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ISOMETRIC ISOMORPHISMS BETWEEN BANACH ALGEBRAS RELATED TO LOCALLY COMPACT GROUPS

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ABSTRACT. Let G_1, G_2 be locally compact groups. We prove in this paper that if T is an isometric isomorphism from the Banach algebra $\mathrm{LUC}(G_1)^*$ (the continuous dual of the Banach space of left uniformly continuous functions on G_1, equipped with Arens multiplication) onto $\mathrm{LUC}(G_2)^*$, then T maps $M(G_1)$ onto $M(G_2)$ and $L^1(G_1)$ onto $L^1(G_2)$. We also prove that any isometric isomorphism from $L^1(G_1)^{**}$ (second conjugate algebra of $L^1(G_1)$) onto $L^1(G_2)^{**}$ maps $L^1(G_1)$ onto $L^1(G_2)$.

0. INTRODUCTION AND PRELIMINARIES

Let G_1, G_2 be locally compact groups. Let $M(G_i)$, $i = 1, 2$, be the Banach algebra of regular Borel measures on G_i. A well-known result of B. E. Johnson [10] asserts that if T is an isometric isomorphism from $M(G_1)$ onto $M(G_2)$, then T maps $L^1(G_1)$ onto $L^1(G_2)$ (and hence G_1 and G_2 must be isomorphic by Wendel's theorem [21]).

In this paper we prove (Theorem 3.1(c)), among other things, that if T is an isometric isomorphism from $L^1(G_1)^{**}$ onto $L^1(G_2)^{**}$, then T maps $L^1(G_1)$ onto $L^1(G_2)$. This answers affirmatively a question raised in [4]. Theorem 3.1(c) was proved for abelian locally compact groups by Lau and Losert in [13], and for compact and discrete groups by Ghahramani and Lau in [4].

Let G be a locally compact group. Let $C(G)$ denote the space of bounded continuous complex-valued functions on G with the sup norm topology, and $\mathrm{LUC}(G)$ denote the closed subspace of bounded left uniformly continuous functions on G, i.e. all $f \in C(G)$ such that the map $x \mapsto l_x f$ from G into $C(G)$ is continuous, where $(l_x f)(y) = f(xy)$, $x, y \in G$. Then $\mathrm{LUC}(G)^*$ is a Banach algebra with the Arens multiplication defined by $\langle nm, f\rangle = \langle n, m_l f\rangle$, $n, m \in \mathrm{LUC}(G)^*$, $f \in \mathrm{LUC}(G)$, where $m_l f(x) = \langle m, l_x f\rangle$, $x \in G$. Furthermore, $M(G)$ may be identified with a closed subspace of $\mathrm{LUC}(G)^*$ by the natural embedding $\langle \mu, f\rangle = \int f(x)\, d\mu(x)$, $f \in \mathrm{LUC}(G)$, $\mu \in M(G)$. It was

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shown by Grosser and Losert [7] that when G is abelian, $M(G)$ is precisely the centre of $\mathrm{LUC}(G)^*$ (see also Lau [12]).

The organization of this paper is as follows. We prove in §1 (Theorem 1.6) that if T is an isometric isomorphism from $\mathrm{LUC}(G_1)^*$ onto $\mathrm{LUC}(G_2)^*$, then T maps $M(G_1)$ onto $M(G_2)$ and $L^1(G_1)$ onto $L^1(G_2)$. In §2 we study the set $\Lambda(G)$ of right identities with norm one in $L^1(G)^{**}$ and the isometric embeddings Γ_E of $M(G)$ into $L^1(G)^{**}$ defined by S. McKilligan [16]. Finally we prove in §3 (and using results established in §§1 and 2) that, if T is an isometric isomorphism from $L^1(G_1)^{**}$ onto $L^1(G_2)^{**}$, then T maps $L^1(G_1)$ to $L^1(G_2)$.

Throughout the paper, G denotes a locally compact group with a fixed left Haar measure λ. Integration with respect to λ will be denoted by $\int \cdots dx$. The spaces $L^1(G)$ $(= L^1(G, \lambda))$ and $L^\infty(G)$ $(= L^\infty(G, \lambda))$ are as defined in [8]. If f and g are measurable functions on G, then

$$(f * g)(x) = \int_G f(y) g(y^{-1}x)\, dy,$$

whenever this makes sense. If f is any function defined on G, then for $x \in G$, the right (resp. left) translate of f by x will be denoted by $r_x f$ (resp. $l_x f$). We denote by $C_{00}(G)$ the functions in $C(G)$ with compact support and by $C_0(G)$ the functions in $C(G)$ which vanish at infinity.

We recall the definition for the (first) Arens product [1] (see also [3]) in the second conjugate of $L^1(G)$: for $f \in L^\infty(G)$ and $\varphi \in L^1(G)$ let $f\varphi \in L^\infty(G)$ be defined by

$$\langle f\varphi, \psi\rangle = \langle f, \varphi * \psi\rangle \qquad (\psi \in L^1(G)).$$

For $m \in L^1(G)^{**}$, let $mf \in L^\infty(G)$ be defined by $\langle mf, \varphi\rangle = \langle m, f\varphi\rangle$. Finally for $m, n \in L^1(G)^{**}$, let $nm \in L^1(G)^{**}$ be defined by $\langle nm, f\rangle = \langle n, mf\rangle$. It is easy to see that for $f \in L^\infty(G)$ and $\varphi \in L^1(G)$, $f\varphi = \tilde{\varphi} * f$, where $\tilde{\varphi}(x) = \Delta(x^{-1})\varphi(x^{-1})$ and Δ denotes the modular function of the group [22]. Also if $f \in \mathrm{LUC}(G)$, $m \in L^1(G)^{**}$, then $mf \in \mathrm{LUC}(G)$ and $(mf)(x) = m_l(f)(x) = \langle m, l_x f\rangle$, $x \in G$. (See [11, Lemma 3].)

