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ISOMETRIC EMBEDDINGS OF BANACH BUNDLES

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Abstract. We show in this paper that every bijective linear isometry between the continuous section spaces of two non-square Banach bundles gives rise to a Banach bundle isomorphism. This is to support our expectation that the geometric structure of the continuous section space of a Banach bundle determines completely its bundle structures. We also describe the structure of an *into* isometry from a continuous section space into an other. However, we demonstrate by an example that a non-surjective linear isometry can be far away from a subbundle embedding.

1. INTRODUCTION

Let $\langle B_X, \pi_X \rangle$ be a Banach bundle over a locally compact Hausdorff space X. For each x in X, denote by $B_x = \pi_X^{-1}(x)$ the fiber Banach space. A *continuous section* f of the Banach bundle $\langle B_X, \pi_X \rangle$ is a continuous function from X into B_X such that $\pi_X(f(x)) = x$, i.e., $f(x) \in B_x$ for all x in X. Denote by Γ_X the Banach space of all continuous sections of $\langle B_X, \pi_X \rangle$ vanishing at infinity, i.e. those f with $\lim_{x\to\infty} \|f(x)\| = 0$.

Let $\langle B_Y, \pi_Y \rangle$ be an other Banach bundle over a locally compact Hausdorff space Y with continuous section space Γ_Y. Assume that Γ_X is isometrically isomorphic to Γ_Y as Banach spaces. We want to assert whether $\langle B_X, \pi_X \rangle$ is isometrically isomorphic to $\langle B_Y, \pi_Y \rangle$ as Banach bundles (see §2 for definitions). In other words, we expect that the geometric structure of the continuous sections of a Banach bundle determines its bundle structure.

Example 1.1. (Trivial line bundles). Let $B_X = X \times \mathbb{K}$ and $B_Y = Y \times \mathbb{K}$, where the underlying field $\mathbb{K}$ is either the real $\mathbb{R}$ or the complex $\mathbb{C}$. The continuous section

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spaces are $C_0(X)$ and $C_0(Y)$, the Banach spaces of continuous scalar functions vanishing at infinity, respectively. The classical Banach-Stone Theorem (see, e.g., [1]) asserts that every linear isometry T from $C_0(X)$ onto $C_0(Y)$ is a weighted composition operator:

$$Tf(y) = h(y)f(\varphi(y)), \quad \forall\ f \in C_0(X),\ y \in Y. \tag{1.1}$$

Here, φ is a homeomorphism from Y onto X, and h is a continuous scalar function on Y with $|h(y)| = 1$, $\forall\, y \in Y$. This induces an isometric bundle isomorphism $\Phi : B_X \to B_Y$ from $B_X = X \times \mathbb{K}$ onto $B_Y = Y \times \mathbb{K}$ defined by

$$\Phi(x, \alpha) = (\varphi^{-1}(x), h(\varphi^{-1}(x))\alpha), \quad \forall\ (x, \alpha) \in X \times \mathbb{K}.$$

Hence, the trivial line bundles $\langle X \times \mathbb{K}, \pi_X\rangle$ and $\langle Y \times \mathbb{K}, \pi_Y\rangle$ are isometrically isomorphic if and only if they have isometrically isomorphic continuous section spaces.

Recall that a Banach space E is *strictly convex* if $\|x + y\| < 2$ whenever $x \neq y$ in E with $\|x\| = \|y\| = 1$. A Banach space E is said to be *non-square* if E does not contain a copy of the two-dimensional space $\mathbb{K} \oplus_\infty \mathbb{K}$ equipped with the norm $\|(a, b)\| = \max\{|a|, |b|\}$. In other words, if x and y are unit vectors in E, at least one of $\|x+y\|$ and $\|x-y\|$ is less than 2. Note that a Banach space E is non-square if E or its dual E^* is strictly convex.

