J. Math. Soc. Japan
Vol. 19, No. 2, 1967

Isometric immersions of Riemannian manifolds

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(Received Oct. 24, 1966)

Introduction.

It seems to be natural in differential geometry to conjecture that many theorems proved under the condition that the underlying manifolds are compact can be extended to manifolds with complete Riemannian metric under some suitable additional conditions.

Now, it is clear that for any smooth function f on a compact manifold there is a point p such that $\nabla_i f(p)=0$ and $\nabla_i \nabla_j f(p)$ is negative semi-definite. It is also clear that if a smooth function $f(x)$ on a real line R has an upper bound, then for any $\varepsilon>0$ there is $x\in R$ such that $f'(x)$ and $f''(x)$ are $<\varepsilon$. This simple fact, however, can not be extended in general to a complete Riemannian manifold. That is, there is a complete Riemannian manifold M and a bounded function f on M such that $m(p)=\{X^i X^j \nabla_i \nabla_j f(p)\,;\,\|X\|=1\}$ is always larger than $a>0$. This example can easily be constructed on R^2 with metric $dr^2+g(r)d\theta^2$ (in the polar coordinate expression). Let $f(r,\theta)=f(r)=\frac{r^2}{1+r^2}$. Since

$$\nabla\nabla f(=\nabla_i \nabla_j f dx^i dx^j)=f''(r)dr^2+\frac{1}{2}f'(r)g'(r)d\theta^2\,,$$

one can choose a suitable function $g(r)$ so that it satisfies (a) $g(r)$ is smooth and $g(r)=r$ for $0\leqq r<1/2$, (b) $g(r)$ is a solution of $g'(r)/g(r)=2c/f'(r)$ (for example $g(r)=\exp\int_1^r c(1-r)^2/r\,dr)$ for $r\geqq 1$. In this example, one can see easily that the sectional curvature has no lower bound.

In this paper, there will be proved first of all a generalization of this example, that is:

THEOREM A. *Let M be a connected and complete Riemannian manifold whose sectional curvature $K(X,Y)$ has a lower bound i.e. $K(X,Y)\geqq -K_0$. If a smooth function f on M has an upper bound, then for any $\varepsilon>0$, there is a point $p\in M$ such that $\|\operatorname{grad} f(p)\|<\varepsilon$ and $m(p)=\max\{X^i X^j \nabla_i \nabla_j f(p)\,;\,\|X\|=1\}$ $<\varepsilon$.*

For an application of this theorem, an isometric immersion of M into the Euclidean N-space R^N will be considered. It is clear that if M is compact, then for any isometric immersion φ of M into R^N, there are a point p and a unit normal ξ at $\varphi(p)$ such that the second fundamental form at $\varphi(p)$ with respect to ξ is positive definite.

In our case, we have the following theorem:

THEOREM B. *Let φ be an isometric immersion of a connected and complete Riemannian manifold M into R^N. Assume that the sectional curvature of M has a lower bound. If there is a unit vector n at the origin of R^N such that*

$$\langle \varphi(p), n\rangle/\|\varphi(p)\| \geqq \delta > 0$$

for all $p \in M$, then there exist a point p_0 and a unit normal vector ξ at $\varphi(p_0)$ such that the second fundamental form at $\varphi(p_0)$ with respect to ξ is positive definite.

From this theorem, we can see immediately the following:

COROLLARY C. *Let φ be an isometric immersion of a connected and complete Riemannian manifold M into R^N. If $\varphi(M)$ is a minimal submanifold in R^N, then for any unit vector n at the origin of R^N and for any positive δ, there exists a point p such that $\langle \varphi(p), n\rangle/\|\varphi(p)\| < \delta$.*

The author would like to acknowledge the kind advice of Professors M. Obata, T. Nagano and T. Takahashi.

1. Geodesic spheres and their second fundamental forms.

Let M be a connected and complete Riemannian C^∞-manifold of dimension n and $g(t)$ be a geodesic going from $g(0)=p$ to $g(h)=q$. The parameter t represents the arc length. Assume that $g(t)$ is not a conjugate point of p with respect to the geodesic g for any $0<t\leqq h$ and that $g(t)$; $0\leqq t\leqq h$ does not intersect itself. Let $S_p(a)=\{X\in T_p(M);\ \|X\|=a\}$ and $T_t=dg(t)/dt$. Then, $T_0\in S_p(1)$.

