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ISOMETRY GROUPS OF MANIFOLDS OF NEGATIVE CURVATURE

W. BYERS

ABSTRACT. Solvable subgroups of the isometry groups of a simply-connected manifold of negative curvature are characterized and this characterization is used to show that the isometry group of the universal Riemannian covering of a compact manifold of negative curvature is either discrete or semisimple.

0. **Introduction.** A number of recent papers have related the geometry of manifolds of negative curvature to the algebra of various groups of isometries (for example [4], [8]). In this paper we study various groups of isometries of a simply-connected manifold M of negative curvature. In Theorem 5 we use results of Bishop and O'Neill [2] to show that a solvable group of isometries either leave a single geodesic invariant, permute a class of asymptotic geodesics, or else have a nonempty fixed point set. If the total isometry group $I(M)$ does not satisfy either of two former conditions, we show in Theorem 7 that there is a compact normal subgroup K such that $I(M)/K$ is semisimple and acts effectively on a closed, connected, totally convex submanifold of M. Using these results we show in Theorem 9 that if M is the universal Riemannian covering of a compact manifold of negative curvature, then the isometry group $I(M)$ is either discrete or semisimple. This may be viewed as an extension of the classical situation where the compact manifold may be considered as a double coset space $\Gamma\backslash G/K$ of a connected semisimple Lie group G and where the symmetric space G/K can be given an invariant metric of nonpositive curvature so that G is isomorphic to the identity component of the isometry group [5].

1. **Preliminaries.** M will denote a simply-connected, complete, Riemannian manifold of sectional curvature $K \leqslant C < 0$. Given any oriented geodesic γ and any point $X \in M$ there exists a unique oriented geodesic through x whose distance from γ tends to zero as t tends to ∞, the *asymptote* to γ through x [2]. Orthogonal trajectories to a family of asymptotic geodesics give a foliation of M by $(n-1)$ planes called *horospheres* [1].

$I(M)$ will denote the Lie group of isometries of M and I_0 its identity component. If ξ is a class of asymptotic geodesics let $S(\xi)$, the stability group of ξ, be the subgroup of isometries which permute the geodesics of ξ (compare

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'ray-subgroup' [2]). Now each element of $S(\xi)$ will also permute the horospheres associated to ξ. Define $H(\xi)$ to be the subgroup of $S(\xi)$, composed of isometries which map each such horosphere into itself. However using the fact that isometries of $S(\xi)$ commute with the geodesic flow restricted to geodesics of ξ we can show

LEMMA 1. *If $\phi \in S(\xi)$ maps one horosphere associated to ξ into itself, then $\phi \in H(\xi)$.*

We need the following characterization of isometries on M.

PROPOSITION 2 (BISHOP-O'NEILL [2]). *Let ϕ be an isometry of M. Then exactly one of the following is true:*

(a) *ϕ has a fixed point.*

(b) *ϕ translates a (unique) geodesic.*

(c) *$f_\phi : M \to R$ defined by $f_\phi(x) = d^2(x, \phi(x))$ has no minimum.*

Following [3], we call the isometries (a), (b), and (c) elliptic, axial, and parabolic, respectively. Parabolic isometrics preserve a unique class of asymptotic geodesics, whereas axial isometrics preserve exactly two such classes. We show

LEMMA 3. *If a parabolic isometry ϕ is in the stability group $S(\xi)$, it must also be in $H(\xi)$.*

PROOF. We show that if $\phi \notin H(\xi)$, then it must be axial. If $\varphi_t : M \to M$ is the geodesic flow along geodesics of ξ and $H(x)$ is the horosphere passing through $x \in M$, then $\phi(H^+(x)) = \phi_{t_0}(H^+(x))$ for some $t_0 \in R$. Now the fact that M has curvature bounded above by a constant less than zero implies (Arnold-Avez [1]) that $\varphi_{t_0} : H^+(x) \to H^+(\varphi_{t_0}x)$ is a metric space contraction mapping (if φ_{t_0} is an expansion we consider φ_{-t_0}). Since ϕ is an isometry, $\phi^{-1} \circ \phi_{t_0} : H^+(x) \to H^+(x)$ is also a contraction and has a unique fixed point by the contraction mapping theorem for complete metric spaces. This fixed point corresponds to the geodesic translated by ϕ.

