representation. On the complement $\mathcal{H}_c = \mathcal{H}_p^\perp$, the restrictions of S_i yield a representation of the Cuntz algebra. Let P_p and P_c denote the projections onto $\mathcal{H}_p$ and $\mathcal{H}_c$ respectively. It is important to note that the projection P_p does not commute with $P_\mathcal{V}$ in general. So we will obtain a method of computing this pure rank directly from the A_i's.

The key technical tool in our analysis shows that $\mathcal{H}_c$ is determined by $\mathcal{V}_c := \mathcal{H}_c \cap \mathcal{V}$. This is not the case for $\mathcal{H}_p$. Let R_k denote the projection onto $\sum_{|v|=k}^{\oplus} S_v \mathcal{W}$, where $\mathcal{W} = \big(\mathcal{V} + \sum_{i=1}^n S_i \mathcal{V}\big) \ominus \mathcal{V}$; and $Q_k = \sum_{j \ge k} R_j$. Notice that

$$Q_k = \sum_{|w|=k} S_w P_\mathcal{V}^\perp S_w^*.$$

On any $\mathfrak{S}$-invariant subspace $\mathcal{M}$ on which the restrictions T_i of S_i are pure, one has for every $x \in \mathcal{M}$

$$\lim_{k\to\infty} \sum_{|w|=k} \|P_\mathcal{M} S_w^* x\|^2 = \lim_{k\to\infty} \sum_{|w|=k} \|T_w^* x\|^2 = 0.$$

In particular, this applies to $\mathcal{H}_p$ and $\mathcal{V}^\perp$. While for $x \in \mathcal{H}_c$, one has

$$\sum_{|w|=k} \|S_w^* x\|^2 = \|x\|^2 \quad \text{for all} \quad k \ge 0.$$

Key Technical Lemma 4.1. *Suppose that $\mathcal{H}_1$ is a reducing subspace for $\mathfrak{S}$ contained in $\mathcal{H}_c$. Let x be a vector such that $P_{\mathcal{H}_1}x \neq 0$. Then the subspace $\mathcal{M} = \mathfrak{S}^*[x]$ contains a vector v in $\mathcal{M} \cap \mathcal{V}_c$ with $P_{\mathcal{H}_1}v \neq 0$.*

Proof. Let P_1 denote the projection of $\mathcal{H}$ onto $\mathcal{H}_1$. Fix $\varepsilon > 0$; and let $x_1 = P_1 x$. By applying the preceding remarks to both $\mathcal{V}^\perp$ and $\mathcal{H}_p$, we may choose an integer k sufficiently large that

$$\sum_{|w|=k} \|P_{\mathcal{V}}^\perp S_w^* x\|^2 = \|Q_k x\|^2 < \varepsilon^2$$
$$\sum_{|w|=k} \|P_{\mathcal{V}}^\perp S_w^* x_1\|^2 = \|Q_k x_1\|^2 < \varepsilon^2$$

and

$$\sum_{|w|=k} \|P_p S_w^* x\|^2 = \sum_{|w|=k} \|S_w^* P_p x\|^2 < \varepsilon^2.$$

Since $\sum_{|w|=k} S_w S_w^* P_1 = P_1$,

$$\begin{aligned}\sum_{|w|=k} \|P_{\mathcal{V}} S_w^* x_1\|^2 &= \sum_{|w|=k} \big(\|S_w^* x_1\|^2 - \|P_{\mathcal{V}}^\perp S_w^* x_1\|^2\big) \\ &= \|x_1\|^2 - \|Q_k x_1\|^2 > \|x_1\|^2 - \varepsilon^2.\end{aligned}$$

Let $\mathcal{E}_1$ denote the set of words w of length k such that

$$\|P_{\mathcal{V}} S_w^* x_1\|^2 > \varepsilon^{-1} \|P_{\mathcal{V}}^\perp S_w^* x\|^2.$$

Likewise let $\mathcal{E}_2$ denote the set of words w of length k such that

$$\|P_{\mathcal{V}} S_w^* x_1\|^2 > \varepsilon^{-1} \|P_p S_w^* x\|^2.$$

The set $\mathcal{E}_1 \cap \mathcal{E}_2$ is relatively large in the sense that

$$\begin{aligned}\sum_{w \in \mathcal{E}_1 \cap \mathcal{E}_2} \|P_{\mathcal{V}} S_w^* x_1\|^2 &> \|x_1\|^2 - \varepsilon^2 - \sum_{w \notin \mathcal{E}_1} \|P_{\mathcal{V}} S_w^* x_1\|^2 - \sum_{w \notin \mathcal{E}_2} \|P_{\mathcal{V}} S_w^* x_1\|^2 \\ &> \|x_1\|^2 - \varepsilon^2 - \sum_{w \notin \mathcal{E}_1} \varepsilon^{-1} \|P_{\mathcal{V}}^\perp S_w^* x\|^2 - \sum_{w \notin \mathcal{E}_2} \varepsilon^{-1} \|P_p S_w^* x\|^2 \\ &> \|x_1\|^2 - \varepsilon^2 - \varepsilon - \varepsilon > \|x_1\|^2/4\end{aligned}$$

for small ε. Now we also have $\sum_{w \in \mathcal{E}_1 \cap \mathcal{E}_2} \|P_{\mathcal{V}} S_w^* x\|^2 \leq \|x\|^2$. Therefore there is a word w in $\mathcal{E}_1 \cap \mathcal{E}_2$ such that

$$\|P_{\mathcal{V}} S_w^* x_1\| > \frac{\|x_1\|}{2\|x\|} \|P_{\mathcal{V}} S_w^* x\|.$$

In this way, construct a sequence of words w_k corresponding to $\varepsilon_k = 1/k$. Hence define unit vectors $y_k = S_{w_k}^* x / \|S_{w_k}^* x\|$ with the properties that

$$\lim_{k\to\infty} \|P_{\mathcal{V}}^{\perp} y_k\| \le \lim_{k\to\infty} \frac{1}{\sqrt{k}} \frac{\|P_{\mathcal{V}} S_{w_k}^* P_1 x\|}{\|S_{w_k}^* x\|} = \lim_{k\to\infty} \frac{1}{\sqrt{k}} \frac{\|P_{\mathcal{V}} P_1 S_{w_k}^* x\|}{\|S_{w_k}^* x\|} = 0.$$

Similarly,

$$\lim_{k\to\infty} \|P_p y_k\| = 0.$$

Also

$$\begin{aligned} \|P_1 y_k\| &= \|S_{w_k}^* x_1\| / \|S_{w_k}^* x\| \ge \|P_{\mathcal{V}} S_{w_k}^* x_1\| / \|S_{w_k}^* x\| \\ &> \frac{\|x_1\|}{2\|x\|} \frac{\|P_{\mathcal{V}} S_{w_k}^* x\|}{\|S_{w_k}^* x\|} = \frac{\|x_1\|}{2\|x\|} \|P_{\mathcal{V}} y_k\|. \end{aligned}$$

By the compactness of the unit ball in $\mathcal{V}$, there is a subsequence of the y_k's which converges to a unit vector v in $\mathcal{V}$. Clearly, $P_p v = 0$, and thus v belongs to $\mathcal{V}_c \cap \mathfrak{S}^*[x]$; whence this subspace is non-zero. By construction, $\|P_1 v\| \ge \|x_1\|/2\|x\|$, and therefore is also non-zero. ∎

Corollary 4.2. *Every non-zero subspace of $\mathcal{H}_c$ which is invariant for $\mathfrak{S}^*$ has non-zero intersection with $\mathcal{V}_c$. In particular $\mathcal{H}_c = \mathfrak{S}[\mathcal{V}_c]$.*

Proof. Let $\mathcal{M}$ be any non-zero $\mathfrak{S}^*$-invariant subspace contained in $\mathcal{H}_c$. If x is any non-zero vector in $\mathcal{M}$, the previous lemma applied to x and $\mathcal{H}_1 = \mathcal{H}_c$ shows that $\mathfrak{S}^*[x]$ intersects $\mathcal{V}_c$ non-trivially.

By Lemma 3.4, $\mathcal{N} = \mathfrak{S}[\mathcal{V}_c]$ reduces $\mathfrak{S}$. We claim that $\mathcal{N} = \mathcal{H}_c$. For otherwise, let $\mathcal{H}_1 = \mathcal{H}_c \cap \mathcal{N}^{\perp}$. By the first paragraph, this reducing subspace for $\mathfrak{S}$ must intersect $\mathcal{V}_c$ non-trivially. So $\mathcal{H}_1$ is not orthogonal to $\mathcal{N}$, contrary to fact. Therefore $\mathcal{H}_1$ must be zero. ∎

Corollary 4.3. *Suppose that $\sum_{i=1}^n A_i A_i^* = I$ and $\mathfrak{A} = \mathcal{B}(\mathcal{V})$. Then every invariant subspace of $\mathfrak{S}^*$ contains $\mathcal{V}$.*

Proof. Since $\mathcal{H} = \mathcal{H}_c$, any $\mathfrak{S}^*$-invariant subspace $\mathcal{M}$ intersects $\mathcal{V}$ in a non-trivial subspace. This subspace is invariant for $\mathfrak{S}^*|_{\mathcal{V}} = \mathfrak{A}^* = \mathcal{B}(\mathcal{V})$. Hence it is all of $\mathcal{V}$. ∎

Let $\mathfrak{B}$ denote the WOT-closed operator algebra on $\mathcal{H} = \mathcal{V} \oplus \mathcal{K}_n^{(\alpha)}$ spanned by $\mathcal{B}(\mathcal{H}) P_{\mathcal{V}}$ and $0_{\mathcal{V}} \oplus \mathfrak{L}_n^{(\alpha)}$.

Lemma 4.4. *Every weak-$*$ continuous functional on $\mathfrak{B}$ is given by a trace class operator of rank at most $d+1$.*

Proof. An element B of $\mathfrak{B}$ is determined by $BP_{\mathcal{V}}$ and $BP_{\mathcal{V}}^{\perp}$. If $e_1, \dots, e_d$ is a basis for $\mathcal{V}$, the former is determined by the vectors Be_j. The latter term is unitarily equivalent to $A^{(\alpha)}$ for some $A \in \mathfrak{L}_n$. Any functional φ is thus determined by a functional φ_0 on $\mathfrak{L}_n^{(\alpha)}$ and by d functionals on $\mathcal{H}$ given by the Riesz Representation Theorem by a vector y_j. By [**15**, Theorem 2.10], the functional on $\mathfrak{L}_n$ is given by a rank one functional $\varphi_0(A) = (A\eta, \zeta)$. Whence

$$\varphi(B) = \sum_{j=1}^{d} (Be_j, y_j) + (B\eta, \zeta). \qquad \blacksquare$$

Corollary 4.5. *The* WOT *and weak-$*$ topologies coincide on $\mathfrak{B}$, and thus also on $\mathfrak{S}$. In particular, the weak-$*$ closed algebra generated by the S_i's coincides with $\mathfrak{S}$.*

5. The Cuntz Case

In this section, we specialize to the Cuntz case: $\sum_{i=1}^{n} A_i A_i^* = I$ for which the isometric dilation yields a representation of the Cuntz algebra.

Example 5.1. We begin with a description of the case in which $\mathcal{V}$ is one dimensional.. A special case of a finite correlated state is a Cuntz state. This is determined by scalars $\eta = (\eta_1, \dots, \eta_n)$ such that $\sum_{i=1}^{n} |\eta_i|^2 = 1$. The state is determined by

$$\varphi_\eta(s_{i_1} \dots s_{i_k} s_{j_1}^* \dots s_{j_l}^*) = \eta_{i_1} \dots \eta_{i_k} \bar{\eta}_{j_1} \dots \bar{\eta}_{j_l}.$$

It is easy to show that the cyclic vector ξ_η from the GNS construction $(\mathcal{H}_\eta, \pi_\eta, \xi_\eta)$ spans a one-dimensional space invariant for every $\pi_\eta(S_i^*)$. Indeed,

$$\begin{aligned} \|\pi_\eta(S_i^*)\xi_\eta - \bar{\eta}_i \xi_\eta\|^2 &= \langle \pi_\eta(S_i^*)\xi_\eta, \pi_\eta(S_i^*)\xi_\eta \rangle - \eta_i \langle \pi_\eta(S_i^*)\xi_\eta, \xi_\eta \rangle \\ &\quad - \bar{\eta}_i \langle \xi_\eta, \pi_\eta(S_i^*)\xi_\eta \rangle + |\eta_i|^2 \\ &= \varphi_\eta(S_i S_i^*) - |\eta_i|^2 = |\eta_i|^2 - |\eta_i|^2 = 0. \end{aligned}$$

The restrictions $A_i^* = S_i^*|_{\operatorname{span}\{\xi_\eta\}} = \overline{\eta_i}$ satisfy $\sum_{i=1}^{n} A_i A_i^* = 1$. They may be dilated to their minimal isometric dilation, which is necessarily the original S_i since ξ_η is a cyclic vector.

Specializing to the case of $\eta = (1, 0, \dots, 0)$, one has $A_1 = 1$ and $A_i = 0$ for $2 \leq i \leq n$. This yields the atomic representation $\sigma_{1,1}$ mentioned in the Background section. In particular, the algebra $\mathfrak{S}$ is unitarily equivalent to $\mathfrak{B}_{n,1}$.

The various Cuntz states are related by the action of the gauge group $\mathcal{U}(n)$ which acts as an automorphism group on $\mathcal{O}_n$ and on the Cuntz–Toeplitz algebra $\mathcal{E}_n$. Indeed, if we write Fock space $\mathcal{K}_n$ as a direct sum $\mathbb{C}\oplus\mathcal{H}_n\oplus\mathcal{H}_n^{\otimes 2}\oplus\mathcal{H}_n^{\otimes 3}\oplus\dots$, where $\mathcal{H}_n$ is an n-dimensional Hilbert space, then each unitary matrix $U\in\mathcal{U}(n)$ determines a unitary operator $\widetilde{U} = I\oplus U\oplus U^{\otimes 2}\oplus U^{\otimes 3}\oplus\dots$ on $\mathcal{K}_n$. Conjugation by $\widetilde{U}$ acts as an automorphism Θ_U of $\mathcal{E}_n$. Moreover, it maps the ideal of compact operators onto itself. So it also induces an automorphism θ_U of $\mathcal{O}_n$. If $U = [u_{ij}]$ is an $n\times n$ unitary matrix, this automorphism can also be seen to be given by

$$\Theta_U(L_j) = \sum_{i=1}^n u_{ij} L_i \quad \text{for} \quad 1\le j\le n.$$

Given η, let U be any unitary with $u_{1j} = \eta_j$. Then it follows that

$$\varphi_\eta(A) = \varphi_{(1,0,\dots,0)}(\theta_U(A)) \quad \text{for all} \quad A\in\mathcal{O}_n.$$

So the corresponding representations are equivalent up to this automorphism. In particular, the *algebras* $\mathfrak{S}_\eta$ generated by these representations are unitarily equivalent even though the representations are not.

A crucial step in the analysis of atomic representations was to show that certain projections lie in the algebra $\mathfrak{S}$. Indeed, this is a major advantage of $\mathfrak{S}$ over the C*-algebra, which contains no non-trivial projections, and over the von Neumann algebra it generates, which contains too many. As a case in point, the projection $P_\eta = \xi_\eta\xi_\eta^*$ belongs to $\mathfrak{S}_\eta$. Indeed, it is the *only* non-trivial projection in the whole algebra $\mathfrak{S}_\eta$.

A crucial point of our analysis is the identification of projections in $\mathfrak{S}$ in greater generality. We begin with the irreducible case.

Theorem 5.2. *Assume that* $\sum_{i=1}^n A_iA_i^* = I$ *and* $\mathfrak{A} = \mathcal{B}(\mathcal{V})$. *Then* $\mathfrak{S}$ *contains the projection* $P_\mathcal{V}$.

