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ISOMETRIES IN ORDERED GROUPS

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In [6], K. L. N. Swamy describes geometric properties of the commutative lattice-ordered groups autometrized by means of the absolute values of their elements. The isometries of lattice-ordered groups are studied by K. L. N. Swamy in [7], [8], [9] for the commutative case and by J. Jakubík in [4], [5] for the general case.

In this paper the notions of the autometry and isometry are generalized to any ordered groups and studied for the class of 2-isolated Riesz groups in particular.

We use the terminology and notation from the books [1] and [3] (in the additive form).

1. We recall some notions and notations used in the paper.

Let $G = (G, +)$ be an ordered group. If $A \subseteq G$, then we denote $L(A) = \{x \in G;$ $x \leqq a$ for each $a \in A\}$ and $U(A) = \{y \in G;\ a \leqq y$ for each $a \in A\}$. For $A = \{a_1, \ldots, a_n\}$ we shall write $L(a_1, \ldots, a_n)$ instead of $L(\{a_1, \ldots, a_n\})$. For each $a \in G$, $|a| = U(a, -a)$. If $a, b \in G$, $a \leqq b$, then $[a, b]$ means $\{x \in G;\ a \leqq x \leqq b\}$.

We say that an ordered group G is 2-isolated if $a \geqq -a$ implies $a \geqq 0$ for each $a \in G$. A Riesz group is any ordered group which is directed and satisfies the Riesz interpolation property, i.e., for each $a_i, b_j \in G$ $(i, j = 1, 2)$ such that $a_i \leqq b_j$ $(i, j = 1, 2)$ there exists $c \in G$ such that $a_i \leqq c \leqq b_j$ $(i, j = 1, 2)$. (See [2], [3].)

In [6], Swamy introduced the notion of an autometrized commutative group as a system $(G, +, \leqq, d)$, where

(i) $(G, +, \leqq)$ is an ordered group,

(ii) $G \times G \to G$ is a mapping such that

(α) $d(a, b) \geqq 0$ with equality iff $a = b$;

(β) $d(a, b) = d(b, a)$;

(γ) $d(a, c) \leqq d(a, b) + d(b, c)$,

and showed that each commutative lattice-ordered group is autometrized by means of $d(a, b) = (a - b) \vee (b - a)$. The following definition describes the notion of autometry in a different way to be applicable for the largest possible classes of ordered groups. However, in the case of lattice-ordered groups autometrized by means of the absolute values of their elements in the sense of lattice-ordered groups or ordered ones, both the definitions are equivalent. Therefore the results obtained in this part of the paper are generalizations of some Swamy's results from [6].

Definition 1.1. An *automerized ordered group* is a system $(G, +, \leqq, d)$, where

(i) $(G, +, \leqq)$ is an ordered group,

(ii) d is a mapping $G \times G \to \exp G$ such that for all $a, b, c \in G$

(α) $d(a, b) \subseteq U(0)$ and $d(a, b) = U(0)$ iff $a = b$;

(β) $d(a, b) = d(b, a)$;

(γ) $d(a, c) \supseteq d(a, b) + d(b, c)$.

Theorem 1.2. *Any 2-isolated commutative Riesz group G is autometrized by $d(a, b) = |a - b|$ for each $a, b \in G$.*

Proof. It is known (see [2, Thm. 2.2], [3, Thm. I.5.13]) that in any Riesz group G

$U(a_1, \ldots, a_n) + U(b_1, \ldots, b_m) = U(a_1 + b_1, a_1 + b_2, \ldots, a_n + b_m)$ for each $a_1, \ldots, a_n, b_1, \ldots, b_m \in G$. Hence if $a, b, c \in G$, then

$$|a - b| + |b - c| = U(a - b, b - a) + U(b - c, c - b) =$$
$$= U(a - b + b - c,\ a - b + c - b,\ b - a + b - c,\ b - a + c - b) \subseteq$$
$$\subseteq U(a - c, -a + c) = |a - c|\,.$$

Since G is 2-isolated, $d(a, b) \subseteq U(0)$ and $d(a, b) = U(0)$ iff $a = b$.

Now we introduce the notion of "betweenness" in an ordered set A as follows: If $a, b, x \in A$, then x lies between a and b iff $x \in U(L(a, b)) \cap L(U(a, b))$. It is clear that if A is a lattice, then x lies between a and b iff $a \wedge b \leqq x \leqq a \vee b$.

If G is an ordered group, then we say that for $a, b, x \in G$, $B(a, x, b)$ holds iff $|a - b| = |a - x| + |x - b|$.