Example 1.2. (Trivial bundles). Let E and F be Banach spaces. We consider the trivial bundles $B_X = X \times E$ and $B_Y = Y \times F$. The continuous section spaces are $C_0(X, E)$ and $C_0(Y, F)$, the Banach spaces of continuous vector-valued functions vanishing at infinity, respectively. If E and F are strictly convex, by a result of Jerison [9] we know that every linear isometry T from $C_0(X, E)$ onto $C_0(Y, F)$ is of the form:

$$Tf(y) = h_y(f(\varphi(y))), \quad \forall\ f \in C_0(X, E),\ y \in Y. \tag{1.2}$$

Here, φ is a homeomorphism from Y onto X, and h_y is a linear isometry from E onto F for all y in Y. Moreover, the map $y \mapsto h_y$ is SOT continuous on Y. In the case both the Banach dual spaces E^* and F^* are strictly convex, Lau gets the same representation (1.2) in [10]. It is further extended that the same conclusion holds whenever E and F are non-square in [8] or the centralizers of E and F are one dimensional in [1]. The representation (1.2) induces an isometric bundle isomorphism $\Phi : B_X \to B_Y$ from $B_X = X \times E$ onto $B_Y = Y \times F$ defined by

$$\Phi(x, e) = (\varphi^{-1}(x), h_{\varphi^{-1}(x)}(e)), \quad \forall\ (x, e) \in X \times E.$$

Hence, the trivial bundles $\langle X \times E, \pi_X\rangle$ and $\langle Y \times F, \pi_Y\rangle$ are isometrically isomorphic if and only if they have isometrically isomorphic continuous section spaces. We note that if E or F is not non-square, the above assertion (1.2) can be false as shown in Example 3.4.

In this paper, we discuss the general Banach bundle case. Motivated by Example 1.2, we call a Banach bundle $\langle B_X, \pi_X \rangle$ *non-square* (resp. *strictly convex*) if every fiber Banach space $B_x = \pi_X^{-1}(x)$ is non-square (resp. strictly convex). The proof of the following theorem will be given in Section .

Theorem 1.3. *Two non-square Banach bundles $\langle B_X, \pi_X \rangle$ and $\langle B_Y, \pi_Y \rangle$ are isometrically isomorphic as Banach bundles if and only if their continuous section spaces Γ_X and Γ_Y are isometrically isomorphic as Banach spaces.*

We also consider the case when the continuous section space Γ_X is embedded into Γ_Y as a Banach subspace. We want to see whether $\langle B_X, \pi_X \rangle$ embedded into $\langle B_X, \pi_X \rangle$ as a subbundle. Assume F is strictly convex. It is shown in [2, 5, 7] that every linear isometry from $C_0(X, E)$ *into* $C_0(Y, F)$ induces a continuous function φ from a nonempty subset Y_1 of Y onto X and a field $y \mapsto h_y$ of norm one linear operators from E into F on Y_1, such that

$$Tf(y) = h_y(f(\varphi(y))), \quad \forall\ f \in C_0(X, E),\ y \in Y_1, \tag{1.3}$$

and

$$\|Tf|_{Y_1}\| = \sup\{\|Tf(y)\| : y \in Y_1\} = \|f\|, \quad \forall\ f \in C_0(X, E). \tag{1.4}$$

When F is not strictly convex, the conclusion does not hold (see [7]).

In Theorem 3.1, we extend the above representation (1.3) and (1.4) to the general strictly convex Banach bundle case. Supposing all h_y are isometries, we can consider $\langle B_X, \pi_X \rangle$ to be embedded into $\langle B_Y, \pi_Y \rangle$ as a subbundle. However, in Example 3.2 we have a linear into isometry between trivial bundles with all fiber maps h_y not being isometric.

