Illinois Journal of Mathematics
Volume 53, Number 2, Summer 2009, Pages 575–580
S 0019-2082

ISOMETRIES BETWEEN NORMED SPACES WHICH ARE SURJECTIVE ON A SPHERE

RUIDONG WANG

ABSTRACT. In this paper, we study the extension of isometric operator between unit spheres of normed spaces, and give an equivalent statement of Tingley problem. We also give another statement of Mazur–Ulam theorem: Let $V : E \to F$ be an isometric operator, and $V|_{S(E)}$ denotes the operator V restricted to the set $S(E)$. If $V|_{S(E)}$ is an onto isometric operator from $S(E)$ to $S(F)$, then V must be linear.

1. Introduction

Throughout this paper, both E and F will denote real normed linear space. We use $S_\delta(E)$ to denote the set $\{z \in E : \|z\| = \delta\}$, where $\delta \in \mathbb{R}^+$. To simplify notation, we write $S(E) := S_1(E)$, i.e., the unit sphere of E. If $x, y \in E$, we denote by $[x, y]$ the set $\{z \in E : z = \lambda x + (1-\lambda)y, 0 \le \lambda \le 1\}$.

A mapping $V : E \to F$ is said to be an isometry if

$$\|Vx - Vy\| = \|x - y\| \quad (\forall x, y \in E).$$

As normed spaces, we say E and F are congruent if there is a linear isometric operator T from E onto F. Mazur and Ulam had shown that any surjective isometry between two real normed spaces must be an affine map. Therefore, two normed spaces are congruent if and only if the two spaces are isometric; and the metric structure determines the linear structure. Mankiewicz [1] extended that any surjective isometry between the convex bodies (or open connected subsets) of two normed spaces can be extended to a surjective affine map between the two spaces. Especially, two normed spaces are congruent if and only if their unit balls are isometric.

Received October 15, 2008; received in final form June 11, 2009.

The author was supported by NSFC (10571090) and the Doctoral Programe Foundation of Institution of Higher Education (20060055010).

2000 *Mathematics Subject Classification.* 46B20.

Tingley proposed the following problem in [2]: Let X and Y be normed spaces with unit spheres $S(X)$ and $S(Y)$. Suppose that $V_0 : S(X) \to S(Y)$ is an onto isometry. Is V_0 necessarily the restriction to $S(X)$ of a linear, or affine transformation on X?

It will be very difficult to answer this question. Until recently, some affirmative results have been obtained, which had been shown in [4].

Since there is not much linear structure in the unit sphere of a normed linear space, it is difficult to answer the above problem. As normed linear space is metric space, we may ask the following question: Let X and Y be normed spaces with unit spheres $S(X)$ and $S(Y)$. Suppose that $V_0 : S(X) \to S(Y)$ is an onto isometry. Is V_0 necessarily the restriction to $S(X)$ of an isometric operator on X?

In [5], Liu and Zhang give the following proposition.

PROPOSITION. *Let E and F be Banach spaces. Assume that $V_0 : S(E) \to S(F)$ is a surjective isometry. If V_0 can be extended to an isometry from E into F, then there exists a linear isometry $\widetilde{V}$ from E onto F such that $\widetilde{V}|_{S(E)} = V_0$.*

In the proof of this proposition, the authors show that if V_0 can be extended to an isometric operator $\widehat{V} : E \to F$, then the positive homogeneous extension $\widetilde{V}(x) = \|x\| V_0(\frac{x}{\|x\|})$ is surjective and isometry, by Mazur–Ulam theorem, $\widetilde{V}$ is an linear isometry.

In this paper, we give a uniqueness theorem. It assumes V_0 is the restriction of some into-isometry $\widehat{V}$, and shows that $\widehat{V}$ is necessarily the positive homogeneous extension $\widetilde{V}$, i.e., $\widehat{V} = \widetilde{V}$, which improves the above proposition. This shows that the above two questions are equivalent, and give another statement of Mazur–Ulam theorem: If $V : E \to F$ is an into-isometry, so that $V(S(E)) = S(F)$, then V is surjective and linear.

