TOKYO J. MATH.
VOL. 21, NO. 1, 1998

Isometric Shift Operators on the Disc Algebra

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Introduction.

The purpose of this note is to study linear isometries on function algebras, especially isometric shift operators on the disc algebra. For a compact Hausdorff space X, we denote by $C(X)$ the Banach space of all complex-valued continuous functions on X. Recently, A. Gutek, D. Hart, J. Jamison and M. Rajagopalan [5] and F. O. Farid and K. Varadarajan [3] have obtained many significant results concerning isometric shift operators on Banach spaces, especially on $C(X)$. Here we investigate linear isometries on function algebras and isometric shift operators on the disc algebra.

In section 1, we give a representation of a codimension 1 linear isometry on a function algebra and in section 2, on the disc algebra A, we establish the form of a codimension 1 linear isometry φ and give equivalent conditions under which φ is a shift operator.

1. Codimension 1 linear isometries on function algebras.

Let E be a Banach space and φ a linear isometry from E into E. Then we call φ a *codimension* 1 *linear isometry* on E if the range of φ has codimension 1. A bounded linear operator φ on E is called a *shift operator* on E if the following conditions are satisfied: (i) φ is injective; (ii) the range of φ has codimension 1; and (iii) $\bigcap_{n=1}^{\infty} \varphi^n(E) = \{0\}$. A linear isometry on E which is a shift operator is an *isometric shift operator* on E.

Let X be a compact Hausdorff space. We say that A is a *function algebra* on X if it is a closed subalgebra of $C(X)$, the Banach algebra of all complex-valued continuous functions on X with the supremum norm, which separates points in X and contains the constants. After now, we consider codimension 1 linear isometries on function algebras and isometric shift operators on the disc algebra.

The following extends a theorem of Gutek, Hart, Jamison and Rajagopalan [5, Theorem 2.1] to the case of the function algebras (cf. [9]).

THEOREM 1.1. *Let A be a function algebra on a compact Hausdorff space X. Suppose*

Received August 2, 1996

the Choquet boundary Ch(A) *for* A *is* X. *Let* φ *be a codimension* 1 *linear isometry on* A. *Then there is a closed subset* F *of* X, *where either,*

(i) $F=X\setminus\{p\}$, *where* p *is an isolated point of* X *or*

(ii) $F=X$

such that there is a continuous map τ *from* F *onto* X *and a unimodular function* $u\in C(F)$ *such that* $(\varphi f)(x)=u(x)f(\tau(x))$ *for any* $f\in A$ *and* $x\in F$.

PROOF. Let $B=\varphi(A)$. Then by Novinger [7], there are a continuous map τ from Ch(B) onto Ch(A) and a unimodular continuous function u on Ch(B) such that $(\varphi f)(x)=u(x)f(\tau(x))$ for $f\in A$ and $x\in \mathrm{Ch}(B)$. Since Ch(A) is closed in X by the hypothesis, Ch(B) is also closed in X [7, Corollary 2]. We here assert that $X\setminus F$ consists of at most one point if we put $F=\mathrm{Ch}(B)$. Otherwise, $X\setminus F$ contains two distinct points p, q. Since $p, q\in X=\mathrm{Ch}(A)$ and A is a function algebra, there are $f, g\in A$ such that $\|f\|=\|g\|=1$, $f(p)=1$, $|f(x)|\le 1/4$ $(x\in F\cup\{q\})$ and $g(q)=1$, $|g(x)|\le 1/4$ $(x\in F\cup\{p\})$ (see [1]). We here show that if $af+bg\in B=\varphi(A)$ $(a, b\in \mathbf{C})$ then $a=b=0$. Since $h=af+bg\in B$, there is a $k\in A$ such that $h=\varphi k$. So $af(x)+bg(x)=(\varphi k)(x)=u(x)k(\tau(x))$ $(x\in F)$. Hence

$$|k(\tau(x))|\le|a||f(x)|+|b||g(x)|\le\frac{1}{4}(|a|+|b|)\qquad x\in F\,.$$

Since τ is surjective, we have

$$\|h\|=\|\varphi k\|=\|k\|\le\frac{1}{4}(|a|+|b|)\,. \tag{1}$$

It implies the following since $h(p)=a+bg(p)$ and $h(q)=af(q)+b$.

$$|a+bg(p)|\le\frac{1}{4}(|a|+|b|)\,,\qquad |af(q)+b|\le\frac{1}{4}(|a|+|b|)\,. \tag{2}$$

Since $|g(p)|\le 1/4$, by the first part of (2) we have $3|a|\le 2|b|$. Similarly, by the latter of (2) we have $3|b|\le 2|a|$ and $a=b=0$. Thus, the codimension of φ is at least two. This contradiction tells us that $X\setminus F$ has at most one point.