2. Preliminaries

Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ be the underlying field. Let X be a locally compact Hausdorff space. A *Banach bundle* (see, e.g. [4]) over X is a pair $\langle B_X, \pi_X \rangle$, where B_X is a topological space and π_X is a continuous open surjective map from B_X onto X, such that, for all x in X, each fiber $B_x = \pi_X^{-1}(x)$ carries a Banach space structure in the subspace topology and satisfying the following conditions:

(1) Scalar multiplication, addition and the norm on B_X are all continuous wherever they are defined.

(2) If $x \in X$ and $\{b_i\}$ is any net in B_X such that $\|b_i\| \to 0$ and $\pi(b_i) \to x$ in X, then $b_i \to 0_x$ (the zero element of B_x) in B_X.

The condition (2) above ensures that the zero section is in Γ_X.

Definition 2.1. ([4], p. 128). A Banach bundle $\langle B_X, \pi_X\rangle$ is said to be *isometrically isomorphic* to a Banach bundle $\langle B_Y, \pi_Y\rangle$ if there are homeomorphisms $\Phi : B_X \to B_Y$ and $\psi : X \to Y$ such that

(a) $\pi_Y \circ \Phi = \psi \circ \pi_X$, i.e., $\Phi(B_x) = B_{\psi(x)},\ \forall\, x \in X;$

(b) $\Phi|_{B_x}$ is a linear map from B_x onto $B_{\psi(x)},\ \forall\, x \in X;$

(c) Φ preserves norm, i.e., $\|\Phi(b)\| = \|b\|,\ \forall\, b \in B_X.$

Clearly, all the fiber linear maps $\Phi|_{B_x}$ are surjective isometries. In fact, an isometrical bundle isomorphism $(\Phi, \psi) : \langle B_X, \pi_X\rangle \to \langle B_Y, \pi_Y\rangle$ induces a linear isometry T from Γ_X onto Γ_Y defined by setting $\varphi = \psi^{-1} : Y \to X$, $h_y = \Phi|_{B_{\varphi(y)}} : B_{\varphi(y)} \to B_y$, and

$$Tf(y) = \Phi(f(\varphi(y))) = h_y(f(\varphi(y))), \quad \forall\ f \in \Gamma_X,\ y \in Y. \tag{2.1}$$

In other words, isometrically isomorphic Banach bundles have isometrically isomorphic continuous section spaces. We want to establish the converse of this observation.

In general, let $\varphi : Y \to X$ be a continuous map, and let $y \mapsto h_y$ be a field of fiber linear maps $h_y : B_{\varphi(y)} \to B_y, \forall\, y \in Y$. We can define a linear map T sending vector sections f in $\langle B_X, \pi_X\rangle$ to vector sections Tf in $\langle B_Y, \pi_Y\rangle$ by setting $Tf(y) = h_y(f(\varphi(y))), \forall y \in Y$. The field $y \mapsto h_y$ is said to be *continuous* if $y_\lambda \to y$ implies $h_{y_\lambda}(f(\varphi(y_\lambda))) \to h_y(f(\varphi(y)))$, and *uniformly bounded* if $\sup_{y\in Y} \|h_y\| < +\infty$. When $B_X = X \times E$ and $B_Y = Y \times F$, the continuity of a field $y \mapsto h_y$ of fiber linear maps reduces to the usual SOT continuity. In general, assuming φ is proper, i.e., $\lim_{y\to\infty} \varphi(y) = \infty$, if the field $y \mapsto h_y$ is uniformly bounded and continuous on Y, then $T(\Gamma_X) \subseteq \Gamma_Y$. Conversely, we will see in Theorem 3.1 that every linear into isometry $T : \Gamma_X \to \Gamma_Y$ defines a continuous field $y \mapsto h_y$ of fiber linear maps with all $\|h_y\| = 1$, provided that $\langle B_Y, \pi_Y\rangle$ is strictly convex.

In terms of Banach bundles, Example 1.1 says that trivial line bundles are completely determined by the geometric structure of its continuous sections. It is also the case for trivial Banach bundles $X \times E$ and $Y \times F$ whenever E and F are non-square, as demonstrated in Example 1.2. In attacking the general Banach bundle case, we need the following result of Fell [4].