A closed linear subspace X of $C(G)$ is left introverted (see Day [2, p. 540]) if $l_a(X) \subseteq X$ for each $a \in G$, and for each $m \in X^*$, $f \in X$, the function $m_l(f)$ on G defined by $m_l(f)(x) = m(l_x f)$, $x \in G$, is also in X. In this case the Arens multiplication on X^* defined by $\langle nm, f\rangle = \langle n, m_l(f)\rangle$ for each $f \in X$, $n, m \in X^*$ makes sense. Furthermore, X^* with this multiplication is a Banach algebra (see [2, §6]). Examples of left introverted subspaces of $C(G)$ include $C_0(G)$, $\mathrm{LUC}(G)$, and the space of almost periodic (resp. weakly almost periodic) functions on G. In the case of $C_0(G)^* = M(G)$, the multiplication on $M(G)$ is precisely the convolution of measures as defined in [8, p. 266]. Furthermore, $\mathrm{LUC}(G)$ is the maximal left introverted subspace of $C(G)$ [17 and 18].

1. Isometric isomorphisms on $\mathrm{LUC}(G)^*$

Let $C_0(G)^{\perp} = \{m \in \mathrm{LUC}(G)^*;\ m(f) = 0 \text{ for all } f \in C_0(G)\}$.

1.1. **Lemma.** *$\mathrm{LUC}(G)^* = C_0(G)^{\perp} \oplus M(G)$. If $m \in \mathrm{LUC}(G)^*$, and $m = m_1 + \mu$ where $m_1 \in C_0(G)^{\perp}$, $\mu \in M(G)$, then $\|m\| = \|m_1\| + \|\mu\|$. Furthermore, $C_0(G)^{\perp}$ is a closed ideal in $\mathrm{LUC}(G)^*$.*

Proof. Clearly $C_0(G)^{\perp} \cap M(G) = \{0\}$. If $m \in \mathrm{LUC}(G)^*$, let μ denote the restriction of m to $C_0(G)$. Let μ also denote the corresponding extension of μ to $\mathrm{LUC}(G)$. Then $m_1 = m - \mu \in C_0(G)^{\perp}$ and $m = m_1 + \mu$. To see that $\|m\| = \|m_1\| + \|\mu\|$, let $\varepsilon > 0$; choose $h \in C_{00}(G)$ such that $\|h\| \le 1$ and $\mu(h) \ge \|\mu\| - \varepsilon$. Let F be a compact set such that $h(x) = 0$ for all $x \notin F$. Let V be an open set with compact closure such that $V \supseteq F$. Let $0 \le g \le 1$ such that $g \equiv 1$ on F and $g(x) = 0$ for all $x \notin V$. Let $k \in \mathrm{LUC}(G)$ such that $\|k\| \le 1$ and $m_1(k) \ge \|m_1\| - \varepsilon$. Define $k' = k - gk + h$. Then $m_1(k') = m_1(k)$ and $\|k'\| \le 1$. Furthermore, $\|\alpha(k - gk) + h\| \le 1$ for any $\alpha \in \mathbf{C}$ with $|\alpha| = 1$. By a proper choice of α, one gets

$$\begin{aligned}\|\mu\| &\ge |\mu(\alpha(k - gk) + h)| = |\mu(k - gk)| + |\mu(h)| \\ &\ge |\mu(k - gk)| + \|\mu\| - \varepsilon.\end{aligned}$$

Hence $|\mu(k - gk)| \le \varepsilon$ and

$$|m(k')| \ge m_1(k') + \mu(h) - |\mu(k - gk)| \ge \|m_1\| + \|\mu\| - 3\varepsilon.$$

So $\|m\| \ge \|m_1\| + \|\mu\|$.

To see that $C_0(G)^{\perp}$ is an ideal, let $h \in C_0(G)$, $\varphi \in L^1(G)$. Then $h\varphi = \tilde{\varphi} * h \in C_0(G)$. Hence if $n \in C_0(G)^{\perp}$, it follows that $\langle nh, \varphi\rangle = \langle n, h\varphi\rangle = 0$, i.e. $nh = 0$. Consequently $mn \in C_0(G)^{\perp}$ for all $m \in \mathrm{LUC}(G)^*$, i.e. $C_0(G)^{\perp}$ is a left ideal in $\mathrm{LUC}(G)^*$.