The following lemma was shown in the case of $C(X)$ in [5, Lemma 2.2], but we observe that this holds true in the case of function algebras.

LEMMA 1.2. *Let* A *be a function algebra on a compact Hausdorff space* X *and suppose that* $\mathrm{Ch}(A)=X$. *Let* φ *be a codimension* 1 *linear isometry on* A *and let* F, τ *and* u *be as in Theorem* 1.1. *Then* $\tau^{-1}(x)$ *has at most two elements for any* $x\in X$. *Furthermore, if* $\tau^{-1}(x_0)$ *has two elements for some* $x_0\in X$, *then* $\tau^{-1}(x)$ *is a singleton for any* $x\in X\setminus\{x_0\}$.

Let $\Gamma=\{z\in\mathbf{C}:|z|=1\}$. If A is a function algebra on Γ, we can have the following

THEOREM 1.3. *Let* A *be a function algebra on* Γ *with* $\mathrm{Ch}(A)=\Gamma$ *and* φ *a codimension* 1 *linear isometry on* A. *Then* τ *is a homeomorphism from* Γ *onto* Γ.

PROOF. Since Γ is connected, F in Theorem 1.1 is equal to Γ. From Lemma 1.2, it suffices to show that there is no element $x_0 \in \Gamma$ such that $\tau^{-1}(x_0)$ has two elements. Suppose $\tau(a)=\tau(b)=x_0$ for some distinct points $a, b \in \Gamma$. Let L_1 be a closed arc on Γ having a, b as the end points. If $t_1 \in L_1, t_1 \neq a, t_1 \neq b$, then $\tau(t_1) \neq x_0$. Since τ is a continuous map from Γ onto Γ, $\tau(L_1)$ is a closed arc on Γ containing x_0 and $\tau(t_1)$. The fact that $\tau^{-1}(x)$ is a singleton for any $x \in \Gamma \setminus \{x_0\}$ follows that $\tau(L_1)=\Gamma$. If L_2 is another closed arc on Γ having a, b as the end points, a similar argument as above implies that $\tau(L_2)=\Gamma$. This is a contradiction since $\tau^{-1}(x)$ is a singleton for any $x \in \Gamma \setminus \{x_0\}$.

2. Isometric shift operators on the disc algebra.

Let $\Gamma=\{z \in \mathbf{C} : |z|=1\}$, $D=\{z \in \mathbf{C} : |z|<1\}$ and $\bar{D}=\Gamma \cup D$. We put $A_0=\{f \in C(\bar{D}) : f$ is analytic on $D\}$ and $A=A_0 | \Gamma$. A is called the *disc algebra*. We here consider isometric shift operators on the disc algebra A. We set $M_a(z)=(z-a)/(1-\bar{a}z)$ for $a \in D$. We begin with the following theorem.

THEOREM 2.1. *Let A be the disc algebra and φ a codimension 1 linear isometry on A. Then there are $\alpha, \beta \in \mathbf{C}$ ($|\alpha|=|\beta|=1$) and $a, b \in D$ such that $(\varphi f)(z)=\alpha M_a(z) f(\beta M_b(z))$ ($f \in A, z \in \Gamma$).*

PROOF. Since A is a function algebra on Γ with Ch(A)$=\Gamma$, by Theorem 1.1 and Theorem 1.3, there are a continuous map u from Γ into Γ and a homeomorphism τ from Γ onto Γ such that $(\varphi f)(z)=u(z)f(\tau(z))$ ($f \in A, z \in \Gamma$). By putting $f=1$, we see that $u \in A$. For every $f \in A$, there is a unique function $f_0 \in A_0$ such that $f_0 | \Gamma = f$. In the rest we also write f instead of f_0.