Since $g(t)$ is not a conjugate point for any $0<t\leqq h$ and $g(t)$, $0\leqq t\leqq h$ does not self-intersect, there exists a neighborhood V of T_0 in $S_p(1)$ such that $\mathrm{Exp}_p: (0, h+\varepsilon)\times V\to M$ is a diffeomorphism for sufficiently small $\varepsilon>0$, where $\mathrm{Exp}_p(t, X)=\mathrm{Exp}_p tX$.

Putting $\mathcal{S}_p(a)=\mathrm{Exp}_p\{S_p(a)\cap(0, h+\varepsilon)\times V\}$, $\mathcal{S}_p(a)$ is an $(n-1)$-dimensional submanifold of M for every $0<a\leqq h$. Let $W=\mathrm{Exp}_p\{(0, h+\varepsilon)\times V\}$.

The following lemma is well-known.

LEMMA 1. *Let $x\in\mathcal{S}_p(a)$, $0<a\leqq h$. For any curve C joining p and x in W, the length of C is no less than a.*

The following lemma is an immediate consequence of Lemma 1.

LEMMA 2. *For any points $x\in\mathcal{S}_p(a)$ and $y\in\mathcal{S}_p(h)$ and for any arc C joining x and y in W, the length of the arc C is no less than $h-a$.*

Let X_t be a unit vector field that is displaced parallel along g and is tangent to $\mathcal{S}_p(a)$ and $\mathcal{S}_p(h)$ at $g(a)$ and $g(h)$ respectively. We consider the variation of the geodesic g by the vector field X in the same way as in §1 of [1].

The following Lemma is clear from the Lemma in §1 of [1] and Lemma 2 above.

LEMMA 3. *The first and second variations of arc length are*

$$L'_X(0)=0,\ L''_X(0)=H_q(X, X)-H_{g(a)}(X, X)-\int_a^h K(T, X)dt \geqq 0,$$

where $K(T, X)$ is the Riemannian sectional curvature corresponding to the 2-plane $T\wedge X$ and $H_{g(a)}(X, X)$ is the second fundamental form for $\mathcal{S}_p(a)$ at $g(a)$ corresponding to the unit normal T, evaluated at the tangent vector X.

Let $ds^2=dt^2+g_{\alpha\beta}d\theta^\alpha d\theta^\beta$ be the metric of M in the polar coordinate expression with radius t and with center p. Since $(t, \theta^1, \cdots, \theta^{n-1})$ can be considered as a coordinate of W, one can see by an elementary calculation that

$$\frac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial t}(q)d\theta^\alpha(X)d\theta^\beta(X)=-H_q(X, X),$$

for every $X\in T_q(\mathcal{S}_p(h))$. Since $H_q(X, Y)=H_{\alpha\beta}d\theta^\alpha(X)d\theta^\beta(Y)$, $H(X, Y)$ can be considered as a 2-tensor field around q in M. That is, for any $X, Y\in T_q(M)$, $H_q(X, Y)$ is defined by $H_{\alpha\beta}(q)d\theta^\alpha(X)d\theta^\beta(Y)$.

On the other hand,

$$(\nabla\nabla t)(x)=\frac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial t}(x)d\theta^\alpha d\theta^\beta.$$

Thus,

$$(\nabla\nabla t^2)(q)(X, X)(=X^iX^j(\nabla_i\nabla_j t^2)(q))=2\{dt(X)\}^2-2hH_q(X, X),$$

for every $X\in T_q(M)$.

LEMMA 4. *Notations being as above, if the sectional curvature $K(X, Y)$ of M has a lower bound, i.e. $K(X, Y)\geqq -K_0$, $K_0>0$, then*

$$(\nabla\nabla t^2)(q)(X, X)\leqq 2\{1+h[K_0(h-a)-H_{g(a)}(X, X)]\},$$

where $X=X_t$ is the vector field that is displaced parallel along the geodesic g.

This Lemma is an immediate result of Lemma 3.

2. Bounded functions of M.

Let M be a connected and complete Riemannian manifold of dimension n and f a C^2-function on M with an upper bound. Let $b=\sup f$. For a fixed point p, we assume without loss of generality that $f(p)=0$.