Proof. Both $\mathfrak{S}$ and $P_\mathcal{V}$ belong to $\mathfrak{B}$. If $P_\mathcal{V}$ were not in $\mathfrak{S}$, Lemma 4.4 would provide a WOT-continuous functional φ which annihilates $\mathfrak{S}$ such that $\varphi(P_\mathcal{V}) = 1$. Represent φ as a functional of rank m in the form $\varphi(B) = \sum_{j=1}^m (Bx_j, y_j)$. This then may be realized as a rank one functional on the m-fold ampliation of $\mathfrak{B}$. Indeed, form the vectors $x = (x_1,\dots,x_m)$ and $y = (y_1,\dots,y_m)$. Then $\varphi(B) = (B^{(m)}x, y)$.

Now the fact that φ annihilates $\mathfrak{S}$ means that x is orthogonal to the subspace $\mathcal{M} = \mathfrak{S}^{*(m)}[y]$. The algebra $\mathfrak{S}^{(m)}$ is generated by isometries $S_i^{(m)}$, which form the minimal dilation of the $A_i^{(m)}$'s. So Corollary 4.2 of the Key Technical Lemma applies, and shows that $\mathcal{M}$ intersects $\mathcal{V}^{(m)}$ in a non-zero subspace $\mathcal{M}_0$ which is invariant for $\mathfrak{S}^{*(m)}$, and thus for $\mathfrak{A}^{*(m)}$.

By hypothesis, $\mathfrak{A}^{*(m)} = \mathcal{B}(\mathcal{V})^{(m)} \simeq \mathcal{B}(\mathcal{V}) \otimes \mathbb{C}^m$, which is a finite dimensional C*-algebra. The invariant subspace $\mathcal{M}_0$ is thus the range of a projection Q in the commutant $\mathbb{C}^d \otimes \mathfrak{M}_m$. Let $\tilde{Q}$ denote the operator in $\mathbb{C}I_{\mathcal{H}} \otimes \mathfrak{M}_m$ acting on $\mathcal{H}^{(m)}$ with the same matrix coefficients as Q. That is, $\tilde{Q}$ is the unique operator in $(\mathcal{B}(\mathcal{H}) \otimes \mathbb{C}^m)'$ such that $P_{\mathcal{V}}^{(m)} \tilde{Q} = Q$.

The projection $\tilde{Q}$ yields a decomposition of $\mathcal{H}^{(m)}$ into $\mathfrak{S}$-reducing subspaces $\mathcal{H}_1 \oplus \mathcal{H}_2$ where $\mathcal{H}_1 = \ker \tilde{Q}$ and $\mathcal{H}_2 = \operatorname{Ran} \tilde{Q}$; and likewise $\mathcal{V}^{(m)} = \mathcal{V}_1 \oplus \mathcal{V}_2$ where

$$\mathcal{V}_1 := \mathcal{H}_1 \cap \mathcal{V}^{(m)} = \ker Q \quad \text{and} \quad \mathcal{V}_2 := \mathcal{H}_2 \cap \mathcal{V}^{(m)} = \operatorname{Ran} Q.$$

Observe that $\mathcal{M}_0$ is contained in $\mathcal{H}_2$. For if we had a vector $x \in \mathcal{M}$ such that $P_{\mathcal{H}_1} x \neq 0$, then our Key Lemma 4.1 implies that there is a non-zero vector v in $\mathcal{M} \cap \mathcal{V}^{(m)} = \mathcal{M}_0$ such that $P_{\mathcal{H}_1} v \neq 0$. But by definition of Q and $\widetilde{Q}$, $\mathcal{M}_0$ is orthogonal to $\mathcal{H}_1$, a contradiction.

In particular, as $y \in \mathcal{M}$, we have $y = \tilde{Q} y$. Thus

$$P_{\mathcal{V}}^{(m)} y = P_{\mathcal{V}}^{(m)} \tilde{Q} y = Q P_{\mathcal{V}}^{(m)} y$$

belongs to $Q\mathcal{V}^{(m)} = \mathcal{M}_0$. Since x is orthogonal to $\mathcal{M}$ and hence to $\mathcal{M}_0$, we see that

$$\varphi(P_{\mathcal{V}}) = (P_{\mathcal{V}}^{(m)} x, y) = (x, P_{\mathcal{V}}^{(m)} y) = 0.$$

Consequently $P_{\mathcal{V}}$ belongs to $\mathfrak{S}$. ■

This immediately yields a structure theorem for $\mathfrak{S}$. Note that this does not classify the associated representations, as they depend on the specific generators, not just the algebra.

Corollary 5.3. *Assume that $\sum_{i=1}^n A_i A_i^* = I$ and $\mathfrak{A} = \mathcal{B}(\mathcal{V})$. Then $\mathfrak{S} \simeq \mathcal{B}(\mathcal{H}) P_{\mathcal{V}} + \big(0_{\mathcal{V}} \oplus \mathfrak{L}_n^{((n-1)d)}\big) \simeq \mathfrak{B}_{n,d}$.*

Proof. By Theorem 5.2, $\mathfrak{S}$ contains $P_{\mathcal{V}}$. Therefore it contains $P_{\mathcal{V}} \mathfrak{S} = \mathcal{B}(\mathcal{V})$. Moreover, it contains $S_i P_{\mathcal{V}}^{\perp} \simeq 0_{\mathcal{V}} \oplus L_i^{(\alpha)}$, where $\alpha = (n-1)d$. Thus $\mathfrak{S}$ contains the WOT-closed algebra that these operators generate, which is evidently $0_{\mathcal{V}} \oplus \mathfrak{L}_n^{(\alpha)}$. Finally, if v is any non-zero vector in $\mathcal{V}$, $\mathfrak{S}[v]$ contains $\mathcal{V}$ by hypothesis. So it is all of $\mathcal{H}$ by minimality of the dilation. Therefore for any $x \in \mathcal{H}$, there are operators $T_k \in \mathfrak{S}$ such that $T_k v$ converges to x. Thus $\mathfrak{S}$ contains $T_k v v^*$, which converge to the rank one operator $x v^*$. So $\mathcal{B}(\mathcal{H}) P_{\mathcal{V}}$ belongs to $\mathfrak{S}$. This is the whole

WOT-closed algebra which we called $\mathfrak{B}$, which trivially contains $\mathfrak{S}$. It is evident that $\mathfrak{B}$ is unitarily equivalent to $\mathfrak{B}_{n,d}$. ■

Now suppose that $\mathfrak{A}$ is a more general subalgebra of $\mathcal{B}(\mathcal{V})$. We wish to determine the structure of $\mathfrak{S}$ from information about $\mathfrak{A}$.

Lemma 5.4. *Assume that $\sum_{i=1}^n A_i A_i^* = I$. Suppose that $\mathcal{V}$ contains a minimal $\mathfrak{A}^*$-invariant subspace $\mathcal{V}_0$ of dimension d_0 which is cyclic for $\mathfrak{A}$. Then $\mathfrak{S}$ contains $\mathcal{B}(\mathcal{H})P_{\mathcal{V}_0}$, and is unitarily equivalent to $\mathfrak{B}_{n,d_0}$.*

Proof. By Burnside's Theorem [**32**, Corollary 8.6], since $\mathfrak{A}^*|_{\mathcal{V}_0}$ has no proper invariant subspaces, it must equal all of $\mathcal{B}(\mathcal{V}_0)$. Let $\mathcal{H}_0 = \mathfrak{S}[\mathcal{V}_0]$. This is a reducing subspace for $\mathfrak{S}$ by Lemma 3.4. We will argue that $\mathcal{H}_0 = \mathcal{H}$.

Suppose that x is a non-zero vector orthogonal to $\mathcal{H}_0$. By Corollary 4.2 of the Key Technical Lemma, $\mathfrak{S}^*[x] \cap \mathcal{V}$ contains a non-zero vector v. Moreover since $\mathcal{H}_0^\perp$ reduces $\mathfrak{S}$, v is orthogonal to $\mathcal{H}_0$. Therefore $\mathfrak{S}^*[v] = \mathfrak{A}^*[v]$ is an $\mathfrak{A}^*$-invariant subspace orthogonal to $\mathcal{V}_0$. Since $\mathfrak{A}\mathcal{V}_0 = \mathcal{V}$, there is an $A \in \mathfrak{A}$ and $v_0 \in \mathcal{V}_0$ such that $Av_0 = v$. So that

$$\|v\|^2 = (Av_0, v) = (v_0, A^*v) = 0.$$

This contradiction establishes our claim.

Now consider the compressions $\widetilde{A}_i = P_{\mathcal{V}_0} A_i|_{\mathcal{V}_0} = (A_i^*|_{\mathcal{V}_0})^*$. Then $\sum_{i=1}^n \widetilde{A}_i \widetilde{A}_i^* = I_{\mathcal{V}_0}$ follows from the $\mathfrak{A}^*$-invariance of $\mathcal{V}_0$. Also by hypothesis, the algebra $\widetilde{\mathfrak{A}}$ generated by the $\widetilde{A}_i$'s is $\mathcal{B}(\mathcal{V}_0)$. The minimal dilation of this n-tuple must be precisely the restriction of S_i to $\mathfrak{S}[\mathcal{V}_0] = \mathcal{H}$, which is S_i. So by Corollary 5.3, it follows that $\mathfrak{S}$ is unitarily equivalent to $\mathfrak{B}_{n,d_0}$. ■

The following corollary is almost immediate from the structure of $\mathfrak{B}_{n,d_0}$. We point it out in order to obtain some non-trivial consequences.

Corollary 5.5. *Assume that $\sum_{i=1}^n A_i A_i^* = I$. If $\mathcal{V}$ contains a subspace $\mathcal{V}_0$ which is cyclic for $\mathfrak{A}$ and is a minimal invariant subspace for $\mathfrak{A}^*$, then $\mathcal{V}_0$ is the unique minimal $\mathfrak{A}^*$-invariant subspace.*

Proof. We have $\mathfrak{A}^* = \mathfrak{S}^*|_{\mathcal{V}}$. So by the previous lemma, $\mathfrak{A}^*$ contains $P_{\mathcal{V}_0}\mathcal{B}(\mathcal{V})$. Consequently, $\mathcal{V}_0$ is contained in every non-zero $\mathfrak{A}^*$-invariant subspace. ■

Remark 5.6. This puts constraints on which subalgebras $\mathfrak{A}$ of $\mathcal{B}(\mathcal{V})$ can be generated by A_i's which satisfy $\sum_{i=1}^n A_i A_i^* = I$. For example, the semisimple algebra of matrices of the form $\mathfrak{A}_t = \left[\begin{smallmatrix} a & 0 \\ (b-a)t & b \end{smallmatrix}\right]$ for a, b in $\mathbb{C}$ and a fixed $t \neq 0$ is similar to the 2×2 diagonal algebra. Note that $\mathfrak{A}_t$ has two independent vectors which are cyclic for $\mathfrak{A}_t$ and eigenvalues for

$\mathfrak{A}_t^*$, namely e_1 and $f_2 = -\bar{t}e_1 + e_2$. By the corollary above, this cannot equal the algebra $\mathfrak{A}$. Indeed, if the generators of our algebra were $A_i = \begin{bmatrix} a_i & 0 \\ (b_i - a_i)t & b_i \end{bmatrix}$, then a computation would show that $\sum_{i=1}^n |a_i|^2 = 1$. Likewise considering the matrix with respect to an orthonormal basis $\{f_1, f_2\}$ would show that $\sum_{i=1}^n |b_i|^2 = 1$. This then forces $\sum_{i=1}^n |a_i - b_i|^2 |t|^2 = 0$. Since $t \neq 0$, this forces all the A_i's to be scalar, and hence they do not generate $\mathfrak{A}_t$.

Example 5.7. Consider a special case of the previous corollary: if $\mathfrak{A}$ has a cyclic vector e which is an eigenvalue for $\mathfrak{A}^*$. Then $\mathfrak{S}$ is unitarily equivalent to $\mathfrak{B}_{n,1}$. The algebra $\mathfrak{A}$ decomposes as $\mathfrak{A} = \mathcal{B}(\mathcal{V})P_e + J\mathfrak{A}_1 P_e^\perp$ where P_e is the orthogonal projection onto $\mathbb{C}e$, J is the injection of $\mathcal{V}_1 = \{e\}^\perp$ into $\mathcal{V}$, and $\mathfrak{A}_1$ is a unital subalgebra of $\mathcal{B}(\mathcal{V}_1)$. It is easy to see that

$$\operatorname{Lat}\mathfrak{A} = \{\mathcal{V}, JM : M \in \operatorname{Lat}\mathfrak{A}_1\}.$$

Hence if $\mathfrak{B}_1 = \operatorname{Alg}\operatorname{Lat}\mathfrak{A}_1$, then

$$\mathfrak{B} := \operatorname{Alg}\operatorname{Lat}\mathfrak{A} = \mathcal{B}(\mathcal{V})P_e + J\mathfrak{B}_1 P_e^\perp.$$

It follows that $\mathfrak{A}$ is reflexive if and only if $\mathfrak{A}_1$ is.

Thus if $\dim \mathcal{V}_1 > 1$, there are non-reflexive examples. For example, consider the non-reflexive algebra $\mathfrak{A}_1 = \{\left[\begin{smallmatrix} a & 0 \\ b & a \end{smallmatrix}\right] : a, b \in \mathbb{C}\}$. Take $n = 3$ and let

$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 0 \\ 0 & 0 & 1/\sqrt{2} \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1/2 & 1/2 & 0 \end{bmatrix} \quad A_3 = \begin{bmatrix} 0 & 0 & 0 \\ 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This can be seen to satisfy $\sum_{i=1}^3 A_i A_i^* = I_3$ and to generate the algebra $\mathfrak{A} = \left\{ \left[\begin{smallmatrix} c & 0 & 0 \\ d & a & 0 \\ e & b & a \end{smallmatrix}\right] : a, b, c, d, e \in \mathbb{C} \right\}$. This is not reflexive.

Nevertheless, $\mathfrak{A}^*$ has a unique minimal invariant subspace, and thus $\mathfrak{S}$ is unitarily equivalent to $\mathfrak{B}_{3,1}$, which is hyper-reflexive. So there is no direct correspondence between the reflexivity of $\mathfrak{A}$ and $\mathfrak{S}$.

Lemma 5.8. *Let $A = (A_1, \dots, A_n)$ be an n-tuple on a finite dimensional space $\mathcal{V}$ such that $\sum_{i=1}^n A_i A_i^* = I$. Let $\mathfrak{A}$ be the unital algebra that they generate. Let $S = (S_1, \dots, S_n)$ be the minimal isometric dilation, and $\mathfrak{S}$ the* WOT*-closed algebra they generate. Then $\mathfrak{S}$ is irreducible if and only if $\mathfrak{A}^*$ has a unique minimal invariant subspace $\mathcal{V}_0$.*

Proof. If $\mathcal{V}_0$ is unique, then it must be cyclic for $\mathfrak{A}$ since $\mathcal{V} \ominus \mathfrak{A}[\mathcal{V}_0]$ is an invariant subspace of $\mathfrak{A}^*$ orthogonal to $\mathcal{V}_0$. So Lemma 5.4 applies. Since $\mathfrak{S}$ contains $\mathcal{B}(\mathcal{H})P_{\mathcal{V}_0}$, it is evidently irreducible.

Indeed, this conclusion follows if there is *any* minimal $\mathfrak{A}^*$-invariant subspace $\mathcal{V}_0$ which is cyclic for $\mathfrak{A}$. By Corollary 5.5, $\mathcal{V}_0$ is necessarily the unique minimal $\mathfrak{A}^*$-invariant subspace.