2. **Solvable and nilpotent subgroups of $I(M)$.** If S is any subset of $I(M)$ we denote by Fix S the set of all common fixed points of elements of S. Then Fix S is known to be a closed totally geodesic submanifold of M.

A set $K \subseteq M$ is totally convex if whenever x and y are two points of K the infinite geodesic joining x to y lies in K. A totally convex submanifold is necessarily connected and totally geodesic.

LEMMA 4. *Let S be a solvable group of elliptic isometries of M. Then* Fix S *is nonempty.*

PROOF. We first prove the lemma for S abelian. Consider closed, totally convex submanifolds of M which are invariant under the isometries of S. Let ϕ be any isometry in S and let $C_\phi =$ Fix ϕ, which is nonempty by assumption. Then C_ϕ is a closed, totally convex submanifold which is invariant under S because S is abelian. Now let C be a submanifold with the above properties which is of minimal dimension. Since C is invariant under ϕ, C must intersect C_ϕ nontrivally [4, Lemma 1]. However, since $C \cap C_\phi$ is a submanifold of C, the minimality of C implies that C is contained in C_ϕ for all $\phi \in S$ and, in particular, that Fix S is nonempty.

In general let $S = S_0 \supset S_1 \supset \cdots \supset S_{k-1} \supset \{1\}$ be the derived series for S. Fix S_{k-1} is nonempty since S_{k-1} is abelian. Now S_{k-2} leaves Fix S_{k-1} invariant because S_{k-1} is normal in S_{k-2}. Thus S_{k-2}/S_{k-1} is an abelian group of isometries of Fix S_{k-1} and the above reasoning implies that Fix S_{k-2} is nonempty. Continuing in this way we show that Fix S is nonempty.

THEOREM 5. *Let S be a solvable group of isometries with no common fixed points. Then either S leaves some geodesics invariant or else S is contained in $S(\xi)$ for some class ξ of asymptotic geodesics.*

PROOF. Let $S = S_0 \supset S_1 \supset \cdots \supset S_{n-1} \supset \{1\}$ be the derived series for S and let S_k be the largest group in the sequence consisting entirely of elliptic isometries. Then Fix S_k is nonempty and invariant under S_{k-1} which must contain either an axial or a parabolic isometry.

Suppose $\phi \in S_{k-1}$ has axis γ. Then γ lies in Fix S_k [4]. Let ψ be any element of S_{k-1}. Then $\psi^{-1}\phi\psi\phi^{-1} = \chi \in S_k$. Thus $\psi^{-1}\phi\psi$ leaves γ invariant and ϕ leaves $\psi(\gamma)$ invariant. However ϕ has a unique axis because the curvature of M is strictly negative and so ψ also leaves γ invariant. Similarly if $\psi \in S_{k-2}$, consideration of $\psi^{-1}\phi\psi\phi^{-1}$ as above will show that ψ leaves γ invariant. Proceeding in this way S must leave γ invariant in this case.

Suppose now that $\phi \in S_{k-1}$ is parabolic and that ξ is the unique class of asymptotes permuted by ϕ. Reasoning exactly as above with ξ taking the place of γ, one can show that S leaves ξ invariant, i.e. that S is contained in $S(\xi)$.

In the case of a nilpotent group we can obtain results which are a little more precise.

THEOREM 6. *Let N be a nilpotent group of isometries with no common fixed points. Then either N contains an axial isometry and leaves its axis invariant or else N is contained in $H(\xi)$ for some ξ.*

PROOF. Consider the series $N = N_0 \supset N_1 \supset \cdots \supset N_{n-1} \supset \{1\}$ where $N_t = [N, N_{t-1}]$ and again suppose that N_k is the largest group for which Fix N_t is nonempty. As above, the existence of an axial isometry ϕ in N_{k-1} implies that N leaves the axis of ϕ invariant.

