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# Isomorphic objects in symmetric monoidal closed categories†

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This paper presents a new and self-contained proof of a result characterizing objects isomorphic in the free symmetric monoidal closed category, *i.e.*, objects isomorphic in every symmetric monoidal closed category. This characterization is given by a finitely axiomatizable and decidable equational calculus, which differs from the calculus that axiomatizes all arithmetical equalities in the language with 1, product and exponentiation by lacking $1^c = 1$ and $(a \cdot b)^c = a^c \cdot b^c$ (the latter calculus characterizes objects isomorphic in the free cartesian closed category). Nevertheless, this calculus is complete for a certain arithmetical interpretation, and its arithmetical completeness plays an essential role in the proof given here of its completeness with respect to symmetric monoidal closed isomorphisms.

## 1. Introduction

In 1981 S. V. Soloviev gave a characterization of objects isomorphic in the cartesian closed category freely generated by an infinite set of objects, and thereby characterized the objects isomorphic in every cartesian closed category. (Two objects $A$ and $B$ are *isomorphic* in a category iff there is an arrow $f$ from $A$ to $B$ and an arrow $f^{-1}$ from $B$ to $A$ such that $f^{-1}f = 1_A$ and $ff^{-1} = 1_B$.) Soloviev's result was proved again by other means in Bruce *et al.* (1992).

This characterization is given by the equational calculus that otherwise axiomatizes all equalities between arithmetical terms built up with variables, 1, product and exponentiation. The specific axioms of this calculus are the equalities of commutative monoids and the following four equalities:

$$c^1 = c, \qquad c^{a \cdot b} = (c^a)^b$$
$$1^c = 1, \qquad (a \cdot b)^c = a^c \cdot b^c.$$

In 1993 Soloviev produced a similar characterization of objects isomorphic in the symmetric monoidal closed category freely generated by an infinite set of objects. As before,

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this yields a characterization of the objects isomorphic in every symmetric monoidal closed category. The equational calculus now in question is obtained just by rejecting the last two equalities displayed above. It is quite easy to see that these equalities cannot hold when interpreted in terms of isomorphisms of an arbitrary symmetric monoidal closed category, and it is as easy to verify that the calculus obtained after rejecting these equalities is sound for this interpretation. Soloviev's and our paper are devoted to proving it is also complete.

Apart from some relatively minor flaws, which we believe can be corrected, the proof of Soloviev (1993) has the drawback that it is not self-contained. It relies essentially on an earlier work of A. A. Babaev (Babaev 1981) about normalization of terms in a specific sort of typed lambda calculus, which plays the role of the internal language of the free symmetric monoidal closed category. This lambda calculus has pairing and projections, but expressions involving pairing and projections are not always granted the status of *terms:* they may happen to be only *quasi-terms*. The normalization of a term may proceed via quasi-terms. Even in the absence of quasi-terms, there have been some uncertainties concerning the normalization of typed lambda terms of that kind (the proofs in Lambek and Scott (1986, Part I, Chapters 13–14) and Mints (1980), a reference in Babaev (1981), are not flawless; we have not checked Babaev (1981) in detail). So there should be a gain in finding for the calculus mentioned above a new, self-contained, completeness proof with respect to symmetric monoidal closed isomorphisms. The aim of this paper is to present such a proof.

Actually, our proof has some similarity with Soloviev's proof of 1981. There, he reduced the characterization of the isomorphic objects of the free cartesian closed category to the characterization of the isomorphic objects of a concrete cartesian closed category – the category of finite sets. Thereby, he found an interpretation for the isomorphisms of the free cartesian closed category in the arithmetic of natural numbers. Similarly, we use the category of finite pointed sets to characterize the isomorphic objects of the free symmetric monoidal closed category, and we find an arithmetical interpretation for these isomorphisms. (The argument of Soloviev (1993) is rather closer to Bruce *et al.* (1992): both papers rely on techniques of the lambda calculus.)

Apart from the purely mathematical interest of characterizing isomorphic objects in such important sorts of categories as cartesian closed and symmetric monoidal closed, results of this kind are also interesting for logic. The free cartesian closed category is the conjunction-implication fragment of intuitionistic logic, and the free symmetric monoidal closed category is the fragment of linear logic with product (multiplicative conjunction), implication and the multiplicative propositional constant $\mathtt{I}$ (sometimes written 1). The philosophical significance of isomorphism in logic is discussed in Došen and Petrić (1994) and Došen (1997, Section 9). An approach to the subject under the imprint, not of logic or mathematics, but of concerns in the zone of computer science, may be found in Di Cosmo (1995), which has a rather useful bibliography.

## 2. The category SyMonCl

In this section we present the symmetric monoidal closed category freely generated by an infinite set of objects, *i.e.*, a graph without arrows whose vertices are these objects. This category, called **SyMonCl**, is the image of this set of objects under the functor left adjoint to the forgetful functor from the category of symmetric monoidal closed categories into the category of graphs.

We take as the set of objects that are the free generators of **SyMonCl** an infinite set of propositional letters, which we call simply *letters*. The objects of the category **SyMonCl** are *formulae* built-up from the letters with the help of the binary connective $\bullet$, called *product* (which is often written $\otimes$), and the binary connective of implication $\rightarrow$; we also have a propositional constant, *i.e.*, nullary connective, $\mathtt{I}$. The letters and $\mathtt{I}$ are the *atomic* formulae. For letters we use the schematic letters $p, q, r, \ldots$, possibly with indices, and for formulae the schematic letters $A, B, C, \ldots$, possibly with indices.

An expression of the form $f : A \vdash B$ will stand for an arrow from $A$ to $B$. (We write $\vdash$ instead of the more usual $\rightarrow$, which we have reserved for implication.) The prefix $f$ in $f : A \vdash B$ is called an *arrow-term*; $A \vdash B$ is the type of this arrow-term. For arrow-terms we use the schematic letters $f, g, h, \ldots$, possibly with indices. The category **SyMonCl** has the following *primitive arrow-terms*, for every formula $A, B$ and $C$:

$$\mathbf{1}_A : A \vdash A$$

$$\mathbf{b}^{\rightarrow}_{A,B,C} : A \bullet (B \bullet C) \vdash (A \bullet B) \bullet C, \qquad \mathbf{b}^{\leftarrow}_{A,B,C} : (A \bullet B) \bullet C \vdash A \bullet (B \bullet C)$$

$$\mathbf{c}_{A,B} : A \bullet B \vdash B \bullet A$$

$$\boldsymbol{\sigma}_A : \mathtt{I} \bullet A \vdash A, \qquad \boldsymbol{\sigma}^i_A : A \vdash \mathtt{I} \bullet A$$

$$\boldsymbol{\delta}_A : A \bullet \mathtt{I} \vdash A, \qquad \boldsymbol{\delta}^i_A : A \vdash A \bullet \mathtt{I}$$

$$\boldsymbol{\varepsilon}_{A,B} : A \bullet (A \rightarrow B) \vdash B, \qquad \boldsymbol{\eta}_{A,B} : B \vdash A \rightarrow (A \bullet B).$$

The remaining arrow-terms of **SyMonCl** are built-up from the primitive arrow-terms with the help of the following binary *operations on arrow-terms*:

$$\frac{f : A \vdash B \qquad g : B \vdash C}{gf : A \vdash C}$$

$$\frac{f : A \vdash B \qquad g : C \vdash D}{f \bullet g : A \bullet C \vdash B \bullet D} \qquad \frac{f : A \vdash B \qquad g : C \vdash D}{f \rightarrow g : B \rightarrow C \vdash A \rightarrow D}$$

Note that the first operation, *composition*, is a partial operation, while the $\bullet$ and $\rightarrow$ operations on arrow-terms are total.

The arrows of the category **SyMonCl** are equivalence classes of arrow-terms; *i.e.*, for every arrow-term $f$ we take the class of arrow-terms equal to $f$ according to the following *equalities* (which we add to the usual equality postulates: reflexivity, symmetry, transitivity and congruence with the operations on arrow-terms):

*categorial equalities*

(cat) For $f : A \vdash B, g : B \vdash C$ and $h : C \vdash D$, $\quad h(gf) = (hg)f$.

(cat**1**) For $f : A \vdash B$, $\quad f\mathbf{1}_A = f$, $\quad \mathbf{1}_B f = f$.

*product equalities*

($\bullet$) For $f_1 : A_1 \vdash B_1, g_1 : B_1 \vdash C_1, f_2 : A_2 \vdash B_2$ and $g_2 : B_2 \vdash C_2$,
$$(g_1 f_1) \bullet (g_2 f_2) = (g_1 \bullet g_2)(f_1 \bullet f_2).$$

($\bullet$**1**) $\mathbf{1}_A \bullet \mathbf{1}_B = \mathbf{1}_{A \bullet B}$

*implication equalities*

$(\to)$ For $f_1 : A_1 \vdash B_1, g_1 : B_1 \vdash C_1, f_2 : A_2 \vdash B_2$ and $g_2 : B_2 \vdash C_2$,
$$(g_1 f_1) \to (g_2 f_2) = (f_1 \to g_2)(g_1 \to f_2).$$

$(\to \mathbf{1})$ $\mathbf{1}_A \to \mathbf{1}_B = \mathbf{1}_{A\to B}$

**b** *equalities*

$(\mathbf{b})$ For $f : A \vdash D, g : B \vdash E$ and $h : C \vdash F$, $\quad ((f \bullet g) \bullet h)\mathbf{b}^{\to}_{A,B,C} = \mathbf{b}^{\to}_{D,E,F}(f \bullet (g \bullet h))$.

$(\mathbf{bb})$ $\mathbf{b}^{\to}_{A,B,C}\mathbf{b}^{\leftarrow}_{A,B,C} = \mathbf{1}_{(A\bullet B)\bullet C}, \quad \mathbf{b}^{\leftarrow}_{A,B,C}\mathbf{b}^{\to}_{A,B,C} = \mathbf{1}_{A\bullet(B\bullet C)}$

$(\mathbf{b}5)$ $\mathbf{b}^{\to}_{A,B\bullet C,D} = (\mathbf{b}^{\leftarrow}_{A,B,C} \bullet \mathbf{1}_D)\mathbf{b}^{\to}_{A\bullet B,C,D}\mathbf{b}^{\to}_{A,B,C\bullet D}(\mathbf{1}_A \bullet \mathbf{b}^{\leftarrow}_{B,C,D})$

**c** *equalities*

$(\mathbf{c})$ For $f : A \vdash C$ and $g : B \vdash D$, $\quad (g \bullet f)\mathbf{c}_{A,B} = \mathbf{c}_{C,D}(f \bullet g)$.