Finally suppose that there is a minimal $\mathfrak{A}^*$-invariant subspace $\mathcal{V}_0$ which is not cyclic. Then as in the first paragraph, $\mathcal{V} \ominus \mathfrak{A}[\mathcal{V}_0]$ is an invariant subspace of $\mathfrak{A}^*$ orthogonal to $\mathcal{V}_0$. Let $\mathcal{V}_1$ be a minimal $\mathfrak{A}^*$-invariant subspace contained therein. Notice that $\mathfrak{S}[\mathcal{V}_i]$ are pairwise orthogonal reducing subspaces for $\mathfrak{S}$ by Lemma 3.4. Hence $\mathcal{H}$ contains proper reducing subspaces, and so $\mathfrak{S}$ is reducible. ■

Now we see how to deal with the case of more than one minimal $\mathfrak{A}^*$-invariant subspace. In this lemma, we do not concern ourselves with questions of uniqueness.

Lemma 5.9. *Assume that $\sum_{i=1}^n A_i A_i^* = I$. There is a family of minimal $\mathfrak{A}^*$-invariant subspaces $\mathcal{V}_j$ of $\mathcal{V}$, $1 \le j \le s$, such that $\mathcal{H}$ decomposes into an orthogonal direct sum of $\mathcal{H}_j = \mathfrak{S}[\mathcal{V}_j]$; and the algebras $\mathfrak{S}|_{\mathcal{H}_j}$ are irreducible.*

Proof. This is just a matter of choosing a maximal family of pairwise orthogonal minimal $\mathfrak{A}^*$-invariant subspaces, say $\mathcal{V}_j$ for $1 \le j \le s$. By Lemma 3.4, the subspaces $\mathcal{H}_j = \mathfrak{S}[\mathcal{V}_j]$ are pairwise orthogonal and reducing for $\mathfrak{S}$. Moreover a direct application of the previous lemma applied to $\mathcal{H}_j$ and $\mathcal{V}_j$ shows that $\mathfrak{S}|_{\mathcal{H}_j}$ is irreducible. Finally we must show that $\sum_{j=1}^{\oplus s} \mathcal{H}_j = \mathcal{H}$. Take any vector x orthogonal to this sum. By Corollary 4.2 of the Key Technical Lemma, $\mathfrak{S}^*[x]$ intersects $\mathcal{V}$ in a non-zero $\mathfrak{A}^*$-invariant subspace orthogonal to all of the $\mathcal{H}_j$'s, and thus orthogonal to all of the $\mathcal{V}_j$'s. This is contrary to construction, and so yields a contradiction. ■

Given an n-tuple $A = (A_1, \dots, A_n)$ such that $\sum_{i=1}^n A_i A_i^* = I$, let us pick a maximal family of mutually orthogonal minimal $\mathfrak{A}^*$-invariant subspaces $\mathcal{V}_j$ of $\mathcal{V}$, $1 \le j \le s$; and let $P_j = P_{\mathcal{V}_j}$. From the minimality of each $\mathcal{V}_j$ as an $\mathfrak{A}^*$-invariant subspace, we know that $P_j \mathfrak{A}^* P_j = \mathcal{B}(\mathcal{V}_j)$. Set $\widetilde{\mathcal{V}} = \sum_{j=1}^{\oplus s} \mathcal{V}_j$. Let $\widetilde{A}_i = P_{\widetilde{\mathcal{V}}} A_i|_{\widetilde{\mathcal{V}}} = (A_i^*|_{\widetilde{\mathcal{V}}})^*$ be the compression of A_i to $\widetilde{\mathcal{V}}$; and let $\widetilde{\mathfrak{A}}$ denote the algebra they generate in $\mathcal{B}(\widetilde{\mathcal{V}})$.

Notice that the minimal isometric dilation of $\widetilde{A} = (\widetilde{A}_1, \dots, \widetilde{A}_n)$ is precisely S. It is evident that S is a joint isometric dilation of $\widetilde{A}$. To show that it is minimal, it suffices to show that $\mathfrak{S}[\widetilde{\mathcal{V}}] = \mathcal{H}$. But this is established above in Lemma 5.9.

Our goal is to show that $\widetilde{\mathfrak{A}}$ is a C*-algebra. For the moment, let us show that it is semisimple. Note that $\widetilde{\mathfrak{A}}$ is contained in $\sum_{1 \le j \le s}^{\oplus} \mathcal{B}(\mathcal{V}_j)$. Moreover the quotient map q_j of compression to $\mathcal{V}_j$ maps $\mathfrak{A}$ onto $\mathcal{B}(\mathcal{V}_j)$.

Thus the kernel of this map is a maximal ideal. Since $\sum^{\oplus} q_j = \mathrm{id}$ is faithful, the intersection of all maximal ideals is $\{0\}$. Hence $\widetilde{\mathfrak{A}}$ is semisimple.

Indeed, there is a minimal family G so that $\sum^{\oplus}_{g\in G} q_g$ is faithful. By the Wedderburn theory, the minimal ideal $\mathfrak{A}_g = \ker \sum^{\oplus}_{h\in G\setminus\{g\}} q_h$ is isomorphic to $\mathcal{B}(\mathcal{V}_g)$. But this kernel will, in practice, be supported on several of the $\mathcal{V}_j$'s. This yields a partition $\widetilde{\mathcal{V}} = \sum^{\oplus}_{g\in G} \mathcal{W}_g$ where $\mathcal{W}_g = \sum^{\oplus}_{j\in G_g} \mathcal{V}_j$ is a sum of those $\mathcal{V}_j$'s equivalent to $\mathcal{V}_g$. Because $\mathcal{B}(\mathcal{V}_g)$ is simple, it follows that there is an algebra isomorphism σ_j of $\mathcal{B}(\mathcal{V}_g)$ onto $\mathcal{B}(\mathcal{V}_j)$ for each $j \in G_g$ such that

$$\widetilde{\mathfrak{A}}|_{\mathcal{W}_g} \simeq \Big\{ \sum_{j\in G_g}^{\oplus} \sigma_j(X) : X \in \mathcal{B}(\mathcal{V}_g) \Big\}.$$

It is well-known that every isomorphism between $\mathcal{B}(\mathcal{V}_g)$ and $\mathcal{B}(\mathcal{V}_j)$ is spatial: $\sigma_j(X) = T_j X T_j^{-1}$ for some invertible operator T_j, which is unique up to a scalar multiple.

We also need to consider the unital completely positive map Φ on $\mathcal{B}(\widetilde{\mathcal{V}})$ given by

$$\Phi(X) = \sum_{i=1}^{n} \widetilde{A}_i X \widetilde{A}_i^*.$$

Suppose that two blocks $\mathcal{V}_1$ and $\mathcal{V}_2$ are related by a similarity as above. Let $B_i := P_{\mathcal{V}_1} A_i|_{\mathcal{V}_1}$ and $C_i := P_{\mathcal{V}_2} A_i|_{\mathcal{V}_2} = T B_i T^{-1}$. Since

$$\sum_{i=1}^{n} B_i B_i^* = I_{\mathcal{V}_1} \qquad \text{and} \qquad \sum_{i=1}^{n} C_i C_i^* = I_{\mathcal{V}_2},$$

we compute that

$$I_{\mathcal{V}_2} = \sum_{i=1}^{n} (T B_i T^{-1})(T B_i T^{-1})^* = T \Phi_1 (T^{-1} T^{*-1}) T^*,$$

where $\Phi_1(X) = \sum_{i=1}^n B_i X B_i^* = P_1 \Phi(P_1 X P_1)|_{\mathcal{V}_1}$. Therefore

$$\Phi_1(T^{-1}T^{*-1}) = T^{-1}T^{*-1}.$$

We now study this completely positive map in order to gain information about the structure of $\widetilde{\mathfrak{A}}$.

Lemma 5.10. *Let $\Phi(X) = \sum_{i=1}^n A_i X A_i^*$ be a unital completely positive map on $\mathcal{B}(\mathcal{V})$, where $\mathcal{V}$ is finite dimensional. If there is a non-scalar operator X such that $\Phi(X) = X$, then $\mathfrak{A}^* = \mathrm{Alg}\{A_1^*, \dots, A_n^*\}$ has two pairwise orthogonal minimal invariant subspaces.*

Proof. Since Φ is self-adjoint and unital, there is a positive non-scalar X such that $\Phi(X) = X$. Let $\|X\| = 1$ and let μ denote the smallest eigenvalue of X. Then $\mathcal{M} = \ker(X - I)$ and $\mathcal{N} = \ker(X - \mu I)$ are pairwise orthogonal non-zero subspaces. For any unit vector $x \in \mathcal{M}$,

$$\begin{aligned} \|x\|^2 = (\Phi(X)x, x) &= \sum_{i=1}^{n} (XA_i^* x, A_i^* x) \\ &\le \sum_{i=1}^{n} (A_i^* x, A_i^* x) = \|x\|^2 \end{aligned}$$

This equality can only hold if each $A_i^* x$ belongs to $\mathcal{M}$. Hence $\mathcal{M}$ is invariant for $\mathfrak{A}^*$.

This argument worked because 1 is an extreme point in the spectrum of X. This is also the case for μ. Hence a similar argument shows that $\mathcal{N}$ is invariant for $\mathfrak{A}^*$. ■

The following is a partial converse to the previous lemma.

Lemma 5.11. *Let $\Phi(X) = \sum_{i=1}^n A_i X A_i^*$ be a unital completely positive map on $\mathcal{B}(\mathcal{V})$, where $\mathcal{V}$ is finite dimensional. Suppose that $A_i = B_i \oplus C_i$ with respect to an orthogonal decomposition $\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_2$. Moreover, suppose that $\mathrm{Alg}\{B_i\} = \mathcal{B}(\mathcal{V}_1)$ and $\mathrm{Alg}\{C_i\} = \mathcal{B}(\mathcal{V}_2)$. If there is an operator X such that $\Phi(X) = X$ and $X_{21} := P_{\mathcal{V}_2} X P_{\mathcal{V}_1} \neq 0$, then there is a unitary operator W such that $C_i = W^* B_i W$. Moreover the fixed point set of Φ consists of all matrices of the form $\begin{bmatrix} a_{11} I_{\mathcal{V}_1} & a_{12} W^* \\ a_{21} W & a_{22} I_{\mathcal{V}_2} \end{bmatrix}$.*

Proof. Since Φ is self-adjoint, we may suppose that $X = X^*$. Then normalize so that $\|X_{21}\| = 1$. Let $\mathcal{M} = \{v \in \mathcal{V}_1 : \|X_{21} v\| = \|v\|\}$. Also let $\mathcal{N} = X_{21}\mathcal{M}$ denote the corresponding subspace of $\mathcal{V}_2$. Write $B = \begin{bmatrix} B_1 & \dots & B_n \end{bmatrix}$ and $C = \begin{bmatrix} C_1 & \dots & C_n \end{bmatrix}$, so that

$$X_{21} v = \Phi(X_{21}) v = C X_{21}^{(n)} B^* v \quad \text{for} \quad v \in \mathcal{M}.$$

Since C and B^* are contractions, and X_{21} achieves its norm on v, it follows that $B^* v$ belongs to the subspace $\mathcal{M}^{(n)}$ on which $X_{21}^{(n)}$ achieves its norm. Consequently each B_i^* leaves $\mathcal{M}$ invariant. But as $\mathrm{Alg}\{B_i\} = \mathcal{B}(\mathcal{V}_1)$, this forces $\mathcal{M} = \mathcal{V}_1$. Similarly, consideration of $X_{12} = X_{21}^*$ shows that $\mathcal{N} = \mathcal{V}_2$. Thus X_{21} and X_{21}^* are isometries; so $W = X_{21}|_{\mathcal{V}_1}$ is a unitary map from $\mathcal{V}_1$ onto $\mathcal{V}_2$.

Further, the identity above now shows that $W = C W^{(n)} B^*$. Hence for all $v \in \mathcal{V}_1$

$$\|v\| = \|Wv\| = \|C W^{(n)} B^* v\| \le \|W^{(n)} B^* v\| \le \|v\|.$$

In particular, C acts as an isometry from the range of $W^{(n)}B^*$ onto the range $\operatorname{Ran} W = \mathcal{V}_2$. Since C is contractive, it must be zero on the orthogonal complement of $\operatorname{Ran} W^{(n)}B^*$. This implies that C^* is an isometry of $\mathcal{V}_2$ onto $\operatorname{Ran} W^{(n)}B^*$. Consequently, $C^*W = W^{(n)}B^*$; or equivalently, $C_i^* = WB_i^*W^*$ for $1 \le i \le n$.

Finally, if $Y \in \mathcal{B}(\mathcal{V}_1, \mathcal{V}_2)$ and $\left[\begin{smallmatrix} 0 & 0 \\ Y & 0 \end{smallmatrix}\right]$ is fixed by Φ, then

$$Y = \sum_{i=1}^{n} C_i Y B_i^* = \sum_{i=1}^{n} W B_i W^* Y B_i^* = W \Phi_1(W^*Y)$$

where $\Phi_1(X) = \sum_{i=1}^n B_i X B_i^*$ acts on $\mathcal{B}(\mathcal{V}_1)$. By Lemma 5.10, W^*Y is scalar; so Y is a multiple of W. A similar analysis works for the other coordinates. ∎

Example 5.12. Let

$$A_1 = \begin{bmatrix} 1/\sqrt{2} & 0 & 0 \\ 1/2\sqrt{2} & 1/2 & 1/2\sqrt{2} \\ 0 & 0 & 1/\sqrt{2} \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 1/\sqrt{2} & 0 & 0 \\ -1/2\sqrt{2} & 1/2 & -1/2\sqrt{2} \\ 0 & 0 & 1/\sqrt{2} \end{bmatrix}.$$

Then the matrix $X = \left[\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right]$ satisfies $\Phi(X) = X$. A calculation shows that the fixed point set of Φ is the set of matrices $X = [x_{ij}]$ such that $x_{12} = x_{21} = x_{23} = x_{32} = 0$ and $x_{11} + x_{13} + x_{31} + x_{33} = 2x_{22}$. In particular, this is not an algebra. The algebra $\mathfrak{A}^*$ has two minimal invariant subspaces, $\mathbb{C}e_1$ and $\mathbb{C}e_3$. Note that the compression of $\mathfrak{A}$ to $\operatorname{span}\{e_1, e_3\}$ consists of scalar matrices, and the fixed point set of the restricted completely positive map is the full 2×2 matrix algebra.

We can now utilize the detailed information about the map Φ to determine the algebra $\widetilde{\mathfrak{A}}$.

Theorem 5.13. *Let $\Phi(X) = \sum_{i=1}^n A_i X A_i^*$ be a unital completely positive map on $\mathcal{B}(\mathcal{V})$, where $\mathcal{V}$ is finite dimensional. Suppose that $\mathcal{V}$ is the orthogonal direct sum of minimal $\mathfrak{A}^*$-invariant subspaces. Then $\mathfrak{A}$ is a C^*-algebra and the fixed point set of Φ coincides with the commutant of $\mathfrak{A}$.*

Proof. Let $\mathcal{V} = \sum_j^{\oplus} \mathcal{V}_j$ be an orthogonal decomposition into minimal $\mathfrak{A}^*$-invariant subspaces. The restriction of $\mathfrak{A}$ to $\mathcal{V}_j$ is all of $\mathcal{B}(\mathcal{V}_j)$ by Burnside's Theorem. Thus the restriction of Φ to $\mathcal{B}(\mathcal{V}_j)$ maps onto the scalars by Lemma 5.10. By the earlier analysis, $\mathfrak{A}$ splits into an algebraic direct sum of minimal ideals which are isomorphic to full matrix algebras. These are determined by certain spatial intertwining relations between some of the summands. If the restriction of A_i's to $\mathcal{V}_1$ and $\mathcal{V}_2$ are related by an intertwining operator T, then we showed that $\Phi_1(T^{-1}T^{*-1}) = T^{-1}T^{*-1}$. But this is scalar by Lemma 5.10. So after scaling T, it becomes a unitary. It follows that $\mathfrak{A}$ is a C*-algebra.