Lemma 1.3. *For any elements a, b of a Riesz 2-isolated group G we have $|a - b| = U(a, b) + U(-a, -b)$.*

Proof. Let $a, b \in G$. Since G is a Riesz group, we have $U(a, b) + U(-a, -b) = U(a - a, a - b, b - a, b - b) = U(0, a - b, b - a)$ and since G is 2-isolated, we obtain $U(0, a - b, b - a) = |a - b|$.

Lemma 1.4. *Let G be a commutative ordered group, $a, b \in G$. Then $L(a, b) = -U(a, b) + a + b$.*

Proof. Let $x \in U(a, b)$. Then $-x + a + b \leqq b$, $-x + a + b \leqq a$, hence $y = -x + a + b \in L(a, b)$. Conversely, let $y \in L(a, b)$, $y = z + a + b$. Then $z + a + b \leqq a$, $z + a + b \leqq b$, thus $z \leqq -b$, $z \leqq -a$, therefore $-z \in U(a, b)$. This means $y \in -U(a, b) + a + b$.

Lemma 1.5. *For any elements a, b of a 2-isolated commutative Riesz group G we have $|a| + |b| = |a + b| \cap |a - b|$.*

Proof. a) Let $x \in |a| + |b|$. Then $x = a_1 + b_1$, $a_1 \geqq a, -a$, $b_1 \geqq b, -b$, hence $x \in |a + b| \cap |a - b|$, and so $|a| + |b| \subseteq |a + b| \cap |a - b|$.

b) Suppose that $y \in |a + b| \cap |a - b|$. Then, by Lemma 1.3, $y \in U(a, -b) + U(-a, b)$ and $y \in U(a, b) + U(-a, -b)$, thus $y = u_1 + u_2 = v_1 + v_2$, where $u_1 \geqq a, -b$, $u_2 \geqq -a, b$, $v_1 \geqq a, b$, $v_2 \geqq -a, -b$. Hence $y \in U(a + b, -a + b, a - b, -a - b) = U(a, -a) + U(b, -b) = |a| + |b|$, therefore $|a + b| \cap |a - b| \subseteq |a| + |b|$.

As an immediate consequence we obtain the following lemma.

Lemma 1.6. *For each a, b of a 2-isolated commutative ordered group we have* $|a \pm b| \supseteq |a| + |b|$.

Theorem 1.7. *If G is a 2-isolated commutative Riesz group, then for any $a, b, x \in G$, $B(a, x, b)$ holds iff x lies between a and b.*

Proof. a) Let $x \in U(L(a, b)) \cap L(U(a, b))$, i.e. $x \geqq c$, for each $c \leqq a, b$, and $x \leqq d$ for each $d \geqq a, b$. Let $z \in |a - b|$. By Lemma 1.3 we have $z = z_1 + z_2$, where $z_1 \geqq a, b$, $-z_2 \leqq a, b$, hence $z_1 \geqq x$, $z_2 \geqq -x$. If we put $a + b = z_1 + z'_1$, then, by Lemma 1.4, $z'_1 \leqq a, b$, thus $x \geqq z'_1$. Therefore $z - (a + b - 2x) = z_1 + z_2 - a - b + 2x = z_2 - z'_1 + 2x = (z_2 + x) + (-z'_1 + x) \geqq 0$. Hence $|a - b| \subseteq U(a + b - 2x)$.

Now, if we consider z'_2 such that $a + b = -z_2 + z'_2$, then, by Lemma 1.4, $z'_2 \geqq a, b$ holds, hence $x \leqq z'_2$. Therefore we obtain $z - (-a - b + 2x) = z + a + b - 2x = z_1 + z_2 - z_2 + z'_2 - 2x = (z_1 - x) + (z'_2 - x) \geqq 0$, and so $|a - b| \subseteq U(-a - b + 2x)$.

This means that $|a - b| \subseteq |a + b - 2x|$.

From $|a - b| \cap |a + b - 2x| = |a - b|$ and from $|a - x| + |x - b| = |a - b| \cap |a + b - 2x|$ (by Lemma 1.5), we obtain $|a - x| + |x - b| = |a - b|$.

b) Let $|a - x| + |x - b| = |a - b|$. Then, by Lemma 1.5, $|a - b| \subseteq |a + b - 2x|$, hence $|a - b| \subseteq U(a + b - 2x)$, $|a - b| \subseteq |-a - b + 2x|$. Let $c_2 \in L(a, b)$. Then for c_1 such that $a + b = c_1 + c_2$ we have $c_1 \geqq a, b$. Since $-c_2 \in U(-a, -b)$, $c = c_1 - c_2 \in |a - b|$. Thus, by Lemma 1.3, $c - (a + b - 2x) \geqq 0$, i.e. $c_1 - c_2 - c_1 - c_2 + 2x \geqq 0$. Therefore $2(-c_2 + x) \geqq 0$ and the hypothesis that G is 2-isolated implies $-c_2 + x \geqq 0$. This means $x \in U(L(a, b))$.