Proposition 2.2. ([4], p. 129). *Let $\{s_i\}$ $(i \in I)$ be a net of elements of B_X and s an element of B_X such that $\pi_X(s_i) \to \pi_X(s)$. Suppose further that for each $\epsilon > 0$ we can find a net $\{u_i\}$ of elements of B_X (indexed by the same I) and an element u of B_X such that:* (1) $u_i \to u$ *in* B_X, (2) $\pi_X(u_i) = \pi_X(s_i)$ *for each* i, (3) $\|s - u\| < \epsilon$, *and* (4) $\|s_i - u_i\| < \epsilon$ *for all large enough* i. *Then* $s_i \to s$ *in* B_X.

3. The Results

First, we discuss the *into* isometry case. We shall write E^* and S_E for the Banach dual space and the unit sphere of a Banach space E, respectively.

Theorem 3.1. *Suppose $\langle B_X, \pi_X\rangle$ and $\langle B_Y, \pi_Y\rangle$ are Banach bundles such that $\langle B_Y, \pi_Y\rangle$ is strictly convex. Let $T : \Gamma_X \to \Gamma_Y$ be a linear into isometry. Then there exist a continuous map φ from a nonempty subset Y_1 of Y onto X, and a field of norm one linear operators $h_y : B_{\varphi(y)} \to B_y$, for all y in Y_1, such that*

$$Tf(y) = h_y(f(\varphi(y))), \quad \forall\ f \in \Gamma_X,\ y \in Y_1,$$

and

$$\|Tf|_{Y_1}\| = \|Tf\|, \quad \forall\ f \in \Gamma_X.$$

Proof. We employ the notations developed in [7, 8]. For x in X, y in Y, μ in $S_{B_x^*}$ and ν in $S_{B_y^*}$, we set

$$S_{x,\mu} = \{f \in \Gamma_X :\ \mu(f(x)) = \|f\| = 1\},$$

$$R_{y,\nu} = \{g \in \Gamma_Y :\ \nu(g(y)) = \|g\| = 1\},$$

$$Q_{x,\mu} = \{y \in Y : T(S_{x,\mu}) \subseteq R_{y,\nu} \text{ for some } \nu \text{ in } S_{B_y^*}\},$$

and

$$Q_x = \bigcup_{\mu \in S_{B_x^*}} Q_{x,\mu}.$$

As in [8], it is not difficult to see that

(a) For all x in X, the set $S_{x,\mu} \neq \emptyset$ for some μ in $S_{B_x^*}$;

(b) If $S_{x,\mu} \neq \emptyset$, then so is $Q_{x,\mu}$.

By the strict convexity of $\langle B_Y, \pi_Y\rangle$, we have

(c) $Q_{x_1} \bigcap Q_{x_2} = \emptyset$ for all $x_1 \neq x_2$. Set

$$Y_1 = \bigcup_{x \in X} Q_x = \bigcup_{x \in X} \bigcup_{\mu \in S_{B_x^*}} Q_{x,\mu}.$$

From (c), we can define a map φ from Y_1 onto X by

$$\varphi(y) = x \quad \text{if} \quad y \in Q_x.$$

Using the strict convexity of $\langle B_Y, \pi_Y\rangle$ again, we also have

(d) $f(\varphi(y)) = 0$ implies $Tf(y) = 0$, i.e. $\ker \delta_{\varphi(y)} \subseteq \ker(\delta_y \circ T)$.