Evidently, Φ fixes the commutant of $\mathfrak{A} = \mathfrak{A}^*$. Suppose that $\Phi(X) = X$. If $\mathcal{V}_k$ and $\mathcal{V}_l$ are not related by a unitary intertwining map, then by Lemma 5.11, $P_kXP_l = 0$. While if they are related by a unitary W_{kl}, then $P_kXP_l = x_{kl}W_{kl}$ belongs to $\mathfrak{A}'$. It follows that the fixed point set is precisely the commutant of $\mathfrak{A}$. ■

Now it is possible to provide a complete description of the algebra $\mathfrak{S}$ in the Cuntz case.

Lemma 5.14. *Let P_g for $g \in G$ denote the minimal central projections in $\widetilde{\mathfrak{A}}$. These projections belong to $\mathfrak{S}$.*

Proof. We follow the lines of Theorem 5.2. We may work in the algebra $\mathfrak{B} = \mathcal{B}(\mathcal{H})P_{\mathcal{V}} + \big(0_{\mathcal{V}} \oplus \mathfrak{L}_n^{(\alpha)}\big)$ which contains $\mathfrak{S}$ and each projection P_g. If a central projection P of $\widetilde{\mathfrak{A}}$ were not in $\mathfrak{S}$, by Lemma 4.4 it could be separated from $\mathfrak{S}$ by a functional of rank $d+1$, which as before we write as $\varphi(A) = (A^{(d+1)}x, y)$. Let $\mathcal{M} = \mathfrak{S}^{*(d+1)}[y]$ and $\mathcal{M}_0 = \widetilde{\mathcal{V}}^{(d+1)} \cap \mathcal{M}$. This subspace $\mathcal{M}_0$ is invariant for the C*-algebra $\widetilde{\mathfrak{A}}^{*(d+1)} = \widetilde{\mathfrak{A}}^{(d+1)}$, and thus is the range of a projection Q in its commutant.

Now $P^{(d+1)}$ lies in the centre of $\widetilde{\mathfrak{A}}^{(d+1)}$, and thus commutes with Q as well. Therefore $\widetilde{\mathcal{V}}^{(d+1)}$ decomposes as

$$P^{(d+1)}Q\widetilde{\mathcal{V}}^{(d+1)} \oplus P^{\perp(d+1)}Q\widetilde{\mathcal{V}}^{(d+1)} \oplus P^{(d+1)}Q^{\perp}\widetilde{\mathcal{V}}^{(d+1)} \oplus P^{\perp(d+1)}Q^{\perp}\widetilde{\mathcal{V}}^{(d+1)} =: \mathcal{M}_{pq} \oplus \mathcal{M}_{p^\perp q} \oplus \mathcal{M}_{pq^\perp} \oplus \mathcal{M}_{p^\perp q^\perp}.$$

This determines an orthogonal decomposition of $\widetilde{\mathcal{V}}^{(d+1)}$ into four reducing subspaces for $\widetilde{\mathfrak{A}}^{(d+1)}$. Note that $\mathcal{M}_0$ is the sum of the first two. Recall the remarks following Lemma 5.9 that S is the minimal isometric dilation of $\widetilde{A}$. So by Lemma 3.4, $\mathcal{H}^{(d+1)}$ has an orthogonal decomposition into the four reducing subspaces for $\mathfrak{S}^{(d+1)}$ generated by these subspaces of $\widetilde{\mathcal{V}}^{(d+1)}$, say

$$\mathcal{H}^{(d+1)} = \mathcal{H}_{pq} \oplus \mathcal{H}_{p^\perp q} \oplus \mathcal{H}_{pq^\perp} \oplus \mathcal{H}_{p^\perp q^\perp}.$$

Moreover,the Key Lemma 4.1 shows as in the proof of Theorem 5.2 that y belongs to $\mathcal{H}_{pq} \oplus \mathcal{H}_{p^\perp q} = \mathfrak{S}^{(d+1)}[\mathcal{M}_0]$.

It is evident from this construction that each of these four subspaces $\mathcal{H}_{ij}$ is mapped onto the corresponding $\mathcal{M}_{ij}$ by the orthogonal projection $P_{\widetilde{\mathcal{V}}}^{(d+1)}$ onto $\widetilde{\mathcal{V}}^{(d+1)}$. Therefore, since $P^{(d+1)}$ is dominated by this projection, it is clear that it maps y into $\mathcal{M}_{pq}$, which is contained in $\mathcal{M}_0$. As before, we obtain that x is orthogonal to $\mathcal{M}_0$, and therefore $\varphi(P) = 0$. Hence we conclude that P belongs to $\mathfrak{S}$. ■

Theorem 5.15. *Let $A_1, \dots, A_n$ be operators on a finite dimensional space $\mathcal{V}$ such that $\sum\limits_{i=1}^{n} A_i A_i^* = I$, and let $S_1, \dots, S_n$ be their joint isometric dilation. Let $\widetilde{\mathcal{V}}$ be the subspace of $\mathcal{V}$ spanned by all minimal $\mathfrak{A}^*$-invariant subspaces. Then the compression $\widetilde{\mathfrak{A}}$ of $\mathfrak{A}$ to $\widetilde{\mathcal{V}}$ is a C^*-algebra. Let $\widetilde{\mathfrak{A}}$ be decomposed as $\sum^{\oplus}_{g\in G} \mathfrak{M}_{d_g} \otimes \mathbb{C}^{m_g}$ with respect to a decomposition $\widetilde{\mathcal{V}} = \sum^{\oplus}_{g\in G} \mathcal{V}_g^{(m_g)}$, where $\mathcal{V}_g$ has dimension d_g and multiplicity m_g. Let P_g denote the projection onto $\mathcal{V}_g$. Then the dilation acts on the space*

$$\mathcal{H} = \sum_{g\in G}{}^{\oplus} \mathcal{H}_g^{(m_g)} = \widetilde{\mathcal{V}} \oplus \mathcal{K}_n^{(\alpha)}$$

where $\mathcal{H}_g = \mathcal{V}_g \oplus \mathcal{K}_n^{(\alpha_g)}$ and $\alpha_g = d_g(n-1)$ and

$$\alpha = \sum_{g\in G} \alpha_g m_g = (n-1) \sum_{g\in G} d_g m_g .$$

The algebra $\mathfrak{S}$ decomposes as

$$\mathfrak{S} \simeq \sum_{g\in G}{}^{\oplus} \big(\mathcal{B}(\mathcal{H}_g)P_g\big)^{(m_g)} \ + \ \big(0_{\widetilde{\mathcal{V}}} \oplus \mathfrak{L}_n^{(\alpha)}\big).$$

Proof. This is now just a matter of putting the pieces together and clearing up some final details. Let $\mathcal{V}_g$, $1 \le g \le s$, be any maximal family of pairwise orthogonal minimal $\mathfrak{A}^*$-invariant subspaces. Let $\widetilde{\mathcal{V}} = \sum^{\oplus}_{1\le g\le s} \mathcal{V}_g$. (Do not worry at this stage about the uniqueness of the definition of $\widetilde{\mathcal{V}}$.) By Lemma 3.4, $\mathcal{H}_g = \mathfrak{S}[\mathcal{V}_g]$ are pairwise orthogonal reducing subspaces of $\mathfrak{S}$. Let $\mathcal{M} = \sum^{\oplus}_{1\le g\le s} \mathcal{H}_g$. We claim that $\mathcal{M} = \mathcal{H}$. Indeed, were there a non-zero vector in $\mathcal{M}^\perp$, then by Corollary 4.2 of the Key Technical Lemma, $\mathcal{M}^\perp \cap \mathcal{V}$ would be an non-zero $\mathfrak{A}^*$-invariant subspace orthogonal to $\widetilde{\mathcal{V}}$, contrary to fact.

It now follows as above that if we compress each A_i to $\widetilde{A}_i$ on $\widetilde{\mathcal{V}}$, then this new n-tuple has the identical joint isometric dilation S_i, and it is the minimal dilation by the previous paragraph. By Theorem 5.13, the algebra $\widetilde{\mathfrak{A}}$ that they generate is self-adjoint. Then applying Lemma 5.14, we deduce that the projection onto $\widetilde{\mathcal{V}}$ belongs to $\mathfrak{S}$, and that $P_{\widetilde{\mathcal{V}}}\mathfrak{S} = \widetilde{\mathfrak{A}}$.

The restriction of $\mathfrak{S}$ to each reducing subspace $\mathcal{H}_g$ is isomorphic to $\mathfrak{B}_{n,d_g}$. Moreover the restriction of $\mathfrak{S}$ to $\widetilde{\mathcal{V}}^\perp$ is canonically isomorphic to $\mathfrak{L}_n^{(\alpha)}$, where by canonical we mean that $u(S)|_{\widetilde{\mathcal{V}}^\perp} \simeq L_u^{(\alpha)}$ when we make the natural identification of $\widetilde{\mathcal{V}}^\perp$ with $\mathcal{K}_n^{(\alpha)}$ as in Lemma 3.1.

Now the finite dimensional C*-algebra $\widetilde{\mathfrak{A}}$ may be decomposed as $\sum_{g\in G}^{\oplus} \mathfrak{M}_{d_g} \otimes \mathbb{C}^{m_g}$. The multiplicities reflect the fact that the restrictions of A_i^* to different $\mathcal{V}_g$'s may be unitarily equivalent. As before, choose a maximal subset G of pairwise inequivalent subspaces $\mathcal{V}_g$, and let $\mathcal{W}_g = \sum_{j\in G_g}^{\oplus} \mathcal{V}_j$ be the sum of all subspaces equivalent to $\mathcal{V}_g$. Then $\mathcal{W}_g$ may be naturally identified with $\mathcal{V}_g \otimes \mathbb{C}^{m_g}$ so that $A_i^*|_{\mathcal{W}_g} \simeq \big(A_i^*|_{\mathcal{V}_g}\big)^{(m_g)}$. This identifies $\widetilde{\mathcal{V}}$ with $\sum_{g\in G}^{\oplus} \mathcal{V}_g^{(m_g)}$.

By the uniqueness of the minimal isometric dilation, it also follows that there is a corresponding unitary equivalence between $\sum_{j\in G_g}^{\oplus} \mathcal{H}_j$ and $\mathcal{H}_g \otimes \mathbb{C}^{(m_g)}$ so that the restriction of S_i is identified with $\big(S_i|_{\mathcal{H}_g}\big)^{(m_g)}$. By Lemma 5.14, the projection $P_{\mathcal{W}_g} \simeq P_g^{(m_g)}$ belongs to $\mathfrak{S}$. Thus we now see that $\mathfrak{S}P_{\widetilde{\mathcal{V}}}$ decomposes as $\sum_{g\in G}^{\oplus} \big(\mathcal{B}(\mathcal{H}_g)P_g\big)^{(m_g)}$. Combining all of the pieces, we obtain the desired structure theory for $\mathfrak{S}$.

It remains to establish the uniqueness of $\widetilde{\mathcal{V}}$. We can now see that $P_{\widetilde{\mathcal{V}}}$ is the unique maximal finite rank projection in $\mathfrak{S}$. Indeed, every operator in $\mathfrak{S}$ has a lower triangular form with respect to the decomposition $\mathcal{H} = \widetilde{\mathcal{V}} \oplus \mathcal{K}_n^{(\alpha)}$. By [**15**, Corollary 1.8], $\mathfrak{L}_n$ contains no proper projections. Therefore all finite rank projections are supported by $\widetilde{\mathcal{V}}$. Now suppose that $\mathcal{V}_0$ is any minimal $\mathfrak{A}^*$-invariant subspace. It may be extended to a maximal family of pairwise orthogonal minimal $\mathfrak{A}^*$-invariant subspaces, and the construction may proceed as above. The same subspace $\widetilde{\mathcal{V}}$ necessarily is obtained by the uniqueness of this maximal projection. In particular, $\widetilde{\mathcal{V}}$ must contain *every* minimal $\mathfrak{A}^*$-invariant subspace. Thus it is the span of all such subspaces. ■

6. The General Finite Dimensional Case

We now return to the problem posed in Section 4. Starting with a contractive n-tuple $A_1, \dots, A_n$ with minimal joint isometric dilation $S_1, \dots, S_n$, we wish to understand the structure of $\mathfrak{S} = \mathrm{Alg}\{S_1, \dots, S_n\}$ in terms of the structure of the n-tuple A and the algebra $\mathfrak{A}$ that it generates.

Recall from the discussion in Section 4 that $\mathcal{H} = \mathcal{H}_p \oplus \mathcal{H}_c$, where $\mathcal{H}_p$ is the pure part determined by the wandering subspace of S, and $\mathcal{H}_c$ is the Cuntz part; and that P_p and P_c denote the orthogonal projections onto these subspaces. We need a method of getting information about this decomposition from A. Corollary 4.2 of the Key Technical Lemma shows that $\mathcal{H}_c = \mathfrak{S}[\mathcal{V}_c]$, so $\mathcal{H}_c$ is recovered if we can compute $\mathcal{V}_c$.

Again we consider the completely positive map $\Phi(X) = \sum_{i=1}^n A_i X A_i^*$. This is no longer unital, since $\Phi(I) = AA^* = \sum_{i=1}^n A_i A_i^* \le I$. But it is completely contractive. Thus the sequence $\Phi^k(I)$ is a decreasing sequence of positive operators, and therefore converges to a limit which we denote as $\Phi^\infty(I)$.

Lemma 6.1. *$\Phi^\infty(I) = P_{\mathcal{V}} P_c P_{\mathcal{V}}$. Hence $\mathcal{V}_c = \ker(I - \Phi^\infty(I))$.*

Proof. If $x \in \mathcal{H}_c$,

$$\sum_{|w|=k} \|S_w^* x\|^2 = \|x\|^2.$$

On the other hand, any vector x in $\mathcal{H}_p$ satisfies

$$\lim_{k\to\infty} \sum_{|w|=k} \|S_w^* x\|^2 = 0.$$

Thus if x is any vector in $\mathcal{H} = \mathcal{H}_c \oplus \mathcal{H}_p$,

$$\lim_{k\to\infty} \sum_{|w|=k} \|S_w^* x\|^2 = \|P_c x\|^2.$$

We write $A_w^* := w(A)^* = S_w^*|_{\mathcal{V}}$. Now if $v \in \mathcal{V}$,

$$\lim_{k\to\infty} \sum_{|w|=k} \|A_w^* v\|^2 = \lim_{k\to\infty} \sum_{|w|=k} \|S_w^* v\|^2 = \|P_c v\|^2.$$

It is evident that $\Phi^k(I) = \sum_{|w|=k} A_w A_w^*$ and thus

$$(\Phi^k(I)v, v) = \sum_{|w|=k} \|A_w^* v\|^2.$$

Therefore

$$(\Phi^\infty(I)v, v) = \|P_c v\|^2 = (P_{\mathcal{V}} P_c P_{\mathcal{V}} v, v).$$

Since a sesquilinear form can be recovered from its quadratic form by the polarization identity, it follows that $\Phi^\infty(I) = P_{\mathcal{V}} P_c P_{\mathcal{V}}$.

In particular, $\ker(I - \Phi^\infty(I)) = \mathcal{V} \cap \mathcal{H}_c = \mathcal{V}_c$. ∎

We have $\mathcal{H}_c = \mathfrak{S}[\mathcal{V}_c]$, and thus the restriction of the S_i's to $\mathcal{H}_c$ are the minimal joint isometric dilations of the compressions of the A_i's to $\mathcal{V}_c$. By the previous section, we know that $\mathfrak{S}|_{\mathcal{H}_c}$ is determined by the restriction of $\mathfrak{A}$ to the span $\widetilde{\mathcal{V}}$ of all $\mathfrak{A}^*$-invariant subspaces contained in $\mathcal{V}_c$. It is desirable to give a definition that is somewhat independent of the definition of $\mathcal{V}_c$. *the space $\widetilde{\mathcal{V}}$ is the span of all $\mathfrak{A}^*$-invariant subspaces $\mathcal{W}$ on which $\sum_{i=1}^n A_i A_i^*|_{\mathcal{W}} = I_{\mathcal{W}}$.* Indeed, the condition

$$I_{\mathcal{W}} = \sum_{i=1}^n A_i A_i^*|_{\mathcal{W}} = \sum_{i=1}^n S_i S_i^*|_{\mathcal{W}}$$

implies that $\mathcal{W}$ is contained in $\mathcal{H}_c$, whence in $\mathcal{H}_c \cap \mathcal{V} = \mathcal{V}_c$. Thus $\mathcal{W}$ is contained in $\widetilde{\mathcal{V}}$ by Theorem 5.15. The converse follows from the description there of $\widetilde{\mathcal{V}}$.