(i) We first assume that u does not have zeros in D. Since $|u|=1$, u is a constant function α. By putting $f=z$ (the coordinate function), we have $u\tau \in A$ and so $\tau \in A$. If τ does not have zeros in D, τ is constant since $|\tau|=1$. Hence τ has (not necessarily distinct) zeros $b_1, b_2, \cdots, b_n$ in D and so $\tau(z)$ is of form $\beta \prod_{i=1}^{n} M_{b_i}(z)$ ($|\beta|=1$). We here assert that $n=1$. Suppose $n \geq 2$. When z turns arround on Γ one time, it is not hard to see that $\tau(z)$ rotates on Γ n-times. This contradicts that τ is a homeomorphism of Γ onto Γ. Thus, φf is of form $\alpha f(\beta M_b)$, and so φ does not have 1 codimension.

(ii) We next assume that u has zeros $a_1, a_2, \cdots, a_m$ in D. Then $u(z)$ is of form $\alpha \prod_{j=1}^{m} M_{a_j}(z)$ ($|\alpha|=1$). By putting $f=z^k$, $u\tau^k \in A$ ($k=1, 2, 3, \cdots$). Since $u(u\tau^2)=(u\tau)^2$, $u\tau \in A$ and $u\tau^2 \in A$, $u\tau$ has zeros in D. From that $|u\tau|=1$, it follows that $(u\tau)(z)$ is of form $\beta \prod_{i=1}^{n} M_{b_i}(z)$ ($|\beta|=1, b_i \in D$). We first consider a_1. Since $u(u\tau^2)=(u\tau)^2$ again, there is some b_j with $b_j=a_1$. Let m_1 be the number of a_i such that $a_i=a_1$ and $a_i \in \{a_j\}_{j=1}^{m}$. From that $u^{k-1}(u\tau^k)=(u\tau)^k$ ($k=1, 2, 3, \cdots$), if n_1 is the number of b_j such that $b_j=a_1$ and $b_j \in \{b_i\}_{i=1}^{n}$, we get $(k-1)m_1 \leq kn_1$. By tending k to ∞, $m_1 \leq n_1$. A similar argument for any a_j implies that $\tau \in A$. Since $\tau \in A$, as we saw in (i), τ is of form βM_b. Finally we show that $m=1$. For otherwise, $m \geq 2$. Suppose first that $\{a_j\}_{j=1}^{m}$ contains two distinct elements; call them a_1 and a_2. If $pM_{a_1}+qM_{a_2} \in \varphi(A)$ ($p, q \in \mathbf{C}$), there is an $f \in A$ such that $pM_{a_1}(z)+qM_{a_2}(z)=\alpha\prod_{j=1}^{m}M_{a_j}(z)f(\beta M_b(z))$. By putting $z=a_1$ and $z=a_2$, $qM_{a_2}(a_1)=0$, $pM_{a_1}(a_2)=0$, and so $p=q=0$. Suppose next that $a_1=a_2=\cdots=a_m=a$. If $p+qM_a(z)\in\varphi(A)$, $p+qM_a(z)=\alpha(M_a(z))^m f(\beta M_b(z))$ for an $f\in A$. By setting $z=a$, we have $p=0$. So $q=\alpha(M_a(z))^{m-1}f(\beta M_b(z))$. By putting $z=a$ again, $q=0$. This means φ has at least 2 codimension either way. This contradiction shows $m=1$ and φf is of form $\alpha M_a f(\beta M_b)$.

We next discuss when a codimension 1 linear isometry on the disc algebra A becomes an isometric shift operator. To do this, we describe the form of φ^n as follows:

Let φ be a codimension 1 linear isometry on the disc algebra A. By Theorem 2.1, there are α, $\beta\in\mathbf{C}$ ($|a|=|b|=1$) and $a, b\in D$ such that

$$(\varphi f)(z)=\alpha M_a(z)f(\beta M_b(z)) \qquad (f\in A,\ z\in\Gamma)\,.$$

Hence,

$$(*) \qquad (\varphi^n f)(z)=\alpha^n M_a(z)M_a(\beta M_b(z))\cdots M_a[(\beta M_b)^{n-1}(z)]f[(\beta M_b)^n(z)]$$

for every positive integer n, $f\in A$ and $z\in\Gamma$, where $(\beta M_b)^k$ denotes the k-times composition of βM_b.

