PACIFIC JOURNAL OF MATHEMATICS
Vol. 40, No. 1, 1972

ISOMETRIC IMMERSIONS OF SPACE FORMS IN SPACE FORMS

JOHN DOUGLAS MOORE

Let M be a connected n-dimensional space form isometrically immersed in a simply connected $(2n\text{-}1)$-dimensional space form of strictly larger curvature. If M is minimal, it is proven that it must be a piece of the flat Clifford torus in the $(2n\text{-}1)$-sphere. If M is complete and simply connected, it is proven that M possesses a global coordinate system whose coordinate vectors are unit-length asymptotic vectors.

Introduction. A well-known theorem of David Hilbert states that a complete two-dimensional riemannian manifold of constant negative curvature cannot be isometrically immersed in three-dimensional euclidean space [5], [7, p. 265]. There is reason to believe that the natural generalization of Hilbert's theorem to higher dimensions would be the following conjecture: A complete n-dimensional riemannian manifold of constant negative curvature cannot be isometrically immersed in E^{2n-1}. If completeness is strengthened to compactness the conjecture is known to be true by work of Chern, Kuiper, and Otsuki [6, vol. 2, p. 29].

The local problem of isometrically immersing a space form in a space form was studied by Élie Cartan [3]. He used his theory of exterior differential systems to show, among other things, that real analytic n-dimensional submanifolds of constant negative curvature in $(2n\text{-}1)$-dimensional euclidean space E^{2n-1} depend upon $n(n\text{-}1)$ functions of a single variable. Cartan also showed that no n-dimensional hyperbolic space form can be isometrically immersed in E^{2n-2}. To construct an explicit example, we choose nonzero real numbers a_i, $1 \leqq i \leqq n-1$, so that $\sum_i a_i^2 = 1$, and we define an immersion from

$$D = \{(y_1, y_2, \cdots, y_n) \in \boldsymbol{R}^n \mid y_n < 0\}$$

into E^{2n-1} with rectangular cartesian coordinates $x_1, x_2, \cdots, x_{2n-1}$ by the equations

$$\begin{aligned} x_{2i-1} &= a_i e^{y_n} \cos(y_i/a_i) , \\ x_{2i} &= a_i e^{y_n} \sin(y_i/a_i) , \qquad 1 \leqq i \leqq n-1, \\ x_{2n-1} &= \int_0^{y_n} (1 - e^{2u})^{1/2} du . \end{aligned}$$

We find that the submanifold metric on D is of constant negative curvature; however D is not complete in this metric.

In §3 of this paper we prove that one of the main steps in the proof of Hilbert's theorem, the construction of a global coordinate system whose coordinate vectors are unit-length asymptotic vectors, can be generalized to the n-dimensional context. Our treatment is based upon a theorem of Cartan, a proof of which is given in § 1. Section 2 is devoted to the local properties of space forms isometrically immersed in space forms, and includes a rigidity theorem for minimal submanifolds of constant curvature.

Unless otherwise stated all manifolds are connected and C^∞.

1. Exteriorly orthogonal symmetric bilinear forms. Let V be an n-dimensional real vector space and let $\Phi^1, \Phi^2, \cdots, \Phi^n$ be n symmetric bilinear forms on V. We say that $\Phi^1, \Phi^2, \cdots, \Phi^n$ are *exteriorly orthogonal* if

$$\sum_{\lambda=1}^{n} [\Phi^\lambda(X, Y)\Phi^\lambda(Z, W) - \Phi^\lambda(X, W)\Phi^\lambda(Z, Y)] = 0$$

for $X, Y, Z, W \in V$.

THEOREM 1. (Élie Cartan [3]). *Suppose that $\Phi^1, \Phi^2, \cdots, \Phi^n$ are n exteriorly orthogonal symmetric bilinear forms on an n-dimensional real vector space V with the following property: if X is a vector in V such that $\Phi^\lambda(X, Y) = 0$ for $1 \leqq \lambda \leqq n$ and for all $Y \in V$, then $X = 0$. Then there exists a real orthogonal matrix (a^λ_μ) and n linear functionals $\varphi^1, \varphi^2, \cdots, \varphi^n$ such that*

$$\Phi^\lambda = \sum_\mu a^\lambda_\mu \varphi^\mu \otimes \varphi^\mu , \qquad 1 \leqq \lambda \leqq n .$$

It follows that $\Phi^1, \Phi^2, \cdots, \Phi^n$ are simultaneously diagonalized with respect to the basis dual to $\{\varphi^1, \varphi^2, \cdots, \varphi^n\}$. Theorem 1 is trivial when $n = 1$ and when $n = 2$ it is a consequence of the following well-known fact: two symmetric bilinear forms, one of which is positive definite, can be simultaneously diagonalized.