Lemma 6.2. *The projection $P_{\widetilde{\mathcal{V}}}$ belongs to $\mathfrak{S}$.*

Proof. We may assume that $P_{\widetilde{\mathcal{V}}} \neq 0$. Decompose $\mathcal{H}$ as $\mathcal{H}_c \oplus \mathcal{H}_p$. Let $\mathfrak{S}_c$ denote the restriction of $\mathfrak{S}$ to $\mathcal{H}_c$. By the Cuntz case, Theorem 5.14, the projection $P_{\widetilde{\mathcal{V}}}$ belongs to the WOT-closure of $\mathfrak{S}_c$. In other words there is a net A_α of polynomials in S such that $A_\alpha|_{\mathcal{H}_c}$ converges in the WOT topology to $P_{\widetilde{\mathcal{V}}}$. Since $\mathcal{H}_c$ contains a wandering vector ξ, the subspace $\mathfrak{S}[\xi]$ is unitarily equivalent to $\mathfrak{L}_n$. Moreover the restriction of A_α to $\mathfrak{S}[\xi]$ converges weakly to 0. Now the restriction of A_α to $\mathcal{H}_p$ is unitarily equivalent to a multiple of $A_\alpha|_{\mathfrak{S}[\xi]}$, and thus it also converges weakly to 0. Combining the two parts, we see that A_α converges to $P_{\widetilde{\mathcal{V}}}$ in $\mathcal{B}(\mathcal{H})$. ∎

Next we wish to compute the pure rank of the dilation. Notice that the proof which follows does not require that the n-tuple act on a finite dimensional space. This fact is used in the development of the non-commutative curvature invariant and Euler characteristic in **[24]**.

Lemma 6.3. *The pure rank of $\mathfrak{S}$ is computed as*

$$\operatorname{pure\,rank}(\mathfrak{S}) = \operatorname{rank}(I - \Phi(I)) = \operatorname{rank}\Big(I - \sum_{i=1}^{n} A_i A_i^*\Big).$$

Proof. The wandering space is $\mathcal{X} = \operatorname{Ran}\big(I - \sum_{i=1}^n S_i S_i^*\big)$ and the pure rank of $\mathfrak{S}$ equals $\dim \mathcal{X}$. The minimality of the dilation means that $\mathcal{X}$ does not intersect $\mathcal{V}^\perp$. Therefore $P_\mathcal{V} P_\mathcal{X} P_\mathcal{V}$ has the same rank as $P_\mathcal{X}$. However it is easy to see that

$$\begin{aligned} P_\mathcal{V} P_\mathcal{X} P_\mathcal{V}|_\mathcal{V} &= P_\mathcal{V}\Big(I_\mathcal{H} - \sum_{i=1}^{n} S_i S_i^*\Big) P_\mathcal{V}|_\mathcal{V} \\ &= I_\mathcal{V} - \sum_{i=1}^{n} A_i A_i^* = I_\mathcal{V} - \Phi(I_\mathcal{V}). \end{aligned}$$

Thus $\operatorname{pure\,rank}(\mathfrak{S}) = \operatorname{rank}\big(I - \Phi(I)\big)$. ∎

Example 6.4. Any subtlety of the preceding lemma is due to fact that $\mathcal{X}$ is not, in general, contained in $\mathcal{V}$. To illustrate this, consider the following example. Let

$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{and} \qquad A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then

$$A_1A_1^* + A_2A_2^* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

It is clear that $\mathbb{C}e_1$ and $\mathbb{C}e_3$ are pairwise orthogonal minimal $\mathfrak{A}^*$-invariant subspaces. The vector e_1 generates the subspace $\mathcal{H}_1 = \overline{\mathfrak{S}e_1}$ on which the representation is equivalent to the atomic representation $\sigma_{1,1}$. Furthermore, e_3 is a wandering vector generating a copy of the left regular representation on $\mathcal{H}_3 = \overline{\mathfrak{S}e_3}$. However e_2 is not orthogonal to $\mathcal{H}_1 \oplus \mathcal{H}_3$. One can show that there is a second wandering vector $\zeta := e_2 - P_{\mathcal{V}}^{\perp} S_2(e_1 + e_3)$. The subspace $\mathcal{H}_2 = \overline{\mathfrak{S}\zeta}$ yields the decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$.

The point here is that this decomposition does not decompose $\mathcal{V}$ into orthogonal pieces. In fact, $\mathcal{H}_2$ has trivial intersection with $\mathcal{V}$; and the vector e_2 has components in all three pieces.

We can now completely describe the algebra $\mathfrak{S}$ determined by the joint isometric dilation of a contractive n-tuple. There is nothing to do except combine the information in Theorem 5.15 with the preceding two lemmas.

Theorem 6.5. *Let $A_1, \dots, A_n$ be a contractive n-tuple on a finite dimensional space $\mathcal{V}$ with joint minimal isometric dilation $S_1, \dots, S_n$ on $\mathcal{H}$. The space $\mathcal{H}$ decomposes as $\mathcal{H}_p \oplus \mathcal{H}_c$ into its pure and Cuntz parts. The multiplicity of $\mathcal{H}_p$ is* pure rank$(\mathfrak{S})$ = rank$\big(I - \sum_{i=1}^n A_iA_i^*\big)$. *The subspace $\widetilde{\mathcal{V}}$ spanned by all minimal $\mathfrak{A}^*$-invariant subspaces $\mathcal{W}$ on which $\sum_{i=1}^n A_iA_i^*|_{\mathcal{W}} = I_{\mathcal{W}}$ determines $\mathcal{H}_c = \mathfrak{S}[\widetilde{\mathcal{V}}]$.*

The compression $\widetilde{\mathfrak{A}}$ of $\mathfrak{A}$ to $\widetilde{\mathcal{V}}$ is a C^*-algebra. Let $\widetilde{\mathfrak{A}}$ be decomposed as $\sum_{g\in G}^{\oplus} \mathfrak{M}_{d_g} \otimes \mathbb{C}^{m_g}$ with respect to a decomposition $\widetilde{\mathcal{V}} = \sum_{g\in G}^{\oplus} \mathcal{V}_g^{(m_g)}$, where $\mathcal{V}_g$ has dimension d_g and multiplicity m_g; and let P_g denote the projection onto $\mathcal{V}_g$. Then the dilation acts on the space

$$\mathcal{H} = \sum_{g\in G}^{\oplus} \mathcal{H}_g^{(m_g)} \oplus \mathcal{H}_p = \widetilde{\mathcal{V}} \oplus \mathcal{K}_n^{(\alpha)}$$

where $\mathcal{H}_g = \mathcal{V}_g \oplus \mathcal{K}_n^{(\alpha_g)}$, $\alpha_g = d_g(n-1)$ and

$$\begin{aligned}\alpha &= \sum_{g\in G} \alpha_g m_g + \text{pure rank}(\mathfrak{S}) \\ &= (n-1)\sum_{g\in G} d_g m_g + \text{rank}\Big(I - \sum_{i=1}^n A_iA_i^*\Big).\end{aligned}$$

The algebra $\mathfrak{S}$ *decomposes as*

$$\mathfrak{S} \simeq \sum_{g\in G}^{\oplus} \big(\mathcal{B}(\mathcal{H}_g)P_g\big)^{(m_g)} \ + \ \big(0_{\widetilde{\mathcal{V}}} \oplus \mathfrak{L}_n^{(\alpha)}\big).$$

We now collect some of the consequences of this theorem. First we obtain simple conditions to determine when the dilation of A is irreducible.

Corollary 6.6. *The algebra* $\mathfrak{S}$ *determined by the joint isometric dilation of a contractive n-tuple* A *on a finite dimensional space* $\mathcal{V}$ *is irreducible if and only if either*

(1) $\operatorname{Ran}(I - \sum_{i=1}^n A_i A_i^*) = \mathbb{C}v \neq 0$ *and* v *is cyclic for* $\mathfrak{A}$. *In this case,* $\mathfrak{S}$ *is unitarily equivalent to* $\mathfrak{L}_n$.

or

(2) $\sum_{i=1}^n A_i A_i^* = I$ *and* $\mathfrak{A}^*$ *has a minimal invariant subspace* $\mathcal{V}_0$ *which is cyclic for* $\mathfrak{A}$. *In this case,* $\mathfrak{S}$ *is unitarily equivalent to* $\mathfrak{B}_{n,d_0}$ *where* $d_0 = \dim \mathcal{V}_0$.

which are respectively equivalent to

(1′) $\operatorname{rank}(I - \Phi(I)) = 1$ *and* $\Phi^\infty(I) = 0$.

or

(2′) $\{X : \Phi(X) = X\} = \mathbb{C}I$.

Proof. $\mathfrak{S}$ is irreducible if and only if either it is pure with pure rank 1, or it has pure rank 0 and, by Lemma 5.8, has a unique minimal $\mathfrak{A}^*$-invariant subspace.

By Lemma 6.3, the pure rank is 1 precisely when $\operatorname{rank}(I-\Phi(I)) = 1$, or equivalently that $\operatorname{Ran}(I-\sum_{i=1}^n A_i A_i^*)$ is a one-dimensional subspace $\mathbb{C}v$. Now $\mathfrak{S}$ is pure precisely when $\mathcal{H}_c = \{0\}$, which by Corollary 4.2 is equivalent to $\mathcal{V}_c = \{0\}$. By Lemma 6.1, this is equivalent to $\Phi^\infty(I) = 0$, which establishes the equivalence with (1′). Now $\mathcal{V}_c$ is $\mathfrak{A}^*$-invariant and orthogonal to v, and therefore orthogonal to $\mathfrak{A}v$. So if v is $\mathfrak{A}$–cyclic, then $\mathfrak{A}[v] = \mathcal{V}$ and $\mathcal{V}_c = \{0\}$. Conversely, if $\mathfrak{A}[v]$ is proper, then $\mathcal{M} = \mathfrak{A}[v]^\perp$ is $\mathfrak{A}^*$-invariant. But $\sum_{i=1}^n A_i A_i^*|_{\mathcal{M}} = I_{\mathcal{M}}$ because of the condition on $\Phi(I)$. So $\mathcal{M}$ is contained in $\mathcal{H}_c$. This verifies the equivalence with (1).

The Cuntz case is synonymous with the condition $\sum_{i=1}^n A_i A_i^* = I$. If $\mathcal{M}$ is a minimal $\mathfrak{A}^*$-invariant subspace, then $\mathfrak{A}[\mathcal{M}]^\perp$ contains another. So if $\mathcal{M}$ is unique, it must be cyclic. Conversely, if it is not unique, then by Theorem 5.15, $\widetilde{\mathcal{V}}$ contains at least two pairwise orthogonal minimal $\mathfrak{A}^*$-invariant subspaces, one of which may be taken to be $\mathcal{M}$; call the other $\mathcal{M}'$. Then $\mathfrak{A}[\mathcal{M}]$ is orthogonal to $\mathcal{M}'$ and thus it is not cyclic for $\mathfrak{A}$. This establishes the equivalence with (2).

Condition (2′) contains the fact that $\Phi(I) = I$, so this is the Cuntz case. If there were more than one minimal $\mathfrak{A}^*$-invariant subspace, then by Theorem 5.13 the fixed point algebra contains non-scalar operators. Conversely, if Φ has non-scalar fixed points, then Lemma 5.10 shows that there are two orthogonal $\mathfrak{A}^*$-invariant subspaces. So (2′) is equivalent to irreducibility. ∎

Corollary 6.7. *The minimal isometric dilation of a finite dimensional n-tuple $A = (A_1, \dots, A_n)$ is pure if and only if $\mathfrak{A}(I - \sum_{i=1}^n A_i A_i^*)\mathcal{V} = \mathcal{V}$ or equivalently that $\Phi^\infty(I) = 0$.*

Proof. The dilation has a Cuntz part if and only if there is a $\mathfrak{A}^*$-invariant subspace $\mathcal{M}$ contained in $\ker(I - \sum_{i=1}^n A_i A_i^*)$. This is equivalent to having the proper $\mathfrak{A}$-invariant subspace $\mathcal{M}^\perp$ containing

$$\big(\ker(I - \sum_{i=1}^n A_i A_i^*)\big)^\perp = \operatorname{Ran}(I - \sum_{i=1}^n A_i A_i^*).$$

The minimal such subspace is clearly $\mathfrak{A}(I - \sum_{i=1}^n A_i A_i^*)\mathcal{V}$. Thus the dilation is pure precisely when $\mathfrak{A}(I - \sum_{i=1}^n A_i A_i^*)\mathcal{V} = \mathcal{V}$.

Evidently, if there is a Cuntz part, then

$$\Phi^\infty(I) \ge \Phi^\infty(P_{\widetilde{\mathcal{V}}}) = P_{\widetilde{\mathcal{V}}}.$$

Conversely, if A is pure, then $\textsc{sot}\text{–}\lim_{k\to\infty} \sum_{|w|=k} S_w S_w^* = 0$. The compression of $S_w S_w^*$ to $\mathcal{V}$ is $A_w A_w^*$, and thus

$$\sum_{|w|=k} P_{\mathcal{V}} S_w S_w^*|_{\mathcal{V}} = \sum_{|w|=k} A_w A_w^* = \Phi^k(I).$$

Since $\mathcal{V}$ is finite dimensional, this converges to 0 in norm. ∎

Our theorem also provides simple complete unitary invariants for the associated finitely correlated representations of $\mathcal{E}_n$ (or of $\mathcal{O}_n$ in the Cuntz case).

Theorem 6.8. *Let $A = (A_1, \dots, A_n)$ and $B = (B_1, \dots, B_n)$ be contractive n-tuples on finite dimensional spaces $\mathcal{V}_A$ and $\mathcal{V}_B$ respectively. Let $S = (S_1, \dots, S_n)$ and $T = (T_1, \dots, T_n)$ be their joint minimal isometric dilations on Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$; and let σ_A and σ_B be the induced representations of $\mathcal{E}_n$. Let $\widetilde{\mathcal{V}}_A$ be the subspace spanned by all minimal $\mathfrak{A}^*$-invariant subspaces $\mathcal{W}$ on which $\sum_{i=1}^n A_i A_i^*|_{\mathcal{W}} = I_{\mathcal{W}}$; and similarly define $\widetilde{\mathcal{V}}_B$. Then σ_A and σ_B are unitarily equivalent if and only if*

1. $\operatorname{rank}(I_{\mathcal{V}_A} - \sum_{i=1}^n A_i A_i^*) = \operatorname{rank}(I_{\mathcal{V}_B} - \sum_{i=1}^n B_i B_i^*)$*; and*
2. $A^*|_{\widetilde{\mathcal{V}}_A}$ *is unitarily equivalent to* $B^*|_{\widetilde{\mathcal{V}}_B}$.