Considering the graph $\Gamma=\{(f(x), x);\ x\in M\}$, Γ is a closed submanifold of $R\times M$, where R is the real line with natural Riemannian metric and $R\times M$ is the product manifold with product metric.

Though many relations exist between M and $R\times M$, we shall use only the following in the remainder of this paper;

LEMMA 5. *Let π be the projection from $R\times M$ onto M.* (i) *If g' is a geodesic in $R\times M$, then so is $g=\pi g'$ in M.* (ii) *Let y, x be two points of a geodesic g' in $R\times M$. x is a conjugate point of y with respect to g' if and only if $\pi(x)$ is a conjugate point of $\pi(y)$ with respect to the geodesic $\pi g'$.* (iii) *Let (k, y) and (a, x) be two points of $R\times M$ and g' a geodesic segment from (k, y) to (a, x); then $L(g')^2=L(\pi g')^2+(k-a)^2$, where $L(g')$ and $L(\pi g')$ are length of g' and $\pi g'$ respectively.*

PROOF. Let $(l, x^1, \cdots, x^n)$ be a local coordinate of $R\times M$ which is a product of local coordinates of R and M. Thus, the equations of a geodesic are

$$\frac{d^2l}{ds'^2}=0 \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (1)$$

$$\frac{d^2x^i}{ds'^2}+\left\{ {i \atop jk} \right\}\frac{dx^j}{ds'}\frac{dx^k}{ds'}=0 \quad \cdots\cdots\cdots\cdots\cdots\cdots \quad (2)$$
$$(1\leqq i\leqq n)$$

where $\left\{ {i \atop jk} \right\}$ is the Christoffel's symbol of M and s' is the arc length. Thus, (i) follows from the equality (2).

Let s', s and l be arc length from (k, y), y and k along g', $\pi g'$ and R respectively. Then, $l=As'$ and $s=as'$ from the equalities (1) and (2). It follows that $s'^2=l^2+s^2$ because of the fact that $A^2+a^2=1$ (obtained from $ds'^2=ds^2+dl^2$).

As for (ii), one can see this easily by the Jacobi equation.

For each positive integer k and a fixed point p in M, consider a point $\hat{p}_k=(kb, p)$ and a geodesic segment $\hat{g}_k$ from $\hat{p}_k$ to Γ which attains the distance between $\hat{p}_k$ and Γ. Let $(f(q_k), q_k)$ be another end point of $\hat{g}_k$.

LEMMA 6. *Notations being as above, there is no conjugate point of $\hat{p}_k$ on $\hat{g}_k$ with respect to $\hat{g}_k$.*

PROOF. Let $\hat{g}_k(t)$ be the geodesic parametrized by the arc length and set $\hat{g}_k(0)=\hat{p}_k$, $\hat{g}_k(l'_k)=(f(q_k), q_k)$, $\hat{g}_k([0, l'_k])=\hat{g}_k$. It is easy to show that $\hat{g}_k(t)$; $0<t<l'_k$ is not a conjugate point of $\hat{p}_k$ with respect to $\hat{g}_k$.

Assume $\hat{g}_k(l'_k)$ is a conjugate point of $\hat{p}_k$ with respect to $\hat{g}_k$. Take a point $\hat{g}_k(l'_k+\varepsilon)$ on the line obtained by extension of $\hat{g}_k$. Since Γ is a differentiable submanifold of $R\times M$, there exists a sufficiently small $\varepsilon>0$ such that the ε-sphere of center $\hat{g}_k(l'_k+\varepsilon)$ is in contact with Γ at one point $\hat{g}_k(l'_k)$. Since $\hat{g}_k(l'_k)$ is a conjugate point, there is a geodesic segment $\hat{g}'$ from $\hat{p}_k$ to $\hat{g}_k(l'_k+\varepsilon)$ such that $L(\hat{g}_k)\leqq l'_k+\varepsilon$. Let $\hat{q}'$ be a point of intersection of Γ and $\hat{g}'$. Since