Now, for $n=0, 1, 2, \cdots$, we take $d_n\in D$ such that $a=(\beta M_b)^n(d_n)$. We call $\{d_n\}$ the *backward orbit* of a by βM_b.

Our final aim is to give equivalent conditions under which a codimension 1 linear isometry φ on the disc algebra A is a shift operator.

We start with the following lemmas.

LEMMA 2.2 (cf. [8], [2]). *Let $D=\{z\in\mathbf{C}\,;\,|z|<1\}$ and $\Gamma=\{z\in\mathbf{C}\,;\,|z|=1\}$ and let m be an analytic automorphism of D. Then it occurs either of the following four cases.*

(i) *m is the identity, that is, $m(z)=z$ ($z\in D$).*

(ii) *m has only one fixed point in D. Then m is said to be elliptic.*

(iii) *m has distinct two fixed points on Γ. Then m is said to be hyperbolic.*

(iv) *m has only one fixed point on Γ. Then m is called parabolic.*

We fix a point $z_0\in D$ and set $z_n=m^n(z_0)$, where m^n denotes the n-times composition of m. Then we obtain the following.

LEMMA 2.3. (a) *If m satisfies* (i) *or* (ii) *of Lemma* 2.2, *then* $\sum_{n=0}^{\infty}(1-|z_n|)=\infty$. (b) *If m satisfies* (iii) *or* (iv) *of Lemma* 2.2, *then* $\sum_{n=0}^{\infty}(1-|z_n|)<\infty$.

PROOF. (a) It is clear if m has (i). Suppose that m satisfies (ii). Let p be the fixed point of m in D. By putting $k(z)=(z-p)/(1-\bar{p}z)$, $h=k\circ m\circ k^{-1}$ is an analytic automorphism of D and $h(0)=0$. Hence $h(z)=\lambda z$ for a $\lambda\in\mathbf{C}$ ($|\lambda|=1$). If we set $w_n=k(z_n)$ ($n=0, 1, 2, \cdots$), then $w_n=k\circ m^n\circ k^{-1}(w_0)=\lambda^n w_0$. Hence $\{w_n\}$ is a relatively compact subset in D, and so is $\{z_n\}$ since k^{-1} is an analytic automorphism on D and $z_n=k^{-1}(w_n)$ ($n=0, 1, 2, \cdots$). It follows that $\sum_{n=0}^{\infty}(1-|z_n|)=\infty$.

(b) Suppose that m satisfies (iii). Let p be the Denjoy-Wolff point of m and q be another fixed point of m on Γ (cf. [2, p. 59]). Let l be a bi-holomorphic map of D onto the upper half plane H of $\mathbf{C}$ such that $l(p)=\infty$ and $l(q)=0$. Then $l\circ m\circ l^{-1}(w)=\alpha w (w\in H)$ for some $\alpha>0$ $(\alpha\neq 1)$ since it is an analytic automorphism on H which fixes 0 and ∞ only [2, p. 59]. Since p is the Denjoy-Wolff point of m, z_n converges to p and so $l(z_n)$ converges to $l(p)=\infty$. If we set $w_n=l(z_n)$ $(n=0, 1, 2, \cdots)$, then $w_n=\alpha^n w_0$. Since $w_n=l(z_n)$ converges to ∞, it follows $\alpha>1$. Therefore,

$$\sum_{n=0}^{\infty}\frac{\operatorname{Im} w_n}{1+|w_n|^2}\leq\sum_{n=0}^{\infty}\frac{1}{|w_n|}=\frac{1}{|w_0|}\sum_{n=0}^{\infty}\frac{1}{\alpha^n}<\infty .$$

It follows that $\{w_n\}$ is the zeros of a Blaschke product defined on H [4, p. 55]. Since $z_n=l^{-1}(w_n)$ $(n=0, 1, 2, \cdots)$ and l is a bi-holomorphic map of D onto H, it guarantees that $\{z_n\}$ is the zeros of a non-zero bounded analytic function on D, and so $\sum_{n=0}^{\infty}(1-|z_n|)<\infty$ (cf. [6]).