If $\mu \in M(G)$, then it is easy to see that $\mu h = h * \mu^*$ (where

$$\int f(t)\, d\mu^*(t) = \int \Delta(t^{-1}) f(t^{-1})\, d\mu(t) \qquad (f \in C_0(G))).$$

In particular $\mu h \in C_0(G)$ for $h \in C_0(G)$. Hence $n \in C_0(G)^{\perp}$ implies $n\mu \in C_0(G)^{\perp}$ (since $\langle n\mu, h\rangle = \langle n, \mu h\rangle$). Now if $m \in \mathrm{LUC}(G)^*$ is arbitrary, it can be written as $m = \mu + m_1$ with $\mu \in M(G)$, $m_1 \in C_0(G)^{\perp}$. If as above $n \in C_0(G)^{\perp}$, then $nm_1 \in C_0(G)^{\perp}$, so $nm = n\mu + nm_1 \in C_0(G)^{\perp}$. Thus, $C_0(G)^{\perp}$ is a right ideal. That completes the proof of the lemma. □

1.2. **Corollary.** *Let $m \in \mathrm{LUC}(G)^*$. Then, the following are equivalent.*

(a) *m is invertible and $\|m\| = \|m^{-1}\| = 1$.*

(b) *There exists $x \in G$, $\alpha \in \mathbf{C}$ with $|\alpha| = 1$ such that $m = \alpha\delta_x$.*

Proof. That (b) $\Rightarrow$ (a) is clear. To prove (a) $\Rightarrow$ (b) write $m = \mu + m_1$, $m^{-1} = \nu + m_2$ with $\mu, \nu \in M(G)$, $m_1, m_2 \in C_0(G)^{\perp}$. Then $\delta_e = \mu * \nu + (\mu m_2 + m_1\nu + m_1 m_2)$ and the part in brackets belongs to $C_0(G)^{\perp}$, by Lemma

1.1. Hence $\|\mu * \nu\| = \|\mu\| = \|\nu\| = 1$, $m_1 = m_2 = 0$ (again by Lemma 1.1). If $h \in C_0(G)$ satisfies $0 \le h \le 1$ and $h(e) = 1$, then $1 = \langle \delta_e, h\rangle = \langle \mu, \nu h\rangle$. Since $0 \le |\nu h| \le 1$, we conclude that $|\nu h(t)| = 1$ for all $t \in \operatorname{supp}\mu$. Since $\nu h(t) = \int h(ts)\, d\nu(s)$, it follows that $h(ts) = 1$ for all $t \in \operatorname{supp}\mu$, $s \in \operatorname{supp}\nu$. From this it follows that $(\operatorname{supp}\mu)(\operatorname{supp}\nu) = \{e\}$. In particular supp μ consists of a single point, i.e. $\mu = \alpha\delta_x$ for some $x \in G$, $\alpha \in \mathbf{C}$, $|\alpha| = 1$. □

1.3. *Remark.* (a) Note that Corollary 1.2 may also be obtained as a consequence of Lemma 2 in [14].

(b) Let X be a left introverted subspace of $C(G)$ containing $C_0(G)$. Then $M(G)$ may also be regarded as a closed subspace of X^* by the isometric embedding: $\rho: M(G) \to X^*$, where $\rho(\mu)(f) = \int f(x)\, d\mu(x)$, $f \in X$, $\mu \in M(G)$. In this case both Lemma 1.1 and Corollary 1.2 remain valid with $\mathrm{LUC}(G)$ replaced by X.

Let $\{m_\alpha\}$ be a net in $\mathrm{LUC}(G)^*$. We say that m_α converges to some $m \in \mathrm{LUC}(G)^*$ *strictly* if $\|m_\alpha\phi - m\phi\| \to 0$, for all $\phi \in L^1(G)$.

1.4. **Lemma.** *Let G_1 and G_2 be locally compact and let T be an isometric isomorphism from $\mathrm{LUC}(G_1)^*$ onto $\mathrm{LUC}(G_2)^*$. Let $\{m_\alpha\}$ be a net in $M(G)$ converging strictly to $m \in M(G)$ and $\|m_\alpha\| = \|m\| = 1$, then $T(m_\alpha)$ converges to $T(m)$ in the weak*-topology of $\mathrm{LUC}(G_2)^*$.*

Proof. Let n be a weak*-cluster point of $\{T(m_\alpha)\}$. By passing to a subnet, if necessary, we may assume that $T(m_\alpha) \to n$ in the w^*-topology. Let $\varphi \in L^1(G_1)$ be fixed. Since $\|m_\alpha\varphi - m\varphi\| \to 0$, it follows that $\|T(m_\alpha)T(\varphi) - T(m)T(\varphi)\| \to 0$. Hence for each $k \in \mathrm{LUC}(G_2)$,

$$\langle T(m)T(\varphi), k\rangle = \lim_\alpha \langle T(m_\alpha)T(\varphi), k\rangle = \langle n, T(\varphi)k\rangle = \langle nT(\varphi), k\rangle,$$

i.e. $T(m)T(\varphi) = nT(\varphi)$ or $m\varphi = T^{-1}(n)\varphi$, for all $\varphi \in L^1(G_1)$. Consequently, if $\varphi \in L^1(G_1)$, $f \in \mathrm{LUC}(G_1)$,

$$\langle m\varphi, f\rangle = \langle T^{-1}(n)\varphi, f\rangle.$$

Hence $\langle m, \varphi f\rangle = \langle T^{-1}(n), \varphi f\rangle$. Consequently, n agrees with $T^{-1}(n)$ on $C_0(G)$. Since $1 = \|m\| \le \|T^{-1}(n)\| = \|n\| \le 1$, it follows that $m = T^{-1}(n)$ or $n = T(m)$ by Lemma 1 in [14]. □