We will find it convenient to regard Φ^λ as a linear transformation from V to the dual space V^* so that it induces a linear map

$$\Phi^\lambda \wedge \Phi^\lambda \colon V \wedge V \to V^* \wedge V^* .$$

Then $\Phi^\lambda \wedge \Phi^\lambda = 0$ if and only if $\Phi^\lambda = \pm \varphi^\lambda \otimes \varphi^\lambda$ for some linear functional φ^λ. We can now restate Theorem 1 as follows: Suppose that $\Phi^1, \Phi^2, \cdots, \Phi^n$ are linear transformations from an n-dimensional real vector space to its dual such that $[\Phi^\lambda(X)](Y) = [\Phi^\lambda(Y)](X)$. If

$$\bigcap_\lambda \ker(\Phi^\lambda) = (0) \quad \text{and} \quad \sum_\lambda \Phi^\lambda \wedge \Phi^\lambda = 0 ,$$

then there exists a real orthogonal matrix (a^λ_μ) such that if

$$\Psi^\lambda = \sum_\mu a^\lambda_\mu \Phi^\mu \,, \quad \text{then} \quad \Psi^\lambda \wedge \Psi^\lambda = 0 \quad \text{for} \quad 1 \leqq \lambda \leqq n \,.$$

The first step in the proof of Theorem 1 consists of showing that there exists a vector X in V such that $\Phi^1(X), \Phi^2(X), \cdots, \Phi^n(X)$ are linearly independent. We prove this by contradiction. If $X \in V$, let $U^*(X)$ be the subspace of V^* generated by $\{\Phi^\lambda(X)\colon 1 \leqq \lambda \leqq n\}$ and let p be the maximum dimension of $U^*(X)$ for $X \in V$. We assume that $p < n$. If M is a vector for which the maximum dimension p is attained, we can assume without loss of generality that

$$\Phi^1(M), \Phi^2(M), \cdots, \Phi^p(M)$$

are linearly independent, and $\Phi^{p+1}(M) = \cdots = \Phi^n(M) = 0$. If Y is any other vector in V, then

$$\sum_{\alpha=1}^{p} \Phi^\alpha(M) \wedge \Phi^\alpha(Y) = 0 \,,$$

so that by Cartan's lemma there exists a $p \times p$ symmetric matrix (c^α_β) such that

$$\Phi^\alpha(Y) = \sum_{\beta=1}^{p} c^\alpha_\beta \Phi^\beta(M) \,, \qquad 1 \leqq \alpha \leqq p \,. \tag{1}$$

If we let W^* be the subspace of V^* generated by

$$\{\Phi^\alpha(X)\colon X \in V,\ 1 \leqq \alpha \leqq p\} \,,$$

then (1) shows that W^* is exactly p-dimensional. Since $p < n$ there exists a nonzero vector Z in V which is annihilated by W^*. But by hypothesis there exists λ, $1 \leqq \lambda \leqq n$, and a vector $N \in V$ such that $\Phi^\lambda(Z, N) \neq 0$. Since Z is annihilated by W^*, $\lambda \geqq p + 1$. If $\varepsilon > 0$ is sufficiently small, $\{\Phi^\alpha[(\cos \varepsilon)M + (\sin \varepsilon)N]\ 1 \leqq \alpha \leqq p\}$ will generate W^* and $\Phi^\lambda[(\cos \varepsilon)M + (\sin \varepsilon)N]$ will be outside of W^*. Hence

$$U^*[(\cos \varepsilon)M + (\sin \varepsilon)N]$$

is at least $(p+1)$-dimensional; this contradicts the definition of p, and the first step is established.