Now suppose that $\phi \in N_{k-1}$ is parabolic with $\phi \in H(\xi)$. As in Theorem 5 we can show that N is contained in $S(\xi)$. Suppose that there exists $\psi \in N$ such that $\psi \notin H(\xi)$. Then (Lemma 3) ψ must be axial with axis γ, say. Now $\phi^{-1}\psi\phi\psi^{-1} = \chi \in N_k$ and so ϕ must leave γ invariant. This is impossible since ϕ is parabolic and thus N is contained in $H(\xi)$.

Note. Theorem 6 is false for solvable groups. For example when SL $(2, r)$ acts as isometries of the Poincaré upper half plane, the upper triangular matrices form a solvable subgroup containing both axial and parabolic isometries.

3. **Invariant manifolds and semisimple groups of isometries.** In this section we shall restrict our attention to metrics of M which have the property that the total isometry group $I(M)$ does not leave invariant any one geodesic and is not equal to any single stability group $S(\xi)$. This condition is satisfied, for example,

when M is the Riemannian covering of a negatively curved manifold N with two distinct closed geodesics and, in particular, when N is compact.

THEOREM 7. *Suppose M has a metric which satisfies the above condition on $I(M)$. Then there exists a compact normal subgroup K of $I(M)$ such that either $I(M)/K$ is discrete or else it is semisimple and acts effectively on a closed, totally convex submanifold of M.*

PROOF. Let M' be a closed, totally convex submanifold of M which is invariant under all isometries and is of minimal dimension. Let K be the subgroup of $I(M)$ made up of all isometries leaving M' pointwise fixed. K is compact since the subgroup of $I(M)$ leaving any given point fixed is compact [5]. Now suppose $\phi \in I(M)$ and $\psi \in K$ and $x \in M'$ are arbitrary. Then $\phi(x) \in M'$ and so $\psi(\phi(x)) = \phi(x)$. Thus $\phi^{-1}\psi\phi(x) = x$ and so $\phi^{-1}\psi\phi \in K$. Thus K is normal in $I(M)$.

Suppose $I(M)/K$ is not discrete and let R be any normal solvable subgroup. Then R acts as a group of isometries of M'. According to Theorem 5 if $\operatorname{Fix} R$ is empty, either R leaves some geodesic of M' invariant or else $R \subseteq S(\xi)$ for some ξ. In the former case R contains an axial isometry ϕ with axis g. The normality of R in I/K implies that ϕ preserves the geodesic $\alpha(g)$ for a given $\alpha \in I$ and thus $\alpha(g) = g$ as in the proof of Theorem 5. Thus $I(M)$ preserves g. In the latter case it follows similarly that $I(M) = S(\xi)$. Since $\operatorname{Fix} R$ is also closed, and totally convex, the minimality of M' implies that $\operatorname{Fix} R = M'$ and so R is trivial. Thus $I(M)/K$ is semisimple.

COROLLARY 8. *If $I(M)$ satisfies the hypothesis of Theorem 7 and $M = M'$ (i.e. except for M there are no closed, totally convex submanifolds invariant under $I(M)$), then $I(M)$ is either semisimple or discrete.*

THEOREM 9. *Let N be a compact Riemannian manifold of negative curvature and M its universal Riemannian covering. Then $I(M)$ is discrete or semisimple.*

PROOF. We consider the fundamental group $\Pi_1(N)$ as a subgroup of $I(M)$. Since N is compact, each covering transformation is axial. If $\Pi_1(N)$ were to leave a single geodesic invariant or if $\Pi_1(N)$ were contained in $S(\xi)$ for some ξ, it would then follow [2] that $\Pi_1(N)$ is isomorphic to the integers, which is impossible for compact N, [6]. Thus $I(M)$ must satisfy the hypothesis of Theorem 7. Now suppose there exists a closed, connected, totally convex submanifold M'—not M itself—which is invariant under $I(M)$ and thus under $\Pi_1(N)$. Let $N' = p(M')$ where $p : M \to N$ is the covering projection. Then N' is closed and totally convex and so N is diffeomorphic to the normal bundle of N' [2, Lemma 3.1] and, in particular, is noncompact. Thus $M' = M$ and the result follows from Corollary 8.

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DEPARTMENT OF MATHEMATICS, SIR GEORGE WILLIAMS UNIVERSITY, MONTREAL 107, CANADA