Then there exists a linear operator $h_y : B_{\varphi(y)} \to B_y$ such that

$$\delta_y \circ T = h_y \circ \delta_{\varphi(y)}, \quad \forall\ y \in Y_1.$$

In other words,

$$Tf(y) = h_y(f(\varphi(y))), \quad \forall\ f \in \Gamma_X,\ y \in Y_1.$$

For all b in $B_{\varphi(y)}$, choose an element f in Γ_X such that $f(\varphi(y)) = b$ and $\|f\| = \|b\|$. It follows

$$\|h_y(b)\| = \|h_y(f(\varphi(y)))\| = \|Tf(y)\| \leq \|Tf\| = \|f\| = \|b\|.$$

Since $y \in Y_1$, there exist x in X, μ in $S_{B_x^*}$ and ν in $S_{B_y^*}$ such that

$$\nu(Tf(y)) = \mu(f(x)) = 1, \quad \forall\ f \in S_{x,\mu},$$

and hence

$$\|h_y(f(x))\| = \|Tf(y)\| = 1.$$

This shows that $\|h_y\| = 1$.

For all f in Γ_X with norm one, $f \in S_{x,\mu}$ for some x and μ. As a result, $Tf \in R_{y,\nu}$ for some y in Y_1 and ν in $S_{B_y^*}$. Thus,

$$\nu(Tf(y)) = \mu(f(x)) = \|f\| = 1.$$

Therefore, $\|Tf|_{Y_1}\| = 1 = \|f\| = \|Tf\|$.

It remains to show that the map φ is continuous. Let y_λ be a net converging to y in Y_1. If $\varphi(y_\lambda)$ does not converge to $\varphi(y)$, then by passing to a subnet if necessary, we can assume it converges to an $x \neq \varphi(y)$ in X_∞. Let U_1 and U_2 be disjoint neighborhoods of x and $\varphi(y)$ in X_∞, respectively. Let f be an element of $S_{\varphi(y),\mu}$ supporting in U_2. Then $f(\varphi(y_\lambda)) = 0$ for large λ. By (d), $Tf(y_\lambda) = 0$ for large λ. The definition of φ implies that there exists a ν in $S_{B_y^*}$ such that $\nu(Tf(y)) = \mu(f(\varphi(y))) = \|f\| = 1$. Hence, $\|Tf(y)\| = 1$, contradicting to the fact $Tf(y_\lambda) = 0$ for large λ. ∎

Example 3.2. For each θ in $[0, 2\pi]$, let $P_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be the orthogonal projection onto the one-dimensional subspace of $\mathbb{R}^2$ spanned by the unit vector $(\cos\theta, \sin\theta)$. Every element f in $C(\{0\}, \mathbb{R}^2)$ is given by the vector $f(0) = (r\cos t, r\sin t)$ for some $r \geq 0$ and $t \in [0, 2\pi]$. Define a linear isometry $T : C(\{0\}, \mathbb{R}^2) \to C([0, 2\pi], \mathbb{R}^2)$ by

$$\begin{aligned} T(f)(\theta) &= P_\theta(f(0)) = P_\theta(r\cos t, r\sin t) \\ &= (r\cos(t-\theta)\cos\theta, r\cos(t-\theta)\sin\theta). \end{aligned}$$

In the notations of Theorem 3.1, $h_\theta = P_\theta$, $Y_1 = Y = [0, 2\pi]$, and

$$Tf(\theta) = h_\theta(f(0)), \quad \forall\, f \in C(\{0\}, \mathbb{R}^2),\ \theta \in [0, 2\pi].$$

Note that $h_\theta = P_\theta$ is not an isometry for every θ in $Y_1 = [0, 2\pi]$.

Here comes the proof of our main result.

Proof of Theorem 1.3. Let $\langle B_X, \pi_X\rangle$ and $\langle B_Y, \pi_Y\rangle$ be two non-square Banach bundles and T a linear isometry from Γ_X onto Γ_Y. Denote by

$$K_X = \bigcup_{x\in X}(\{x\} \times U_{B_x^*}) \quad \text{and} \quad K_Y = \bigcup_{y\in Y}(\{y\} \times U_{B_y^*}),$$

the disjoint unions of the compact sets $\{x\} \times U_{B_x^*}$ and $\{y\} \times U_{B_y^*}$, respectively. Note that both the Hausdorff spaces K_X and K_Y are locally compact. Define a linear isometry $\Psi : \Gamma_Y \to C_0(K_Y)$ by

$$\Psi(g)(y, \nu) = \nu(g(y)), \quad \forall\, g \in \Gamma_Y, (y, \nu) \in K_Y.$$