$(\mathbf{cc})$ $\mathbf{c}_{B,A}\mathbf{c}_{A,B} = \mathbf{1}_{A\bullet B}$

$(\mathbf{bc}6)$ $\mathbf{c}_{A\bullet B,C} = \mathbf{b}^{\leftarrow}_{C,A,B}(\mathbf{c}_{A,C} \bullet \mathbf{1}_B)\mathbf{b}^{\to}_{A,C,B}(\mathbf{1}_A \bullet \mathbf{c}_{B,C})\mathbf{b}^{\leftarrow}_{A,B,C}$

$\boldsymbol{\sigma\delta}$ *equalities*

$(\boldsymbol{\sigma})$ For $f : A \vdash B$, $f\boldsymbol{\sigma}_A = \boldsymbol{\sigma}_B(\mathbf{1}_{\mathrm{I}} \bullet f)$. $\quad(\boldsymbol{\delta})$ For $f : A \vdash B$, $f\boldsymbol{\delta}_A = \boldsymbol{\delta}_B(f \bullet \mathbf{1}_{\mathrm{I}})$.

$(\boldsymbol{\sigma\sigma}^{\mathrm{i}})$ $\boldsymbol{\sigma}_A\boldsymbol{\sigma}^{\mathrm{i}}_A = \mathbf{1}_A, \quad \boldsymbol{\sigma}^{\mathrm{i}}_A\boldsymbol{\sigma}_A = \mathbf{1}_{\mathrm{I}\bullet A}$ $\quad(\boldsymbol{\delta\delta}^{\mathrm{i}})$ $\boldsymbol{\delta}_A\boldsymbol{\delta}^{\mathrm{i}}_A = \mathbf{1}_A, \quad \boldsymbol{\delta}^{\mathrm{i}}_A\boldsymbol{\delta}_A = \mathbf{1}_{A\bullet\mathrm{I}}$

$(\boldsymbol{\sigma\delta})$ $\boldsymbol{\sigma}_{\mathrm{I}} = \boldsymbol{\delta}_{\mathrm{I}}$

$(\boldsymbol{\sigma\delta}b)$ $\mathbf{b}^{\to}_{A,\mathrm{I},D} = \boldsymbol{\delta}^{\mathrm{i}}_A \bullet \boldsymbol{\sigma}_D$

$(\boldsymbol{\sigma\delta}c)$ $\mathbf{c}_{\mathrm{I},C} = \boldsymbol{\delta}^{\mathrm{i}}_C\boldsymbol{\sigma}_C$

$\boldsymbol{\varepsilon\eta}$ *equalities*

$(\boldsymbol{\varepsilon})$ For $f : A \vdash B$, $\quad f\boldsymbol{\varepsilon}_{C,A} = \boldsymbol{\varepsilon}_{C,B}(\mathbf{1}_C \bullet (\mathbf{1}_C \to f))$.

$(\boldsymbol{\eta})$ For $f : A \vdash B$, $\quad (\mathbf{1}_C \to (\mathbf{1}_C \bullet f))\boldsymbol{\eta}_{C,A} = \boldsymbol{\eta}_{C,B}f$.

$(\bullet\boldsymbol{\varepsilon\eta})$ For $f : A \vdash B$, $\quad \boldsymbol{\varepsilon}_{A,B\bullet C}(\mathbf{1}_A \bullet (f \to \mathbf{1}_{B\bullet C}))(\mathbf{1}_A \bullet \boldsymbol{\eta}_{B,C}) = f \bullet \mathbf{1}_C$.

$(\to\boldsymbol{\varepsilon\eta})$ For $f : A \vdash B$, $\quad (\mathbf{1}_A \to \boldsymbol{\varepsilon}_{B,C})(\mathbf{1}_A \to (f \bullet \mathbf{1}_{B\to C}))\boldsymbol{\eta}_{A,B\to C} = f \to \mathbf{1}_C$.

The product and implication equalities say that $\bullet$ and $\to$ are bifunctors. The **b** and **c** equalities say that the $\mathbf{b}^{\to}$ and **c** arrows are natural isomorphisms satisfying the coherence conditions (**b**5) and (**bc**6) (which amount to Mac Lane's pentagonal and hexagonal diagrams; see Mac Lane (1971, VII.1, p. 158, VII.7, p. 180)). The $\boldsymbol{\sigma\delta}$ equalities say that the $\boldsymbol{\sigma}$ and $\boldsymbol{\delta}$ arrows are natural isomorphisms satisfying the coherence conditions $(\boldsymbol{\sigma\delta})$, $(\boldsymbol{\sigma\delta}\mathbf{b})$ and $(\boldsymbol{\sigma\delta}\mathbf{c})$ (see Mac Lane (1971, pp. 159, 180); the second condition amounts to Mac Lane's triangular diagram). The $\boldsymbol{\varepsilon\eta}$ equalities say that the $\boldsymbol{\varepsilon}$ and $\boldsymbol{\eta}$ arrows are natural transformations that are the counit and unit of an adjunction.

## 3. Linkage and diversification

In the first, preparatory, part of our proof we establish a lemma – the Diversification Lemma – with the help of a technique derived from Eilenberg and Kelly (1966) (see also Kelly and Mac Lane (1971); this technique is related to Gentzen's notion of a *cluster* – *cf.* Došen and Petrić (1994, Section 3)). This part of our proof is not entirely new. Soloviev establishes a lemma essentially identical to ours in Soloviev (1993, Section 7), with a similar technique. However, because Soloviev works in a somewhat different context (he has a sequent system *à la* Gentzen where we have a category introduced in a more standard way), and because we want to make our proof self-contained, we shall not skip this preparatory part of the proof.

We shall define below when for an arrow-term $f$ of type $A \vdash B$ of **SyMonCl** two occurrences of the same letter in $A$ or $B$ are *linked* by $f$. Formally, we define a set of unordered pairs $\{x, y\}$ such that for some letter $p$, the elements $x$ and $y$ are distinct occurrences of $p$ in $A$ or $B$, each pair $\{x, y\}$ being called a *link* of $f$. When $A$ is the same formula as $B$, we nevertheless consider the set of occurrences of letters in $A$ as disjoint from the set of occurrences of letters in $B$.

Every link of $f$ will be of one of the following sorts:

(1) it is *straight* iff it links an occurrence of a letter in $A$ to an occurrence of this letter in $B$;

(2.1) it is *crooked* in $A$ iff it links an occurrence of a letter in $A$ to another occurrence of this letter in $A$;

(2.2) it is *crooked* in $B$ iff it links an occurrence of a letter in $B$ to another occurrence of this letter in $B$.

The *word* $w(A)$ of a formula $A$ is obtained by deleting in $A$ all occurrences of the connectives $\mathtt{I}$, $\bullet$ and $\rightarrow$, and all parentheses. For example, $w(((p \bullet \mathtt{I}) \rightarrow q) \rightarrow p)$ is $pqp$. For $f : A \vdash B$, the *word figure* of $f$ will be $w(A)$ written above $w(B)$. For example, the word figure of $\mathbf{b}^{\rightarrow}_{p \bullet q, \mathtt{I}, p}$ is

$$\begin{array}{c} pqp \\ pqp \end{array}$$

Word figures are only an auxiliary device. Links defined for them are to be transferred to formulae from which the words in the word figure were obtained. After the links are made, we just put back the connectives and parentheses. In examples below we shall draw immediately the links as they are to be found in formulae, assuming it is clear how these links were established through the word figures associated with the formulae.

We can now proceed with our definition of links by induction on the complexity of the arrow-term $f$ of type $A \vdash B$.

If $f$ is $\mathbf{1}_A$, $\mathbf{b}^{\rightarrow}_{D,E,C}$, $\mathbf{b}^{\leftarrow}_{D,E,C}$, $\boldsymbol{\sigma}_B$, $\boldsymbol{\sigma}^{\mathrm{i}}_A$, $\boldsymbol{\delta}_B$ or $\boldsymbol{\delta}^{\mathrm{i}}_A$, the $i$-th symbol of $w(A)$ is linked to the $i$-th symbol of $w(B)$, the words $w(A)$ and $w(B)$ being two copies of the same word. For example, with $\mathbf{b}^{\rightarrow}_{p \bullet q, \mathtt{I}, p}$ we have the links

$$\begin{array}{c} (p \bullet q) \bullet (\mathtt{I} \bullet p) \\ |\quad\;|\qquad\quad\;| \\ ((p \bullet q) \bullet \mathtt{I}) \bullet p \end{array}$$

If $f$ is $\mathbf{c}_{C,D}$, we have the word figure

$$\begin{array}{c} w(C)w(D) \\ w(D)w(C) \end{array}$$

($w(C)w(D)$ is $w(C \bullet D)$, that is, $w(C)$ concatenated with $w(D)$). Then the $i$-th symbol of $w(C)$ above is linked to the $i$-th symbol of $w(C)$ below, and the $i$-th symbol of $w(D)$ above is linked to the $i$-th symbol of $w(D)$ below. For example, with $\mathbf{c}_{p \bullet q, p \rightarrow \mathtt{I}}$ we have the links

$$(p \bullet q) \bullet (p \to \mathtt{I})$$

$$(p \to \mathtt{I}) \bullet (p \bullet q)$$

All the links defined up to this point are straight.

If $f$ is $\boldsymbol{\varepsilon}_{C,B}$, we have the word figure

$$w(C)w(C)w(B)$$
$$w(B)$$

Then the $i$-th symbol of the left $w(C)$ above is linked to the $i$-th symbol of the right $w(C)$ above, and the $i$-th symbol of $w(B)$ above is linked to the $i$-th symbol of $w(B)$ below. For example, with $\boldsymbol{\varepsilon}_{p \bullet q, \mathtt{I} \to (p \bullet q)}$ we have the links

$$(p \bullet q) \bullet ((p \bullet q) \to (\mathtt{I} \to (p \bullet q)))$$

$$\mathtt{I} \to (p \bullet q)$$

Note that here, besides straight links, we have links crooked in $A$, that is $C \bullet (C \to B)$.