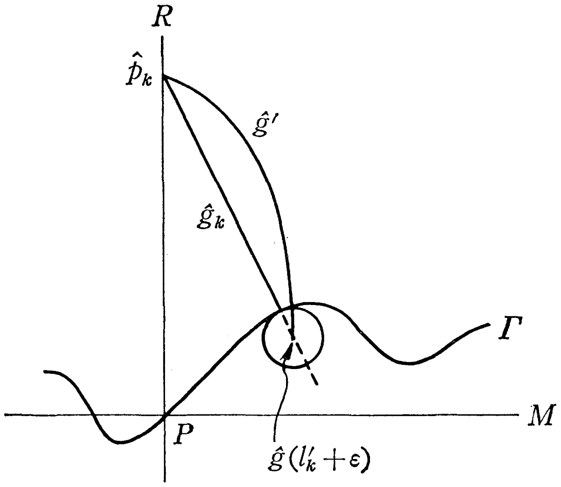

Fig. 1.

dist. $(\hat{q}', \hat{g}_k(l'_k+\varepsilon)) > \varepsilon$ we have dist. $(\hat{p}_k, \hat{q}') < l'_k$. This is a contradiction.

Now, notations being as above, $\pi(\hat{g}_k(t))$ is not a conjugate point of p with respect to $g_k = \pi\hat{g}_k$ for any $0 < t \leqq l'_k$, and g_k does not intersect itself. Let T_k be the unit tangent vector of g_k at p. There is a neighborhood W_k of T_k such that Exp_p is a diffeomorphism from $(0, h_k+\delta_k)\times W_k$ into M, where $h_k = L(\pi\hat{g}_k)$. Set $\phi(x) = \mathrm{Exp}_p^{-1}(x)$ for $x \in \mathrm{Exp}_p\{(0, h_k+\delta_k)\times W_k\}$ and define the function $F_k(x)$ by

$$F_k(x) = kb - \sqrt{l'^2_k - \|\phi(x)\|^2}\,.$$

There is no difficulty in verifying the following:

LEMMA 7. (i) *$f(x) \leqq F_k(x)$ and $f(q_k) = F_k(q_k)$ for every $k \geqq 1$.*

(ii) $$(\nabla\nabla F_k)(q_k) = \frac{1}{2}\frac{(\nabla\nabla\|\phi\|^2)(q_k)}{kb-f(q_k)} + \frac{1}{4}\frac{(\nabla\|\phi\|^2)(q_k)(\nabla\|\phi\|^2)(q_k)}{(kb-f(q_k))^3}\,.$$

PROOF. From (iii) of Lemma 5, we see that $l'^2_k \leqq (kb-f(x))^2 + \|\phi(x)\|^2$. Thus, $f(x) \leqq kb - \sqrt{l'^2_k - \|\phi(x)\|^2}$. (i) follows from this. (ii) is a direct calculation.

Assume $K(X, Y) \geqq -K_0$; then, combining with Lemma 4, we see that

$$\begin{aligned}(\nabla\nabla f)(q_k)(X, X) &\leqq (\nabla\nabla F_k)(q_k)(X, X)\\ &\leqq \frac{1}{kb-f(q_k)}\{1+h_k[K_0(h_k-a) - H_{g_k(a)}(X, X)]\}\\ &\quad + \frac{1}{(k-1)^3 b^3} h_k^2 \|X\|^2\,, \qquad k \geqq 2\,.\end{aligned}$$

Since all g_k are geodesic segments from p to q_k in M, all $g_k(a)$ are points of $\tilde{S}_p(a) = \{\mathrm{Exp}_p X;\ \|X\| = a\}$. For sufficiently small a, $\tilde{S}_p(a)$ is a differentiable submanifold of M. Let $m=\min\{H_q(X, X); q\in\tilde{S}_p(a), \|X\|=1\}$. Then,

$$(\nabla\nabla f)(q_k)(X, X)\leqq\frac{1}{kb-f(q_k)}\{K_0h_k^2-(aK_0+m)h_k+1\}+\frac{h_k^2}{(k-1)^3b^3} \quad\cdots\cdots (3)$$

for all $\|X\|=1$.