Then $\widetilde{T} = \Psi \circ T$ is a linear isometry from Γ_X into $C_0(K_Y)$. By Theorem 3.1, there exist a continuous map $\widetilde{\varphi}$ from a nonempty subset A_Y of K_Y onto X and bounded linear functionals $\widetilde{h}_{(y,\nu)} \in B^*_{\widetilde{\varphi}(y,\nu)}$ such that

$$\text{(3.1)} \qquad \widetilde{T}f(y, \nu) = \nu(Tf(y)) = \widetilde{h}_{(y,\nu)}(f(\widetilde{\varphi}(y, \nu))), \quad \forall\, f \in \Gamma_X,\ (y, \nu) \in A_Y.$$

Applying the same argument to T^{-1}, there exist a continuous map $\widetilde{\psi}$ from a subset A_X of K_X onto Y and bounded linear functionals $\widetilde{k}_{(x,\mu)}$ in $B^*_{\widetilde{\psi}(x,\mu)}$ such that

$$\mu(T^{-1}g(x)) = \widetilde{k}_{(x,\mu)}(g(\widetilde{\psi}(x, \mu))), \quad \forall\, g \in \Gamma_Y,\ (x, \mu) \in A_X.$$

Let

$$C_y = \{\nu \in S_{B_y^*} : (y, \nu) \in A_Y\},$$

$$X_I = \{x \in X : \text{ there exists a } \mu \text{ in } S_{B_x^*} \text{ such that } (x, \mu) \in A_X\},$$

and

$$Y_I = \{y \in Y : \text{ there exists a } \nu \text{ in } S_{B_y^*} \text{ such that } (y, \nu) \in A_Y\}.$$

We make the following easy observations:

(I) $X_I = X$ and $Y_I = Y$;

(II) C_y is total in B_y^*, for all y in Y.

By modifying the arguments in [8], it is not difficult to show that if $\langle B_Y, \pi_Y\rangle$ is non-square, $\widetilde{\varphi}(y, \nu_1) = \widetilde{\varphi}(y, \nu_2)$ for all ν_i in C_y and for all y in Y. Consequently, we can define a continuous map $\varphi : Y \to X$ by

$$\varphi(y) = \widetilde{\varphi}(y, \nu), \text{ for some } \nu \in C_y.$$

In view of (3.1) and (II), we have $f(\varphi(y)) = 0$ implies $Tf(y) = 0$. Then there exists a bounded linear operator $h_y : B_{\varphi(y)} \to B_y$ such that

$$Tf(y) = h_y(f(\varphi(y))), \ \forall\ f \in \Gamma_X,\ y \in Y. \tag{3.2}$$

By symmetry, T^{-1} also carries a form

$$T^{-1}g(x) = k_x(g(\psi(x))), \ \forall\ g \in \Gamma_Y,\ x \in X,$$

for some continuous map ψ from X onto Y, and bounded linear operators k_x from $B_{\psi(x)}$ into B_x, for all x in X. Consequently,

$$f(x) = (T^{-1}(Tf))(x) = k_x(Tf(\psi(x))) = k_x h_{\psi(x)}(f(\varphi(\psi(x)))).$$

This implies that φ is a homeomorphism with inverse ψ, and h_y are bijective linear isometries with inverses $k_{\varphi(y)}$ for all y in Y.

Let $\Phi = (h_y^{-1})_{y\in Y}$, i.e. $\Phi|_{B_y} = h_y^{-1}$. Then it defines a map from B_Y onto B_X as follows: for all b in B_Y and $\pi_Y(b) = y_0$. Choose a continuous section g in Γ_Y such that $g(y_0) = b$. Then

$$\Phi(b) = h_{y_0}^{-1}(g(y_0)) = T^{-1}(g)(\varphi(y_0)).$$

We show that Φ is a homeomorphism from B_Y to B_X. By symmetry, it suffices to prove that Φ is continuous. We shall make use of Proposition 2.2 in below.