2. Main result

LEMMA 1. *Let E be a two-dimensional space whose unit sphere S contains a nontrivial face $[u,v]$. Fix $\alpha > 1$ and a point a point $z \in E$ with $\|z\| = \alpha$ so that $z/\alpha \in [u,v]$. Then $S \cap S_{\alpha-1}(z)$ is a subinterval $I_z \in [u,v]$.*

Moreover, if $\alpha \le 2$, then I_z is a proper subinterval of $[u,v]$ unless $\alpha = 2$ and $z = u + v$.

Proof. It follows from the triangle inequality that $S \cap S_{\alpha-1}(z) \subset [u,v]$. Write $z/\alpha = \lambda u + (1-\lambda)v$ with $0 \le \lambda \le 1$, and assume $x \in S \cap S_{\alpha-1}(z)$. It follows that $x = su + (1-s)v$ for some $0 \le s \le 1$.

From $x \in S_{\alpha-1}(z)$ and the fact that $\frac{z-x}{\alpha-1}$ lies on the line passing through u and v (see the formula below), it follows that $\frac{z-x}{\alpha-1} \in [u,v]$. Writing

$$\frac{z-x}{\alpha-1} = \frac{\alpha\lambda u + \alpha(1-\lambda)v - su - (1-s)v}{\alpha-1} = \frac{\alpha\lambda - s}{\alpha-1}u + \left(1 - \frac{\alpha\lambda - s}{\alpha-1}\right)v$$

we see that s must also satisfy the condition $0 \le \frac{\alpha\lambda-s}{\alpha-1} \le 1$, or

$$\alpha\lambda - \alpha + 1 \le s \le \alpha\lambda.$$

Thus, $x \in S \cap S_{\alpha-1}(z)$ is the interval I_z, which corresponds to the values of s in the interval

$$[\max\{0, \alpha\lambda - \alpha + 1\}, \min\{1, \alpha\lambda\}] \subset [0,1].$$

If $\alpha \le 2$ and $v \in I_z$, then $\alpha\lambda \ge 1$, hence $\alpha\lambda - \alpha + 1 \ge 2 - \alpha \ge 0$ with equality $\alpha\lambda - \alpha + 1 = 0$ (i.e., $u \in I_z$) only when $\alpha = 2$ and $\lambda = \frac{1}{2}$. □

LEMMA 2. *Let E be a normed space, and assume that $\|z_1\| = \|z_2\| = \alpha \le 2$ and that $\|z_1 - x\| = \|z_2 - x\|$ for every $x \in S(E)$. Then $z_1 = z_2$.*

Proof. Assume $z_1 \ne z_2$ and $\alpha \ge 1$ (the case $\alpha < 1$ is trivial, just take as z one of the endpoints of the cord through z_1 and z_2). Passing to the subspace spanned by z_1 and z_2 we may assume E is two-dimensional, and it then follows from the triangle inequality that $S \cap S_{\alpha-1}$ contains the segment $[z_1/\alpha, z_2/\alpha]$. It follows that $S(E)$ has a face $[u,v]$ parallel to the line passing through z_1 and z_2.

By Lemma 1, $S \cap S_{\alpha-1}(z_j)$ are intervals I_{z_j}, and when $\alpha \le 2$ and $z_1 \ne z_2$ the explicit formula for the intervals shows that they are different. But this means that there is a $z \in S$ with $\|z_1 - z\| = \alpha - 1$ and $\|z_2 - z\| \ne \alpha - 1$ (or vice-versa) contradicting the assumption of the lemma. □

In Lemma 2, it is necessary for us to assume that $\alpha \le 2$. If $\alpha > 2$, we have the following counter example.

EXAMPLE 1. If $\alpha > 2$, let $z_1 = (\alpha, \alpha - 2), z_2 = (\alpha, \frac{\alpha-2}{2}) \in l^{\infty}_{(2)}$, where $l^{\infty}_{(2)}$ denotes linear space $\mathbb{R}^2$ with sup-norm, i.e., for any $(\beta_1, \beta_2) \in l^{\infty}_{(2)}$, $\|(\beta_1, \beta_2)\| = \max\{|\beta_1|, |\beta_2|\}$. Then $\|z_1\| = \|z_2\| = \alpha$, and for any $z = (\alpha_1, \alpha_2) \in S(l^{\infty}_{(2)})$, we have $\|z_1 - z\| = \|z_2 - z\|$.