LEMMA 8. *Notations being as above, if h_k is bounded for $k\to\infty$, then there exists a point q_∞ in M such that $(\mathrm{grad} f)(q_\infty)=0$ and $(\nabla\nabla f)(q_\infty)$ is negative semi-definite.*

PROOF. Since $\mathrm{grad} f(q_k)=\mathrm{grad} F_k(q_k)$, we see that

$$\|\mathrm{grad} f(q_k)\|=\frac{h_k}{kb-f(q_k)} . \quad\cdots\cdots (4)$$

Since $\{h_k\}$ is bounded, there exists a subsequence $\{q_{k'}\}$ which converges to q_∞. Thus, from the equality (4) and the inequality (3), we have that $(\mathrm{grad} f)(q_\infty)=0$ and $(\nabla\nabla f)(q_\infty)$ is negative semi-definite.

Thus, in the remainder of this section, assume that $h_k\to\infty$ for $k\to\infty$.

Since $K_0>0$, we see that $K_0h_k^2-(aK_0+m)h_k+1\geqq 0$ for sufficiently large k. It follows

$$(\nabla\nabla f)(q_k)(X, X)\leqq\frac{1}{(k-1)b}\{K_0h_k^2-(aK_0+m)h_k+1\}+\frac{h_k^2}{(k-1)^3b^3} \quad\cdots\cdots (3)'$$

By the assumption $f(p)=0$ and the definition of l_k' we see that $l_k'\leqq kb$ and thus we can show easily that $f(q_k)\geqq 0$ for all $k\geqq 1$.

Since $l_k'\leqq\mathrm{dist.}(\hat{p}_k, (f(q_j), q_j))$, one obtains

$$(kb-f(q_k))^2+h_k^2\leqq(kb-f(q_j))^2+h_j^2 .$$

Thus, $h_k^2-h_j^2\leqq 2kb(f(q_k)-f(q_j))-(f(q_k)^2-f(q_j)^2)$.

Since $h_k\to\infty$, for $k\to\infty$, there exists k' for every k such that $k'>k$, $h_{k'}\geqq h_k$ and $h_{k''}<h_k$ for every k'', $k<k''<k'$. If $f(q_k)>f(q_{k'})$, then by the above inequality, we see that $h_k>h_{k'}$. Thus, one can choose a subsequence $\{k_i\}$ such that (i) $k_i<k_{i+1}$, (ii) $h_{k_{i+1}}\geqq h_{k_i}$, (iii) $h_{k'}<h_{k_i}$ for any $k_i<k'<k_{i+1}$ and (iv) $f(q_{k_{i+1}})\geqq f(q_{k_i})$.

Since

$$(k_{i+1}b-f(q_{k_{i+1}}))^2+h_{k_{i+1}}^2\leqq(k_{i+1}b-f(q_{k_i}))^2+h_{k_i}^2$$

we see easily that

$$h_{k_{i+1}}^2-h_{k_i}^2\leqq 2k_{i+1}b(f(q_{k_{i+1}})-f(q_{k_i})) .$$

It follows that

$$\sum_{i=1}^{\infty}\frac{h_{k_{i+1}}^2-h_{k_i}^2}{k_{i+1}}\leqq 2b\sum_{i=1}^{\infty}(f(q_{k_{i+1}})-f(q_{k_i}))\leqq 2b^2 .$$

Thus,

$$\sum_{i=1}^{\infty}\frac{1}{k_{i+1}}(h_{k_{i+1}}^2-h_{k_i}^2)$$

is absolutely convergent.

LEMMA 9. *Notations being as above,* $\sum_{k=1}^{\infty}\frac{1}{k(k+1)}h_k^2$ *is absolutely convergent.*

PROOF. Since $h_{k'}^2 \leqq h_{k_i}^2$ for any $k_i \leqq k' < k_{i+1}$, one obtains that

$$\sum_{i=1}^{\infty}\frac{1}{k_{i+1}}(h_{k_{i+1}}^2-h_{k_i}^2)-\frac{1}{k_1}h_{k_1}=\sum_{i=1}^{\infty}\left(\frac{1}{k_i}-\frac{1}{k_{i+1}}\right)h_{k_i}^2$$

$$=\sum_{i=1}^{\infty}\left(\sum_{k=k_i}^{k_{i+1}-1}\frac{1}{k(k+1)}\right)h_{k_i}^2 \geqq \sum_{k=1}^{\infty}\frac{1}{k(k+1)}h_k^2 .$$

It follows that $\sum_{k=1}^{\infty}\frac{1}{k(k+1)}h_k^2$ is absolutely convergent.