Proof. The two representations are equivalent if and only if they have the same pure rank and the Cuntz parts are unitarily equivalent. By Theorem 6.5, the algebra $\mathfrak{S}$ contains the projection onto $\widetilde{\mathcal{V}}_A$. It is the unique maximal finite rank projection in $\mathfrak{S}$. Therefore the restriction $A^*|_{\widetilde{\mathcal{V}}_A}$ is a unitary invariant. Conversely, if these two conditions hold, then the unitary identifying $A^*|_{\widetilde{\mathcal{V}}_A}$ and $B^*|_{\widetilde{\mathcal{V}}_B}$ extends to a unitary equivalence between the dilations S_A of $\widetilde{A} := P_{\widetilde{\mathcal{V}}_A} A|_{\widetilde{\mathcal{V}}_A}$ and S_B of $\widetilde{B} := P_{\widetilde{\mathcal{V}}_B} B|_{\widetilde{\mathcal{V}}_B}$ because of the uniqueness of the minimal isometric dilation. This identifies the restriction of S_A to $\mathfrak{S}[\widetilde{\mathcal{V}}_A] = \mathcal{H}_{Ac}$, namely the Cuntz part of S_A, with the corresponding Cuntz part of S_B. The pure rank condition allows a unitary equivalence between the two pure parts. ∎

Bratteli and Jorgensen [**9**] give a detailed analysis of representations of the Cuntz algebra which has a lot in common with our results. They look somewhat different since they concentrate on the state and not on the restriction to the subspace $\mathcal{V}$. In particular, their contractions are not the same as ours. They point out the relationship in the discussion preceding their Theorem 5.3. They obtain our Corollary 6.6 in the Cuntz case, and in particular recognize the role of the completely positive map Φ. Again however, their different normalization results in a different map. But they do not appear to classify these representations up to unitary equivalence. The reason they do not succeed is that they did not identify the subspace which we call $\widetilde{\mathcal{V}}$, and instead work with a subspace they call $\mathcal{V}_k$ which is often strictly larger. The space $\widetilde{\mathcal{V}}$ does not occur in their hierarchy of invariant subspaces. Instead, they specialize in section 7 to a smaller class which they call diagonalizable shifts. These they do completely classify up to unitary equivalence. We have not determined in this case how their special invariants relate to ours.

Corollary 6.9. *The algebra $\mathfrak{S}$ determined by the joint isometric dilation of a contractive n-tuple on a finite dimensional space is hyper-reflexive with distance constant at most 5.*

Proof. This follows immediately from [**15**, Theorem 3.14] since the algebra $\mathfrak{S}$ is unitarily equivalent to the algebra of certain atomic representations. Indeed, the projection $P = P_{\widetilde{\mathcal{V}}}$ belongs to $\mathfrak{S}$ and $\mathfrak{S}P = \mathfrak{W}P$ where $\mathfrak{W}$ is a type I von Neumann algebra containing the projection P. Thus by Christensen's result [**12**] which shows that type I von Neumann algebras have distance constant at most 4, we obtain the same for our slice. The upper bound for the distance constant of $\mathfrak{L}_n$ was improved by Bercovici [**6**] to 3 from the original 51. Arguing as in [**15**], we obtain a distance constant no larger than $(3^2 + 4^2)^{1/2} = 5$. ∎

7. Similarity

Now consider the question of when two contractive n-tuples are similar, and the effect on their dilations. The first step is to show that the Cuntz parts must be unitarily equivalent. Thus the question of similarity reduces to the pure parts. First we need a variant of Lemma 5.10.

Lemma 7.1. *Suppose that an n-tuple $(A_1, \dots, A_n)$ acts on a finite dimensional space $\mathcal{V}$, and generates $\mathcal{B}(\mathcal{V})$ as an algebra. Moreover suppose that $\Phi(X) = \sum_{i=1}^n A_i X A_i^*$ is unital. Then the only self-adjoint operators X satisfying $\Phi(X) \le X$ are scalar, and in particular are fixed points.*

Proof. Since $\Phi(I) = I$, we may translate X so that $X \ge 0$ and 0 belongs to its spectrum. Let $\mathcal{M} = \ker X$. This is a non-zero subspace. Let $x \in \mathcal{M}$. Then

$$\begin{aligned} 0 = (\Phi(0)x, x) &\le (\Phi(X)x, x) \\ &= \sum_{i=1}^n (A_i X A_i^* x, x) = \sum_{i=1}^n \|X A_i^* x\|^2 \le (Xx, x) = 0. \end{aligned}$$

It follows that $\mathcal{M}$ is invariant for each A_i^*. But by hypothesis, the A_i^*'s generate the full matrix algebra, and thus have no proper invariant subspaces. So $\mathcal{M} = \mathcal{V}$ and $X = 0$ is scalar. ∎

Corollary 7.2. *Suppose that $A = (A_1, \dots, A_n)$ and $B = (B_1, \dots, B_n)$ are similar contractive n-tuples in the finite dimensional algebra $\mathcal{B}(\mathcal{V})$. Let $\widetilde{\mathcal{V}}_A$ and $\widetilde{\mathcal{V}}_B$ denote the subspaces spanned by the minimal $\mathfrak{A}^*$ and $\mathfrak{B}^*$-invariant subspaces $\mathcal{M}$ on which $AA^*|_{\mathcal{M}} = I_{\mathcal{M}}$ and $BB^*|_{\mathcal{M}} = I_{\mathcal{M}}$, respectively. Then $P_{\widetilde{\mathcal{V}}_A} A|_{\widetilde{\mathcal{V}}_A}$ and $P_{\widetilde{\mathcal{V}}_B} B|_{\widetilde{\mathcal{V}}_B}$ are unitarily equivalent.*

Proof. Let T be the similarity such that $B = TAT^{-1}$. Then $B^* = T^{*-1} A^* T^*$. So T^{*-1} carries $\mathfrak{A}^*$-invariant subspaces onto $\mathfrak{B}^*$-invariant subspaces. Also T^{*-1} preserves minimality. However it is not immediately evident that it preserves the condition that $AA^*|_{\widetilde{\mathcal{V}}_A} = I_{\widetilde{\mathcal{V}}_A}$.

Let $\mathcal{M}$ be a minimal $\mathfrak{A}^*$-invariant subspace of $\widetilde{\mathcal{V}}_A$ on which $AA^*|_{\mathcal{M}} = I_{\mathcal{M}}$. Then $T^{*-1}\mathcal{M} = \mathcal{N}$ is invariant for $\mathfrak{B}^*$. It follows that if $\bar{A}_i^*$,and $\bar{T}^{*-1}$ are the restrictions of A_i^* and T^{*-1} to $\mathcal{M}$, and $\bar{B}_i^*$ is the restriction of B_i^* to $\mathcal{N}$, then $\bar{B}_i^* = \bar{T}^{*-1} \bar{A}_i^* \bar{T}^*$. Let $\bar{\Phi}(X) = \sum_{i=1}^n \bar{A}_i X \bar{A}_i^*$ on $\mathcal{B}(\mathcal{M})$. It is easy to verify that $\bar{\Phi}$ is unital.

Now compute that

$$I_{\mathcal{N}} \ge \sum_{i=1}^n \bar{B}_i \bar{B}_i^* = \sum_{i=1}^n \bar{T} \bar{A}_i \bar{T}^{-1} \bar{T}^{*-1} \bar{A}_i^* \bar{T}^* = \bar{T} \Phi(\bar{T}^{-1} \bar{T}^{*-1}) \bar{T}^*.$$

Therefore $\Phi(\bar{T}^{-1}\bar{T}^{*-1}) \le \bar{T}^{-1}\bar{T}^{*-1}$. By Lemma 7.1, it follows that $\bar{T}^{-1}\bar{T}^{*-1}$ is scalar. So up to a scaling factor, $\bar{T}$ is unitary.

This shows that the restrictions of $\mathfrak{A}^*$ to each minimal $\mathfrak{A}^*$-invariant subspace $\mathcal{M}$ of $\widetilde{\mathcal{V}}_A$ on which $AA^*|_{\mathcal{M}} = I_{\mathcal{M}}$ is unitarily equivalent to the corresponding subspace of $\mathfrak{B}^*$. Since $\widetilde{\mathcal{V}}_A$ and $\widetilde{\mathcal{V}}_B$ are each the orthogonal direct sum of such subspaces, it follows that the restriction to these larger subspaces are unitarily equivalent (although T itself need not be a multiple of a unitary on the whole space). Thus $A^*|_{\widetilde{\mathcal{V}}_A}$ is unitarily equivalent to $B^*|_{\widetilde{\mathcal{V}}_B}$. Equivalently, the compressions $P_{\widetilde{\mathcal{V}}_A}A|_{\widetilde{\mathcal{V}}_A}$ and $P_{\widetilde{\mathcal{V}}_B}B|_{\widetilde{\mathcal{V}}_B}$ are unitarily equivalent. ■

Corollary 7.3. *Suppose that $A = (A_1,\dots,A_n)$ and $B = (B_1,\dots,B_n)$ are similar contractive n-tuples in the finite dimensional algebra $\mathcal{B}(\mathcal{V})$. Let S_i and T_i be their respective minimal joint isometric dilations. Then the Cuntz parts of S_i and T_i are unitarily equivalent.*

Proof. This is immediate from the proposition above and the fact that the Cuntz part of S_i and T_i are determined by the compressions of A_i and B_i to the subspaces $\widetilde{\mathcal{V}}_A$ and $\widetilde{\mathcal{V}}_B$ respectively by Corollary 6.8. ■

Example 7.4. Now we show through a couple of examples that the pure part of the dilation is not preserved by similarity. This first example shows that one dilation can be strictly Cuntz type while a similarity can introduce a pure part. Consider

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad \text{and} \qquad A_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

This is of Cuntz type since $A_1A_1^* + A_2A_2^* = I$. Moreover there is a unique minimal $\mathfrak{A}^*$-invariant subspace, $\mathbb{C}e_1$. The dilation of this pair is thus irreducible by Corollary 6.6, and is determined by the 1-dimensional restrictions 1 and 0 of A_1^* and A_2^* to $\mathbb{C}e_1$. In fact, this is easily seen to be the atomic representation $\sigma_{1,1}$.

However, this pair is similar via $T = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}$ to

$$B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad \text{and} \qquad B_2 = \begin{bmatrix} 0 & 0 \\ 1/2 & 0 \end{bmatrix}.$$

The restrictions of B_i^* to the unique minimal $\mathfrak{B}^*$-invariant subspace are still 1 and 0 respectively; and they determine a dilation which has the representation $\sigma_{1,1}$ as a summand. However, since

$$\operatorname{rank}(I - B_1B_1^* - B_2B_2^*) = \operatorname{rank} \begin{bmatrix} 0 & 0 \\ 0 & 3/4 \end{bmatrix} = 1,$$

the pure rank of this representation is 1.

Example 7.5. A second easy example shows that even in the pure case, the pure rank is not a similarity invariant. Fix an orthonormal basis $e_1, \dots, e_n$ for $\mathcal{V}$. Let $A_1 = \frac{1}{2} e_1 e_1^*$ and $A_i = e_i e_1^*$ for $2 \le i \le n$. Then $I - \sum_{i=1}^n A_i A_i^* = \frac{3}{4} e_1 e_1^*$ is rank 1, and its range $\mathbb{C}e_1$ is $\mathfrak{A}$-cyclic. So by Corollary 6.6, this yields an irreducible pure dilation.

However, this is similar via $T = I + e_1 e_1^*$ to $B_1 = A_1$ and $B_i = \frac{1}{2} A_i$ for $2 \le i \le n$. This n-tuple satisfies $I - \sum_{i=1}^n B_i B_i^* = \frac{3}{4} I$, which has rank n. So this dilation has pure rank n.

We wish to provide more detail about the effect of similarity on pure representations. By Popescu[**26**], the dilation is pure if and only if

$$\operatorname{WOT-}\lim_{k\to\infty} \sum_{|w|=k} A_w A_w^* = 0.$$

He calls these n-tuples C_0 contractions, and provides a WOT-continuous functional calculus in [**28**]. We can analyze this using the theory of representations of $\mathfrak{L}_n$ developed in [**16**].

Let $A = (A_1, \dots, A_n)$ be a C_0-contraction on $\mathcal{V}$ with pure minimal isometric dilation $S_i \simeq L_i^{(s)}$. This determines a WOT-continuous representation Φ_A of $\mathfrak{L}_n$ which sends X to $P_{\mathcal{V}} X^{(s)}|_{\mathcal{V}}$. In particular, $\Phi_A(L_w) = A_w := A_{i_1} \dots A_{i_k}$ for every word $w = i_1 \dots i_k$ in $\mathbb{F}_n^+$. The kernel $\mathfrak{J} = \ker \Phi_A$ is a WOT-closed ideal of $\mathfrak{L}_n$. By [**16**, Theorem 2.1], this ideal is determined by its range $\mathcal{M} = \overline{\mathfrak{J}\mathcal{K}_n}$, which is an invariant subspace for both $\mathfrak{L}_n$ and its commutant $\mathfrak{R}_n$. The representation of compression of $\mathfrak{L}_n$ to $\mathcal{M}^\perp$ has the same kernel. We wish to determine to what extent A can be recovered from the compression of L to $\mathcal{M}^\perp$.

To get a feeling for the situation, consider the case in which the A_i are $d \times d$ matrices which generate $\mathfrak{M}_d$ as an algebra. Then Φ_A maps $\mathfrak{L}_n$ onto $\mathfrak{M}_d$. The kernel $\mathfrak{J}$ will then have codimension d^2, and therefore the dimension of $\mathcal{M}^\perp$ is also d^2. The compression homomorphism to $\mathcal{M}^\perp$ factors through Φ_A. Since $\mathfrak{M}_d$ has only one irreducible representation up to similarity, the compression to $\mathcal{M}^\perp$ must be similar to the direct sum of d copies of Φ_A. In particular, $\mathcal{M}^\perp$ will decompose into a (non-orthogonal) direct sum of d subspaces which are $\mathfrak{L}_n^*$-invariant such that the compression of L is similar to A.

Nevertheless, Φ_A need not occur as a compression of L to some $\mathfrak{L}_n^*$-invariant subspace. This could occur only if Φ_A has pure rank 1, which need not be the case. However, this shows that there are representations similar to Φ_A which do have pure rank 1. Moreover it turns out that in a certain sense, these similarities of pure rank 1 are the extreme points of those representations similar to Φ_A. This will be established by showing that Φ_A can be recovered as a C*-convex combination of pure rank 1 representations.

Theorem 7.6. *Let $A = (A_1, \dots, A_n)$ be a C_0-contraction on a d-dimensional space $\mathcal{V}$. Let $\mathfrak{J}$ be the kernel of the* WOT*-continuous representation Φ_A of $\mathfrak{L}_n$ that it determines. Then Φ_A is unitarily equivalent to the compression of $\mathfrak{L}_n$ to a semi-invariant subspace $\mathcal{S} = \mathcal{N}_1 \ominus \mathcal{N}_2$, where $\mathcal{N}_1 = \mathfrak{L}_n[\mathcal{S}]$ and $\mathcal{N}_2 = \mathcal{N}_1 \ominus \mathcal{S}$ belong to* $\operatorname{Lat} \mathfrak{L}_n$.

Let $\mathcal{M} = \overline{\mathfrak{J}\mathcal{K}_n}$ be the corresponding $\mathfrak{L}_n$ and $\mathfrak{R}_n$-invariant subspace associated to $\mathfrak{J}$. Then there are at most d wandering vectors ζ_j, say for $1 \le j \le s$ where $s \le d$, with $\mathfrak{L}_n[\zeta_i]$ pairwise orthogonal, such that

$$\mathcal{N}_1 = \sum_{j=1}^{s}{}^{\oplus} R_{\zeta_j}\mathcal{K}_n \qquad \textit{and} \qquad \mathcal{N}_2 \supset \sum_{j=1}^{s}{}^{\oplus} R_{\zeta_j}\mathcal{M}.$$

Moreover, the subspaces $\mathcal{M}_j = R_{\zeta_j}^*\mathcal{S}$ are $\mathfrak{L}_n^*$-invariant subspaces of $\mathcal{M}^\perp$, and $\dim(\mathcal{M}_j) \le d$. The contractive n-tuples $B_j = (B_{j1}, \dots, B_{jn})$ obtained by compression of L to $\mathcal{M}_j$ have pure rank 1 and $\ker \Phi_{B_j} \supset \mathfrak{J}$. There is an isometry X mapping $\mathcal{S}$ into $\sum_{j=1}^{\oplus s} \mathcal{M}_j$ so that

$$\Big(\sum_{j=1}^{s}{}^{\oplus} B_{ji}^*\Big) X = X A_i^* \quad \textit{for} \quad 1 \le i \le n.$$

Thus A^* is unitarily equivalent to the restriction of $\sum_{j=1}^{\oplus s} B_j^*$ to an invariant subspace. Consequently, $A = X^* \sum_{j=1}^{\oplus s} B_j X$ is a C^*-convex combination of the B_j's.