Similarly, if d_1 is an arbitrary element of $U(a, b)$, then for d_2 such that $a + b = d_1 + d_2$, we have $d_2 \in L(a, b)$, hence also $d = d_1 - d_2 \in |a - b|$. Thus $d - (-a - b + 2x) \geqq 0$, i.e. $d_1 - d_2 - d_1 + d_2 - 2x \geqq 0$, and by the fact that G is 2-isolated, we obtain $d_1 \geqq x$, and so $x \in L(U(a, b))$.

But this means that x lies between a and b.

Theorem 1.8. *If G is a (non-commutative) 2-isolated Riesz group, $a, b, x \in G$, $a \leqq b$, then the following conditions are equivalent:*

(i) $x \in [a, b]$,

(ii) $|a - b| = |b - x| + |x - a|$.

Proof. (i) $\Rightarrow$ (ii): If $x \in [a, b]$, then $|a - b| = U(b - a)$, $|b - x| = U(b - x)$, $|x - a| = U(a - x)$. Furthermore, in any ordered group the identity $U(b - x) + U(x - a) = U(b - a)$ holds.

(ii) $\Rightarrow$ (i): Let $a, b, x \in G$, $a \leqq b$ and $|a - b| = |b - x| + |x - a|$. G is a Riesz group, hence $U(b - a) = U(b - x, x - b) + U(x - a, a - x) = U(b - a, b - x + a - x, x - b + x - a, x - b + a - x)$. Suppose that $x \notin [a, b]$. Then at least one of the following cases occurs:

a) $x < a$, b) $b < x$, c) $a \parallel x$, d) $b \parallel x$.

a) If $x < a$, then $-x < -x + a - x$ and $-a < -x$. Hence $-a < -x + a - x$, thus $b - a < b - x + a - x$. But then $|b - x| + |x - a| \subset |a - b|$, a contradiction.

b) If $b < x$, then similarly $b - a < x - b + x - a$, hence $|b - x| + |x - a| \subset |a - b|$, a contradiction.

c) Let $a \parallel x$ and $b - x + a - x \leqq b - a$. Then $-(x - a) \leqq x - a$ and the fact that G is 2-isolated yields $x - a \geqq 0$. Hence $b - x + a - x \nleqq b - a$, therefore $|b - x| + |x - a| \subset |a - b|$, a contradiction.

d) If $b \parallel x$, then analogously $x - b + x - a \nleqq b - a$, hence $|b - x| + |x - a| \subset |a - b|$, a contradiction.

Note. It is clear that for a commutative case, Theorem 1.8 is an immediate consequence of Theorem 1.7.

For autometrized ordered groups we shall now consider two types of linearity that are generalizations of the corresponding notions for lattice-ordered groups. (See [6].)

In the following the expression $d(a, b)$ means $|a - b|$.

Definition 1.9. If A is an n-element set ($n \geqq 3$) of mutually distinct elements of an autometrized ordered group G, then A is called:

a) *D-linear*, if there exists a labelling $(p_1, \ldots, p_n)$ of A such that

$$d(p_1, p_n) = \sum_{i=1}^{n-1} d(p_i, p_{i+1})\,;$$

b) *B-linear*, if there exists a labelling $(p_1, \ldots, p_n)$ of A such that $B(p_i, p_j, p_k)$ for all $1 \leqq i < j < k \leqq n$.

Lemma 1.10. *For any elements a, b of a commutative ordered group G we have $|a \pm b| \subseteq |a| - |b|$.*

Proof. Let $x \in |a + b|$. Suppose that $z \in |b|$ and $y = x + z$. Since $x \geqq a + b$, $-a - b$, $z \geqq b, -b$, we have $y \geqq a + b - b = a$, $y \geqq -a - b + b = -a$, hence $y \in |a|$, and so $x \in |a| - |b|$. Similarly for $|a - b|$.