If $f$ is $\boldsymbol{\eta}_{C,A}$, we have the word figure

$$w(A)$$
$$w(C)w(C)w(A)$$

Then the $i$-th symbol of the left $w(C)$ below is linked to the $i$-th symbol of the right $w(C)$ below, and the $i$-th symbol of $w(A)$ above is linked to the $i$-th symbol of $w(A)$ below. Note that here, besides straight links, we may have links crooked in $B$, that is $C \to (C \bullet A)$.

We obtain the links of the arrow-term $hg$ of type $A \vdash B$ from the links of $g$ of type $A \vdash C$ and the links of $h$ of type $C \vdash B$ in the following manner. A sequence of links $\{x_1, x_2\}$, $\{x_2, x_3\}$, $\{x_3, x_4\}$, …, $\{x_{n-1}, x_n\}$, with $n \geqslant 2$, makes a *chain* of links. A link of $hg$ is a pair $\{x_1, x_n\}$ obtained from a chain $\{x_1, x_2\}$, …, $\{x_{n-1}, x_n\}$ of alternating $g$ and $h$ links, with at least one $g$ or $h$ link, where

(1) $x_1$ is in $A$ and $x_n$ is in $B$, or
(2.1) $x_1$ is in $A$ and $x_n$ is in $A$, or
(2.2) $x_1$ is in $B$ and $x_n$ is in $B$.

If $n \geqslant 3$, then for every $i$ in $\{2, ..., n-1\}$ we must have $x_i$ in $C$. Note that all these chains are maximal (they are not proper parts of longer chains). In Case (1), where we define the straight links of $hg$, we have chains made of an even number of alternating $g$ and $h$ links – at least two of them. In Case (2.1), where we define the links of $hg$ crooked in $A$, we have chains made of an odd number of alternating $g$ and $h$ links, starting with a $g$ link and finishing with a $g$ link – so we have at least one $g$ link. And in Case (2.2), where we define the links of $hg$ crooked in $B$, we have chains made of an odd number of alternating $g$ and $h$ links, starting with an $h$ link and finishing with an $h$ link – so we have at least one $h$ link. (We have no use for closed chains $\{x_1, x_2\}$, ..., $\{x_{2n-1}, x_{2n}\}$, $\{x_{2n}, x_1\}$, with $n \geqslant 1$, made of an even number of alternating $g$ and $h$ links, where for every $i$ in $\{1, \ldots, 2n\}$ we have that $x_i$ is in $C$. Anyway, by a result of Kelly and Mac Lane (1971, Theorem 2.2), such chains cannot arise in the free category **SyMonCl**.)

For the links of the arrow-terms $f_1 \bullet f_2$ of type $A_1 \bullet A_2 \vdash B_1 \bullet B_2$ and $f_1 \rightarrow f_2$ of type $B_1 \rightarrow A_2 \vdash A_1 \rightarrow B_2$, we take the disjoint union of exact copies of the links of $f_1$ of type $A_1 \vdash B_1$ and of the links of $f_2$ of type $A_2 \vdash B_2$. This concludes our definition of links.

The links of an arrow-term in which there is no composition are called *atomic*. Such links are made of minimal chains, with only one pair of occurrences of letters. We find longer chains only behind links of arrow-terms with composition (though not all links of such arrow-terms must be obtained from chains with more than one atomic link).

Note that for $f$ of type $A \vdash B$ every occurrence of a letter in $A$ or $B$ is linked by exactly one link of $f$ to another occurrence of the same letter in $A$ or $B$; that is there are no chains of $f$ links with more than one link. The links establish a one–one correspondence between half of the occurrences of letters in $A$ or $B$ and the remaining half.

Let a subformula of a formula $A$ be an *even-floor* subformula of $A$ iff it occurs in an even number of antecedents of implications of $A$, and let it be an *odd-floor* subformula of $A$ iff it occurs in an odd number of antecedents of implications of $A$ (zero is an even number: so every formula is an even-floor subformula of itself; even-floor and odd-floor subformulae are sometimes called, respectively, *positive* and *negative*, or *consequent* and *antecedent* subformulae). It is not difficult to conclude that for $f$ of type $A \vdash B$ every straight link of $f$ links either an even-floor subformula of $A$ with an even-floor subformula of $B$, or an odd-floor subformula of $A$ with an odd-floor subformula of $B$; a link crooked in $A$ links an even-floor subformula of $A$ with an odd-floor subformula of $A$, and, analogously, with links crooked in $B$. In this paper we have no need for this distinction between even-floor and odd-floor subformulae (but we need the related notion of floor; see Section 10).

We now establish that our definition of links, which is given with respect to the arrow-terms of **SyMonCl**, may also be given with respect to the arrows of **SyMonCl**.

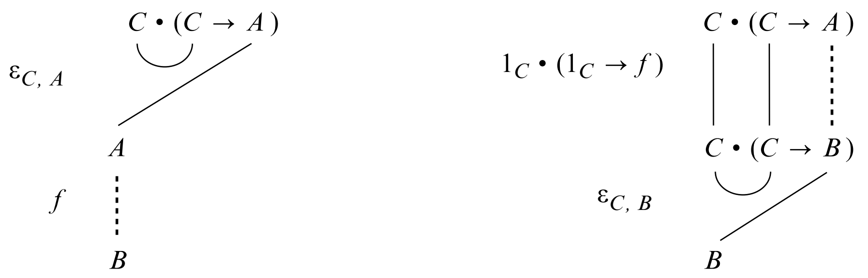
**Lemma 3.1.** If $f = g$ in **SyMonCl**, then the links of $f$ are equal to the links of $g$.

*Proof.* We make an induction on the length of the derivation of the equality $f = g$ in **SyMonCl**. This reduces to checking only the basis of this induction (namely, all the equalities of Section 2), because the induction step (namely, checking that the rules for equality preserve the equality of links) is trivial. Let us consider as examples a few cases in the basis of the induction, which might not be immediately obvious.

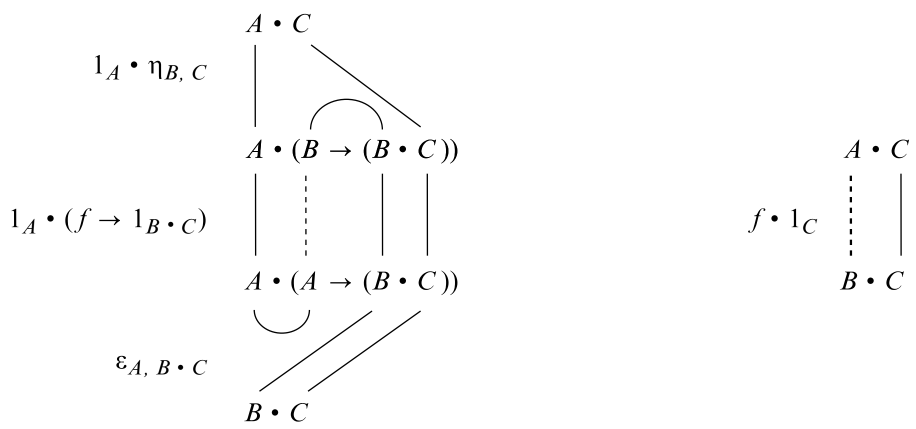
For the associativity of composition (cat) we have to check that every chain of alternating $f$ links, $g$ links and $h$ links that is a link of $h(gf)$ is also a link of $(hg)f$, and *vice versa*. If from such a chain we delete all the $h$ links, we obtain a collection of $gf$ links, whereas if we delete all the $f$ links, then we obtain a collection of $hg$ links.

For the equality $(\bullet)$ it is enough to note that a link of $f_1$ can never make a chain with a link of $g_2$, and a link of $f_2$ can never make a chain with a link of $g_1$. We reason similarly for the equality ($\rightarrow$).

For the equality ($\varepsilon$) it is enough to consider the following figures, where lines stand for finite, possibly empty, sets of links: full straight lines stand for sets of straight links; crooked lines for sets of crooked links; while the dotted line of $f$ stands for a set of links that may be either straight or crooked.




For the equality ($\bullet\varepsilon\eta$) we have the figures




We proceed similarly for the equalities ($\eta$) and ($\rightarrow \varepsilon\eta$). □

**Lemma 3.2.** Take the arrows $f : A \vdash B$ and $g : B \vdash A$. If every link of $gf$ and $fg$ is straight, then every link of $f$ and $g$ is straight.

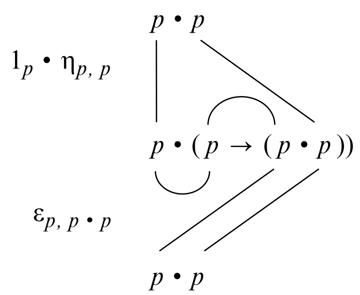
*Proof.* If a link of $f$ is crooked in $A$, then a link of $gf : A \vdash A$ is crooked in the left $A$. If a link of $f$ is crooked in $B$, then a link of $fg : B \vdash B$ is crooked in the right $B$. Similarly, if a link of $g$ is crooked in $A$, then a link of $gf : A \vdash A$ is crooked in the right $A$. If a link of $g$ is crooked in $B$, then a link of $fg : B \vdash B$ is crooked in the left $B$. □

The formula $A$ is an instance of the formula $A'$ iff $A$ is the result of uniformly substituting the formula $B_1$ for the letter $p_1, \ldots$, the formula $B_n$ for the letter $p_n$ in $A'$, with $n \geqslant 0$; the letters $p_1, \ldots, p_n$ are mutually distinct, but the formulae $B_1, \ldots, B_n$ need not be mutually distinct. Similarly, the arrow-term $f$ is an instance of the arrow-term $f'$ iff $f$ is obtained by uniformly substituting $B_1$ for $p_1$, …, $B_n$ for $p_n$ in the indices of $f'$. It is easy to check that the indices of arrow-terms are such that if $f$ of type $A \vdash B$ is an instance of $f'$ of type $A' \vdash B'$, then $A$ is an instance of $A'$ and $B$ is an instance of $B'$. We speak of *letter-for-letter* instances when substitution is letter-for-letter; *i.e.*, the substitute formulae $B_1, \ldots, B_n$ are all letters.