Indeed, since $0 \le |\alpha_1|, |\alpha_2| \le 1$, we have

$$\alpha - \alpha_1 > 1, \quad -1 < \alpha - 2 - \alpha_2 \le \alpha - \alpha_1$$

and

$$-1 < \frac{\alpha-2}{2} - \alpha_2 \le \alpha - \alpha_1.$$

From the three inequalities above, we obtain that

$$\|z_1 - z\| = \|z_2 - z\| = \alpha - \alpha_1.$$

But it is obviously that $z_1 \ne z_2$.

From the following theorem, we can know exactly the way we construct the above counter example.

THEOREM 3. *Let X be a Banach space*, $\dim(X)=2$. *If $[u,v]\subset S(X)$ and*

$$\beta=\max_{x=\xi_1u+\xi_2v\in S(X)}\{|\xi_1|,|\xi_2|\}.$$

Then for any $\alpha>2\beta$, there exists $z_1,z_2\in S_\alpha(X)$, such that for any $x\in S(X)$, we have $\|z_1-x\|=\|z_2-x\|$, but $z_1\neq z_2$.

Proof. Letting $z_1=\frac{\alpha}{2}u+\frac{\alpha}{2}v$ and $z_2=(\alpha-\beta)u+\beta v$. Then for any $x=\xi_1u+\xi_2v\in S(X)$, we have

$$z_1-x=\left(\frac{\alpha}{2}-\xi_1\right)u+\left(\frac{\alpha}{2}-\xi_2\right)v$$

and

$$z_2-x=(\alpha-\beta-\xi_1)u+(\beta-\xi_2)v.$$

Since $[u,v]\subset S(X)$, $\alpha>2\beta>0$ and $\beta>\max\{\xi_1,\xi_2\}$, it is easy to see that $\|z_1\|=\|z_2\|=\alpha$ and $z_1\neq z_2$. We also obtain that

$$\|z_1-x\|=\left\|\left(\frac{\alpha}{2}-\xi_1\right)u+\left(\frac{\alpha}{2}-\xi_2\right)v\right\|=\alpha-\xi_1-\xi_2$$

and

$$\|z_2-x\|=\|(\alpha-\beta-\xi_1)u+(\beta-\xi_2)v\|=\alpha-\xi_1-\xi_2.$$

So $\|z_1-x\|=\|z_2-x\|$. □

THEOREM 4. *Let $V:E\to F$ be a isometric operator, and $V|_{S(E)}$ denotes the operator V restricted to the set $S(E)$. If $V|_{S(E)}$ is an onto isometric operator from $S(E)$ to $S(F)$, then V must be linear.*

Proof. First, we will show that $V(\theta)=\theta$.

Indeed, if $V(\theta)\neq\theta$, since $V|_{S(E)}$ is an onto isometric operator from $S(E)$ to $S(F)$, there exists $x\in S(E)$ such that $V(x)=-\frac{V(\theta)}{\|V(\theta)\|}$. Then

$$\left\|\frac{V(\theta)}{\|V(\theta)\|}+V(\theta)\right\|=\|V(x)-V(\theta)\|=\|x-\theta\|=1.$$

But

$$\left\|\frac{V(\theta)}{\|V(\theta)\|}+V(\theta)\right\|=1+\|V(\theta)\|.$$

Contradict to $V(\theta)\neq\theta$.

Second, we show that $\overline{\operatorname{span}\{V(E)\}}=F$. Indeed,

$$\overline{\operatorname{span}\{V(E)\}}\supset\overline{\operatorname{span}\{V(S(E))\}}\supset\overline{\operatorname{span}\{S(F)\}}=F.$$

By Figiel's theorem ([3, p. 401, Theorem 9.4.2]), there is a continuous linear operator T mapping F into E and such that the superposition TV is an identity on E. The operator T is uniquely determined and it has norm one.