Let $\{v_1, v_2, \cdots, v_n\}$ be a basis for V such that

$$\Phi^1(v_1), \Phi^2(v_1), \cdots, \Phi^n(v_1)$$

are linearly independent. Then we can apply Cartan's lemma to the equation

$$\sum_\lambda \Phi^\lambda(v_1) \wedge \Phi^\lambda(v_i) = 0$$

and conclude that there exists a symmetric matrix $C(i) = (c(i)^\lambda_\mu)$ such that

$$\Phi^\lambda(v_i) = \sum_\mu c(i)^\lambda_\mu \Phi^\mu(v_1) , \qquad 1 \leqq \lambda \leqq n .$$

(Notice that $C(1)$ is the identity matrix.) We next observe that it follows from the equation

$$\sum_\lambda \Phi^\lambda(v_i) \wedge \Phi^\lambda(v_j) = 0$$

that the matrices $C(i)$ and $C(j)$ commute with each other. By a well-known theorem from linear algebra there exists an orthogonal matrix $A = (a^\lambda_\mu)$ such that $A[C(i)][{}^tA]$ is diagonal for $1 \leqq i \leqq n$. If we let $\Psi^\lambda = \sum_\mu a^\lambda_\mu \Phi^\mu$ then $\Psi^\lambda(v_i)$ is a constant multiple of $\Psi^\lambda(v_1)$ for $1 \leqq i \leqq n$, so that

$$\Psi^\lambda(v_i) \wedge \Psi^\lambda(v_j) = 0 , \qquad 1 \leqq i, j, \lambda \leqq n .$$

It follows that $\Psi^\lambda \wedge \Psi^\lambda = 0$, $1 \leqq \lambda \leqq n$, and Theorem 1 is proven.

An examination of the above proof shows that $\Psi^1, \Psi^2, \cdots, \Psi^n$ are uniquely determined up to a permutation. Hence the linear functionals $\varphi^1, \varphi^2, \cdots, \varphi^n$ are uniquely determined up to changes of sign and a possible permutation.

2. Submanifolds of constant curvature: local theory. In the rest of this paper, our setup will be as follows: we will let M be an n-dimensional riemannian manifold of constant curvature k isometrically immersed in a $(2n-1)$-dimensional riemannian manifold N of constant curvature K. We will use the following conventions on ranges of indices:

$$1 \leqq i, j, k, l \leqq n , \quad n+1 \leqq \lambda, u \leqq 2n-1 , \quad 1 \leqq A, B, C \leqq 2n-1 .$$

Let $e_1, e_2, \cdots, e_{2n-1}$ be a moving oriented orthonormal frame on an open set U in N, chosen so that at points of a suitable open subset V of the submanifold M the first n frame vectors are tangent to M. Let $\theta^1, \theta^2, \cdots, \theta^{2n-1}$ be the dual orthonormal coframe. A fundamental theorem of riemannian geometry states that there exists a unique collection of 1-forms θ^A_B on U which satisfy the structure equation

$$(2) \qquad d\theta^A = -\sum_B \theta^A_B \wedge \theta^B, \; \theta^A_B = -\theta^B_A .$$

The fact that N has constant curvature K is expressed by the equation

$$(3) \qquad d\theta^A_B = -\sum_C \theta^A_C \wedge \theta^C_B + K\theta^A \wedge \theta^B .$$

If we restrict these equations to the open subset V of M and make use of the fact that $\theta^\lambda = 0$ on V, we obtain from (2) the equations

(4) $$d\theta^i = -\sum_k \theta^i_k \wedge \theta^k,\ 0 = -\sum_k \theta^\lambda_k \wedge \theta^k .$$

The second of these implies via Cartan's lemma that

(5) $$\theta^\lambda_i = \sum_j b^\lambda_{ij}\theta^j,\ b^\lambda_{ij} = b^\lambda_{ji} ,$$

where the b^λ_{ij}'s are differentiable functions on V called the components of the second fundamental forms. From equation (3) we obtain the equation

$$d\theta^i_j = -\sum_k \theta^i_k \wedge \theta^k_j - \sum_\lambda \theta^i_\lambda \wedge \theta^\lambda_j + K\theta^i \wedge \theta^j .$$

Since M is of constant curvature k

$$-\sum_\lambda \theta^i_\lambda \wedge \theta^\lambda_j = (k - K)\theta^i \wedge \theta^j ,$$

or equivalently

(6) $$\sum_\lambda (b^\lambda_{ij}b^\lambda_{kl} - b^\lambda_{il}b^\lambda_{kj}) = (k - K)(\delta_{ij}\delta_{kl} - \delta_{il}\delta_{kj}) ,$$

where δ_{ij} is the usual Kronecker delta.