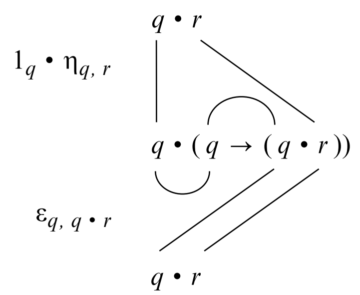
We say that an arrow-term $f$ has *diversified links* if each link of $f$ is made of occurrences of a letter different from the letters of any other link of $f$.

**Lemma 3.3.** Every arrow-term of **SyMonCl** is a letter-for-letter instance of an arrow-term of **SyMonCl** with diversified links.

*Proof.* Take the links of an arrow-term $f$ of **SyMonCl**. Each such link is obtained from a chain of atomic links. For every such chain, take a separate new letter and replace the letter of the chain by the new letter throughout the chain. This replacement yields an arrow-term $f'$ of **SyMonCl** with diversified links. For example, let $f$ be $\boldsymbol{\varepsilon}_{p,p\bullet p}(\mathbf{1}_p \bullet \boldsymbol{\eta}_{p,p})$, whose links are shown in the following figure:




This yields $\boldsymbol{\varepsilon}_{q,q\bullet r}(\mathbf{1}_q \bullet \boldsymbol{\eta}_{q,r})$ for $f'$, according to the figure




(Note that though $\boldsymbol{\varepsilon}_{q,q\bullet r}(\mathbf{1}_q \bullet \boldsymbol{\eta}_{q,r})$ has diversified links, this is not the case for either $\mathbf{1}_q \bullet \boldsymbol{\eta}_{q,r}$ or $\boldsymbol{\varepsilon}_{q,q\bullet r}$.) □

**Lemma 3.4.** If $f$ is a letter-for-letter instance of $f'$, then every link of $f$ is straight iff every link of $f'$ is straight.

*Proof.* It is enough to note that letter-for-letter substitution cannot change the sort of links: a straight link made of occurrences of the letter $p_i$ engenders a straight link made of occurrences of the substitute letter $q_i$; similarly, a crooked link engenders a crooked link. (If the substitution were not letter-for-letter, some links of $f'$ could simply disappear in $f$, by substituting for $p_i$ a formula $B_i$ without letters, and the left-to-right direction of the 'iff' statement would be invalidated.) □

Let $let(A)$ be the set of letters occurring in the formula $A$. The formula $A$ is diversified iff every letter in $let(A)$ occurs in $A$ exactly once. Now we can prove the main lemma of this section.

**Diversification Lemma.** For every isomorphism $f : A \vdash B$ of **SyMonCl** there is an isomorphism $f' : A' \vdash B'$ of **SyMonCl** such that $f$ is an instance of $f'$, and $A'$ and $B'$ are diversified.

*Proof.* Suppose $f : A \vdash B$ is an isomorphism of **SyMonCl**. So there is an arrow $g : B \vdash A$ of **SyMonCl** such that $gf = \mathbf{1}_A$ and $fg = \mathbf{1}_B$. Since every link of $\mathbf{1}_A$ and $\mathbf{1}_B$ is straight, by Lemma 3.1, every link of $gf$ and $fg$ is straight. By Lemma 3.2, every link of $f$ and $g$ is straight.

Then we apply the procedure of the proof of Lemma 3.3 to $gf$ in order to obtain $g'f'$ with diversified links whose letter-for-letter instance is $gf$, where $f$ is a letter-for-letter instance of $f'$ of type $A' \vdash B'$ and $g$ a letter-for-letter instance of $g'$ of type $B' \vdash A'$ (we could as well have applied this procedure to $fg$ in order to obtain $f'g'$). Since every link of $gf$ is straight, every link of $g'f'$ is straight, by Lemma 3.4, and since $g'f'$ has diversified links, $A'$ is diversified. Since every link of $f$ is straight, again by Lemma 3.4, every link of $f'$ is straight, and hence $B'$ is diversified, too (we could as well have appealed to the fact that every link of $g'$ is straight). We compute that $g'f' = \mathbf{1}_{A'}$ and $f'g' = \mathbf{1}_{B'}$ exactly as we computed that $gf = \mathbf{1}_A$ and $fg = \mathbf{1}_B$. For that it is enough to check that none of the equalities of **SyMonCl** ceases to hold after the replacements made in chains according to the procedure in the proof of Lemma 3.3. □

It is clear that in the Diversification Lemma we actually have that $f$ is a letter-for-letter instance of $f'$, and we also have that $let(A') = let(B')$.

## 4. The system S

We shall now introduce the system that characterizes the isomorphic objects of **SyMonCl** in the sense of the completeness theorem of the next section. This system, called **S**, is an equational calculus whose theorems are of the form $A \cong B$, where $A$ and $B$ are formulae, as introduced in Section 2 (we write $\cong$ rather than $=$ because of the intended interpretation of **S** in terms of isomorphisms). The axiom schemata and rules of **S** are:

*specific axioms*

| | |
|---|---|
| $(\bullet\mathtt{I})$ | $\mathtt{I} \bullet C \cong C$ |
| $(\bullet\text{ass})$ | $(A \bullet B) \bullet C \cong A \bullet (B \bullet C)$ |
| $(\bullet\text{com})$ | $A \bullet B \cong B \bullet A$ |
| (S1) | $\mathtt{I} \to C \cong C$ |
| (S2) | $(A \bullet B) \to C \cong B \to (A \to C)$ |

*equality postulates*

$A \cong A$
From $A \cong B$, infer $B \cong A$.
From $A \cong B$ and $B \cong C$, infer $A \cong C$.
From $A \cong B$ and $C \cong D$, infer $A \bullet C \cong B \bullet D$.
From $A \cong B$ and $C \cong D$, infer $B \to C \cong A \to D$.

(Note that the axiom schema $A \cong A$ can be derived with the symmetry and transitivity of $\cong$ either from $(\bullet\mathtt{I})$ or from (S1).)

When letters stand for numerical variables, $\mathtt{I}$ is read as 1, the product $A \bullet B$ is read as arithmetical product, $A \to B$ is read as $B^A$ and $\cong$ is replaced by $=$, all the equalities of **S** hold in the arithmetic of natural numbers. Two equalities are missing to get a complete axiomatization of the equalities in 1, product and exponentiation that hold in natural numbers:

(C1) $\quad C \to \mathtt{I} \cong \mathtt{I}$
(C2) $\quad C \to (A \bullet B) \cong (C \to A) \bullet (C \to B)$.

The system **S** plus (C1) and (C2), which we call **C**, characterizes objects isomorphic in the cartesian closed category freely generated by an infinite set of objects, as was demonstrated by Soloviev (1981).

The subsystem of **S** in the language without $\to$ that has only the axiom schemata $(\bullet\mathtt{I})$, $(\bullet\text{ass})$ and $(\bullet\text{com})$, and where the last equality postulate is omitted, axiomatizes commutative monoids. This system characterizes objects isomorphic in the symmetric monoidal category freely generated by an infinite set of objects, as well as objects isomorphic in the cartesian category freely generated by an infinite set of objects.

We shall need, in particular, the subsystem of **S** that has only the axiom schemata $(\bullet\text{ass})$ and $(\bullet\text{com})$ and the equality postulates. We call this system **AssCom**. That $A \cong B$ is provable in **AssCom** means that the formulae $A$ and $B$ are identical up to associativity and commutativity of the product.

## 5. The Main Completeness Theorem

Our ultimate goal in this paper is to prove the following completeness theorem for **S**.

**Main Completeness Theorem.** $A \cong B$ is provable in **S** iff $A$ and $B$ are isomorphic in **SyMonCl**.

If two objects are isomorphic in **SyMonCl**, then their images under a symmetric monoidal closed functor are isomorphic in every symmetric monoidal closed category. Such a functor always exists because of the freedom of **SyMonCl**. So, according to the Main Completeness Theorem, the system **S** characterizes the forms of objects isomorphic in every symmetric monoidal closed category. Since **SyMonCl** is freely generated from infinitely many propositional letters, there is no bound on the number of distinct objects partaking in these forms. (If **SyMonCl** were generated by a finite set of letters, there would be such a bound; in the extreme case, if **SyMonCl** were generated from an empty set of letters, we would only learn from it something about the isomorphisms involving the unit objects of symmetric monoidal closed categories.)

It is straightforward to check the soundness part of the Main Completeness Theorem, *i.e.*, the direction from left to right. The bulk of this paper is devoted to establishing the converse, *i.e.*, the completeness part.

## 6. Normal forms

For our proof of the Main Completeness Theorem we need a notion of normal form for formulae, which we proceed to define in this section.

Let us first define the following $\mathtt{I}$-*reductions* of formulae, which consist in replacing subformulae of the forms on the left-hand side by formulae on the right-hand side:

| $\mathtt{I}$-*redexes* | *contracta* |
|---|---|
| $\mathtt{I} \bullet A$ | $A$ |
| $A \bullet \mathtt{I}$ | $A$ |
| $\mathtt{I} \rightarrow A$ | $A$ |

The $\mathtt{I}$-reductions are strongly normalizing (*i.e.*, every sequence of $\mathtt{I}$-reductions is finite), because the number of occurrences of $\mathtt{I}$ diminishes. Moreover, it is easy to show that $\mathtt{I}$-reductions have the Church–Rosser property, so that all formulae have a unique $\mathtt{I}$-*normal form*, that is one without $\mathtt{I}$-redexes. In a formula in $\mathtt{I}$-normal form occurrences of $\mathtt{I}$ can only remain in subformulae of the form $A \rightarrow \mathtt{I}$.

Let us also define the $\rightarrow$-*reductions* of formulae, which consist of replacing subformulae of the forms on the left-hand side by formulae on the right-hand side:

| $\rightarrow$-*redex* | *contractum* |
|---|---|
| $B \rightarrow (A \rightarrow C)$ | $(A \bullet B) \rightarrow C$ |

The $\rightarrow$-reductions are also strongly normalizing, because the number of occurrences of $\rightarrow$ diminishes. However, $\rightarrow$-*normal forms*, that is those without $\rightarrow$-redexes, are not necessarily unique, as the following example shows:

$$\begin{tikzcd}
 & C \rightarrow (B \rightarrow (A \rightarrow D)) \arrow[dl] \arrow[dr] & \\
(B \bullet C) \rightarrow (A \rightarrow D) \arrow[d] & & C \rightarrow ((A \bullet B) \rightarrow D) \arrow[d] \\
(A \bullet (B \bullet C)) \rightarrow D & & ((A \bullet B) \bullet C) \rightarrow D
\end{tikzcd}$$

Though the $\rightarrow$-normal form of a formula is not unique, it is unique up to associativity of the product; *i.e.*, all $\rightarrow$-normal forms of a formula are in the relation $\cong$ of the system that has only the axiom schema $(\bullet \mathrm{ass})$ and the equality postulates.