For any $x, y \in S(E), 0 \le \lambda \le 1$, we have

$$\begin{aligned} \|x-\lambda y\| &= \|TV(x)-\lambda TV(y)\| \\ &= \|TV(x)-T(\lambda V(y))\| \\ &\le \|V(x)-\lambda V(y)\|. \end{aligned}$$

We now show that the inequality above must be an equality, if not, there exist $x_0, y_0 \in S(E)$ and $0 < \lambda_0 < 1$ such that

$$\|x_0-\lambda_0 y_0\| < \|V(x_0)-\lambda_0 V(y_0)\|.$$

The line $L(V(x_0), \lambda_0 V(y_0))$, which contains $V(x_0)$ and $\lambda_0 V(y_0)$, and $S(F)$ intersect in a point $\widetilde{z}_0$, which is different from the point $V(x_0)$. Since $V|_{S(E)}$ is an onto isometric operator from $S(E)$ to $S(F)$, it follows that there exists $z_0 \in S(E)$ such that $V(z_0) = \widetilde{z}_0$, and we have

$$\|V(x_0)-V(z_0)\| = \|V(x_0)-\lambda_0 V(y_0)\| + \|V(z_0)-\lambda_0 V(y_0)\|,$$

and

$$\|z_0-\lambda_0 y_0\| \le \|V(z_0)-\lambda_0 V(y_0)\|.$$

So

$$\begin{aligned} \|x_0-z_0\| &\le \|x_0-\lambda_0 y_0\| + \|z_0-\lambda_0 y_0\| \\ &< \|V(x_0)-\lambda_0 V(y_0)\| + \|V(z_0)-\lambda_0 V(y_0)\| \\ &= \|V(x_0)-V(z_0)\| \\ &= \|x_0-z_0\|. \end{aligned}$$

The above inequality is impossible, so $\|V(x)-\lambda V(y)\| = \|x-\lambda y\|$.

For any $\eta > 1$ and $x, y \in S(E)$, we can still get the similar equality as above, since

$$\begin{aligned} \|V(x)-\eta V(y)\| &= \eta \left\| \frac{1}{\eta} V(x) - V(y) \right\| \\ &= \eta \left\| \frac{1}{\eta} x - y \right\| \\ &= \|x-\eta y\|. \end{aligned}$$

If $\gamma \le 2$, since V is isometry and by the above equality, for any $x, y \in S(E)$, we obtain that

$$\|V(x)-\gamma V(y)\| = \|x-\gamma y\| = \|V(x)-V(\gamma y)\|.$$

As $V|_{S(E)}$ is an onto isometric operator from $S(E)$ to $S(F)$ and by Lemma 2, for any $y \in S(E)$, we have

$$V(\gamma y) = \gamma V(y).$$

Thus, we obtain that $V|_{S_\gamma(E)}$ is an onto isometric operator from $S_\gamma(E)$ to $S_\gamma(F)$, for any $\gamma \leq 2$.

By a similar argument as above, since $V|_{S_2(E)}$ is an onto isometric operator from $S_2(E)$ to $S_2(F)$, we obtain that

$$V(\zeta x) = \zeta V(x),$$

for any $x \in S(E)$ and $\zeta \leq 2^2$.

By induction, we obtain that $V(\xi x) = \xi V(x)$, for any $\xi \in \mathbb{R}^+$ and $x \in S(E)$. Since $V|_{S(E)}$ is an onto isometric operator from $S(E)$ to $S(F)$, so V is an onto isometric operator form E to F. Our assertion is immediate from Mazur–Ulam theorem. □

Acknowledgments. I express my deep gratitude to my PhD advisor, Professor Ding Guanggui, for his encouragement and advice. I wish to thank the referee for making several suggestions which improved this paper.

References

[1] P. Mankiewicz, *On extension of isometries in normed linear spaces*, Bull. Acad. Polon. Sci. Math. Astronomy Phys. **20** (1972), 367–371. MR 0312214

[2] D. Tingley, *Isometries of the unit sphere*, Geometriae Dedicata **22** (1987), 371–378. MR 0887583

[3] S. Rolewicz, *Metric linear spaces*, D. Reidel Publishing Company, 1972. MR 0438074

[4] G. G. Ding, *The isometric extension of the into mapping from the $\mathfrak{L}^\infty$-type space to some normed space E*, Illinois J. Math. **52** (2007), 445–453. MR 2342668

[5] R. Liu and L. Zhang, *On extension of isometries and approximate isometries between unit spheres*, J. Math. Anal. Appl. **352** (2009), 749–761. MR 2501920

Ruidong Wang, School of Mathematics, Nankai University, Tianjin, 300071, China and Department of Mathematics, Tianjin University of Technology, Tianjin 300191, China

E-mail address: wangruidong@mail.nankai.edu.cn