LEMMA 10. *For any* $k>1$, *there is* $k'>k$ *such that* $h_{k'}^2 \leqq \frac{k'+1}{\log k'}$.

PROOF. Assume there is k_0 such that $h_k^2 > \frac{k+1}{\log k}$ for any $k \geqq k_0$. Then, $\sum_{k=1}^{\infty}\frac{h_k^2}{k(k+1)} > \sum_{k=k_0}^{\infty}\frac{1}{k\log k}=\infty$.

PROOF OF THEOREM A. It is easy to show that $\|\mathrm{grad}\, f(q_k)\| \leqq \frac{h_k}{(k-1)b}$. From inequality (3)′,

$$(\nabla\nabla f)(q_k)(X, X) \leqq \frac{1}{(k-1)b}\{K_0 h_k^2-(aK_0+m)h_k+1\}+\frac{h_k^2}{(k-1)^3 b^3},$$

for sufficiently large k.

From Lemma 10 above, there exists a subsequence $\{k_j\}$ such that $h_{k_j}^2 \leqq (k_j+1)/\log k_j$. It follows that

$$\lim_{j\to\infty}\frac{h_{k_j}}{(k_j-1)b} \leqq \lim_{j\to\infty}\frac{k_j+1}{(k_j-1)b\log k_j}=0$$

$$\overline{\lim_{j\to\infty}}\,(\nabla\nabla f)(q_{k_j})(X, X) \leqq \lim_{j\to\infty}\frac{(k_j+1)K_0}{(k_j-1)b\log k_j}+\lim_{j\to\infty}\frac{k_j+1}{(k_j-1)^3 b^3\log k_j}=0 .$$

This completes the proof of Theorem A.

From this proof, one can see more precisely the following:

THEOREM A′. *Let* f *be a* C^2*-function on a complete and connected Riemannian manifold having an upper bound. Assume that the sectional curvature has a lower bound. Then, for an arbitrarily fixed point* p *and for any* $\varepsilon>0$, *there exists a point* q *depending on* p *such that* (i) $\|\mathrm{grad}\, f(q)\|<\varepsilon$, (ii) $(\nabla\nabla f)(q)(X, X)<\varepsilon$ *for* $\|X\|=1$ *and* (iii) $f(q) \geqq f(p)$.

3. Isometric immersions of a complete and connected Riemannian manifold.

As an application of Theorem A or A$'$, one considers an isometric immersion φ of a complete and connected Riemannian manifold M into Euclidean space R^N with natural Euclidean metric. In this section, Theorem B will be proved.

Assume there exists a unit vector n at the origin of R^N such that $\langle\varphi(x), n\rangle/\|\varphi(x)\| \geqq a'$ for a fixed $a'>0$. Let R^{N-1} be the subspace of R^N which is orthogonal to n. Set $p=p(x)=\varphi(x)$ and denote by p' the R^{N-1}-component of p.

Assume without loss of generality that $\langle p, n\rangle^2 - a'^2\langle p', p'\rangle \geqq 1$ for every $p=p(x)$, $x\in M$. For a positive a, $a'>a>0$, set

$$f_a(x) = -\langle p(x), n\rangle + \sqrt{a^2\langle p'(x), p'(x)\rangle + 1}\,.$$

The meaning of this function is clear, if one changes the equality of the definition to $(\langle p, n\rangle + f_a(x))^2 - a^2\langle p', p'\rangle = 1$. Let $p_0 = p(x_0)$ for some fixed point $x_0\in M$. It is easy to see that $\{p(x)\,;\,f_a(x)\geqq f_a(x_0)\}$ is contained in a compact