We say that a formula is in *normal form*, tout court, iff it is in $\mathtt{I}$-normal form and $\rightarrow$-normal form. We can easily prove the following lemma.

**Normal Form Lemma.** For every formula $A$ there is a formula $A'$ in normal form such that $A \cong A'$ is provable in $\mathbf{S}$.

*Proof.* We can reduce every formula $A$ to a formula $A'$ in normal form by reducing $A$ first to the $\mathtt{I}$-normal form and then reducing this $\mathtt{I}$-normal form to an $\rightarrow$-normal form. Note that $\rightarrow$-reductions cannot produce new $\mathtt{I}$-redexes in formulae in $\mathtt{I}$-normal form. Of course, $A \cong A'$ is provable in **S**, because both $\mathtt{I}$-reductions and $\rightarrow$-reductions are covered by theorems of **S**. □

The proof of this lemma yields a decision procedure for **S**. To verify whether $A \cong B$ is provable in **S**, reduce $A$ and $B$ to normal forms $A'$ and $B'$. It remains to check whether $A' \cong B'$ is provable in **AssCom**, which is done by deleting parentheses around products and making all possible permutations of factors in these products – a finite task.

We have mentioned here the uniqueness of normal form up to associativity of the product and we have shown that **S** is decidable, but we need neither this uniqueness of normal form nor the decidability of **S** to prove the Main Completeness Theorem. For that the Normal Form Lemma is enough.

## 7. The category FinSet*

As the system **C**, that is, **S** plus (C1) and (C2), has an arithmetical interpretation for which it is complete, we may wonder whether **S** itself has such an arithmetical interpretation. We introduce such an interpretation in this section, and establish in subsequent sections that **S** is complete for this interpretation under the condition that we deal with diversified formulae. This will be the essential ingredient of our proof of the Main Completeness Theorem.

First we introduce a concrete symmetric monoidal closed category. The objects of the category **FinSet*** of *finite pointed sets* are finite nonempty sets each of which contains a distinguished element $*$. Arrows are functions $f$ from finite pointed sets to finite pointed sets that preserve $*$; that is, $f(*) = *$. For two finite pointed sets $X$ and $Y$, the product is the *smash product*

$$X \bullet Y =_{\text{def.}} \{*\} \cup \{\langle a, b\rangle : a \in X \,\&\, b \in Y \,\&\, a \neq * \,\&\, b \neq *\}$$

and $X \rightarrow Y$ is the set of all functions from $X$ to $Y$ that preserve $*$, the constant function with value $*$ in $Y$ being the element $*$ of $X \rightarrow Y$. In **FinSet*** the object $\mathtt{I}$ is a set $\{*, 0\}$ with two elements. The arrows $\mathbf{1}_X$ are identity functions, composition is composition of functions, and for the remaining functions of the symmetric monoidal closed setting we have

$$\mathbf{b}^{\rightarrow}(\langle a, \langle b, c\rangle\rangle) = \langle\langle a, b\rangle, c\rangle, \qquad \mathbf{b}^{\leftarrow}(\langle\langle a, b\rangle, c\rangle) = \langle a, \langle b, c\rangle\rangle$$

$$\mathbf{c}(\langle a, b\rangle) = \langle b, a\rangle$$

$$\boldsymbol{\sigma}(\langle 0, a\rangle) = a, \qquad \boldsymbol{\sigma}^i(a) = \langle 0, a\rangle, \text{ when } a \text{ is not } *$$

$$\boldsymbol{\delta}(\langle a, 0\rangle) = a, \qquad \boldsymbol{\delta}^i(a) = \langle a, 0\rangle, \text{ when } a \text{ is not } *$$

$$\boldsymbol{\varepsilon}(\langle a, f\rangle) = f(a), \qquad (\boldsymbol{\eta}(b))(a) = \langle a, b\rangle, \text{ when } a \text{ and } b \text{ are not } *$$

$$(f \bullet g)(\langle a, b\rangle) = \langle f(a), g(b)\rangle, \text{ when } f(a) \text{ and } g(b) \text{ are not } *$$

$$(f \rightarrow g)(h) = ghf.$$

It is straightforward to verify that **FinSet*** is a symmetric monoidal closed category (*cf.* Eilenberg and Kelly (1966a, Chapter IV.1, pp. 548–549)). Actually, we have in **FinSet*** projection arrows and a pairing operation on arrows with respect to the smash product, but this category fails to be cartesian with this product because projections are not natural transformations.

Two finite pointed sets $X$ and $Y$ are isomorphic iff they are of the same cardinality, that is, $card(X) = card(Y)$. Let us now see how the operation of smash product and the operation $\rightarrow$ on finite pointed sets behave with respect to cardinality. It is easy to calculate that

$$card(X \bullet Y) = (card(X) - 1)(card(Y) - 1) + 1$$
$$card(X \rightarrow Y) = card(Y)^{card(X)-1}.$$

Of course, $card(\mathtt{I}) = 2$. It is useful to have special symbols for the corresponding operations on $\mathbb{N}^+$, that is, the set of natural numbers strictly greater than 0. We shall write

$$m \odot n =_{\text{def.}} (m-1)(n-1) + 1$$
$$m^{\underline{n}} =_{\text{def.}} m^{n-1}.$$

These definitions can be explained as follows. Take any one–one map $f$ from the set of natural numbers $\mathbb{N}$ into $\mathbb{N}$. Then $f$ defines a one–one correspondence between $\mathbb{N}$ and $f(\mathbb{N})$, the image of $\mathbb{N}$ under $f$. Let $f^{-1}$ be the inverse of $f$ from $f(\mathbb{N})$ to $\mathbb{N}$. We can define on $f(\mathbb{N})$ an operation $\odot$ such that $f$ is an isomorphism from the commutative monoid $\mathbb{N}$ with ordinary multiplication and 1 onto $f(\mathbb{N})$ with $\odot$ and $f(1)$. Since we want $f(mn) = f(m) \odot f(n)$, by replacing $m$ by $f^{-1}(m)$ and $n$ by $f^{-1}(n)$ we obtain

$$m \odot n = f(f^{-1}(m)f^{-1}(n)).$$

By putting $f(k) = k + 1$, we obtain the definition of $\odot$ above, and $f(1) = 2$. For $m^{\underline{n}}$ we have

$$m^{\underline{n}} = m^{f^{-1}(n)}$$

which yields the equalities corresponding to (S1) and (S2):

$$m^{\underline{f(1)}} = m$$
$$m^{\underline{n \odot k}} = (m^{\underline{n}})^{\underline{k}}$$

but not the equalities corresponding to (C1) and (C2).

So, in the symmetric monoidal closed category **FinSet*** we find an arithmetic of positive natural numbers with the operations $\odot$ and $\underline{\ }$. In the following sections we make this arithmetic work for our proof of the Main Completeness Theorem.

The category **FinSet*** is isomorphic to the category whose objects are finite sets and whose arrows are partial maps. The set $X \bullet Y$ is the cartesian product of the sets $X$ and $Y$, while $X \rightarrow Y$ is the set of all *nonempty* partial maps from $X$ to $Y$; the set $\mathtt{I}$ is a singleton. The arrows and operations of the symmetric monoidal closed setting are defined analogously to what we had in **FinSet***. The cardinality of $X \bullet Y$ is the ordinary product of the cardinalities of $X$ and $Y$, while

$$card(X \rightarrow Y) = (card(Y) + 1)^{card(X)} - 1.$$

Let $^{=}$ be the corresponding operation on $\mathbb{N}$; that is, $m^{\overset{n}{=}}$ is short for $(m+1)^n - 1$. The structure $\langle \mathbb{N}^+, \odot, ^-, 2\rangle$ is isomorphic to $\langle \mathbb{N}, \cdot, ^=, 1\rangle$, where $\cdot$ is ordinary product, and instead of the first structure we might as well have used the latter structure for our proof of the Main Completeness Theorem in the following sections.

## 8. Some arithmetical inequalities

We shall need, in particular, the following inequality.

**Lemma 8.1.** If $m, n > 2$, then $m^{\underline{n}} > m \odot 2^{\underline{n}}$.

*Proof.* We have

$$m^{\underline{n}} - 1 = (m-1)(m^{n-2} + m^{n-3} + \ldots + m^1 + 1)$$
$$(m \odot 2^{\underline{n}}) - 1 = (m-1)(2^{n-2} + 2^{n-3} + \ldots + 2^1 + 1). \qquad \square$$

As a corollary we have another inequality.

**Lemma 8.2.** If $m, n > 2$, then $2^{\underline{m\odot n}} > 2^{\underline{m}} \odot 2^{\underline{n}}$.

*Proof.* Put $2^{\underline{m}}$ for $m$ in the inequality of Lemma 8.1. Then $2^{\underline{m\odot n}} = (2^{\underline{m}})^{\underline{n}} > 2^{\underline{m}} \odot 2^{\underline{n}}$. $\square$

Let us introduce the following notation by recursion:

$$ч_0(m) = m$$
$$ч_{k+1}(m) = 2^{\underline{ч_k(m)}}$$

(so the number $ч_k(m)$ is

$$2^{\underline{2}^{\cdot^{\cdot^{\cdot^{2^{\underline{m}}}}}}}$$

with 2 appearing $k$ times). It is easy to see that

$$ч_k(ч_h(m)) = ч_{k+h}(m)$$
$$ч_{k+1}(m)^{\underline{ч_h(n)}} = ч_{h+1}(n)^{\underline{ч_k(m)}}.$$

We can now establish the following generalization of Lemma 8.2.