Let $\tau: G_1 \to G_2$ be a (topological) isomorphism of G_1 onto G_2 and let $\alpha: G_1 \to \mathbf{T}$ (where $\mathbf{T} = \{\lambda \in \mathbf{C}: |\lambda| = 1\}$) be a continuous character on G_1. Define $\tau_\alpha: C_0(G_2) \to C_0(G_1)$ by $\tau_\alpha(f)(x) = \alpha(x)f(\tau(x))$ for all $x \in G_1$, $f \in C_0(G_2)$. Then τ_α is an isometric isomorphism mapping $C_0(G_2)$ onto $C_0(G_1)$. Furthermore, $T_{\tau,\alpha} = \tau_a^*$ is an isometric algebra isomorphism from $M(G_1)$ onto $M(G_2)$ such that $T_{\tau,\alpha}(\delta_x) = \alpha(x)\delta_{\tau(x)}$, $x \in G_1$.

For each $\mu \in M(G_1)$, let $\mu^\tau \in M(G_2)$ be defined by

$$\langle \mu^\tau, f\rangle = \int_{G_1} f(\tau(x))\, d\mu(x), \qquad f \in C_0(G_2).$$

Also let

$$\hat{\mu}(\alpha) = \int_{G_1} \alpha(x)\, d\mu(x).$$

1.5. **Lemma.** *Let T be an isometric isomorphism from* $\mathrm{LUC}(G_1)^*$ *onto* $\mathrm{LUC}(G_2)^*$ *such that $T(\delta_x) = T_{\tau,\alpha}(\delta_x)$ for each $x \in G_1$. Then $T(\mu) = \hat{\mu}(\alpha)\mu^\tau$ for each $\mu \in M(G_1)$. In particular T maps $M(G_1)$ onto $M(G_2)$ in* $\mathrm{LUC}(G_2)^*$ *and $L^1(G_1)$ onto $L^1(G_2)$.*

Proof. The equation

$$T(\mu) = \hat{\mu}(\alpha)\mu^\tau \tag{1}$$

clearly holds for all $\mu = \delta_x$, $x \in G_1$, and hence all convex combinations of all such measures. Let $\mu \geq 0$ and $\|\mu\| = 1$. There exists a net $\mu_\beta = \sum_{i=1}^{n_\beta} \lambda_i^\beta \delta_{x_i}$ of convex combination of δ_x's, $x \in G_1$, such that μ_β converges to μ in the w^*-topology. Since $\|\mu_\beta\| = \|\mu\| = 1$ for each β, μ_β must converge to μ strictly (see [5 or 15]). Hence by Lemma 1.4, $T(\mu_\beta)$ must converge to $T(\mu)$ in the weak*-topology. Now the net $\hat{\mu}_\beta(\alpha)\mu_\beta^\tau \to \hat{\mu}(\alpha)\mu^\tau$ in the weak*-topology also. Hence (1) holds for all $\mu \geq 0$, $\|\mu\| = 1$. Consequently (1) must hold for all $\mu \in M(G)$.

The last statement follows from [10]. However it also follows directly from the well-known fact that $L^1(G)$ can be identified with all $\mu \in M(G)$ such that the map $a \mapsto \delta_a * \mu$ from G into $(M(G), \|\cdot\|)$ is continuous. □

We are now ready to prove the main theorem of this section.

1.6. **Theorem.** *Let G_1 and G_2 be locally compact groups and T be an isometric isomorphism from* $\mathrm{LUC}(G_1)^*$ *onto* $\mathrm{LUC}(G_2)^*$*, then T maps $M(G_1)$ onto $M(G_2)$ and $L^1(G_1)$ onto $L^1(G_2)$.*

Proof. Indeed for each $x \in G_1$, $T(\delta_x)$ is invertible and $\|T(\delta_x)\| = \|T(\delta_x)^{-1}\| = 1$. Hence by Corollary 1.2 there exist $\alpha(x) \in \mathbf{C}$, $|\alpha(x)| = 1$ and $\gamma(x) \in G_2$ such that $T(\delta_x) = \alpha(x)\delta_{\gamma(x)}$. Clearly α is a character and γ is an algebraic isomorphism of G_1 onto G_2. Furthermore, if $x_i \to x$, $x_i, x \in G_1$, then $\delta_{x_i} \to \delta_x$ strictly. Hence by Lemma 1.4 $T(\delta_{x_i}) \to T(\delta_x)$ in the weak*-topology of $\mathrm{LUC}(G_2)^*$. Consequently $\alpha(x_i) \to \alpha(x)$ and $\gamma(x_i) \to \gamma(x)$, i.e. both α and γ are continuous. Hence $T(\delta_x) = T_{\tau,\alpha}(\delta_x)$ for each $x \in G_1$. The theorem now follows from Lemma 1.5. □

1.7. *Remark.* Lemmas 1.4, 1.5 and Theorem 1.6 are valid when $\mathrm{LUC}(G_i)$, $i = 1, 2$ are replaced by left introverted subspaces X_i of $C(G_i)$ containing $C_0(G_i)$ (see [14, Theorem 1]). When $X_1 = C_0(G_1)$ and $X_2 = C_0(G_2)$, this provides an alternative proof to the main result in [10].