Assume now that $k < K$. Equation (6) then states that the second fundamental forms $\Phi^\lambda = \sum_i \theta^\lambda_i \otimes \theta^i$ and the symmetric bilinear form

$$\Psi = \sqrt{K - k}\,(\sum_i \theta^i \otimes \theta^i)$$

are exteriorly orthogonal, and Theorem 1 implies that they can be simultaneously diagonalized by a basis for the tangent space to M. Since the basis diagonalizes Ψ it can be chosen to be orthonormal, and hence we can assume that the moving frame $e_1, e_2, \cdots, e_{2n-1}$ chosen in the preceding paragraphs satisfies the equations $b^\lambda_{ij} = 0$ for $i \neq j$. In view of the remark at the end of § 1, any two diagonalizing orthonormal bases differ at most by changes of sign and a possible permutation. Hence if M is simply connected we can choose a global moving frame $e_1, e_2, \cdots, e_n$ on M which diagonalizes the second fundamental forms. In particular, the universal covering space of M is parallelizable.

In terms of the diagonalizing moving frame, equation (6) takes the simpler form

(7) $$\sum_\lambda b^\lambda_{ii}b^\lambda_{jj} = (k - K) , \qquad i \neq j .$$

We claim that it follows from this equation that there exist unique positive functions $x_1, x_2, \cdots, x_n$ such that

(8) $$\sum_i b^\lambda_{ii}x_i^2 = 0 \text{ and } \sum_i x_i^2 = 1 .$$

Indeed, such functions need to satisfy the equation

$$0 = \sum_{i,\lambda} b^\lambda_{jj} b^\lambda_{ii} x_i^2 = \sum_\lambda b^\lambda_{jj} b^\lambda_{jj} x_j^2 - (K - k)(1 - x_j^2) ,$$

from which it follows that

$$\sum_\lambda b^\lambda_{ii} b^\lambda_{ii} = (K - k)(1 - x_i^2)/x_i^2 . \tag{9}$$

We can solve for x_i to obtain the expression

$$x_i = \left[\left(\sum_\lambda b^\lambda_{ii} b^\lambda_{ii}\right)\Big/(K - k) + 1\right]^{-1/2} , \tag{10}$$

and check that the functions defined by this equation satisfy equations (8). A slight modification of this argument shows that any $n - 1$ of the "principal normal curvature vectors" $\sum_\lambda b^\lambda_{ii} e_\lambda$ are linearly independent.

A restatement of what we proved in the preceding paragraph is that there exist exactly 2^n unit-length vectors on which all the second fundamental forms vanish simultaneously. They are all of the form

$$\pm x_1 e_1 \pm x_2 e_2 \pm \cdots \pm x_n e_n , \tag{11}$$

where the signs can be chosen in 2^n ways, and they are called *asymptotic vectors*.

We remark that the normal bundle of M in N has zero curvature because the curvature forms of the normal bundle are $-\sum_i \theta^\lambda_i \wedge \theta^i_\mu$ and both θ^λ_i and θ^i_μ are multiples of θ^i. Hence without loss of generality we will assume that $e_{n+1}, \cdots, e_{2n-1}$ have been chosen so that $\theta^\lambda_\mu = 0$.

Our next objective is to find an expression for the differential 1-forms θ^i_j in terms of the functions x_i. For this purpose we will use the tensor b^λ_{ijk} defined by the following equation

$$db^\lambda_{ij} + \sum_\mu b^\mu_{ij} \theta^\lambda_\mu - \sum_k b^\lambda_{kj} \theta^k_i - \sum_k b^\lambda_{ik} \theta^k_j = \sum_k b^\lambda_{ijk} \theta^k . \tag{12}$$