**Lemma 8.3.** If $m, n > 2$, then for every $k \geqslant 1$ we have $ч_k(m \odot n) > ч_k(m) \odot ч_k(n)$.

*Proof.* We make an induction on $k$. For $k = 1$, we have Lemma 8.2. For $k > 1$, we have

$$\begin{aligned} ч_k(m\odot n) &= 2^{\underline{ч_{k-1}(m\odot n)}} \\ &> 2^{\underline{ч_{k-1}(m)\odot ч_{k-1}(n)}}, \text{by the induction hypothesis} \\ &> 2^{\underline{ч_{k-1}(m)}} \odot 2^{\underline{ч_{k-1}(n)}}, \text{by Lemma 8.2} \\ &= ч_k(m) \odot ч_k(n). \end{aligned} \qquad \square$$

## 9. The Arithmetical Completeness Theorem

Let a *valuation* of the set of formulae in $\langle \mathbb{N}^+, \odot, ^-, 2\rangle$ be a function $v$ that assigns to letters some arbitrary numbers from $\mathbb{N}^+$ and satisfies

$$v(\mathtt{I}) = 2$$

$$v(A \bullet B) = v(A) \odot v(B)$$
$$v(A \to B) = v(B)^{\underline{v(A)}}.$$

We shall prove in Section 11 the following completeness theorem for **S**, which we shall use in the same section to prove the Main Completeness Theorem.

**Arithmetical Completeness Theorem.** If

(D) $A$ and $B$ are diversified,

then $A \cong B$ is provable in **S** iff for every valuation $v$ we have $v(A) = v(B)$.

The soundness part of this theorem, *i.e.*, the direction from left to right of the 'iff' statement, is easily established by induction on the length of proof of $A \cong B$ in **S**. Actually, we do not need condition (D) to establish this part of the theorem. We use (D) for the converse, *i.e.*, the completeness part. We leave for another occasion the question of whether the Arithmetical Completeness Theorem also holds without condition (D), because we do not need to answer this question to prove the Main Completeness Theorem.

## 10. The Auxiliary Theorem

In this section we shall prove the following theorem, which in Section 11 we shall use for the proof of the Arithmetical Completeness Theorem.

**Auxiliary Theorem.** If

(D) $A$ and $B$ are diversified,

(N) $A$ and $B$ are in normal form,

(V) for every valuation $v$ we have $v(A) = v(B)$,

then $A \cong B$ is provable in **AssCom**.

First we introduce the following terminology. The *floor* of a subformula in a formula $A$ is the number of antecedents of implications of $A$ in which this subformula occurs. For example, in $(((p \to q) \to p) \bullet r) \to p$ the floor of the leftmost $p$ is 3 and the floor of the subformula $p \to q$ is 2; the floor of $p$ in $p$ is 0. (The floor of an even-floor subformula is even and the floor of an odd-floor subformula is odd; *cf.* Section 3.) In Soloviev (1981) the floor of a subformula is called its *implicational depth* (in Russian; we prefer climbing up because in the exponential notation the floor of an exponent is indeed the level above the ground line on which the exponent is written).

We denote the floor of a letter $p$ in a diversified formula $A$ by $p \uparrow A$. (This notation would be ambiguous if $A$ were not diversified.) If we take a valuation $v$ such that $v(p) = n$ for some $n > 2$ and $v(q) = 2$ for every letter $q$ different from $p$, then it is easy to calculate that

$$v(A) = \text{ч}_{p \uparrow A}(n).$$

We shall often use this fact in the following series of lemmata, where for the sake of definiteness we take that $n$ is 3. In these lemmata, (D), (N) and (V) will always be as in the Auxiliary Theorem; $let(A)$ is, as before, the set of letters occurring in $A$.

**Lemma 10.1.** If (V), then $let(A) = let(B)$.

*Proof.* Suppose $p \in let(A)$ and $p \notin let(B)$. Take a valuation $v$ such that $v(p) \neq 2$ and $v(q) = 2$ for every letter $q$ different from $p$. If $v(p) = 1$, then $v(A) = 1$, and if $v(p) > 2$, then $v(A) > 2$. Since $v(B) = 2$, we get a contradiction with (V). (Of course, we proceed analogously when $p \notin let(A)$ and $p \in let(B)$.) $\square$

**Lemma 10.2.** If (D) and (V), then for every $p$ in $let(A)$ we have $p \uparrow A = p \uparrow B$.

*Proof.* Suppose $p \in let(A)$. Then, by Lemma 10.1, $p \in let(B)$. Take the valuation $v$ such that $v(p) = 3$ and $v(q) = 2$ for every letter $q$ different from $p$. If $p \uparrow A \neq p \uparrow B$, then

$$v(A) = \text{ч}_{p\uparrow A}(3) \neq \text{ч}_{p\uparrow B}(3) = v(B)$$

which contradicts (V). $\square$

Let us call a formula *prime* iff it is either atomic or an implication. Every formula $A$ is of the form $A_1 \bullet \ldots \bullet A_n$, for some $n \geqslant 1$, where for every $i$ in $\{1, \ldots, n\}$ the subformula $A_i$ is prime and parentheses are distributed in some manner in the outermost products of $A_1 \bullet \ldots \bullet A_n$ (if there are such products). Let us call the schema $A_1 \bullet \ldots \bullet A_n$ the *prime representation* of $A$. We can then state and prove the following lemma.

**Lemma 10.3.** Suppose (D), (N) and (V), and let $A_1 \bullet \ldots \bullet A_n$, for some $n \geqslant 1$, be the prime representation of $A$, and $B_1 \bullet \ldots \bullet B_m$, for some $m \geqslant 1$, the prime representation of $B$. If for some $i$ and $j$ in $\{1, \ldots, n\}$ such that $i \neq j$ we have $p \in let(A_i)$ and $q \in let(A_j)$, then there is no $k$ in $\{1, \ldots, m\}$ such that $p \in let(B_k)$ and $q \in let(B_k)$.

*Proof.* Suppose for some $i$ and $j$ in $\{1, \ldots, n\}$ we have $i \neq j$. So $n \geqslant 2$. Suppose, moreover, $p \in let(A_i)$, $q \in let(A_j)$ and for some $k$ in $\{1, \ldots, m\}$, $p \in let(B_k)$ and $q \in let(B_k)$. Since $B_k$ is prime and has more than one letter ($p$ and $q$ are different because of (D)), it must be of the form $B_k' \rightarrow B_k''$. Now four cases are possible:

(Case 1) $p \in let(B_k')$ and $q \in let(B_k'')$,
(Case 2) $p \in let(B_k'')$ and $q \in let(B_k')$,
(Case 3) $p \in let(B_k')$ and $q \in let(B_k')$,
(Case 4) $p \in let(B_k'')$ and $q \in let(B_k'')$.

*Case 1.* Take the valuation $v$ such that $v(p) = v(q) = 3$ and $v(s) = 2$ for every letter $s$ different from $p$ and $q$. Then

$$v(A) = \text{ч}_{p\uparrow A}(3) \odot \text{ч}_{q\uparrow A}(3) = \text{ч}_{q\uparrow A}(3) \odot 2^{\underline{\text{ч}_{p\uparrow A-1}(3)}}$$

$$v(B) = \text{ч}_{q\uparrow B}(3)^{\underline{\text{ч}_{p\uparrow B-1}(3)}}.$$

By Lemma 10.2, $p \uparrow A = p \uparrow B \geqslant 1$ and $q \uparrow A = q \uparrow B$, while by Lemma 8.1, $v(B) > v(A)$, which contradicts (V).
For Case 2 we proceed as in Case 1.

*Case 3*. Take the valuation $v$ of Case 1. Then for some $h \geqslant 1$, we have one of

(3.1) $\quad v(B) = \text{ч}_h(\text{ч}_{q\uparrow B-h}(3)^{\text{ч}_{p\uparrow B-h-1}(3)})$, or

(3.2) $\quad v(B) = \text{ч}_h(\text{ч}_{p\uparrow B-h}(3)^{\text{ч}_{q\uparrow B-h-1}(3)})$, or

(3.3) $\quad v(B) = \text{ч}_h(\text{ч}_{p\uparrow B-h}(3) \odot \text{ч}_{q\uparrow B-h}(3))$,

while

$$v(A) = \text{ч}_h(\text{ч}_{p\uparrow A-h}(3)) \odot \text{ч}_h(\text{ч}_{q\uparrow A-h}(3)).$$

By Lemma 10.2, $p \uparrow A = p \uparrow B \geqslant 1$ and $q \uparrow A = q \uparrow B \geqslant 1$. If (3.1), then

$$\begin{aligned} v(B) &> \text{ч}_h(\text{ч}_{q\uparrow B-h}(3) \odot 2^{\text{ч}_{p\uparrow B-h-1}(3)}), \text{ by Lemma 8.1} \\ &> v(A), \text{ by Lemma 8.3.} \end{aligned}$$

We proceed analogously with (3.2), whereas with (3.3) we simply apply Lemma 8.3 to get $v(B) > v(A)$. So we have a contradiction with (V).

*Case 4*. Here, because of (N), there must be a letter $r$ in $let(B_k')$; otherwise, $B$ would not be in normal form. By Lemma 10.1, $r \in let(A)$. Now three subcases are possible:

(Subcase 4.1) $\quad r \in let(A_i)$,

(Subcase 4.2) $\quad r \in let(A_j)$,

(Subcase 4.3) $\quad r \in let(A_h)$, where $h \neq i$ and $h \neq j$.

In Subcase 4.1 we take the valuation $v$ such that $v(r) = v(q) = 3$ and $v(s) = 2$ for every letter $s$ different from $r$ and $q$, and we obtain a situation analogous to Case 1: the letter $r$ now plays the role of $p$.

In Subcase 4.2 we take the valuation $v$ such that $v(p) = v(r) = 3$ and $v(s) = 2$ for every letter $s$ different from $p$ and $r$, and we obtain a situation analogous to Case 2: the letter $r$ now plays the role of $q$.

In Subcase 4.3 we take either the valuation of Subcase 4.1 or the valuation of Subcase 4.2. $\square$

It is clear that we can also prove the version of Lemma 10.3 where its second sentence is replaced by: 'If for some $i$ and $j$ in $\{1, \ldots, m\}$ such that $i \neq j$ we have $p \in let(B_i)$ and $q \in let(B_j)$, then there is no $k$ in $\{1, \ldots, n\}$ such that $p \in let(A_k)$ and $q \in let(A_k)$.' As a corollary of Lemma 10.3 we have the following lemma.

**Lemma 10.4.** Suppose (D), (N) and (V), and let $A_1 \bullet \ldots \bullet A_n$, for some $n \geqslant 1$, be the prime representation of $A$, and $B_1 \bullet \ldots \bullet B_m$, for some $m \geqslant 1$, the prime representation of $B$. Then $n = m$.