2. The embeddings $\Gamma_E: M(G) \to L^1(G)^{**}$

Let $\Lambda(G)$ denote the set of weak*-cluster points of the canonical images of the bounded approximate identities, bounded by 1, of $L^1(G)$ in $L^1(G)^{**}$. We first observe that the set $\Lambda(G)$ coincides with the sets K and K_1 considered in [9, Theorem 3.2] for compact groups:

2.1. **Proposition.** *Let* $E \in L^1(G)^{**}$. *The following are equivalent*:

(a) $E \in \Lambda(G)$.

(b) $\|E\| = 1$ *and* $E(f) = f(e)$ *for all* $f \in C_0(G)$.

(c) $E \geq 0$, $E\psi = \psi E = \psi$ *for all* $\psi \in L^1(G)$.

(d) $\|E\| = 1$ *and* E *is a right identity of* $L^1(G)^{**}$.

Proof. (a) $\Rightarrow$ (b). If $E \in \Lambda(G)$, then $\|E\| \leq 1$. Let $\mu \in M(G)$ be the restriction of E to $C_0(G)$. Then μ is the identity of $M(G)$ (by weak*-weak* continuity of multiplication in $M(G)$). So $\mu = \delta_e$, where $\delta_e(f) = f(e)$, $f \in C_0(G)$. Hence (b) holds.

(b) $\Rightarrow$ (c). Let m denote the restriction of E to $\mathrm{LUC}(G)$. Then $m(f) = f(e)$ for all $f \in \mathrm{LUC}(G)$ by Lemma 1.1 and its proof. Hence $\|E\| = E(1) = 1$. So $E \geq 0$ [20, p. 9]. Now if $\psi \in L^1(G)$, $f \in L^\infty(G)$, then

$$\langle E\psi, f\rangle = \langle E, \psi f\rangle = \langle E, f * \check{\psi}\rangle = (f * \check{\psi})(e) = \int f(t)\psi(t)\,dt = \langle \psi, f\rangle,$$

i.e. $E\psi = \psi$ (where $\check{\psi}(t) = \psi(t^{-1})$, $t \in G$). Similarly $\psi E = \psi$.

(c) $\Rightarrow$ (a). We first observe that $\|E\| = 1$ (since $E(1) = E(\psi \cdot 1) = E\psi(1) = \psi(1) = 1$, when $\psi \in L^1(G)$, $\psi \geq 0$, $\|\psi\|_1 = 1$). Let $P_1(G)$ denote all $\psi \in L^1(G)$, $\psi \geq 0$, $\|\psi\|_1 = 1$. Let $\{\theta_\alpha\}$ be a net in $P_1(G)$ converging to E in the weak*-topology. Then $\{\theta_\alpha\}$ is a weak approximate identity for $L^1(G)$. Then an argument similar to that in the proof of [2, Theorem 1, p. 524] shows that we can find a net $\{e_\lambda\}$ consisting of convex combinations of elements in $\{\theta_a\}$ such that

(i) $\|e_\lambda\psi - \psi\| \to 0$, for each $\psi \in L^1(G)$.

(ii) $\{e_\lambda\}$ is far out in $\{\theta_\alpha\}$, i.e. for each α_0, there exists λ_0 such that if $\lambda \geq \lambda_0$, and $e_\lambda = \sum_{i=1}^n a_i\theta_{\alpha_i}$, $a_i > 0$, $\sum_{i=1}^n a_i = 1$, then each $\alpha_i \geq \alpha_0$.

Then $\{e_\lambda\}$ is a left approximate identity in $L^1(G)$ converging in the weak*-topology of $L^1(G)^{**}$ to E. Furthermore, $\{e_\lambda\}$ is also a weak right approximate identity in $L^1(G)$. Indeed, if $\psi \in L^1(G)$ and $f \in L^\infty(G)$, choose α_0 such that $|\langle f, \psi\theta_\alpha - \psi\rangle| < \varepsilon$ for all $\alpha \geq \alpha_0$. Let λ_0 be as chosen in (ii). Then for all $\lambda \geq \lambda_0$,

$$\begin{aligned} |\langle f, \psi e_\lambda - \psi\rangle| &= \left|\left\langle f, \psi\left(\sum_{i=1}^n a_i\theta_{\alpha_i}\right) - \psi\right\rangle\right| \\ &\leq \sum_{i=1}^n a_i|\langle f, \psi\theta_{\alpha_i} - \psi\rangle| < \varepsilon\|f\|. \end{aligned}$$

Again, repeating the argument in the proof of [2, Theorem 1, p. 524], we can find a net $\{f_\mu\}$ consisting of convex combinations of elements in $\{e_\lambda\}$ such that (i)′ $\{f_\mu\}$ is a right approximate identity of $L^1(G)$.

(ii)′ $\{f_\mu\}$ is far out in $\{e_\lambda\}$.

Necessarily, $\{f_\mu\} \subseteq P_1(G)$ and is also a left approximate identity of $L^1(G)$ converging in the weak*-topology of $L^1(G)^{**}$ to E by (ii)′.

(b) ⇒ (d). If $E \in \Lambda(G)$, then E is a right identity of $L^1(G)^{**}$ by (a).

