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ISOMETRIC IMMERSION OF A COMPACT RIEMANNIAN MANIFOLD INTO A EUCLIDEAN SPACE

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We show that an isometric immersion of an n-dimensional compact Riemannian manifold of non-negative Ricci curvature with scalar curvature always less than $n(n-1)\lambda^{-2}$ into a Euclidean space of dimension $n+1$ can never be contained in a ball of radius λ.

Jacobowitz [2] proved that an isometric immersion of an n-dimensional compact Riemannian manifold with sectional curvature always less than some constant λ^{-2} into Euclidean space of dimension $2n-1$ can never be contained in a ball of radius λ. This result generalised the results of Tompkins [4], Chern and Kuiper [1] and Otsuki [3]. However it is not known whether such a nonimmersibility theorem holds with the condition on sectional curvature replaced by a suitable condition on Ricci curvature or the scalar curvature. In this note we prove the following co-dimension one nonimmersibility theorem.

THEOREM. *Let E be Euclidean space of dimension $n+1$ and M be a compact n-dimensional Riemannian manifold of non-negative Ricci curvature whose scalar curvature is less than some constant $n(n-1)\lambda^{-2}$. Then no isometric immersion of M into E is contained in a ball of radius λ.*

PROOF: Assume that $\psi\colon M \to E$ is the isometric immersion such that $\psi(M)$ is contained in a ball of radius λ. Thus we can assume that $\|\psi\| \leqslant \lambda$, where $\|\,,\|$ is the norm on E with respect to the Euclidean metric $\langle\,,\rangle$. Let N be the unit normal vector field to M. Then the support function $\rho\colon M \to R$ is defined by $\rho = \langle \psi, N\rangle$, and we have the Minkowski's formula

$$\int_M (1+\rho\alpha)dv = 0, \tag{1}$$

where α is the mean curvature of M.

Denote by g, ∇ and A the Riemannian metric, the Riemannian connection and the Weingarten map on M respectively. The position vector field ψ can be expressed

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as $\psi = t + \rho N$, where t is a vector field on M. Then using Gauss and Weingarten formulae we immediately get

$$\nabla_X t = X + \rho AX, \tag{2}$$

for any vector field X on M.

If η is a 1-form dual to t, then using equation (2), we obtain

$$d\eta = 0, \qquad \delta\eta = n(1 + \rho\alpha). \tag{3}$$

Also, we obtain

$$\|\nabla t\|^2 = n + 2\rho n\alpha + \rho^2 \operatorname{tr} A^2.$$

Now using $\|t\|^2 + \rho^2 = \|\psi\|^2$ and the expression for the scalar curvature S, $S = n^2\alpha^2 - \operatorname{tr} A^2$, we get

$$\|\nabla t\|^2 - (\delta\eta)^2 = \|t\|^2 S + n(n-1) - \|\psi\|^2 S - 2n(n-1)(1 + \rho\alpha). \tag{4}$$

For a compact Riemannian manifold the following integral formula is known (see [5], p.41)

$$\int_M \left\{ \operatorname{Ric}(t, t) - \frac{1}{2}\|d\eta\|^2 + \|\nabla t\|^2 - (\delta\eta)^2 \right\} dv = 0,$$

where Ric is the Ricci tensor of M.

Using (1), (3) and (4) in the above integral formula we get

$$\int_M \left\{ \operatorname{Ric}(t, t) + \|t\|^2 S + \left(n(n-1) - \|\psi\|^2 S \right) \right\} dv = 0. \tag{5}$$

From the hypothesis of the Theorem it follows that $\operatorname{Ric}(t, t) \geqslant 0$, $\|t\|^2 S \geqslant 0$ and $\|\psi\|^2 S \leqslant \lambda^2 S < n(n-1)$ which contradicts (5). This proves the Theorem. $\square$

References

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