The exterior derivative of equation (5) shows that the tensor b^λ_{ijk} is symmetric in its lower indices. If we make use of the facts that $b^\lambda_{ij} = 0$ for $i \neq j$ and $\theta^\lambda_\mu = 0$, we can simplify (12) and obtain the equations

$$db^\lambda_{ii} = \sum_k b^\lambda_{iik} \theta^k , \tag{13}$$

$$(b^\lambda_{jj} - b^\lambda_{ii}) \theta^i_j = \sum_k b^\lambda_{ijk} \theta^k , \qquad i \neq j . \tag{14}$$

If we choose e_{n+1} at a point $x \in M$ so that $b^{n+2}_{11}(x) = \cdots = b^{2n-1}_{11}(x) = 0$, then it follows from equation (7) that

$$b_{ii}^{n+1}(x) = (k - K)/b_{11}^{n+1}(x) .$$

Equation (14) therefore implies that $b_{ijk}^{n+1}(x) = 0$ for $i, j, 1$ distinct. It follows that $b_{ijk}^{n+1}(x) = 0$ for i, j, k distinct, and since the principal normal curvature vectors span the normal space, $b_{ijk}^{\lambda}(x) = 0$ for i, j, k distinct. Since x is arbitrary, equation (14) now becomes

$$(b_{jj}^{\lambda} - b_{ii}^{\lambda})\theta_j^i = b_{iji}^{\lambda}\theta^i + b_{ijj}^{\lambda}\theta^j , \qquad i \neq j .$$

We multiply this last equation by b_{ii}^{λ} and sum with respect to λ to conclude that

$$(k - K - \sum_{\lambda} b_{ii}^{\lambda}b_{ii}^{\lambda})\theta_j^i = \sum_{\lambda} b_{ii}^{\lambda}b_{iij}^{\lambda}\theta^i + \sum_{\lambda} b_{ii}^{\lambda}b_{jji}^{\lambda}\theta^j , \qquad i \neq j .$$

We now need to use the following fact which is a consequence of (9):

$$2 \sum_{\lambda} b_{ii}^{\lambda}b_{iij}^{\lambda} = (K - k)e_j[(1 - x_i^2)/x_i^2] . \tag{15}$$

We can use this to derive the following equation for the 1-forms θ_j^i:

$$\theta_j^i = (1/x_i)e_j(x_i)\theta^i + (\text{something})\theta^j .$$

Using skew-symmetry we conclude that

$$\theta_j^i = (1/x_i)e_j(x_i)\theta^i - (1/x_j)e_i(x_j)\theta^j . \tag{16}$$

As an application of these ideas we prove the following theorem closely related to recent work of do Carmo and Wallach [2]:

THEOREM 2. *Let M be a connected n-dimensional riemannian manifold of constant curvature k isometrically and minimally immersed in a simply connected $(2n - 1)$-dimensional riemannian manifold N of constant curvature K. Then either M is totally geodesic or it is flat. In the flat case it is immersed as a piece of the n-dimensional Clifford torus in the $(2n - 1)$-sphere.*

The proof is local. The fact that the immersion is minimal is expressed by the equation

$$\sum_{i} b_{ii}^{\lambda} = 0 \tag{17}$$

which together with equation (6) implies that

$$\sum_{i,\lambda} b_{ij}^{\lambda}b_{ik}^{\lambda} = (n - 1)(K - k)\delta_{jk} .$$

Hence $k \leqq K$ and if $k = K$ then the submanifold M is totally geodesic. Therefore we assume without loss of generality that $k < K$.

In the case where $k < K$ we will actually prove a little more than the theorem states: if the hypothesis that M be minimal is replaced by the weaker condition that its mean curvature vector be parallel, it still follows that M is flat.

Since the normal moving frame vectors are parallel, the mean curvature vector is parallel if and only if there exist constants c^λ such that

$$\sum_i b_{ii}^\lambda = c^\lambda .$$

On the other hand, equations (13) and (7) imply that

$$\sum_i b_{iij}^\lambda = 0, \text{ and } \sum_\lambda b_{iij}^\lambda b_{kk}^\lambda = -\sum_\lambda b_{ii}^\lambda b_{kkj}^\lambda \text{ if } i \neq k .$$

Hence we conclude that

$$\sum_{i,\lambda} b_{iij}^\lambda b_{ii}^\lambda = -\sum_{\substack{\lambda \\ k\neq i}} b_{kkj}^\lambda b_{ii}^\lambda = \sum_{\substack{\lambda \\ k\neq i}} b_{kk}^\lambda b_{iij}^\lambda$$
$$= \sum_{i,\lambda} c^\lambda b_{iij}^\lambda - \sum_{i,\lambda} b_{ii}^\lambda b_{iij}^\lambda = -\sum_{i,\lambda} b_{ii}^\lambda b_{iij}^\lambda .$$

It follows that $\sum_{i,\lambda} b_{iij}^\lambda b_{ii}^\lambda = 0$, and hence equation (15) implies that $e_j(x_i) = 0$. Now by equation (16) the differential forms θ_j^i vanish, proving that M is flat.