(d) ⇒ (b). Let θ denote the restriction of E to $C_0(G)$. Then θ is a right identity in $C_0(G)^*$. It suffices to show θ is also a left identity. Let $\{e_j\}$ denote a bounded weak right approximate identity in $L^1(G)$ converging to E in the weak*-topology. Let $f \in C_0(G)$. Then $f = g\psi$ for some $g \in C_0(G)$, $\psi \in L^1(G)$ (by Cohen's factorization theorem). Hence for each $m \in C_0(G)^*$,

$$\begin{aligned}\langle \theta m, f\rangle &= \lim_j \langle m, fe_j\rangle = \lim_j \langle m, g\psi e_j\rangle \\ &= \lim_j \langle m, g(\psi e_j)\rangle = \langle m, g\psi\rangle = \langle m, f\rangle,\end{aligned}$$

since $\varphi \mapsto \langle m, g\varphi\rangle$ defines a bounded linear functional on $L^1(G)$. □

2.2. *Remark.* $\Lambda(G)$ does not change if one uses (weak) approximate identities, bounded by 1, in the definition. It also does not change if one uses weak*-cluster points of positive bounded approximate identities in $L^1(G)$.

Let $E = w^*\text{-}\lim e_j$, where (e_j) is a bounded approximate identity bounded by 1. For $\mu \in M(G)$, let $\rho_\mu: L^1(G) \to L^1(G)$ be defined by $\rho_\mu(\nu) = \nu * \mu$, and let $\Gamma_E(\mu) = \rho_\mu^{**}(E)$, where ρ_μ^{**} is the second adjoint of ρ_μ. Then

2.3. **Proposition.** (i) $\langle \Gamma_E(\mu), f\rangle = \int f\, d\mu$ $(f \in \mathrm{LUC}(G), \mu \in M(G))$.

(ii) $\Gamma_E(\mu) = \mu$, *if* $\mu \in L^1(G)$.

(iii) $\langle \Gamma_E(\mu)f, \varphi\rangle = \langle \mu, \tilde{\varphi} * f\rangle$ $(f \in L^\infty(G),\ \varphi \in L^1(G),\ \mu \in M(G))$. *In particular,* $\Gamma_E(\mu)f = \rho_\mu^* f$, *for each* $f \in L^\infty(G)$.

(iv) $\Gamma_E(\delta_x)f = r_x f$ $(f \in L^\infty(G), x \in G)$, *where* δ_x *is the Dirac measure at* x.

(v) Γ_E *is an isometric embedding of the algebra* $M(G)$ *into* $L^1(G)^{**}$, *which extends the canonical embedding of* $L^1(G)$ *into* $L^1(G)^{**}$.

Proof. (i) Let $\mu \in M(G)$ and $f \in \mathrm{LUC}(G)$. Then by a version of Cohen's factorization theorem [8, 32.45(b)], there exists $g \in L^1(G)$ and $h \in L^\infty(G)$ such that $f = g * h$. Hence, with $\tilde{\varphi}(x) = \Delta(x^{-1})\varphi(x^{-1})$, we have

$$\langle \Gamma_E(\mu), f\rangle = \lim_j \langle f, e_j * \mu\rangle = \lim_j \langle \tilde{e}_j * f, \mu\rangle = \langle f, \mu\rangle;$$

since $\{\tilde{e}_j\}$ is also a bounded approximate identity of $L^1(G)$ [22, Lemma 3.3], $\|\tilde{e}_j * f - f\|_\infty \to 0$, by another application of Cohen's factorization theorem.

(ii) follows directly from

$$\langle \Gamma_E(\mu), f\rangle = \lim_j \langle f, e_j * \mu\rangle.$$

(iii) We have

$$\begin{aligned}\langle \Gamma_E(\mu)f, \varphi\rangle &= \langle \Gamma_E(\mu), f\varphi\rangle = \langle \Gamma_E(\mu), \tilde{\varphi} * f\rangle = \langle \mu, \tilde{\varphi} * f\rangle = \langle \varphi\mu, f\rangle \\ &= \langle \rho_\mu(\varphi), f\rangle = \langle \varphi, \rho_\mu^*(f)\rangle\end{aligned}$$

by part (i).

(iv) follows from (iii) with $\mu = \delta_x$ and a direct computation.

(v) From the definition of $\Gamma_E(\mu)$ it follows that

$$\|\Gamma_E(\mu)\| = \|\rho_\mu^{**}(E)\| \le \|\rho_\mu^{**}\| \, \|E\| = \|\mu\|.$$

This together with (i) shows that $\mu \mapsto \Gamma_E(\mu)$ is obviously linear. To prove that it is multiplicative we note that for $\mu, \nu \in M(G)$ and $f \in L^\infty(G)$,

$$\begin{aligned}\langle \Gamma_E(\mu)\Gamma_E(\nu), f\rangle &= \langle \Gamma_E(\mu), \Gamma_E(\nu)f\rangle = \langle \rho_\mu^{**}(E), \rho_\nu^*(f)\rangle \\ &= \langle E, \rho_\mu^*\rho_\nu^*(f)\rangle = \langle E, \rho_{\mu*\nu}^* f\rangle \\ &= \langle \Gamma_E(\mu * \nu), f\rangle \quad \text{by (iii).} \quad \square\end{aligned}$$

2.4. **Proposition.** (i) $E_2\Gamma_{E_1}(\mu) = \Gamma_{E_2}(\mu)$, *for any* $E_1, E_2 \in \Lambda(G)$, *and* $\mu \in M(G)$.

(ii) *A measure* μ *belongs to* $L^1(G)$ *if and only if* $\Gamma_{E_1}(\mu) = \Gamma_{E_2}(\mu)$, *for any* $E_1, E_2 \in \Lambda(G)$.