*Proof.* Suppose $n > m$ (when $n < m$, we proceed analogously). So $n \geqslant 2$. Because of (N), for every $i$ in $\{1, \ldots, n\}$ there must be a letter $p_i$ in $let(A_i)$. By Lemma 10.1, for every $i$ in $\{1, \ldots, n\}$ we have $p_i \in let(B)$. By the Pigeonhole Principle, for some $i$ and $j$ in $\{1, \ldots, n\}$ such that $i \neq j$ there is a $k$ in $\{1, \ldots, m\}$ such that $p_i \in let(B_k)$ and $p_j \in let(B_k)$. But this contradicts Lemma 10.3. $\square$

We shall now start on the proof of the Auxiliary Theorem by induction on the number of binary connectives in $A$. The basis of this induction is covered by the following proposition.

**Basis.** If (D), (N) and (V), and $A$ is an atomic formula, then $A \cong B$ is provable in **AssCom**.

*Proof.* If $A$ is $\mathtt{I}$, then, because of Lemma 10.1 and (N), we must have that $B$ is $\mathtt{I}$, too. If $A$ is $p$, then, because of Lemmata 10.1 and 10.2, and (D) and (N), we must have that $B$ is $p$, too. □

For the next two propositions, which cover the induction step of our proof, we need the following *Induction Hypothesis*:

> The Auxiliary Theorem holds when $A$ is replaced by an $A'$ in which there are strictly less binary connectives than in $A$.

**Induction Step with →.** Suppose the Induction Hypothesis. If (D), (N) and (V), and $A$ is an implication, then $A \cong B$ is provable in **AssCom**.

*Proof.* If $A$ is an implication, then, because of Lemmata 10.1 and 10.2, and (D) and (N), we cannot have that $B$ is an atomic formula. Because of Lemma 10.4, the formula $B$ cannot be a product. So $B$ must be an implication, too. Let $A$ be $A_1 \to A_2$ and $B$ be $B_1 \to B_2$. We shall show that $let(A_1) = let(B_1)$.

Suppose for some $p$ we have $p \in let(A_1)$ and $p \notin let(B_1)$ (of course, we proceed analogously when $p \notin let(A_1)$ and $p \in let(B_1)$). By Lemma 10.1, $p \in let(B_2)$. Because of (N), there must be a letter $q$ in $let(B_1)$, and by Lemma 10.1, $q \in let(A)$. Then we have the following two cases:

(Case 1) $\quad q \in let(A_1)$,
(Case 2) $\quad q \in let(A_2)$.

*Case 1.* There must be an implication in $B_2$, because, by Lemma 10.2, $p \uparrow B \geqslant 1$. We cannot have that $B$ is $B_1 \to (B_2' \to B_2'')$, because then $B$ would not be in normal form. So $B$ must be $B_1 \to (B_2' \bullet B_2'')$ (and the implication is in $B_2'$ or $B_2''$). Suppose $p \in let(B_2')$ (when $p \in let(B_2'')$, we proceed analogously). Because of (N), there must be a letter $r$ in $let(B_2'')$. By Lemma 10.1, $r \in let(A)$. Then there are two subcases:

(Subcase 1.1) $\quad r \in let(A_1)$,
(Subcase 1.2) $\quad r \in let(A_2)$.

*Subcase 1.1.* Take the valuation $v$ such that $v(p) = v(r) = 3$ and $v(s) = 2$ for every letter $s$ different from $p$ and $r$. Then for some $h \geqslant 1$ either

(1.11) $\quad v(A) = \text{ч}_h(\text{ч}_{r\uparrow A-h}(3)^{\underline{\text{ч}}_{p\uparrow -h-1}\underline{(3)}})$, or
(1.12) $\quad v(A) = \text{ч}_h(\text{ч}_{p\uparrow A-h}(3)^{\underline{\text{ч}}_{r\uparrow A-h-1}\underline{(3)}})$, or
(1.13) $\quad v(A) = \text{ч}_h(\text{ч}_{p\uparrow A-h}(3) \cdot \underline{\text{ч}}_{r\uparrow A-h}(3))$,

while

$$v(B) = \text{ч}_{p\uparrow B}(3) \odot \text{ч}_{r\uparrow B}(3).$$

By Lemma 10.2, $p \uparrow A = p \uparrow B \geqslant 1$ and $r \uparrow A = r \uparrow B \geqslant 1$. If (1.11) or (1.12), we use Lemmata 8.1 and 8.3, and if (1.13), we use Lemma 8.3, to get $v(A) > v(B)$, which contradicts (V) (*cf.* Case 3 of the proof of Lemma 10.3).

*Subcase 1.2.* Take the valuation $v$ of Subcase 1.1. Then

$$v(A) = \text{ч}_{r\uparrow A}(3)^{\underline{\text{ч}}_{p\uparrow A-1}\underline{(3)}}.$$

By Lemma 10.2, $p \uparrow A = p \uparrow B \geqslant 1$ and $r \uparrow A = r \uparrow B$, while by Lemma 8.1, $v(A) > v(B)$, which contradicts (V) (*cf.* Case 1 of the proof of Lemma 10.3).

*Case 2.* There must be an implication in $A_2$, because, by Lemma 10.2, $q \uparrow A \geqslant 1$. We cannot have that $A$ is $A_1 \to (A_2' \to A_2'')$, because then $A$ would not be in normal form. So $A$ must be $A_1 \to (A_2' \bullet A_2'')$ (and the implication is in $A_2'$ or $A_2''$). Suppose $q \in let(A_2')$ (when $q \in let(A_2'')$, we proceed analogously). Because of (N), there must be a letter $r$ in $let(A_2'')$. By Lemma 10.1, $r \in let(B)$. Then there are two subcases:

(Subcase 2.1) $\quad r \in let(B_1)$,
(Subcase 2.2) $\quad r \in let(B_2)$.

We treat Subcase 2.1 as Subcase 1.1 and Subcase 2.2 as Subcase 1.2, replacing $p$ by $q$ and interchanging $A$ and $B$.
So $let(A_1) = let(B_1)$. From (D) and Lemma 10.1, it follows that $let(A_2) = let(B_2)$. Moreover, $let(A_1)$ and $let(A_2)$ are disjoint, and $let(B_1)$ and $let(B_2)$ are disjoint, too. From (V), it follows that for every valuation $v$ such that $v(p) = 2$ for every $p$ in $let(A_2)$

$$v(A_2)^{\underline{v(A_1)}} = v(B_2)^{\underline{v(B_1)}}$$

that is

$$2^{\underline{v(A_1)}} = 2^{\underline{v(B_1)}}$$

which implies $v(A_1) = v(B_1)$. Since $let(A_1)$ and $let(A_2)$ are disjoint, it follows that ($V_1$) for every valuation $v$ we have $v(A_1) = v(B_1)$. We also have that ($D_1$) $A_1$ and $B_1$ are diversified (a subformula of a diversified formula is diversified), and ($N_1$) $A_1$ and $B_1$ are in normal form (a subformula of a formula in normal form is in normal form). Since $A_1$ has strictly less binary connectives than $A$, we may apply the Induction Hypothesis to obtain that $A_1 \cong B_1$ is provable in **AssCom**. Quite analogously, we obtain that $A_2 \cong B_2$ is provable in **AssCom**. So $A_1 \to A_2 \cong B_1 \to B_2$ is provable in **AssCom**. □

**Induction Step with $\bullet$.** Suppose the Induction Hypothesis. If (D), (N) and (V), and $A$ is a product, then $A \cong B$ is provable in **AssCom**.

*Proof.* Let $A_1 \bullet \ldots \bullet A_n$, for some $n \geqslant 2$, be the prime representation of $A$ and $B_1 \bullet \ldots \bullet B_m$, for some $m \geqslant 1$, the prime representation of $B$. By Lemma 10.4, we have $n = m$.

First we show that for every $i$ in $\{1, \ldots, n\}$ there is a $k$ in $\{1, \ldots, n\}$ such that $let(A_i) = let(B_k)$. Take an $A_i$ and let $p \in let(A_i)$. By Lemma 10.1, there is a $B_k$ such that $p \in let(B_k)$. If $let(A_i) = let(B_k)$, we are done. If $let(A_i) \neq let(B_k)$, then there are two cases:

(Case 1) $\quad q \notin let(A_i)$ and $q \in let(B_k)$,
(Case 2) $\quad q \in let(A_i)$ and $q \notin let(B_k)$.

In Case 1, by Lemma 10.1, for some $j$ different from $i$ we have $q \in let(A_j)$. Since $p \in let(B_k)$ and $q \in let(B_k)$, we get a contradiction with Lemma 10.3.

In Case 2 we similarly get a contradiction with Lemma 10.3 (or, rather, with the version of this lemma mentioned immediately after the proof of Lemma 10.3).

We infer easily from (D), (N) and (V) that $A_i$ and $B_k$ are diversified, that they are in normal form, and that for every valuation $v$ we have $v(A_i) = v(B_k)$ (*cf.* the last paragraph of the proof of the Induction Step with $\to$). Since $A_i$ has strictly less binary connectives than $A$, we may apply the Induction Hypothesis to obtain that $A_i \cong B_k$ is provable in **AssCom**. We obtain similarly that

$$A_1 \bullet \ldots \bullet A_{i-1} \bullet A_{i+1} \bullet \ldots \bullet A_n \cong B_1 \bullet \ldots \bullet B_{k-1} \bullet B_{k+1} \bullet \ldots \bullet B_n$$

is provable in **AssCom**, and with the help of $(\bullet\text{ass})$ and $(\bullet\text{com})$ we may infer that $A \cong B$ is provable in **AssCom**. □

This concludes the proof of the Auxiliary Theorem.

## 11. Proof of the completeness of S

First we prove the Arithmetical Completeness Theorem. Suppose

(D) $A$ and $B$ are diversified,
(V) for every valuation $v$ we have $v(A) = v(B)$.

Then there are formulae $A'$ and $B'$ such that $A \cong A'$ and $B \cong B'$ are provable in **S**,

(D′) $A'$ and $B'$ are diversified,
(N′) $A'$ and $B'$ are in normal form,
(V′) for every valuation $v$ we have $v(A') = v(B')$.