Theorem 1.11. *If G is a 2-isolated commutative Riesz group, then its n-element subset A is B-linear if and only if A is D-linear.*

Proof. a) The assertion, that any B-linear subset is also D-linear can be proved formally in the same manner as in [6, Theorem 3] for lattice-ordered groups.

b) Let A be a D-linear and let $(p_1, \ldots, p_n)$ be its labelling such that

$$|p_1 - p_n| = \sum_{i=1}^{n-1} |p_i - p_{i+1}| .$$

If $1 \leqq i < k \leqq n$, then by Lemmas 1.6 and 1.10

$$|p_i - p_k| = |\sum_{t=i}^{k-1} (p_t - p_{t+1})| \supseteq \sum_{t=i}^{k-1} |p_t - p_{t+1}| =$$

$$= |p_1 - p_n| - (\sum_{t=1}^{i-1} |p_t - p_{t+1}| + \sum_{t=\kappa}^{n-1} |p_t - p_{t+1}|) \supseteq$$

$$\supseteq |(p_1 - p_n) - (\sum_{t=1}^{i-1} (p_t - p_{t+1}) + \sum_{t=k}^{n-1} (p_t - p_{t+1})| = |p_k - p_i| ,$$

i.e.

$$|p_i - p_k| = \sum_{t=i}^{k-1} |p_t - p_{t+1}| .$$

Now, if $i < j < k$, then

$$|p_i - p_j| = \sum_{t=i}^{j-1} |p_t - p_{t+1}| , \quad |p_j - p_k| = \sum_{t=j}^{k-1} |p_t - p_{t+1}| ,$$

hence $|p_i - p_j| + |p_j - p_k| = |p_i - p_k|$, therefore $B(p_i, p_j, p_k)$ holds.

2. Swamy introduced the notion of isometry in a commutative lattice-ordered group in [7] and studied properties of the isometries especially in [8] and [9]. Jakubík studied the isometries of any (non-commutative) lattice-ordered groups in [4] and [5].

In this part of the paper, we shall show that the notion of isometry can be generalized to any ordered groups. Further, we shall obtain some properties of isometries of ordered groups, in particular, of 2-isolated Riesz groups.

Definition 2.1. a) If G is an ordered group, then a bijection $f : G \to G$ is said to be an *isometry* in G if

(1) $\forall a, b \in G;\ |a - b| = |f(a) - f(b)|$.

b) An isometry f is called *strong* if

(2) $\forall a, b \in G;\ f(U(L(a, b)) \cap L(U(a, b))) = U(L(f(a), f(b))) \cap L(U(f(a), f(b)))$.

Theorem 2.2. *Each isometry in a 2-isolated commutative Riesz group G is strong.*

Proof. Let $a, b, x \in G$ and let f be an isometry in G. Then $x \in f(U(L(a, b)) \cap \cap L(U(a, b)))$ holds if and only if $x = f(y)$, where y lies between a and b, and (by Thm. 1.7) this is equivalent to $B(a, y, b)$, i.e., to $|a - b| = |a - y| + |y - b|$. Since f is an isometry, the last condition is equivalent to $|f(a) - f(b)| = |f(a) -$

$- f(y)| + |f(y) - f(b)| = |f(a) - x| + |x - f(b)|$, and this holds if and only if $B(f(a), x, f(b))$.

Note. Jakubík proved in [5] that if a bijection f of a lattice-ordered non-commutative group G onto G satisfies the condition (1) (where $|x| = x \vee -x$), then

$$\forall x, y \in G\,; \quad f([x \wedge y, x \vee y]) = [f(x) \wedge f(y), f(x) \vee f(y)]\,.$$

The question if the condition (1) implies the condition (2) also in the case of an arbitrary (non-commutative) 2-isolated Riesz group G is open.

Theorem 2.3. *If f is an isometry in a 2-isolated Riesz group G, $a, b \in G$, $a \leqq b$, $f(a) \leqq f(b)$, then $f([a, b]) = [f(a), f(b)]$.*

Proof. It is clear that f^{-1} is an isometry in G as well. Hence, by Theorem 1.8, $f(x) \in [f(a), f(b)]$ if and only if $|f(a) - f(b)| = |f(b) - f(x)| + |f(x) - f(a)|$ and this is equivalent to the condition $|a - b| = |b - x| + |x - a|$ and (by Theorem 1.8) to the condition $x \in [a, b]$. Therefore $x \in [a, b]$ if and only if $f(x) \in [f(a), f(b)]$.

Theorem 2.4. *If f is an isometry in a commutative 2-isolated Riesz group G, $a, b \in G$, $a \leqq b$, $f(b) \leqq f(a)$, then $f([a, b]) = [f(b), f(a)]$.*

Proof. If $x \in [a, b]$, then $|a - b| = |b - x| + |x - a|$, hence $|f(b) - f(a)| = |f(a) - f(x)| + |f(x) - f(a)|$, and so $f(x) \in [f(b), f(a)]$. The inclusion $[f(b), f(a)] \subseteq f([a, b])$ can be obtained similarly by means of f^{-1}.