Proof. (i) Suppose $h \in L^\infty(G)$. Then

$$\langle E_2\Gamma_{E_1}(\mu), h\rangle = \langle E_2, \Gamma_{E_1}(\mu)h\rangle = \langle E_2, \rho_\mu^*(h)\rangle = \langle \rho_\mu^{**}(E_2), h\rangle = \langle \Gamma_{E_2}(\mu), h\rangle,$$

by (iii) of Proposition 2.4.

(ii) The "only if" part being obvious, we assume that $\mu \notin L^1(G)$. We then may (and do) assume that μ is real and $\mu \neq 0$. We will construct two bounded approximate identities (e_i) and (f_j) both bounded by 1 such that for a w^*-cluster point E_1 of (e_i) and a w^*-cluster point E_2 of (f_j), $\Gamma_{E_1}(\mu) \neq \Gamma_{E_2}(\mu)$. By [19, Theorem 2], there exists a continuous function f such that the function $h: x \mapsto \int f(xy)\, d\mu(y)$ is not (equal almost everywhere to) a function continuous at the identity e. We can also assume that f, and hence h, is real. We may further assume that for each neighbourhood V of e there are sets $A, B \subseteq V$ of positive Haar measure with $h \ge 1$ on A and $h \le 0$ on B. By the method of the proof of [9, Lemma 2.3] there exists bounded approximate identities (e_i) and (f_i) of $L^1(G)$ bounded by 1, with $\langle e_i, h\rangle \ge 1$ and $\langle f_j, h\rangle \le 0$. Now let $E_1 = w^*\text{-}\lim e_i'$ and $E_2 = w^*\text{-}\lim f_j'$ where (e_i') is a subnet of (e_i) and (f_j') is a subnet of (f_j). Then

$$\langle \Gamma_{E_1}(\mu), f\rangle = \lim_i \langle f, \mu * e_i'\rangle = \lim_i \langle e_i', h\rangle \ge 1,$$

while

$$\langle \Gamma_{E_2}(\mu), f\rangle = \lim_j \langle f, \mu * f_j'\rangle = \lim_j \langle f_j', h\rangle \le 0. \quad \square$$

2.5. **Proposition.** *Let* $m \in L^1(G)^{**}$ *and* $E \in \Lambda(G)$. *Then the following are equivalent*:

(a) $m = \Gamma_E(\mu)$, *for some* $\mu \in M(G)$.

(b) *As a functional* m *is an extension of* $\mu \in C_0(G)^*$ *with* $\|m\| = \|\mu\|$ *and* $Em = m$.

Proof. (a) $\Rightarrow$ (b) follows from parts (i) and (v) of Proposition 2.3 together with part (i) of Proposition 2.4. To prove (b) $\Rightarrow$ (a) let m be an extension of μ with $\|m\| = \|\mu\|$. Then the norm of the restriction of m to LUC(G) will also be equal to $\|m\|$. Then from [14, Lemma 1] it follows that for $f \in$ LUC(G), $\langle m, f\rangle = \int f\,d\mu$. Hence by Proposition 2.3(i) $\langle m, f\rangle = \langle \Gamma_E(\mu), f\rangle$ for every $f \in \mathrm{LUC}(G)$. Now if E is the w^*-limit of (e_j), then from $Em = m$ we have

$$\begin{aligned}\langle m, f\rangle &= \langle Em, f\rangle = \langle E, mf\rangle = \lim_j \langle e_j, mf\rangle \\ &= \lim_j \langle mf, e_j\rangle = \lim_j \langle m, fe_j\rangle = \lim_j \langle \mu, fe_j\rangle \quad (\text{since } fe_j \in \mathrm{LUC}(G)) \\ &= \lim_j \langle \mu, \tilde{e}_j * f\rangle = \lim_j \langle f, e_j * \mu\rangle = \langle \Gamma_E(\mu), f\rangle. \quad \square\end{aligned}$$

In the following propositions the canonical image of $L^1(G)$ in $L^1(G)^{**}$ will be denoted by the same symbol.

2.6. **Proposition.** *Let* $\Delta(G) = \bigcap EL^1(G)^{**}$, *where* E *ranges in* $\Lambda(G)$. *Then* $\Delta(G)$ *is a closed right ideal of* $L^1(G)^{**}$ *containing* $L^1(G)$. *Furthermore,* $L^1(G)$ *is an ideal in* $\Delta(G)$ *if and only if* G *is compact, in which case* $\Delta(G) = L^1(G)$.

Proof. Since for each $E \in \Lambda(G)$, $E^2 = E$ (by Proposition 2.1(d)), each $EL^1(G)^{**}$ is a closed right ideal, whence $\Delta(G)$ is a closed right ideal. If G is a compact group, then an argument similar to the one of [9, 3.3, v] shows that $\Delta(G) = L^1(G)$. Suppose conversely that $L^1(G)$ is an ideal in $\Delta(G)$. Let $m \in L^1(G)^{**}$ and $\psi \in L^1(G)$. Then $\psi m \in \Delta(G)$. Let (φ_α) be a bounded approximate identity of $L^1(G)$. Then $\varphi_\alpha \psi m \to \psi m$, in norm. So $\psi m \in L^1(G)$. Therefore, $L^1(G)$ is a right ideal in $L^1(G)^{**}$. Hence G is a compact group [6]. $\square$

2.7. **Proposition.** *The intersection of all* $\Gamma_E(M(G))$ *when* E *ranges in* $\Lambda(G)$ *is equal to* $L^1(G)$.