This is an immediate consequence of the Normal Form Lemma of Section 6, of the fact that a normal form of a diversified formula is a diversified formula, and of the soundness of **S** with respect to our arithmetical models (if $A \cong A'$ and $B \cong B'$ are provable in **S**, then for every valuation $v$ we have $v(A) = v(A')$ and $v(B) = v(B')$).

Then, by the Auxiliary Theorem, we have that $A' \cong B'$ is provable in **AssCom**, and hence $A' \cong B'$ is provable in **S**. Combining this with $A \cong A'$ and $B \cong B'$, which are provable in **S**, we obtain that $A \cong B$ is provable in **S**, and we have established the Arithmetical Completeness Theorem.

Next we prove the Main Completeness Theorem. Suppose $A$ and $B$ are isomorphic in **SyMonCl**. By the Diversification Lemma of Section 3, there is an isomorphism $f' : A' \vdash B'$ of **SyMonCl** such that $A'$ and $B'$ are diversified, and $A$ and $B$ are instances of $A'$ and $B'$, respectively, with the same substitution. Since $f'$ is an isomorphism of **SyMonCl**, for every symmetric monoidal closed functor $F$ from **SyMonCl** into **FinSet*** we must have that $F(f')$ is an isomorphism in **FinSet***. This implies that $card(F(A')) = card(F(B'))$.

For every valuation $v$ of the set of formulae in $\langle \mathbb{N}^+, \odot, ^-, 2\rangle$ there is a symmetric monoidal closed functor $F$ from **SyMonCl** into **FinSet*** such that for every formula $C$, $card(F(C)) = v(C)$ (it is enough to ensure that for every letter $p$, $card(F(p)) = v(p)$, and this we can always do because of the freedom of **SyMonCl**). It follows that for every valuation $v$ we have $v(A') = v(B')$. By the Arithmetical Completeness Theorem, it follows that in **S** we can prove $A' \cong B'$, and hence also its instance $A \cong B$.

## 12. A short proof of a restricted completeness theorem for S

Our proof of the Main Completeness Theorem is not short. (If the proof of Soloviev (1993) were written in sufficient detail, we suppose it would be longer than ours, especially if Babaev (1981) were incorporated.) So it might be worth presenting as an appendix to our paper a shorter proof of a restricted, but still quite general, version of the theorem. However, this proof is not self-contained: it relies on the result of Soloviev (1981).

Let us call a formula *consequential* iff for each of its subformulae of the form $A \to B$, if some letter occurs in $A$, then some letter occurs in $B$. The restricted version of the

Main Completeness Theorem excludes formulae that are not consequential – the same sort of troublesome formulae that were excluded in Kelly and Mac Lane (1971) for proving coherence in symmetric monoidal closed categories. So one may surmise that the difficulties encountered in proving the unrestricted Main Completeness Theorem are due to the presence of these formulae.

Consider the subsystem of the system **S** of Section 4 obtained by rejecting the axiom schema

(S2) $(A \bullet B) \rightarrow C \cong B \rightarrow (A \rightarrow C)$.

That $A \cong B$ is provable in **S** minus (S2) means that the formulae $A$ and $B$ are identical up to associativity and commutativity of the product and the deleting of $\mathtt{I}$ in products and antecedents.

Let $[A]$ be the set of all formulae $B$ such that $A \cong B$ is provable in **S** minus (S2). We call the equivalence classes $[A]$ *formula-terms*, and we define on them the following operations:

$$[A_1] \circ \ldots \circ [A_n] =_{\text{def.}} [(\ldots(A_1 \bullet A_2) \bullet \ldots \bullet A_n)], \text{ for } n \geqslant 2$$

$$[A] \Rightarrow [B] =_{\text{def.}} [A \rightarrow B].$$

We use the schematic letters $a, b, c, \ldots$, possibly with indices, for formula-terms.

Next we define the following reductions of formula-terms, which consist of replacing subterms of the forms on the left-hand side by formula-terms on the right-hand side:

| S2-*redex* | *contractum* |
|---|---|
| $b \Rightarrow (a \Rightarrow c)$ | $(a \circ b) \Rightarrow c$ |
| C2-*redex* | *contractum* |
| $b \Rightarrow (((d_1 \circ a) \Rightarrow c_1) \circ \ldots \circ ((d_n \circ a) \Rightarrow c_n))$ | $(a \circ b) \Rightarrow ((d_1 \Rightarrow c_1) \circ \ldots \circ (d_n \Rightarrow c_n))$ |

To exclude trivial reductions, we take S2-redexes with the proviso that $a \neq [\mathtt{I}]$ and $b \neq [\mathtt{I}]$, and C2-redexes with the proviso that $a \neq [\mathtt{I}]$.

In Section 4, we called **C** the system **S** extended with the axiom schemata

(C1) $C \rightarrow \mathtt{I} \cong \mathtt{I}$

(C2) $C \rightarrow (A \bullet B) \cong (C \rightarrow A) \bullet (C \rightarrow B)$.

The S2-*reductions*, that is, those involving S2-redexes only, are justified by (S2), and hence by the system **S**, while, in the presence of (S2), the C2-*reductions*, that is, those involving C2-redexes only, are justified by (C2). We call the reductions involving both S2-redexes and C2-redexes C-*reductions*, because they are justified by theorems of the system **C**.

Note that if we interpret $a_1 \circ \ldots \circ a_n$ as $a_1$ for $n = 1$, we obtain S2-reductions as a particular case of C2-reductions for $n = 1$ and $d_1 = [\mathtt{I}]$. In fact, **S** allows a somewhat more general kind of reduction than S2-reductions, which is obtained from C2-reductions by putting $n = 1$ and requiring that $b \neq [\mathtt{I}]$ (for $n = 1$ and $b = [\mathtt{I}]$, C2-reductions would be exactly converse to S2-reductions). If we interpret $a_1 \circ \ldots \circ a_n$ as $[\mathtt{I}]$ for $n = 0$, we obtain reductions justified by (C1) as a particular case of C2-reductions for $n = 0$ (with $b = [\mathtt{I}]$, this corresponds exactly to (C1); with $b \neq [\mathtt{I}]$ we also use (S2)). So the equality of the C2-redex with its contractum sums up all of (S2), (C1) and (C2) for $n \geqslant 0$. However, we use here the notation $a_1 \circ \ldots \circ a_n$ only for $n \geqslant 2$, as it was officially introduced.

A formula-term in which there are no S2-redexes is in S-*normal form*, while a formula-term in which there are neither S2-redexes nor C2-redexes is in C-*normal form*. We can establish that both S2-reductions and C-reductions are strongly normalizing and that they have the Church–Rosser property (in connection with the S-normal form, *cf.* the $\rightarrow$-normal form of Section 6). So S-normal forms and C-normal forms of formula-terms are unique.

We can then prove three lemmata, which yield the Restricted Completeness Theorem below.

**Lemma 12.1.** Suppose $A$ and $B$ are consequential. If $A \cong B$ is provable in **C**, then $[A]$ and $[B]$ have the same C-normal form.

*Proof.* Let $A \cong B$ be provable in **C**. Since $A$ and $B$ are consequential, we can prove $A \cong B$ in **C** without using the axiom schema (C1). (To show this, replace by $\mathtt{I}$ every subformula of the form $C \rightarrow D$ where some letter occurs in $C$ and no letter occurs in $D$, in the proof of $A \cong B$ in **C**, and iterate the replacement until no such subformulae are left; the result can be transformed into a proof of $A \cong B$ in **C** without (C1).) Then we can pass from $[A]$ to $[B]$ by a chain of C-reductions or replacements converse to C-reductions. Because of the Church–Rosser property of C-reductions, this yields that $[A]$ and $[B]$ have the same C-normal form. □

**Lemma 12.2.** Suppose $A$ and $B$ are consequential and diversified. If $A \cong B$ is provable in **C**, and $[A]$ and $[B]$ are in S-normal form, then $[A]$ is equal to $[B]$.

*Proof.* Let $A \cong B$ be provable in **C** and let $[A]$ and $[B]$ be in S-normal form. By Lemma 12.1, $[A]$ and $[B]$ have the same C-normal form. But no C-reduction can apply to these formula-terms: S2-reductions are excluded because $[A]$ and $[B]$ are in S-normal form, while C2-reductions are excluded because $A$ and $B$ are diversified. So $[A]$ and $[B]$ are in C-normal form, and hence they are equal. □

**Lemma 12.3.** Suppose $A$ and $B$ are consequential and diversified. If $A \cong B$ is provable in **C**, then $A \cong B$ is provable in **S**.

*Proof.* Let $A \cong B$ be provable in **C**. Then for some $A'$ and $B'$, consequential and diversified, we have that $A \cong A'$ and $B \cong B'$ are provable in **S**, $A' \cong B'$ is provable in **C**, and $[A']$ and $[B']$ are in S-normal form. By Lemma 12.2, $[A']$ is equal to $[B']$, so $A' \cong B'$ is provable in **S**. It follows that $A \cong B$ is provable in **S**. □

**Restricted Completeness Theorem.** If $A$ and $B$ are consequential, then $A \cong B$ is provable in **S** iff $A$ and $B$ are isomorphic in **SyMonCl**.

*Proof.* We need to establish only the completeness part of the 'iff' statement. So suppose that $A$ and $B$ are consequential and that they are isomorphic in **SyMonCl**. By the Diversification Lemma of Section 3, there is an isomorphism $f' : A' \vdash B'$ of **SyMonCl** such that $A'$ and $B'$ are diversified, and $A$ and $B$ are instances of $A'$ and $B'$, respectively, with the same substitution. Since this substitution is letter-for-letter, $A'$ and $B'$ are consequential, too. It is clear that $A'$ and $B'$ are isomorphic in the cartesian closed category freely generated by the same letters that have generated **SyMonCl**, and so, by the completeness result of Soloviev (1981), $A' \cong B'$ is provable in **C**. But then, by Lemma 12.3, $A' \cong B'$ is provable in **S**, and hence its instance $A \cong B$ is provable in **S**. □

Lemma 12.3 is of an independent interest – besides serving for the proof of the last theorem. It says that for consequential and diversified formulae the systems **S** and **C** coincide. Since **C** axiomatizes all the equalities that hold between arithmetical terms in the language with 1, product and exponentiation, this gives another restricted arithmetical completeness for the system **S**.

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