An isometry f in an ordered group G is called a 0-*isometry if* $f(0) = 0$.

Theorem 2.5. *Let f be a 0-isometry in an ordered group G. Then for each $a \in G$*

(i) $a, f(a) \geqq 0 \Rightarrow f(a) = a$;

(ii) $a, -f(a) \geqq 0 \Rightarrow f(a) = -a$;

(iii) $a, f(a) \leqq 0 \Rightarrow f(a) = a$;

(iv) $a, -f(a) \leqq 0 \Rightarrow f(a) = -a$.

Proof. (i) $U(a) = |a| = |a - 0| = |f(a) - f(0)| = |f(a)| = U(f(a))$, thus $a = f(a)$.

(ii) $U(a) = U(-f(a))$, hence $a = -f(a)$.

(iii) $U(-a) = |a| = |a - 0| = |f(a) - f(0)| = |f(a)| = U(-f(a))$, thus $a = f(a)$

(iv) $U(-a) = U(f(a))$, and so $a = -f(a)$.

Theorem 2.6. *Let f be a strong 0-isometry in G. If $0 \leqq a \in G$, then*

(i) $f(a) = a \Leftrightarrow f(-a) = -a$;

(ii) $f(a) = -a \Leftrightarrow f(-a) = a$.

Proof. a) "$\Rightarrow$": Since f is a strong isometry, we have $f(U(L(a, -a)) \cap L(U(a, -a))) = U(L(f(a), f(-a))) \cap L(U(f(a), f(-a)))$. Moreover, $|f(-a)| = |f(-a) - 0| = |f(-a) - f(0)| = |-a - 0| = |-a| = U(a)$, that is $a = f(-a) \vee -f(-a)$. Thus if $f(a) = a$, then $U(L(f(a), f(-a))) = U(L(a, f(-a))) =$

$= U(f(-a))$, $L(U(f(a), f(-a))) = L(U(a, f(-a))) = L(a)$, and hence $f(U(-a) \cap \cap L(a)) = U(f(-a)) \cap L(a)$. But $0 \in U(-a) \cap L(a)$, thus $0 = f(0) \in U(f(-a)) \cap \cap L(a)$, therefore $f(-a) \leqq 0$. Thus, by Theorem 2.5 (iii), $f(-a) = -a$ holds.

"$\Leftarrow$": Analogously from $-a = f(a) \wedge -f(a)$ and from Theorem 2.5 (i).

b) Equivalence (ii) can be proved similarly.

A *translation* of a group G is any mapping $f_g : G \to G$ (where $g \in G$) such that $x \mapsto x + g$ for each $x \in G$.

Theorem 2.7. *Any translation of an ordered group G is a strong isometry in G.*

Proof. a) The condition (1) from the definition of isometry is trivially satisfied.

b) Let $a, b, x, g \in G$. Then $x \in f_g(U(L(a, b))) \cap L(U(a, b)))$ if and only if $x - g \in \in U(L(a, b)) \cap L(U(a, b))$, this holds if and only if $x \in U(L(a + g, b + g)) \cap \cap L(U(a + g, b + g))$ and this is equivalent to $x \in U(L(f_g(a), f_g(b))) \cap L(U(f_g(a), f_g(b)))$.

Theorem 2.8. *If f is an isometry (strong isometry) in an ordered group G, then there exists a unique 0-isometry (a strong 0-isometry) h in G such that $f(x) = = h(x) + f(0)$ for each $x \in G$.*

Proof. It is clear that the bijection h is a 0-isometry. Let f be a strong isometry. Then the assertion follows from the fact that the composition of any two strong mappings is also a strong mapping, and from Theorem 2.7.

If G is an ordered group, then the set of all isometries (strong isometries, 0-isometries, strong 0-isometries, translations) in G is an ordered group with respect to the composition of mappings and with respect to the order relation "$\leqq$" such that $f \leqq g$ if and only if $f(x) \leqq g(x)$ for each $x \in G$.

Let us denote by $I(G)$ the ordered group of all isometries and by $T(G)$ its subgroup of all translations of G.

Theorem 2.9. *Let G and G' be ordered groups and let there exists a bijective mapping $\varphi : I(G) \to I(G')$ such that*

(a) *φ is a group isomorphism;*

(b) *φ is an isomorphism of ordered sets;*

(c) *the restriction φ on $T(G)$ is a mapping of $T(G)$ onto $T(G')$.*

Then the ordered groups G and G' are isomorphic.

Proof is analogous to that of Theorem 2 in [9] for lattice-ordered groups.

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