In particular, when $\mathfrak{A} = \mathcal{B}(\mathcal{V})$, each subspace $\mathcal{M}_j$ is d-dimensional and each n-tuple B_j is similar to A.

Proof. The isometric dilation $S = (S_1, \dots, S_n)$ of A has pure rank $s = \operatorname{rank}(I - \sum_{i=1}^n A_i A_i^*) \le d$. The identity representation of $\mathfrak{L}_n$ contains many invariant subspaces with infinite dimensional wandering space; and thus an infinite multiple of the identity representation is contained in $\mathfrak{L}_n$. So we may assume that Φ_A is the compression of $\mathfrak{L}_n$ to a semi-invariant subspace $\mathcal{S}$ of $\mathfrak{L}_n$ itself. The minimal choice of a pair of $\mathfrak{L}_n$-invariant subspaces with difference $\mathcal{S}$ is given by $\mathcal{N}_1 = \mathfrak{L}_n[\mathcal{S}]$ and $\mathcal{N}_2 = \mathcal{N}_1 \ominus \mathcal{S}$ [**33**].

Now $\mathcal{N}_1$ has a wandering space $\mathcal{W}$ of dimension s. Choose an orthonormal basis ζ_j, $1 \le j \le s$, for $\mathcal{W}$. By [**15**, Theorem 2.1], there is an isometry R_{ζ_j} in $\mathfrak{R}_n$ with range equal to the cyclic $\mathfrak{L}_n$-invariant subspace $\mathfrak{L}_n[\zeta_j]$. Then $\mathcal{N}_1 = \sum_{j=1}^{\oplus s} R_{\zeta_j}\mathcal{K}_n$. Since the kernel of the compression to $\mathcal{S}$ is $\mathfrak{J}$,

$$\begin{aligned}\mathcal{N}_2 \supset \mathfrak{J}\mathcal{S} &= \mathfrak{J}\mathfrak{L}_n\mathcal{S} = \mathfrak{J}\mathcal{N}_1 \\ &= \sum_{j=1}^{s} \mathfrak{J} R_{\zeta_j}\mathcal{K}_n = \sum_{j=1}^{s} R_{\zeta_j}\mathfrak{J}\mathcal{K}_n = \sum_{j=1}^{s} R_{\zeta_j}\mathcal{M}.\end{aligned}$$

The subspaces $\mathcal{M}_j = R_{\zeta_j}^*\mathcal{S}$ are contained in $\mathcal{M}^\perp$, and have dimension at most $d = \dim \mathcal{S}$. Moreover, they are $\mathfrak{L}_n^*$-invariant because of the identity

$$\mathfrak{L}_n^* R_{\zeta_j}^*\mathcal{S} = R_{\zeta_j}^*\mathfrak{L}_n^*\mathcal{S} \subset R_{\zeta_j}^*\mathcal{N}_2^\perp = R_{\zeta_j}^*\mathcal{S}.$$

Let B_j denote the contractive n-tuple obtained by compression of L to $\mathcal{M}_j$. Clearly L is an isometric dilation of B_j. The minimal dilation is obtained by restricting L to $\mathfrak{L}_n[\mathcal{M}_j]$. However by Lemma 3.4, this is a reducing subspace of $\mathcal{K}_n$. Since the commutant $\mathfrak{R}_n$ of $\mathfrak{L}_n$ contains no idempotents [**15**, Corollary 1.8], this space must be all of $\mathcal{K}_n$. Thus the n-tuple B_j has pure rank 1 for each $1 \le j \le s$. Since $\mathcal{M}_j$ is contained in $\mathcal{M}^\perp$, it follows that $\ker \Phi_{B_j}$ contains the ideal $\mathfrak{J}$.

Now notice that $R_{\zeta_j}\mathcal{M}_j$ are pairwise orthogonal subspaces, and

$$\sum_{j=1}^{s}{}^{\oplus} R_{\zeta_j}\mathcal{M}_j = \sum_{j=1}^{s}{}^{\oplus} R_{\zeta_j}R_{\zeta_j}^*\mathcal{S} \supset \Big(\sum_{j=1}^{s} R_{\zeta_j}R_{\zeta_j}^*\Big)\mathcal{S} = P_{\mathcal{N}_1}\mathcal{S} = \mathcal{S}.$$

This allows us to identify $\mathcal{S}$ isometrically with a subspace of $\sum_{j=1}^{\oplus s}\mathcal{M}_j$. Let $X_j = R_{\zeta_j}^* P_{\mathcal{S}}$ be considered as a map from $\mathcal{S}$ into $\mathcal{M}_j$. Define X to be the column matrix $\begin{bmatrix} X_1 & \dots & X_s\end{bmatrix}^t$. Then

$$X^*X = \sum_{j=1}^{s} P_{\mathcal{S}} R_{\zeta_j}R_{\zeta_j}^* P_{\mathcal{S}} = P_{\mathcal{S}}P_{\mathcal{N}_1}P_{\mathcal{S}} = P_{\mathcal{S}}.$$

So X is an isometry of $\mathcal{S}$ into $\sum_{j=1}^{\oplus s}\mathcal{M}_j$. One may compute

$$R_{\zeta_j}^* L_i^* P_{\mathcal{S}} = \big(R_{\zeta_j}^* P_{\mathcal{N}_1}\big)\big(P_{\mathcal{N}_2}^\perp L_i^* P_{\mathcal{S}}\big) = R_{\zeta_j}^* P_{\mathcal{S}} L_i^* P_{\mathcal{S}}.$$

Therefore identifying A_i with $P_{\mathcal{S}}L_iP_{\mathcal{S}}$, we obtain

$$\begin{aligned}\big(\sum_{j=1}^{s}{}^{\oplus} B_{ji}^*\big)X &= \sum_{j=1}^{s} L_i^* R_{\zeta_j}^* P_{\mathcal{S}} = \sum_{j=1}^{s} R_{\zeta_j}^* L_i^* P_{\mathcal{S}} \\ &= \sum_{j=1}^{s} R_{\zeta_j}^* P_{\mathcal{S}} L_i^* P_{\mathcal{S}} = XA_i^*.\end{aligned}$$

From this it is evident that the range of X is invariant for $\sum_{j=1}^{\oplus s} B_{ji}^*$, and implements a unitary equivalence between A_i^* and this restriction of $\sum_{j=1}^{\oplus s} B_{ji}^*$. Consequently, $X^*\big(\sum_{j=1}^{\oplus s} B_{ji}X\big) = A_i$ for $1 \le i \le n$.

This expresses A as a C*-convex combination of the pure rank one contractions B_j.

When $\mathfrak{A}$ is isomorphic to $\mathcal{B}(\mathcal{V}) \simeq \mathfrak{M}_d$, then $\mathfrak{L}_n/\mathfrak{J}$ is likewise isomorphic to $\mathfrak{M}_d$ and the compression of $\mathfrak{L}_n$ to $\mathcal{M}^\perp$ is a representation of $\mathfrak{M}_d$ on a subspace of dimension d^2. The only representations of $\mathfrak{M}_d$ are multiples of the identity representation up to similarity, and the compression to $\mathcal{M}^\perp$ has multiplicity d. Thus the $\mathfrak{L}_n^*$-invariant subspaces of $\mathcal{M}^\perp$ have dimension which is a multiple of d. As $\mathcal{M}_j$ are non-zero and have dimension at most d, they are all exactly d-dimensional and each B_j^* is similar to A^*, whence B_j is similar to A. ∎

Say that the n-tuple A is *irreducible* if it generates $\mathfrak{A} = \mathcal{B}(\mathcal{V})$, or equivalently $\mathfrak{A}$ has no proper invariant subspaces. When A is an irreducible C_0-contraction, we see that the compression representation to $\mathcal{M}^\perp$ takes the generators to an n-tuple similar to the direct sum of d copies of A. In particular, this occurs if $\|A\| < 1$. So we obtain a complete similarity invariant for an *arbitrary* irreducible n-tuple of matrices (after scaling appropriately).

Restricting to the irreducible case is not just a matter of convenience. Simple examples show that multiplicity cannot be detected from the set of polynomial identities that an n-tuple satisfies. For example, with $n = 1$, take $A = J_2 \oplus 0^{(3)}$ and $B = J_2^{(2)} \oplus 0$ where J_2 is the 2×2 nilpotent Jordan matrix and 0 is a one-dimensional zero. These two matrices satisfy exactly the same polynomial identities. The natural way to distinguish them is to use rank. Indeed, familiar invariants for similarity of single matrices shows that the ranks of various polynomials can be used to determine the multiplicity function.

Corollary 7.7. *Suppose that $A = (A_1, \dots, A_n)$ and $B = (B_1, \dots, B_n)$ are two n-tuples of $d \times d$ matrices which are irreducible and strictly contractive, $\|A\| < 1$ and $\|B\| < 1$. Then A and B are similar if and only if* $\ker \Phi_A = \ker \Phi_B$.

Proof. Clearly two similar n-tuples give rise to representations with the same kernel. Conversely, if they are irreducible, the kernel determines the subspace $\mathcal{M}$. We adopt the notation from the proof of Theorem 7.6. A minimal $\mathfrak{L}_n^*$-invariant subspace $\mathcal{M}_j$ of $\mathcal{M}^\perp$ yields a compression representation Φ_j which is similar to Φ_A by Theorem 7.6. Likewise, B determines the same subspaces, and thus Φ_B is also similar to Φ_j, and hence to Φ_A. ∎

The similarity question for n-tuples of matrices is an old one, and the solution is complicated. Friedland **[21]** provides an algorithm for checking whether two n-tuples $A = (A_1, \dots, A_n)$ and $B = (B_1, \dots, B_n)$

of $d \times d$ matrices are similar. This is quite involved even for two 2×2 matrices, which he calculates explicitly. The situation simplifies when the two matrices are not simultaneously triangularizable—which in the 2×2 case is the same as irreducibility. In this case, the pairs A and B are similar if and only if these five identities hold:

$$\begin{aligned} \mathrm{Tr}(A_1) &= \mathrm{Tr}(B_1) & \mathrm{Tr}(A_1^2) &= \mathrm{Tr}(B_1^2) \\ \mathrm{Tr}(A_2) &= \mathrm{Tr}(B_2) & \mathrm{Tr}(A_2^2) &= \mathrm{Tr}(B_2^2) \\ \mathrm{Tr}(A_1A_2) &= \mathrm{Tr}(B_1B_2). \end{aligned}$$

In general, there is no explicit list of polynomials to check.

In our case, we do obtain a fairly small *finite* list of invariants for an irreducible n-tuple. Unfortunately, at this point, we do not have an explicit method for computing these invariants. Nor are they polynomials. The natural invariants in our setting are isometries in $\mathfrak{L}_n$. Polynomials can be obtained by a simple approximation argument, but are no longer canonical. In the case of two 2×2 matrices, we obtain exactly five conditions.

Theorem 7.8. *Let $A = (A_1, \dots, A_n)$ be an irreducible n-tuple of $d \times d$ matrices with $\|A\| < 1$. The ideal $\mathfrak{J} = \ker \Phi_A$ determines its range space $\mathcal{M} = \overline{\mathfrak{J}\mathcal{K}_n}$ with wandering dimension $1 + (n-1)d^2$. Thus there are $1 + (n-1)d^2$ isometries X_j in $\mathfrak{L}_n$ so that an n-tuple B of $d \times d$ matrices with $\|B\| < 1$ is similar to A if and only if $\Phi_B(X_j) = 0$ for $1 \le j \le 1 + (n-1)d^2$.*

Moreover there is a set of $m = 1 + (n-1)d^2$ polynomials p_j in n non-commuting variables such that an n-tuple B of $d \times d$ matrices with $\|B\| < 1$ is similar to A if and only if $p_j(B) = 0$ for $1 \le j \le m$.

Proof. The space $\mathcal{M}$ has the same codimension as $\mathfrak{J}$, which is d^2 since $\mathfrak{L}_n/\mathfrak{J}$ is isomorphic to $\mathrm{Alg}\{A_1, \dots, A_n\} = \mathfrak{M}_d$. Its wandering space is

$$\begin{aligned} \mathcal{W} &= \mathcal{M} \ominus \sum_{i=1}^n L_i \mathcal{M} \\ &= \mathcal{K}_n \ominus \big(\mathcal{M}^\perp \oplus \sum_{i=1}^n L_i(\mathcal{K}_n \ominus \mathcal{M}^\perp)\big) \\ &= \Big(\big(\mathcal{K}_n \ominus \sum_{i=1}^n L_i \mathcal{K}_n\big) \oplus \sum_{i=1}^n L_i \mathcal{M}^\perp\Big) \ominus \mathcal{M}^\perp \\ &= \Big(\mathbb{C}\xi_e \oplus \sum_{i=1}^n L_i \mathcal{M}^\perp\Big) \ominus \mathcal{M}^\perp. \end{aligned}$$

This has dimension $m = 1 + (n-1)\dim \mathcal{M}^\perp = 1 + (n-1)d^2$.

Now $\mathcal{M}$ is invariant for both $\mathfrak{L}_n$ and its commutant $\mathfrak{R}_n$. Since it is the latter, it decomposes [**15**, Theorem 2.1] as the direct sum of m cyclic $\mathfrak{R}_n$-invariant subspaces; and each is the range of an isometry X_j in $\mathfrak{L}_n$. Thus by [**16**, Lemma 2.5], we obtain that $\mathfrak{J} = \sum_{j=1}^{m} X_j \mathfrak{L}_n$.

Therefore $\ker \Phi_B$ contains $\mathfrak{J}$ if and only if $\Phi_B(X_j) = 0$ for $1 \leq j \leq m$. Moreover, since A is irreducible, $\mathfrak{J}$ is a maximal ideal. Thus this condition ensures that $\ker \Phi_B = \mathfrak{J}$. In particular, $\mathrm{Alg}\{B_1, \dots, B_n\}$ is isomorphic to $\mathfrak{L}_n/\mathfrak{J} \simeq \mathfrak{M}_d$; and hence B is also irreducible. Therefore B and A are similar by Corollary 7.7.

To obtain polynomials, we notice that the algebra $\mathfrak{P}$ which is the algebraic span of $\{L_w : w \in \mathbb{F}_n^+\}$ is WOT-dense in $\mathfrak{L}_n$. Let $\mathfrak{I} = \mathfrak{J} \cap \mathfrak{P}$ be the ideal of all polynomials which annihilate A. The algebra (without closure) generated by the A_i's is $\mathfrak{M}_d$. So the map Φ_A takes $\mathfrak{P}$ onto $\mathfrak{M}_d$ with kernel $\mathfrak{I}$; and takes $\mathfrak{L}_n$ onto $\mathfrak{M}_d$ with kernel $\mathfrak{J}$. It follows from the Hahn-Banach theorem that $\mathfrak{I}$ is WOT-dense in $\mathfrak{J}$.

Let $\varepsilon = (1+nd^2)^{-1}$. For each $1 \leq j \leq m$, choose polynomials $p_j \in \mathfrak{I}$ such that $\|p_j(L) - X_j\| < \varepsilon$. We claim that $p_j(L)$ generate $\mathfrak{J}$ as a norm-closed right ideal. For let $J \in \mathfrak{J}$. By [**16**, Lemma 2.5], there are elements $Y_j \in \mathfrak{L}_n$ such that $J = \sum_{j=1}^{m} X_j Y_j =: XY$. Moreover, the row operator $X = \begin{bmatrix} X_1 & \dots & X_m \end{bmatrix}$ is an isometry. Hence the column operator $Y = \begin{bmatrix} Y_1 & \dots & Y_m \end{bmatrix}^t$ has $\|Y\| = \|J\|$. Let $P = \begin{bmatrix} p_1(L) & \dots & p_m(L) \end{bmatrix}$. It follows that

$$\|J - PY\| \leq \|X - P\| \|Y\| < \frac{m}{1+nd^2} \|J\|.$$

Since $m/(1+nd^2) < 1$, the right ideal generated by P is norm dense in $\mathfrak{J}$ as claimed.