Let $b_i \to b$ in B_Y. We show that $\Phi(b_i) \to \Phi(b)$ in B_X. Since π_Y and φ are continuous, we have $\pi_Y(b_i) \to \pi_Y(b)$ and $\varphi(\pi_Y(b_i)) \to \varphi(\pi_Y(b))$. Let $s_i = \Phi(b_i)$ and $s = \Phi(b)$. Choose a continuous section g in Γ_Y such that $g(\pi_Y(b)) = b$. Then, for all $\epsilon > 0$, we have $\|g(\pi_Y(b_i))-b_i\| < \epsilon$ for all large enough i. The fact $\pi_X\circ\Phi = \varphi \circ \pi_Y$ (this follows from (3.2)) implies that $\pi_X(s_i) = \pi_X(\Phi(b_i)) = \varphi(\pi_Y(b_i))$ approaches $\varphi(\pi_Y(b)) = \pi_X(\Phi(b)) = \pi_X(s)$. Let $u_i = \Phi(g(\pi_Y(b_i)))$ and $u = \Phi(g(\pi_Y(b)))$. Since $\Phi|_{B_y}$ is an isometry, we have $\|u_i - s_i\| = \|g(\pi_Y(b_i))-b_i\| < \epsilon$, for all large enough i. And

$$u_i = \Phi(g(\pi_Y(b_i))) = h^{-1}_{\pi_Y(b_i)}(g(\pi_Y(b_i))) = f(\varphi(\pi_Y(b_i))),$$

for some f in Γ_X, which converges to

$$f(\varphi(\pi_Y(b))) = h^{-1}_{\pi_Y(b)}(g(\pi_Y(b))) = \Phi(b) = u$$

in B_X. By Proposition 2.2, we have $\Phi(b_i) = s_i \to s = \Phi(b)$ in B_X. This shows that Φ is continuous and complete the proof of Theorem 1.3. ∎

Corollary 3.3. *Assume* $\langle B_X, \pi_X \rangle$ *and* $\langle B_Y, \pi_Y \rangle$ *are two non-square Banach bundles over locally compact Hausdorff spaces with isometrically isometric continuous sections. If* $\langle B_X, \pi_X \rangle$ *is locally trivial, then so is* $\langle B_Y, \pi_Y \rangle$.

The following example shows that the conclusion in Theorem 1.3 might not hold if $\langle B_X, \pi_X \rangle$ or $\langle B_Y, \pi_Y \rangle$ is not non-square.

Example 3.4. Let π_i be the i-th coordinate map of $\mathbb{R} \oplus_\infty \mathbb{R}$, $i = 1, 2$. Each element f in $C(\{0\}, \mathbb{R} \oplus_\infty \mathbb{R})$ is given by the vector $f(0)$ in $\mathbb{R} \oplus_\infty \mathbb{R}$. Define a linear map $T : C(\{0\}, \mathbb{R} \oplus_\infty \mathbb{R}) \to C(\{1, 2\}, \mathbb{R})$ by

$$Tf(i) = \pi_i(f(0)), \quad \forall\ f \in C(\{0\}, \mathbb{R} \oplus_\infty \mathbb{R}),\ i = 1, 2.$$

It is easy to see that T is an isometrical isomorphism, but $\mathbb{R} \oplus_\infty \mathbb{R}$ and $\mathbb{R}$ are not isomorphic as Banach spaces. In particular, $\mathbb{R} \oplus_\infty \mathbb{R}$ is not isometrically isomorphic to $\{1, 2\} \times \mathbb{R}$ as Banach bundles, although they have isometrically isomorphic continuous section spaces.

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