To finish the proof of the theorem, we notice that if M is minimal the principal normal curvature vectors (i.e., the b_{ii}^λ's) are determined up to a rotation of $e_{n+1}, \cdots, e_{2n-1}$ by equations (7) and (17). Since the b_{ii}^λ's determine the θ_i^λ's and $\theta_j^i = 0 = \theta_\mu^\lambda$, it follows from the classical rigidity theorem [1, p. 202] that locally there is at most one minimal flat n-dimensional submanifold of N, up to a rigid motion. Therefore M must be a piece of the Clifford torus, and the theorem is proven.

3. The global existence of asymptotic coordinates.

If M is complete and simply connected, then any choice of signs in expression (11) determines a globally defined unit-length asymptotic vector field on M. If n unit-length asymptotic vector fields are linearly independent at one point, they are linearly independent everywhere.

THEOREM 3. *If M is a complete simply connected riemannian manifold of constant curvature k isometrically immersed in a $(2n-1)$-dimensional riemannian manifold N of constant curvature $K > k$, then any n linearly independent unit-length asymptotic vector fields $Z_1, Z_2, \cdots, Z_n$ determine a global coordinate system whose coordinate vectors are the Z_i's.*

First we establish local existence. Because of the theorem of Frobenius, it suffices to show that the Lie bracket of any two asymptotic vector fields is zero. But

$$\begin{aligned}\theta^i([x_je_j, x_ke_k]) &= x_je_j(\theta^i(x_ke_k)) - x_ke_k(\theta^i(x_je_j)) - 2d\theta^i(x_je_j, x_ke_k)\\ &= x_je_j(\theta^i(x_ke_k)) - x_ke_k(\theta^i(x_je_j)) + 2\sum_l \theta^i_l \wedge \theta^l(x_je_j, x_ke_k)\\ &= \delta_{ik}x_je_j(x_k) - \delta_{ij}x_ke_k(x_j) + \delta_{ij}x_ke_k(x_j) - \delta_{ik}x_je_j(x_k)\\ &= 0 .\end{aligned}$$

In this derivation we have used equations (4) and (16). Since the asymptotic vectors are sums of $\pm x_ie_i$, local existence is proven.

To prove global existence, we let $\varphi_i(x, t)$, $x \in M$, $t \in \boldsymbol{R}$ be the one-parameter group of transformations corresponding to Z_i. Since Z_i is a vector field of unit length, it follows from the theory of ordinary differential equations [4, p. 15] that $\varphi_i(x, t)$ is defined for all values of x and t. Let x_0 be a fixed point in M and define a function F: $\boldsymbol{R}^n \rightarrow M$ by

$$F(t_1, t_2, \cdots, t_n) = \varphi_n(\varphi_{n-1}(\cdots\varphi_2(\varphi_1(x_0, t_1), t_2), \cdots), t_n) .$$

Since the Lie bracket $[Z_i, Z_j]$ vanishes, the one-parameter groups φ_i and φ_j commute. Using this fact we can verify the following equation:

$$(18)\quad F(s_1 + t_1, \cdots, s_n + t_n) = \varphi_n(\varphi_{n-1}(\cdots\varphi_1(F(s_1, \cdots, s_n), t_1), \cdots), t_n) .$$