Next suppose that m has (iv). Let p be the unique fixed point of m on Γ. Let l be the bi-holomorphic map of D onto H such that $l(p)=\infty$ and $l(-p)=0$. Then $l\circ m\circ l^{-1}(w)=w+\gamma$ $(w\in H)$ for some non-zero real number γ, since it is an analytic automorphism on H which fixes ∞ only [2, p. 59]. If we set $w_n=l(z_n)$, then $w_n=w_0+n\gamma$ $(n=0, 1, 2, \cdots)$. Since $\sum_{n=0}^{\infty}\operatorname{Im} w_n/(1+|w_n|^2)\leq\alpha\sum_{n=1}^{\infty}1/n^2<\infty$ for some $\alpha>0$, $\{w_n\}$ is the zeros of a Blaschke product defined on H and so $\{z_n\}$ is the zeros of a non-zero bounded analytic function on D. It follows that $\sum_{n=0}^{\infty}(1-|z_n|)<\infty$.

Let β be a complex number with $|\beta|=1$. Then $(1+\beta)/\sqrt{\beta}$ is real. Since the trace of βM_b is $(1+\beta)/\sqrt{\beta(1-|b|^2)}$, we have the following by [8, Theorem, p. 5].

LEMMA 2.4. *Let β be a complex number with $|\beta|=1$ and $b\in D$. Then βM_b is elliptic if and only if $(1+\beta)/\sqrt{\beta(1-|b|^2)}<2$, where a branch of $\sqrt{\beta(1-|b|^2)}$ is chosen so that $(1+\beta)/\sqrt{\beta(1-|b|^2)}$ is non-negative.*

We are now in a position to discuss conditions under which a codimention 1 linear isometry on the disc algebra is a shift operator.

THEOREM 2.5. *Let φ be a codimension 1 linear isometry on the disc algebra A and $\varphi f=\alpha M_a f(\beta M_b)$ for $f\in A$. Let $\{d_n\}$ be the backward orbit of a by βM_b. Then the following four conditions are mutually equivalent.*

(a) *φ is a shift operator.*

(b) *βM_b is the identity or elliptic.*

(c) $\sum_{n=0}^{\infty}(1-|d_n|)=\infty$.

(d) *$\beta=1$ and $b=0$, or $(1+\beta)/\sqrt{\beta(1-|b|^2)}<2$, where a branch of $\sqrt{\beta(1-|b|^2)}$ is chosen so that $(1+\beta)/\sqrt{\beta(1-|b|^2)}$ is non-negative.*

PROOF. The equivalence of (b) and (d) follows from Lemma 2.4. By Lemma 2.2 and 2.3, (b) and (c) are equivalent.

(c)→(a). If $f \in \bigcap_{n=1}^{\infty} \varphi^n(A)$, by (∗), we get $f(d_n)=0$ $(n=0, 1, 2, \cdots)$. Since $\sum_{n=0}^{\infty}(1-|d_n|)=\infty$ and f is bounded and analytic on D, it follows $f=0$.

To prove the theorem, it remains only to show that (a)→(b). Suppose that βM_b is hyperbolic or parabolic. Then $m=(\beta M_b)^{-1}$ is also hyperbolic or parabolic and $d_n = m^n(d_0)$ $(n=0, 1, 2, \cdots)$. Hence d_n converges to the Denjoy-Wolff point d of m and $\sum_{n=0}^{\infty}(1-|d_n|)<\infty$ by (b) of Lemma 2.3.

Let B be the Blaschke product having $\{d_n\}$ as its zeros. If we put $f(z)=(z-d)B(z)$, then $f \in A$, $f \neq 0$ and $f(d_n)=0$ $(n=0, 1, 2, \cdots)$. Hence by (∗), for any positive integer n we can find a $g \in A$ such that $f=\varphi^n g$. Thus $\bigcap_{n=1}^{\infty} \varphi^n(A) \neq \{0\}$ and φ is not a shift operator. The proof is completed.

EXAMPLES. Let φ be a codimension 1 linear isometry on the disc algebra A and $\varphi f = \alpha M_a f(\beta M_b)$ for $f \in A$. From Theorem 2.5, the following are immediate.

(a) If $\beta=1$, then φ is a shift operator on A if and only if $b=0$.
(b) If $\beta=-1$, then φ is always a shift operator on A.
(c) If $\beta=\pm i$, then φ is a shift operator on A if and only if $|b|^2<1/2$.

ACKNOWLEDGEMENTS. The authors would like to express their thanks to Professor O. Hatori for helping them with valuable suggestions in the latter half of section 2.

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