Proof. Let Ω denote the intersection of all $\Gamma_E(M(G))$, where E ranges in $\Lambda(G)$. Suppose $m \in \Omega$, and let E_1 and E_2 belong to $\Lambda(G)$. Then for some $\mu, \nu \in M(G)$, $m = \Gamma_{E_1}(\mu) = \Gamma_{E_2}(\nu)$. Hence $\Gamma_{E_1}(\mu) = E_1\Gamma_{E_1}(\mu) = E_1\Gamma_{E_2}(\nu) = \Gamma_{E_1}(\nu)$, by Proposition 2.4(i). Hence $\mu = \nu$, and we have $\Gamma_{E_1}(\mu) = \Gamma_{E_2}(\mu)$ for every E_1 and E_2 in $\Lambda(G)$. From Proposition 2.4(ii), it now follows that $\mu \in L^1(G)$. $\square$

Let $E \in \Lambda(G)$ and let π_E be the map which associates to any functional in $EL^1(G)^{**}$ its restriction to LUC(G). Then π_E is an isometric isomorphism from $EL^1(G)^{**}$ onto LUC$(G)^*$ (see [4]).

2.8. **Proposition.** *Let* $E \in \Lambda(G)$. *For each* $\mu \in M(G)$, *we have* $\pi_E^{-1}(\mu) = \Gamma_E(\mu)$.

Proof. Let $m \in L^1(G)^{**}$ be an extension of μ. Let $\{e_j\}$ be an approximate identity in $L^1(G)$ bounded by 1 converging to E in the weak*-topology (see the proof of Proposition 2.1). Then for each $f \in L^\infty(G)$,

$$\begin{aligned}\langle \Gamma_E(\mu), f\rangle &= \lim_j \langle \mu, \tilde{e}_j * f\rangle = \lim_j \langle m, fe_j\rangle \\ &= \lim_j \langle mf, e_j\rangle = \langle E, mf\rangle = \langle Em, f\rangle,\end{aligned}$$

i.e. $\Gamma_E(\mu) \in EL^1(G)^{**}$. Since $\Gamma_E(\mu)$ extends μ by Proposition 2.3(i), $\pi_E(\Gamma_E(\mu)) = \mu$, i.e. $\Gamma_E(\mu) = \pi_E^{-1}(\mu)$. □

3. Isometric isomorphisms on $L^1(G)^{**}$

We are now ready to prove our next main result.

3.1. **Theorem.** *Let* G_1 *and* G_2 *be locally compact groups and let* T *be an isometric isomorphism from* $L^1(G_1)^{**}$ *onto* $L^1(G_2)^{**}$. *Then*

(a) $T(\Lambda(G_1)) = \Lambda(G_2)$.

(b) *For each* $E \in \Lambda(G_1)$, *there exists a continuous character* $\alpha: G_1 \to \mathbf{T}$ *and a bicontinuous isomorphism* $\tau: G_1 \to G_2$ *such that for each* $\mu \in M(G_1)$

$$T(\Gamma_E(\mu)) = \hat{\mu}(\alpha)\Gamma_{T(E)}(\mu^\tau).$$

(c) T *maps* $L^1(G_1)$ *onto* $L^1(G_2)$.

Proof. (a) follows immediately from Proposition 2.1(d).

(b) Let $E \in \Lambda(G_1)$. Let $\tilde{T} = \pi_{T(E)} \circ T \circ \pi_E^{-1}$. Then $\tilde{T}$ is an isometric isomorphism from $\mathrm{LUC}(G_1)^*$ onto $\mathrm{LUC}(G_2)^*$ (see [4]). So by the proof of Lemma 1.5, there exist a continuous character α on G_1 and a bicontinuous isomorphism $\tau: G_1 \to G_2$ such that $\tilde{T}(\mu) = \hat{\mu}(\alpha)\mu^\tau$, for each $\mu \in M(G)$. In particular,

$$T \circ \pi_E^{-1}(\mu) = \hat{\mu}(\alpha)\pi_{T(E)}^{-1}(\mu^\tau).$$

So $T(\Gamma_E(\mu)) = \hat{\mu}(\alpha)\Gamma_{T(E)}(\mu^\tau)$ by Proposition 2.8.

(c) follows from (b) and Proposition 2.3(ii). □

For each $m \in L^1(G)^{**}$, let Q_m denote the map from $L^1(G)^{**} \to L^1(G)^{**}$ defined by $Q_m(n) = mn$, $n \in L^1(G)^{**}$.

3.2. **Corollary.** *Let* G_1 *and* G_2 *be locally compact groups and let* T *be an isometric isomorphism from* $L^1(G)^{**}$ *onto* $L^1(G_2)^{**}$. *Let* $m \in L^1(G_1)^{**}$. *Then* Q_m *is weak*-weak* continuous if and only if* $Q_{T(m)}$ *is weak*-weak* continuous.*

Proof. This follows from Theorem 1 in [13] and Theorem 3.1 above. □

3.3. *Remark.* Note that if G_1 and G_2 are abelian, then Q_m is weak*-weak* continuous if and only if m is in the centre of $L_1(G)^{**}$ (see [13, Lemma 7]).

Hence in this case Corollary 3.2 holds even when T is an algebraic isomorphism.

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