We claim that F is a covering map. Let x be a point in the manifold M and let U_x be an open neighborhood of x on which local asymptotic coordinates $z_1, z_2, \cdots, z_n$ exist, and we can assume that $z_1(x) = z_2(x) = \cdots = z_n(x) = 0$. For $\delta > 0$, let

$$B_\delta(x) = \{y \in U_x\colon |z_i(y)| < \delta\}$$

and choose ε so small that $(z_1, z_2, \cdots, z_n)$ give a diffeomorphism from $B_{2\varepsilon}(x)$ onto an open ball of radius 2ε in $\boldsymbol{R}^n$. Let $\tilde{x}_\alpha$, $\alpha \in A$, be the points in $F^{-1}(x)$, and let $B_\delta(\tilde{x}_\alpha)$ denote the open ball of radius δ around x_α. To show that F is a covering map, it suffices to check the following facts:

1. $F \mid B_{2\varepsilon}(\tilde{x}_\alpha)$ is a diffeomorphism from $B_{2\varepsilon}(\tilde{x}_\alpha)$ onto $B_{2\varepsilon}(x)$ for $\alpha \in A$.
2. $B_\varepsilon(\tilde{x}_\alpha) \cap B_\varepsilon(\tilde{x}_\beta) = \phi$ if $\tilde{x}_\alpha \neq \tilde{x}_\beta$.
3. $\tilde{y} \in F^{-1}(B_\varepsilon(x)) \Rightarrow \tilde{y} \in B_\varepsilon(\tilde{x}_\alpha)$ for some $\alpha \in A$.

To prove 1, we need only check that the local asymptotic coordinates define an inverse to $F|B_{2\varepsilon}(\tilde{x}_\alpha)$ using equation (18). 2 follows from 1, and 3 follows from the fact that $\tilde{y} - (z_1(F(\tilde{y})), \cdots, z_n(F(\tilde{y})))$ goes to x under F.

Thus F is a covering map, and since M is simply connected it is a diffeomorphism. Therefore F defines a global coordinate system whose coordinate vectors are the Z_i's and Theorem 3 is proven.

A straightforward modification of the above proof establishes the existence of "principal coordinates" whose coordinate vectors are $x_1e_1, x_2e_2, \cdots, x_ne_n$.

Since $\boldsymbol{R}^n$ is not a covering space for the n-sphere when $n > 1$, we obtain the positive curvature analogue of our conjecture:

COROLLARY. *A complete n-dimensional riemannian manifold of constant positive curvature k cannot be isometrically immersed in a* $(2n - 1)$*-sphere of constant curvature* $K > k$.

The corresponding local assertion is false, as Cartan proved in [3]. An n-sphere of constant curvature can be isometrically immersed in a $(2n + 1)$-sphere of constant curvature by first embedding it in E^{n+1} in the usual fashion, and then immersing E^{n+1} in the $(2n + 1)$-sphere as a flat torus.

If M is a complete simply connected space form as in Theorem 3, we will use the term "asymptotic surface" to denote a complete two-dimensional submanifold generated by two unit-length asymptotic vector fields. Every asymptotic surface possesses a global Tchebychef net ([7], p. 198) and it follows from the formula of Hazzidakkis that the integral of the Gaussian curvature over any parallelogram of the Tchebychef net is bounded in absolute value by 2π.

REFERENCES

1. R. L. Bishop and R. J. Crittenden, *Geometry of manifolds*, Academic Press, New York, 1964.
2. M. P. do Carmo and N. J. Wallach, *Minimal immersions of spheres into spheres*, Ann. of Math., **93** (1971), 43-62.
3. É. Cartan, *Sur les variétés de courbure constante d'un espace euclidien ou non-euclidien*, Bull. Soc. Math. France, **47** (1919), 125-160 and **48** (1920), 132-208.
4. E. Coddington and N. Levinson, *Theory of ordinary differential equations*, McGraw-Hill, New York, 1955.
5. D. Hilbert, *Über Flächen von konstanter Gausscher Krümmung*, Trans. Amer. Math. Soc., **2** (1901), 87-99.
6. S. Kobayashi and K. Nomizu, *Foundations of differential geometry*, 2 vols., Interscience, New York, 1969.
7. J. J. Stoker, *Differential Geometry*, Wiley-Interscience, New York, 1969.

Received October 30, 1970. This paper is a revised version of part of the author's dissertation written under the direction of Professor Kobayashi at the University of California, Berkeley. The author wishes to express his gratitude to Professors do Carmo and Kobayashi for their encouragement and helpful suggestions.

UNIVERSITY OF CALIFORNIA, SANTA BARBARA