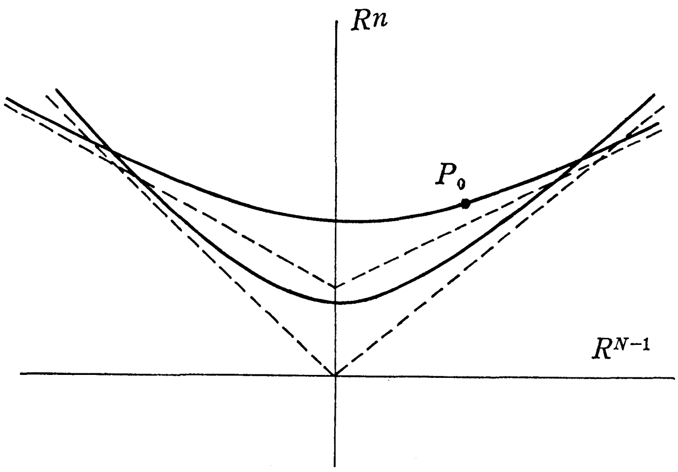

Fig. 2.

subset K for any a, $0<a<a'-\delta'$. Thus, there is a real number a such that $a^2\langle p(x), p(x)\rangle \leqq \frac{1}{2}$ for any x satisfying $f_a(x)\geqq f_a(x_0)$. Choose such a number a and fix it throughout the remainder of this paper. Put $f(x)=f_a(x)-f_a(x_0)$. Clearly, $f(x)$ has an upper bound and $f(x_0)=0$.

By the definition of $f(x)$, one has directly that

$$\nabla_i\nabla_j f=-\langle p_{i,j}, n\rangle+\frac{a^2(\langle p_{i,j}, p\rangle-\langle p, n\rangle\langle p_{i,j}, n\rangle)}{\sqrt{a^2(\langle p, p\rangle-\langle p, n\rangle^2)+1}}$$

$$+\frac{a^4}{\sqrt{a^2(\langle p, p\rangle-\langle p, n\rangle^2)+1}}\Big\{(\langle p_i, p_j\rangle-\langle p_i, n\rangle\langle p_j, n\rangle)\Big(\langle p, p\rangle-\langle p, n\rangle^2+\frac{1}{a^2}\Big)$$

$$-(\langle p_i, p\rangle-\langle p, n\rangle\langle p_i, n\rangle)(\langle p_j, p\rangle-\langle p, n\rangle\langle p_j, n\rangle)\Big\}$$

where $p_i=\nabla_i p$, $p_{i,j}=\nabla_j\nabla_i p$ for some coordinate neighborhood of M. Let $\hat{H}_{ij}$ be the first and second terms of $\nabla_i\nabla_j f$ and R_{ij} the third term. Let $\underset{\xi}{N}$ be the normal vectors of M in R^N, where $\xi=1, 2, \cdots, N-n$.

Since $p_{i,j}=\sum_{\xi}\underset{\xi}{H}_{ij}\underset{\xi}{N}$, we see easily that $\hat{H}(X, X)=\hat{H}_{ij}X^iX^j=\sum_{\xi}\underset{\xi}{H}(X, X)\langle\underset{\xi}{N}, \hat{m}\rangle$, where

$$\hat{m}=\frac{a^2p+a^2\langle p, n\rangle n}{\sqrt{a^2\langle p', p'\rangle+1}}-n\,.$$

For X, $\|X\|=1$, we have that

$$R(X, X)=\frac{a^4}{\sqrt{a^2\|p'\|^2+1}}(1-\langle X, n\rangle^2)\Big(\langle p, p\rangle-\langle p, n\rangle^2+\frac{1}{a^2}\Big)$$

$$-(\langle X, p\rangle-\langle p, n\rangle\langle X, n\rangle)^2\,,$$

where $R(X, X)=R_{ij}X^iX^j$.

By the definition of $f(x)$ and Theorem A′, there exists a sequence $\{x_n\}$ in M such that $f(x_n)\geqq 0$ and

$$\lim_{n\to\infty}\|\operatorname{grad} f(x_n)\|=\lim_{n\to\infty,\|X\|=1}\max\Big\{-\langle X, n\rangle+\frac{a^2\langle X, p\rangle-\langle p, n\rangle\langle X, n\rangle}{\sqrt{a^2\|p'(x_n)\|^2+1}}\Big\}=0 \quad ..(5)$$

$$\lim_{n\to\infty}(\nabla\nabla f)(x_n)(X, X)=\lim_{n\to\infty}\sum_{\xi}\underset{\xi}{H}_{x_n}(X, X)\langle\underset{\xi}{N}, \hat{m}\rangle+\lim_{n\to\infty}R_{x_n}(X, X)=0\,, \quad ....(6)$$

for $X\in T_{x_n}(M)$ and $\|X\|=1$, where the tangent space $T_{x_n}(M)$ at x_n is identified with the subspace of R^N by the immersion φ.