Therefore the condition that $p_j(B) = 0$ for $1 \leq j \leq m$ is equivalent to the condition $\Phi_B(X_j) = 0$, and thus is equivalent to joint similarity to A. ∎

While the X_j's are needed to generate $\mathfrak{J}$ as a WOT-closed *right* ideal, there will generally be redundancies as generators for $\mathfrak{J}$ as a two-sided ideal. So $1+(n-1)d^2$ is an upper bound on the number of test elements needed. It would be interesting to have better bounds on the number of generators for a two-sided ideal.

We observe that the existence of a determining set of polynomials for an irreducible n-tuple can be deduced directly by elementary means. One can write down polynomials in A representing the matrix units of $d \times d$ matrices and their relations. In fact $O(d^2)$ generators and relations suffice. Then each A_i can be expressed as a combination of matrix units. This requires only $n + O(d^2)$ polynomials, which is somewhat better than our bound. In many concrete cases, this simple bare hands approach is the best.

On the other hand, our result provides an algorithm for obtaining a set of generators for the ideal $\mathfrak{J}$. Perhaps this will prove to be of some use.

Example 7.9. This example illustrates parts of the previous two theorems. Consider the pair of 2×2 matrices

$$A_1 = \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix} \qquad \text{and} \qquad A_2 = \begin{bmatrix} 1/2 & 0 \\ 0 & 0 \end{bmatrix}.$$

Since $I - A_1A_1^* - A_2A_2^* = \left[\begin{smallmatrix} 1/2 & 0 \\ 0 & 3/4 \end{smallmatrix}\right]$ has rank 2, this determines a pure isometric dilation of pure rank 2. The algebra $\mathfrak{A} = \mathfrak{M}_2$, and thus the representation Φ_A of $\mathfrak{L}_2$ is irreducible. The kernel will be a WOT-closed maximal ideal $\mathfrak{J}$ of codimension 4. Therefore the subspace $\mathcal{M} = \overline{\mathfrak{J}\mathcal{K}_2}$ will be codimension 4.

The matrices A_1 and A_2 satisfy certain relations that express the fact that

$$\mathfrak{M}_2 = \mathrm{Alg}\{A_1, A_2\} = \mathrm{span}\{I, A_1, A_2, A_1A_2\}.$$

A natural and sufficient list is

(1) $4A_1^2 = I$
(2) $2A_2^2 = A_2$
(3) $A_2A_1A_2 = 0$
(4) $8A_1A_2A_1 = I - 2A_2$
(5) $2A_1A_2 + 2A_2A_1 = A_1$

However, (5) and (2) can be derived from the others. So (1), (3) and (4) are sufficient.

The ideal $\mathfrak{J}$ is therefore generated as a two-sided ideal by the set

$$\mathcal{J} = \{I - 4L_1^2,\ L_2L_1L_2,\ I - 2L_2 - 8L_1L_2L_1\},$$

since the quotient will be $\mathfrak{M}_2$. Therefore the range $\mathcal{M}$ is the $\mathfrak{L}_2\mathfrak{R}_2$-invariant subspace generated by $\mathcal{J}\xi_e$,

$$\begin{aligned}\mathcal{M} = \overline{\mathfrak{J}\mathcal{K}_2} &= \overline{\mathfrak{L}_2\mathfrak{R}_2\mathcal{J}\xi_e} \\ &= \mathrm{span}\{\xi_{uv} - 4\xi_{u11v},\ \xi_{u212v},\ \xi_{uv} - 2\xi_{u2v} - 8\xi_{u121v} : u, v \in \mathbb{F}_2^+\}.\end{aligned}$$

We wish to determine $\mathcal{M}^\perp$. To this end, define Ω_1 to be the set of all words in $11 = 1^2$ and 2,

$$\Omega_1 = \{w = 1^{2k_0}21^{2k_1}2\dots21^{2k_s} : k_i \geq 0, s \geq 0\}.$$

Let $\Omega_2 = \Omega_1 1$, $\Omega_3 = 1\Omega_1$ and $\Omega_4 = \{e\} \cup 1\Omega_1 1$. Define

$$x_i = \sum_{w \in \Omega_i} 2^{-|w|} \xi_w \quad \text{for} \quad 1 \le i \le 4.$$

Then a computation shows that $\mathcal{M}^\perp = \operatorname{span}\{x_1, x_2, x_3, x_4\}$. These vectors are not orthogonal, nor of constant length. Indeed,

$$\|x_1\|^2 = 16/11\,, \qquad \|x_2\|^2 = \|x_3\|^2 = 4/11 \quad \text{and} \quad \|x_4\|^2 = 12/11.$$

The pair $\{x_1, x_4\}$ is orthogonal to $\{x_2, x_3\}$, but

$$(x_1, x_4) = 16/15 \qquad \text{and} \qquad (x_2, x_3) = 4/15.$$

Another matrix calculation relative to the ordered basis $\{x_1, \dots, x_4\}$ for $\mathcal{M}^\perp$ shows that

$$L_1^*|_{\mathcal{M}^\perp} = \begin{bmatrix} 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 \\ 1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \end{bmatrix} \quad \text{and} \quad L_2^*|_{\mathcal{M}^\perp} = \begin{bmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Let Φ_c denote the representation of compression to $\mathcal{M}^\perp$. This calculation shows that Φ_c is similar (but not unitarily equivalent) to the direct sum of two copies of Φ_A. Thus the compression to any two dimensional $\mathfrak{L}_2^*$-invariant subspace of $\mathcal{M}^\perp$ is similar to Φ_A. As noted in the proof of Theorem 7.6, these representations all have pure rank 1. In particular, Φ_A does not occur as such a compression. It is also a fact that Φ_c is not unitarily equivalent to an orthogonal direct sum of two representations.

Next, we compute the 2-dimensional $\mathfrak{L}_2^*$-invariant subspaces of $\mathcal{M}^\perp$. Examining the representation Φ_c, we observe that the two subspaces $\mathcal{M}_1 := \operatorname{span}\{x_1, x_3\}$ and $\mathcal{M}_2 := \operatorname{span}\{x_2, x_4\}$ are invariant for $\mathfrak{L}_2^*$. Setting $\eta_i = x_i / \|x_i\|$, we find that $\{\eta_1, \eta_3\}$ and $\{\eta_2, \eta_4\}$ are orthonormal bases for $\mathcal{M}_1$ and $\mathcal{M}_2$ respectively. Compute

$$L_1^*|_{\mathcal{M}_1} \simeq B_1^* = \begin{bmatrix} 0 & 1 \\ 1/4 & 0 \end{bmatrix} \qquad \text{and} \qquad L_2^*|_{\mathcal{M}_1} \simeq B_2^* = \begin{bmatrix} 1/2 & 0 \\ 0 & 0 \end{bmatrix},$$

$$L_1^*|_{\mathcal{M}_2} \simeq C_1^* = \begin{bmatrix} 0 & 1/\sqrt{12} \\ \sqrt{3}/2 & 0 \end{bmatrix} \qquad \text{and} \qquad L_2^*|_{\mathcal{M}_2} \simeq C_2^* = \begin{bmatrix} 1/2 & 0 \\ 0 & 0 \end{bmatrix}.$$

These must be pairs which have pure rank 1, as is verified by computing the ranks of

$$I - B_1 B_1^* - B_2 B_2^* = \begin{bmatrix} 11/16 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad I - C_1 C_1^* - C_2 C_2^* = \begin{bmatrix} 0 & 0 \\ 0 & 11/12 \end{bmatrix}.$$

The representation Φ_c factors through a representation of $\mathfrak{M}_2$ of multiplicity 2. Thus every 2-dimensional $\mathfrak{L}_2^*$-invariant subspace is the cyclic subspace determined by its intersection with the range of $L_2^*|_{\mathcal{M}^\perp} =$

$\mathrm{span}\{\eta_1,\eta_2\}$, namely $\mathbb{C}(\alpha\eta_1+\beta\eta_2)$ for $|\alpha|^2+|\beta|^2=1$. The second vector spanning the subspace must be the image under $2L_1^*$,

$$2L_1^*(\alpha\eta_1+\beta\eta_2)=2\alpha B_1^*\eta_1+2\beta C_1^*\eta_2=\tfrac{\alpha}{2}\eta_3+\sqrt{3}\beta\eta_4.$$

A typical subspace of this form is

$$\mathcal{M}_{\alpha,\beta}=\mathrm{span}\{\alpha\eta_1+\beta\eta_2,\tfrac{\alpha}{2}\eta_3+\sqrt{3}\beta\eta_4\}.$$

However it is sufficient just to use $\mathcal{M}_1$ and $\mathcal{M}_2$, as they correspond to a particular choice of a basis for the wandering subspace of $\mathfrak{L}_2[\mathcal{S}]$.

Since rank considerations show that the subspace $\mathcal{S}$ cannot be $\mathfrak{L}_2^*$-invariant, we must write $\mathcal{S}$ as the difference of two $\mathfrak{L}_2$-invariant subspaces of multiplicity 2. We will look for an orthonormal set $\{\zeta_1,\zeta_2\}$ to span $\mathcal{S}$ of the form

$$\zeta_1=\alpha R_1\eta_1+\beta R_2\eta_2 \quad\text{and}\quad \zeta_2=\tfrac{\alpha}{2}R_1\eta_3+\sqrt{3}\beta R_2\eta_4.$$

They are always orthogonal, so the condition that they be norm one requires that

$$|\alpha|^2+|\beta|^2=1=\tfrac{1}{4}|\alpha|^2+3|\beta|^2.$$

This has the solution $|\alpha|^2=8/11$ and $|\beta|^2=3/11$. Therefore set

$$\zeta_1=\sqrt{8/11}R_1\eta_1+\sqrt{3/11}R_2\eta_2 \ \text{ and } \ \zeta_2=\sqrt{2/11}R_1\eta_3+\sqrt{9/11}R_2\eta_4.$$

Another computation shows that

$$\begin{aligned} L_1^*\zeta_1&=\tfrac{1}{2}\zeta_2+\tfrac{1}{\sqrt{2}}\xi_e & \qquad L_2^*\zeta_1&=\tfrac{1}{2}\zeta_1\\ L_1^*\zeta_2&=\tfrac{1}{2}\zeta_1 & \qquad L_2^*\zeta_2&=0+\tfrac{1}{2}\xi_e \end{aligned}$$

Thus we see that $\mathcal{S}:=\mathrm{span}\{\zeta_1,\zeta_2\}$ is a semi-invariant subspace $\mathcal{S}=\mathcal{N}_1\ominus\mathcal{N}_2$ where $\mathcal{N}_1=\{\xi_e\}^\perp=\mathfrak{L}_2[\xi_1,\xi_2]$ and $\mathcal{N}_2=\{\xi_e,\zeta_1,\zeta_2\}^\perp$. Moreover these identities show that the compression of L_i to $\mathcal{S}$ is unitarily equivalent to A_i for $i=1,2$.

Let us compute the operators X_1 and X_2 promised in Theorem 7.6. The projection onto $\mathcal{S}$ is given by $P_{\mathcal{S}}=\zeta_1\zeta_1^*+\zeta_2\zeta_2^*$. Then

$$X_1=R_1^*P_{\mathcal{S}}=\sqrt{\tfrac{8}{11}}\eta_1\zeta_1^*+\sqrt{\tfrac{2}{11}}\eta_3\zeta_2^*$$

and

$$X_2=R_2^*P_{\mathcal{S}}=\sqrt{\tfrac{3}{11}}\eta_2\zeta_1^*+\sqrt{\tfrac{9}{11}}\eta_4\zeta_2^*.$$

So recalling the matrix forms for B_i and C_i, we obtain

$$\sum_{i=1}^{2} X_i^* L_1 X_i = \begin{bmatrix} \sqrt{8/11} & 0 \\ 0 & \sqrt{2/11} \end{bmatrix} \begin{bmatrix} 0 & 1/4 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{8/11} & 0 \\ 0 & \sqrt{2/11} \end{bmatrix} + \begin{bmatrix} \sqrt{3/11} & 0 \\ 0 & \sqrt{9/11} \end{bmatrix} \begin{bmatrix} 0 & \sqrt{3}/2 \\ 1/\sqrt{12} & 0 \end{bmatrix} \begin{bmatrix} \sqrt{3/11} & 0 \\ 0 & \sqrt{9/11} \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 1/11 \\ 4/11 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 9/22 \\ 3/22 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix} = A_1$$

and

$$\sum_{i=1}^{2} X_i^* L_2 X_i = \begin{bmatrix} \sqrt{8/11} & 0 \\ 0 & \sqrt{2/11} \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{8/11} & 0 \\ 0 & \sqrt{2/11} \end{bmatrix} + \begin{bmatrix} \sqrt{3/11} & 0 \\ 0 & \sqrt{9/11} \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{3/11} & 0 \\ 0 & \sqrt{9/11} \end{bmatrix}$$
$$= \begin{bmatrix} 4/11 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 3/22 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ 0 & 0 \end{bmatrix} = A_2.$$

The wandering space for $\mathcal{M}$ has dimension 5, and as in the proof of Theorem 7.8 is given by

$$\mathcal{W} = \operatorname{span}\{\xi_e, L_1 x_j, L_2 x_j : 1 \le j \le 4\} \ominus \operatorname{span}\{x_j : 1 \le j \le 4\}.$$

We do not compute a basis for this space, as the result is not particularly illuminating. But such a basis corresponds to 5 isometries $X_1, \dots, X_5$ in $\mathfrak{L}_n$ such that $\mathfrak{J} = \sum_{j=1}^{5} X_j \mathfrak{L}_n$. Thus $\ker \Phi_B = \mathfrak{J}$ if and only if $\Phi_B(X_j) = 0$ for $1 \le j \le 5$. But as we noted earlier in our remarks, finding generators for $\mathfrak{J}$ as a *right* ideal is overkill. It suffices to use generators for $\mathfrak{J}$ as a two sided ideal. Thus it is sufficient to verify the 3 polynomial conditions:

(1) $4B_1^2 = I$
(3) $B_2 B_1 B_2 = 0$
(4) $8B_1 B_2 B_1 = I - 2B_2$

It is an easy exercise to verify directly that these conditions suffice to determine the pair up to similarity.

Pursuing this example further, let us consider other contractive pairs which are similar to A. A calculation shows that these pairs are unitarily equivalent to a pair of the form

$$\widetilde{A}_1 = \begin{bmatrix} -at & \frac{1}{4t} - a^2 t \\ t & at \end{bmatrix} \quad \widetilde{A}_2 = \begin{bmatrix} 1/2 & a/2 \\ 0 & 0 \end{bmatrix} \quad \text{where } t > 0 \text{ and } a \in \mathbb{C}.$$

Since this is a contractive pair, $Z := I - \widetilde{A}_1 \widetilde{A}_1^* - \widetilde{A}_2 \widetilde{A}_2^* \geq 0$. It suffices to check that $Z_{22} \geq 0$ and $\det Z \geq 0$. In other words,

$$(1) \qquad (1+|a|^2)t^2 \leq 1 \quad \text{and}$$
$$(2) \qquad 12(1+|a|^2)^2 t^4 - (13 - 4|a|^2 + 8\,\mathrm{Re}\, a^2)t^2 + 1 \leq 0.$$

The dilation of this pair has pure rank 1 if and only if the determinant is 0, which requires an equality in (2). Notice that when $a = 0$, one obtains the inequality $1/\sqrt{12} \leq t \leq 1$. The extremes yield the two pure index 1 pairs (B_1, B_2) and (C_1, C_2) obtained above.

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