Thus, we have only to show that $R_{x_n}(X, X)\geqq\delta'>0$ for sufficiently large n.

Let $\hat{X}$ be the R^{N-1}-component of X. Thus, $X=\lambda n+\hat{X}$, $\lambda=\langle X, n\rangle$ and then $\langle X, p\rangle-\langle p, n\rangle\langle X, n\rangle=\langle\hat{X}, p\rangle$. It follows that

$$\frac{1}{\|p\|^2}(1-\langle X, n\rangle^2)\Big(\langle p, p\rangle-\langle p, n\rangle^2+\frac{1}{a^2}\Big)-(\langle X, p\rangle-\langle p, n\rangle\langle X, n\rangle)^2$$

$$=(1-\lambda^2)\Big(1-\mu^2+\frac{1}{a^2\|p\|^2}\Big)-\langle\hat{X}, p_0\rangle^2\,,$$

where $\mu=\langle p, n\rangle/\|p\|$ and $p_0=p/\|p\|$. Since $\langle\hat{X}, p_0\rangle\leqq\|\hat{X}\|^2=1-\mu^2$, one obtains that

$$R(X, X)\geqq\frac{a^4\|p\|^2}{\sqrt{a^2\|p'\|^2+1}}(1-\lambda^2)\Big(\frac{1}{a^2\|p\|^2}-\mu^2\Big)\,,$$

for all $x \in M$. Since $f(x_n) \geqq 0$, we see that $a^2\|p(x_n)\|^2 < 1/2$. It follows that $1/a^2\|p\|^2 - \mu^2 \geqq \delta > 0$ for all x_n.

Assume that $1-\lambda^2$ is not larger than $\delta'' > 0$ for every x_n and for every element of $T_{x_n}(M)$ satisfying $\|X\|=1$. Then, there is a subsequence $\{x_{n'}\}$ of $\{x_n\}$ such that, for every $x_{n'}$, there exists $X_{n'} \in T_{x_{n'}}(M)$ and $1-\langle X_{n'}, n\rangle^2$ converges to 0 for $n' \to \infty$. Put $\lambda_{n'} = \langle X_{n'}, n\rangle$. Without loss of generality, we assume $\lambda_{n'} \leqq 0$. Since $X_{n'}$ converges to n, one obtains that $\lim \|\mathrm{grad} f(x_{n'})\| = 1$, contradicting the equality (5). Thus, there exists an $\varepsilon > 0$ such that $1-\lambda^2 \geqq \varepsilon > 0$ for sufficiently large n and for all $X \in T_{x_n}(M)$ satisfying $\|X\|=1$. It follows that

$$R_{x_n}(X, X) \geqq \frac{a^4\|p_n\|^2\delta\varepsilon}{\sqrt{a^2\|p(x_n)\|^2+1}} \quad \text{for a sufficiently large } n\,.$$

Since $\|p_n\|$ is bounded, $\dfrac{a^4\|p_n\|^2}{\sqrt{a^2\|p(x_n)\|^2-1}} \geqq C$ for some positive constant. Thus, we see that

$$\lim_{n\to\infty} (\nabla\nabla f)(x_n)(X, X) \geqq \lim_{n\to\infty} \sum_{\xi} \underset{\xi}{H}_{x_n}(X, X)\langle \underset{\xi}{N}, \hat{m}\rangle + C\delta\varepsilon\,.$$

It follows that $\sum_{\xi} \underset{\xi}{H}_{x_n}(X, X)\langle \underset{\xi}{N}, \hat{m}\rangle$ is negative definite for sufficiently large n, since $\sum_{\xi} \langle \underset{\xi}{N}, \hat{m}\rangle \underset{\xi}{H}_{x_n}$ is the second fundamental form with respect to the normal vector $\hat{N}$ at x_n with the coefficient $\langle \hat{N}, \underset{\xi}{N}\rangle = \langle \underset{\xi}{N}, \hat{m}\rangle$. This completes the proof of Theorem B.

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Reference

[1] T. Frankel, Manifolds with positive curvature, Pacific J. Math., 11 (1961), 165–174.